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WEYL GROUP COVERS FOR BRIESKORN'S RESOLUTIONS IN ALL CHARACTERISTICS AND THE INTEGRAL COHOMOLOGY OF G/P

N. I. SHEPHERD-BARRON

Abstract

Given an affine surface X with rational singularities and minimal resolution X' , the covering of the Artin component of the deformation space of X where simultaneous resolutions are achieved is Galois and the Galois group is the Weyl group W associated to the configuration of (-2) -curves on X' . This gives the existence of actions of W on polynomial rings over \mathbb{Z} where the ring of invariants is also polynomial. In turn this leads to a description of the integral cohomology rings of flag varieties of type ADE that extends the known description of the rational cohomology rings as rings of co-invariants for actions of W .

1 Introduction

Artin proved [Ar74] that, over a field k of any characteristic, for a given affine surface X_s over k with only rational singularities there is a unique irreducible component A (“the Artin component”) of the deformation space of X_s that contains all those deformations that can be simultaneously resolved after some finite covering of the base. This extended earlier constructions by Brieskorn that concerned rational double points (these are also called du Val singularities, Kleinian singularities, simple singularities, ...) in characteristic zero. Burns and Rapoport conjectured ([BR75], Conjecture 7.4) that this covering of A is Galois and that its Galois group is the Weyl group W corresponding to the configuration of (-2) -curves in the minimal resolution X'_s of X_s .

The main results of this paper are these.

Theorem 1.1 (= Theorem 2.11 1) *For any field k the conjecture of Burns and Rapoport is true over k and over any coefficient ring Λ for k .*

When restricted to the context of certain RDPs defined over \mathbb{Z} Theorem 1.1 can be stated and proved over \mathbb{Z} and then we learn something about the invariant theory of Weyl groups over \mathbb{Z} and about the integral cohomology rings $H^*(G/P, \mathbb{Z})$, where G/P is a partial flag variety, as follows.

Recall that if L is a weight lattice then W acts on the polynomial \mathbb{Z} -algebra $\mathbb{Z}[L]$ and, over a ring R in which a certain integer m is invertible (for example, $m = 30$ in the case of E_8), the corresponding ring of invariants $R[L]^W$ is a polynomial R -algebra [Dm73]. This is false for E_8 over \mathbb{Z} or over a field of

characteristic 2, 3 or 5. However, we prove that, for the corresponding root lattice M rather than for the weight lattice, it is stably true in the ADE case, in that it becomes true after “polynomial enlargement”.

Theorem 1.2 (= Theorem 4.9) *There is an effective graded action of W on a graded polynomial $\mathbb{Z}[M]$ -algebra \mathcal{O} such that the ring \mathcal{O}^W of invariants is a polynomial \mathbb{Z} -algebra.*

However, we do not know how to write down the W -action. That is, we do not have formulae for the action of any of the reflexions in W , not even the simple ones. Nor do we have formulae that describe generators of \mathcal{O}^W in terms of the given generators of \mathcal{O} .

Let G denote a simple algebraic group of type ADE . Using the results and techniques due to Bernstein, Gel’fand and Gel’fand [BGG73] and Demazure [Dm73] we are lead to a description of the integral cohomology ring $H^*(G/B, \mathbb{Z})$ (which ring equals the Chow ring, up to a doubling of degrees, since the flag variety G/B is paved by affine spaces) that is an integral version of Borel’s description of $H^*(G/B, \mathbb{Q})$ as the ring of co-invariants associated to the action of the Weyl group on the \mathbb{Q} -polynomial algebra $\mathbb{Q}[M]$.

Corollary 1.3 (= Corollary 5.10) *$H^*(G/B, \mathbb{Z})$ is isomorphic, as a graded ring with an action of W , to the ring $\mathcal{O}/(\mathcal{O}_+^W \cdot \mathcal{O})$ of co-invariants.*

Via the results of [BGG73] this leads to a similar description of $H^*(G/P, \mathbb{Z})$ where P is a parabolic subgroup G that corresponds to a given set Θ of simple roots. Let W_Θ denote the corresponding subgroup of W .

Corollary 1.4 (= Corollary 6.3) *\mathcal{O}^{W_Θ} is also a polynomial ring over \mathbb{Z} and $H^*(G/P, \mathbb{Z})$ is isomorphic to the quotient ring $\mathcal{O}^{W_\Theta}/(\mathcal{O}_+^W \cdot \mathcal{O}^{W_\Theta})$.*

These results complement earlier explicit descriptions of $H^*(G/B, \mathbb{Z})$ in terms of generators and relations. See, for example, [TW74] for $G = D_n$ and [Na10] and [DZ15] for $G = E_8$.

We also explain, in Theorem 7.1, some consequences for the local moduli of Enriques surfaces in characteristic two.

We now give a more detailed sketch of the background. The conjecture of [BR75] was proved by Wahl [Wa79] in characteristic zero; his proof depends upon the fact that this conjecture had already been proved, if the characteristic is either zero or bigger than the relevant Coxeter number, when X_s is an RDP in earlier work by many people (Brieskorn [Br70], Tyurina [Ty70], Slodowy [Sl80]).

They showed that one way of achieving simultaneous resolution for du Val singularities is to embed the picture into the corresponding simply connected simple algebraic group. This approach also gives a complete description of the finite covering that is required, in terms of the associated Weyl group and its monodromy action. Much later this was extended to good characteristics [SB01]. All of this depends upon knowing that the formal, or étale, equivalence class of the singularity is determined by the combinatorial structure of the exceptional locus; that is, the singularities are *taut*. However, in bad characteristics they are

not taut (although Artin, following work of Lipman [Li69], showed [Ar77], by giving a complete and explicit list of equations, that there are only finitely many isomorphism classes of each combinatorial type).

For types A and D Tyurina also showed that simultaneous resolution could be achieved without introducing algebraic groups into the picture by manipulating explicit invariant polynomials under the Weyl group and then making a suitable blow-up; this relied on knowing the equation defining the singularity. For type E this approach has not been carried out; instead, Brieskorn and Tyurina independently used the idea of embedding the singularity into a del Pezzo surface. All of this too depends upon tautness. Each approach reveals that the necessary finite cover can be taken to have the corresponding Weyl group as Galois group.

Artin's approach requires neither that the singularities be taut, nor that they be realized on the unipotent variety of a simply connected simple algebraic group, nor that they be embeddable in a del Pezzo surface. In fact, there are du Val singularities of type E_8 in characteristic 2 which have none of these properties. This is because there are five types of E_8 singularity, while only three exist on a del Pezzo surface and localizing the unipotent variety of the simple algebraic group E_8 at the generic point of the subregular locus gives, of course, a unique singularity. (In fact, although this plays no rôle here, in any characteristic the unipotent singularity of E_8 lies on a del Pezzo surface D of degree 1 where the j -invariant of the anti-canonical curves on D is non-constant, and this property, subject to the presence of an E_8 singularity on D , specifies D uniquely. In turn, this pins down the singularity of the unipotent variety: in the notation of [Ar77] it is E_8^r where $r = 4, 2, 1$ when the characteristic is 2, 3, 5. For the details, see [GS].)

Rather, Artin's approach only depends upon the singularity being rational. However, according to Artin, "This more precise result [concerning the Weyl group] does not follow directly from our method". The point of this paper is that in fact Artin's methods do give this result. The idea is merely to embellish the definition of Artin's functor Res so as to include a suitable marking and polarization.

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2 Simultaneous resolution

Fix a field k and a surface X_s/k with rational singularities. Fix also a Noetherian ring Λ with a surjection $\Lambda \rightarrow k$; assume that there is a prime ℓ that is invertible in Λ .

Let $X'_s \rightarrow X_s$ denote the minimal resolution, F the exceptional locus in X'_s and $E \subseteq F$ the maximal sublocus of F on which $K_{X'_s}$ is trivial. Note that, since X_s has rational singularities, F has normal crossings. We shall assume that E

consists of copies of \mathbb{P}_k^1 , so that E consists of the (-2) -curves E_1, \dots, E_l in X'_s . Of course, this assumption is vacuous if k is algebraically closed.

Define the *simple roots* to be the classes of the (-2) -curves E_1, \dots, E_l and set $Q = \bigoplus \mathbb{Z}E_i$, the *root lattice*. Put $P = \text{Hom}(Q, \mathbb{Z})$, the *weight lattice*. Then there is a natural embedding $\iota : Q \hookrightarrow \text{Pic}_{X'_s}$ which extends uniquely to an embedding $\iota : P \hookrightarrow \text{Pic}_{X'_s}$ such that the intersection pairing identifies Q with a sublattice of P . There is a dual basis $\{\varpi_i\}$ of P with $\deg \iota(\varpi_i)|_{E_j} = (\varpi_i, E_j) = \delta_{ij}$; the ϖ_i are the *fundamental dominant weights*.

If Σ is the set of simple roots then the reflexions in the elements of Σ give a Coxeter system (W, Σ) that acts on $P_{\mathbb{R}}$ and tessellates $P_{\mathbb{R}}$ into chambers which are permuted simply transitively by W .

If \mathcal{D} is any such chamber, Γ is the subgroup of the orthogonal group $O_{P_{\mathbb{R}}}$ that preserves P and Q and $\text{Stab}_{\mathcal{D}}$ is the subgroup of Γ that preserves \mathcal{D} , then Γ is a semi-direct product $\Gamma = W \rtimes \text{Stab}_{\mathcal{D}}$.

Now suppose that S is a scheme of finite type over Λ , that s is a closed point of S with $k(s) = k$, that $f : X \rightarrow S$ is flat, that X_s is the fibre over s and that the relative singular locus is finite over S .

Fix a chamber \mathcal{D} as above.

We shall consider Artin's functor $\text{Res}_{X/S}$ and two stacks Res_P and $\text{Res}_{P, \mathcal{D}}$. Here are their definitions.

- (1) A T -point of $\text{Res}_{X/S}$ is an isomorphism class of minimal resolutions

$$\begin{array}{ccccc} \widetilde{X}_T & \xrightarrow{\pi} & X_T & \longrightarrow & X \\ & \searrow \Phi & \downarrow \alpha & & \downarrow \\ & & T & \longrightarrow & S \end{array}$$

in particular, Φ is smooth, π is proper and is birational, in the sense that $\pi_* \mathcal{O}_{\widetilde{X}_T} = \mathcal{O}_{X_T}$, and π induces a minimal resolution of each geometric fibre of α .

Given a T -point of $\text{Res}_{X/S}$, define $\text{NS}_{\ell}(\widetilde{X}_T/T)$ to be the image of $\text{Pic}(\widetilde{X}_T)$ in $R^2\Phi_*\mathbb{Z}_{\ell}(1)$. This is a finitely generated \mathbb{Z} -module.

- (2) An object of $\text{Res}_P(T)$ consists of a T -point of $\text{Res}_{X/S}$ and a homomorphism $\tilde{\psi} : P \rightarrow \text{Pic}(\widetilde{X}_T)/\text{Pic}(T)$ such that the composite $\psi : P \rightarrow \text{NS}(\widetilde{X}_T/T)$ satisfies the following three conditions:

- (i) $\psi(Q)$ lies in the image of $R^2\Phi_c\mathbb{Z}_{\ell}(1)$ in $R^2\Phi\mathbb{Z}_{\ell}(1)$;
- (ii) $\psi(Q)$ is orthogonal to the relative canonical class $K_{\widetilde{X}_T/T}$;
- (iii) the composed pairing

$$P \times Q \rightarrow R^2\Phi_*\mathbb{Z}_{\ell}(1) \times R^2\Phi_c\mathbb{Z}_{\ell}(1) \xrightarrow{\alpha} \mathbb{Z}_{\ell}$$

factors through $P \times Q \xrightarrow{\beta} \mathbb{Z} \xrightarrow{\gamma} \mathbb{Z}_\ell$, where α is the cup product, β is the given pairing between P and Q , and γ is the natural inclusion.

- (3) Let $\text{Res}_{P,\mathcal{D}}$ denote the stack obtained from Res_P by adding a fourth condition to the objects of $\text{Res}_{P,\mathcal{D}}(T)$:
- (iv) the cone $\langle \mathcal{D}, K \rangle$ spanned by $\psi_{\mathbb{R}}(\mathcal{D})$ and the canonical class $K = K_{\widetilde{X}_T/T}$ lies in the nef cone $\text{NE}(\widetilde{X}_T/X_T)$ of π .

The morphisms between the objects in each of these stacks are the obvious ones, and it is routine to verify that appropriate morphisms can be composed.

An object of $\text{Res}_{P,\mathcal{D}}(T)$ is a family of P -marked, $\langle \mathcal{D}, K \rangle$ -polarized surfaces and an object of $\text{Res}_P(T)$ is a family of P -marked surfaces.

For each \mathcal{D} there is a forgetful morphism $j_{\mathcal{D}} : \text{Res}_{P,\mathcal{D}} \rightarrow \text{Res}_P$, which is an open immersion. There are also morphisms $q : \text{Res}_P \rightarrow \text{Res}_{X/S}$ and $r_{\mathcal{D}} = q \circ j_{\mathcal{D}} : \text{Res}_{P,\mathcal{D}} \rightarrow \text{Res}_{X/S}$.

According to [Ar74] the functor $\text{Res}_{X/S}$ is represented by a locally quasi-separated algebraic space R over S such that, for every field K , $R \times \mathbf{Spec} K \rightarrow S \times \mathbf{Spec} K$ is an isomorphism.

Let s also denote the unique point of R over s .

Suppose that $S^0 \subset S$ is the complement of the discriminant locus δ_S and R^0 its inverse image in R : then Artin proves also that $R^0 \rightarrow S^0$ is an isomorphism.

Lemma 2.1 *Assume that $H^2(X'_s, \mathcal{O}_{X'_s}) = 0$. Then the forgetful morphism $q : \text{Res}_P \rightarrow \text{Res}_{X/S}$ is a torsor under Γ over some Zariski neighbourhood U of s in S .*

PROOF: The action of Γ on Res_P is given by the left action of Γ on the set of homomorphisms $\phi : P \rightarrow \text{NS}$ that satisfy conditions (i) and (ii): $\gamma(\phi) = \phi \circ \gamma^{-1}$. This makes it clear that q is a pseudo-torsor under Γ .

The fibre of q over $s \in S$ is non-empty so it is enough to prove that q is dominant. For this it is enough to show that, given a henselian local S -scheme (T, s) , every T -point of $\text{Res}_{X/S}$ lies in the image of q .

Suppose given a minimal resolution $\pi : \widetilde{X}_T \rightarrow X_T$. The resolution $\widetilde{X}_s \rightarrow X_s$ is minimal, so there is a unique homomorphism $\widetilde{\phi}_0 : P \rightarrow \text{Pic}_{\widetilde{X}_s}$ such that $\widetilde{\phi}(E_i) = E_i$. (Recall that Q is regarded as a subgroup of P .) Since T is henselian, the obstruction to extending $\widetilde{\phi}_0(\varpi_i)$ to a class in $\text{NS}(\widetilde{X}_T/T)$ lies (after passing to formal completions and then algebraizing) in $H^2(X'_s, \mathcal{O}_{X'_s})$, which vanishes. Therefore $\widetilde{\phi}_0$ extends to $\widetilde{\phi} : P \rightarrow \text{Pic}_{\widetilde{X}_T}$. \square

Corollary 2.2 *The definition of Res_P is independent of ℓ .*

PROOF: Suppose that ℓ' is another prime. Let $\text{Res}_{P,\ell}$ and $\text{Res}_{P,\ell'}$ be the corresponding functors. Then define a third functor Res'_P as follows: an object of

Res'_P consists of a pair $(\widetilde{X}_T, \widetilde{\phi})$ as in the definition of Res_P but demand that the induced homomorphism

$$\phi' : P \rightarrow \text{NS}_\ell(\widetilde{X}_T/T) \oplus \text{NS}_{\ell'}(\widetilde{X}_T/T)$$

satisfies

- (1) $\phi'(Q)$ lies in the image of $R^2\Phi_c\mathbb{Z}_\ell(1) \oplus R^2\Phi_c\mathbb{Z}_{\ell'}(1)$,
- (2) $\phi(Q)$ is orthogonal to $K_{\widetilde{X}_T/T}$ and
- (3) the composed pairing

$$\begin{aligned} P \times Q &\rightarrow (R^2\Phi_*\mathbb{Z}_\ell(1) \oplus R^2\Phi_*\mathbb{Z}_{\ell'}(1)) \times (R^2\Phi_*\mathbb{Z}_\ell(1) \oplus R^2\Phi_*\mathbb{Z}_{\ell'}(1)) \\ &\rightarrow \mathbb{Z}_\ell \oplus \mathbb{Z}_{\ell'} \end{aligned}$$

factors through $\mathbb{Z} \rightarrow \mathbb{Z}_\ell \oplus \mathbb{Z}_{\ell'}$.

There are obvious forgetful morphisms $\alpha : \text{Res}'_P \rightarrow \text{Res}_{P,\ell}$ and $\beta : \text{Res}'_P \rightarrow \text{Res}_{P,\ell'}$; by Lemma 2.1 each of these morphisms is a morphism of Γ -torsors over $\text{Res}_{X/S}$ and so is an isomorphism. \square

For the rest of this section assume that the groups $H^2(X'_s, \mathcal{O}_{X'_s})$ and $H^2(X'_s, T_{X'_s}(-\log F))$ both vanish. Of course, these assumptions hold if X_s is affine, or a partial resolution of an affine surface.

There is a unique maximal neighbourhood U given by Lemma 2.1. From now on we replace S by U , so that q is a Γ -torsor over S .

Lemma 2.3 *The stacks Res_P and $\text{Res}_{P,\mathcal{D}}$ are representable by locally quasi-separated algebraic spaces R_P and $R_{P,\mathcal{D}}$ over S . These spaces are étale over R .*

PROOF: By Lemma 2.1, the forgetful morphisms $\text{Res}_P \xrightarrow{q} \text{Res}_{X/S}$ and $\text{Res}_{P,\mathcal{D}} \xrightarrow{r_{\mathcal{D}}} \text{Res}_{X/S}$ are étale. Since the functor $\text{Res}_{X/S}$ is represented by R , the representability of the stacks Res_P and $\text{Res}_{P,\mathcal{D}}$ follows. \square

These spaces fit into a diagram

$$\begin{array}{ccc} R_{P,\mathcal{D}} & \xrightarrow{j_{\mathcal{D}}} & R_P \\ & \searrow r_{\mathcal{D}} & \downarrow q \\ & & R \longrightarrow S \end{array}$$

where the circle in the arrow indicates that it is an open embedding.

Lemma 2.4 (1) $R_{P,\mathcal{D}} \rightarrow S$ is separated and quasi-finite.

(2) $R_{P,\mathcal{D}}$ is a scheme.

$$(3) R_P = \cup_{\mathcal{D}} R_{P,\mathcal{D}}.$$

PROOF: In (1) the separatedness is a consequence of the fact that a cone $\langle \mathcal{D}, K \rangle$ of polarizations is specified in the data that define a point of $\text{Res}_{P,\mathcal{D}}$. The quasi-finiteness is immediate, as is (2).

(3) follows from the fact that any resolution $\widetilde{X}_T \rightarrow X_T$ is, locally in the étale topology on T , a projective morphism. \square

For any $T \rightarrow S$, let $T(s)$ denote the closed fibre over s and let S^h denote the henselization of S at s .

Lemma 2.5 *Suppose that Y is a quasi-finite separated S^h -scheme of finite type. Then*

- (1) *there is a unique minimal open subscheme Y^1 of Y that contains $Y(s)$;*
- (2) *Y^1 is henselian, semi-local and finite over S^h ;*
- (3) *Y^1 is also the maximal open subscheme of Y that is finite over S^h ;*
- (4) *Y^1 is the union of those connected components of Y that meet $Y(s)$.*

PROOF: By Grothendieck's version of Zariski's Main Theorem (EGA IV₄, 18.12.13) \blacksquare Y can be identified with an open subscheme of a finite S^h -scheme T . Then $T = \sqcup T_i$, a finite disjoint union of finite henselian local S -schemes T_i .

Suppose that T_i meets $Y(s)$ non-trivially. Then $T_i \cap Y$ is open in T_i , $(T_i \cap Y)(s) = T_i(s) \cap Y(s) \neq \emptyset$ and $T_i(s)$ is connected. Therefore $(T_i \cap Y)(s) = T_i(s)$.

It follows that $T_i \cap Y = T_i$, so that $T_i \subset Y$. Then take Y^1 to be the union of these T_i ; that is, Y^1 is the union of the connected components T_i of T that meet $Y(s)$ non-trivially.

It is clear that Y^1 is finite over S^h and open in Y , and that $Y^1(s) = Y(s)$. So Y^1 satisfies (2) and (4).

Suppose W is open in Y^1 and $W(s)$ contains $Y(s)$. Then $W(s) = Y^1(s)$, so that $(Y^1 - W)(s)$ is empty. Moreover, $Y^1 - W$ is closed in Y^1 , so finite over S^h . Then $Y^1 - W$ is empty, so that $W = Y^1$. Therefore Y^1 satisfies (1).

Finally, suppose that Y^1 is an open subscheme of V and that V is open in Y and finite over S^h . Then $V - Y^1$ is closed in V , so finite over S^h . However, $(V - Y^1)(s)$ is empty, so $V - Y^1$ is empty, so that Y^1 satisfies (3). \square

Writing $(R_{P,\mathcal{D}} \times_S S^h)^1$ as the disjoint union of its connected components gives a decomposition

$$(R_{P,\mathcal{D}} \times_S S^h)^1 = \sqcup_{g \in \text{Stab}_{\mathcal{D}}} R_{P,\mathcal{D},g}^0,$$

where each $R_{P,\mathcal{D},g}^0$ is a local henselian scheme that is finite over S^h and $\text{Stab}_{\mathcal{D}}$ acts freely on $(R_{P,\mathcal{D}} \times_S S^h)^1$ by permuting the various connected components $R_{P,\mathcal{D},g}^0$.

Lemma 2.6 $R_P \times_S S^h = \cup_{\mathcal{D}} (R_{P,\mathcal{D}} \times_S S^h)^1$.

PROOF: $R_P \times_S S^h = \cup_{\mathcal{D}} (R_{P,\mathcal{D}} \times_S S^h)$, so that $\cup_{\mathcal{D}} (R_{P,\mathcal{D}} \times_S S^h)^1$ is the unique minimal open subscheme of the non-separated scheme $R_P \times_S S^h$ that contains $R_P(s)$, which is a Γ -torsor over a point. Since $R_P \times_S S^h \rightarrow R \times_S S^h$ is a Γ -torsor the result follows. \square

Lemma 2.7 (1) If $T \rightarrow S^h$ is finite then every point $T \rightarrow R$ factors through R^h .
 (2) R^h represents the restriction of $\text{Res}_{X/S}$ to the category of finite S^h -schemes.

PROOF: According to [Ar74], p. 332, the co-ordinate ring \mathcal{O}_{R^h} is the direct limit $\lim_{\rightarrow} \Gamma(V, \mathcal{O}_V)$ as V runs over all affine étale neighbourhoods of s in R . So $R^h = (R \times_S S^h)^h$. Since $R \rightarrow S$ is quasi-finite and surjective, $R^h \rightarrow S^h$ is finite; therefore this system of rings is cofinal with the subsystem consisting of those rings $\Gamma(V, \mathcal{O}_V)$ that are finite over S^h and (1) follows. (2) is an immediate consequence. \square

In each chamber \mathcal{D} , choose a fundamental domain \mathcal{S} for the action of $\text{Stab}_{\mathcal{D}}$ on \mathcal{D} such that

- (1) \mathcal{D} is tessellated by copies of \mathcal{S} that are permuted simply transitively by $\text{Stab}_{\mathcal{D}}$ and
- (2) the corresponding tessellation of $P_{\mathbb{R}}$ is preserved by Γ .

For example, take the case of D_4 and label the central vertex as 2. Then \mathcal{D} is the \mathbb{R} -span of $\varpi_1, \dots, \varpi_4$ and we can take \mathcal{S} to be the cone spanned by $\varpi_2, \varpi_1, \varpi_1 + \varpi_4, \varpi_1 + \varpi_3 + \varpi_4$.

Then an *orientation* of the nef cone \mathcal{C} of $X'_s \rightarrow X_s$ is the choice of fundamental domain \mathcal{S} for the action of $\text{Stab}_{\mathcal{D}}$ on \mathcal{D} where \mathcal{D} is the positive chamber defined by the irreducible (-2) -curves E_1, \dots, E_r on X'_s when these curves are regarded as simple roots.

Consider the functor Res_+ defined on the category of finite S^h -schemes T as follows: its T -points are elements of $R^h(T)$ together with an orientation of the nef cone of $\widetilde{X}_T \rightarrow X_T$.

The natural morphism $\epsilon : \text{Res}_+ \rightarrow R^h$ is a torsor under $\text{Stab}_{\mathcal{D}}$, so Res_+ is represented by a semi-local scheme $R_+^h = \sqcup_{g \in \text{Stab}_{\mathcal{D}}} R_g^h$, a disjoint union of copies of R^h .

Lemma 2.8 Each forgetful morphism $p_{\mathcal{D}} : (R_{P, \mathcal{D}} \times_S S^h)^1 \rightarrow R_+^h$ and each forgetful morphism $p_{\mathcal{D}, g} : R_{P, \mathcal{D}, g}^0 \rightarrow R_g^h$ is an isomorphism.

PROOF: From the definition of R_+^h and the construction of $(R_{P, \mathcal{D}} \times_S S^h)^1$ as an open subspace of $(R_{P, \mathcal{D}} \times_S S^h)$ and from consideration of the functors that they represent, it is clear that $p_{\mathcal{D}}$ is étale. Since both $(R_{P, \mathcal{D}} \times_S S^h)^1$ and R_+^h are semi-local and $p_{\mathcal{D}}$ is an isomorphism on the fibers over s the result follows for $p_{\mathcal{D}}$. Since $p_{\mathcal{D}, g}$ is the restriction of $p_{\mathcal{D}}$ to connected components the lemma is proved for it too. \square

By construction, there is a family of P -marked, \mathcal{D} -polarized surfaces over $R_{P, \mathcal{D}} \times_S S^h$ whose fibers are resolutions of the fibers of $X \times_S S^h \rightarrow S^h$; via the previous lemma there is then, for each chamber \mathcal{D} , a family $X'_{\mathcal{D}} \rightarrow R^h$ of P -marked, \mathcal{D} -polarized surfaces whose closed fiber is X'_s .

Note that, for any chambers \mathcal{D}, \mathcal{E} , the isomorphisms $p_{\mathcal{D}}$ and $p_{\mathcal{E}}$ agree on the overlap $(R_{P,\mathcal{D}} \times_S S^h)^1 \cap (R_{P,\mathcal{E}} \times_S S^h)^1$ and so glue to a morphism

$$p : \cup_{\mathcal{D}} (R_{P,\mathcal{D}} \times_S S^h)^1 = R_P \times_S S^h \rightarrow R_+^h.$$

Given an algebraic space Y , we say that a morphism $Y \rightarrow Q$ of algebraic spaces is the *separated quotient* of Y if Q is separated and every morphism from Y to a separated algebraic space factors uniquely through Q .

Lemma 2.9 $p : R_P \times_S S^h \rightarrow R_+^h$ is étale and is the separated quotient of $R_P \times_S S^h$.

PROOF: This follows from the fact that for each chart $(R_{P,\mathcal{D}} \times_S S^h)^1$ of $R_P \times_S S^h$ the morphism $(R_{P,\mathcal{D}} \times_S S^h)^1 \rightarrow R_+^h$ is an isomorphism. \square

Remark: Even when it exists, the quotient morphism to a separated quotient is not always quasi-finite. For example, the separated quotient of the universal family over $R_P \times_S S^h$ is the pullback of the given family over S .

We gather this information into a diagram

$$\begin{array}{ccccccc} (R_{P,\mathcal{D}} \times_S S^h)^1 & \hookrightarrow & \cup_{\mathcal{D}} (R_{P,\mathcal{D}} \times_S S^h)^1 & \xrightarrow{=} & R_P \times_S S^h & \longrightarrow & R_P \\ & & \searrow \cong \scriptstyle p_{\mathcal{D}} & & \downarrow p & \searrow & \downarrow q \\ & & & & R_+^h & \xrightarrow{\epsilon} & R^h \longrightarrow R \longrightarrow S \end{array}$$

where R^h is a henselian local scheme, $(R_{P,\mathcal{D}} \times_S S^h)^1$ and R_+^h are henselian semi-local schemes, each finite and étale over R^h , $R_P \times_S S^h$ is a non-separated scheme, R_P and R are locally quasi-separated algebraic spaces, q is a torsor under Γ , ϵ is a torsor under $\text{Stab}_{\mathcal{D}}$ and p is a separated quotient.

Lemma 2.10 (1) R_+^h is the normalization of R^h in $R_P \times_S S^h$. If S^h is normal then R_+^h is also the normalization of S^h in $R_P \times_S S^h$.

(2) Γ acts on R_+^h .

PROOF: (1) is clear from Lemma 2.9. (2) follows from (1) and the fact that Γ acts on R_P . \square

Now suppose that $X \rightarrow S$ is versal at s , with respect to deformations over Λ . Assume also that

- (1) either X_s is affine
- (2) or the miniversal deformation space Def_{X_s} of X_s is formally smooth over Λ . Note that this latter condition holds if X_s is affine with only RDPs.

Then [Ar74] the image of $R \times_S S^h$ in S^h is an irreducible component A of S^h , and $A = S^h$ in the second case. Let $\tilde{A} \rightarrow A$ denote the normalization. Then all spaces appearing in the diagram above, except maybe S , are smooth in the appropriate sense.

Theorem 2.11 (1) *There is an effective action of W on R^h such that the geometric quotient $R^h//W$ is naturally isomorphic to \tilde{A} .*

(2) *R^h is smooth over Λ and is the base of a versal deformation of the minimal resolution X'_s .*

(3) *For every positive root r with corresponding reflexion σ_r , the fixed locus of σ_r is the locus D_r^h in R^h that consists of deformations of X'_s where the root r survives as an effective Cartier divisor class. This locus D_r^h is a divisor and is smooth over Λ . If $\{r_1, \dots, r_l\}$ is the set of simple roots corresponding to the chamber \mathcal{D} then the divisors D_{r_1}, \dots, D_{r_l} are transverse relative to Λ . That is, they form a divisor with normal crossings relative to Λ .*

(4) *If $\text{Def}_{X'_s}$ is formally smooth over Λ then $S^h = R^h//W$.*

PROOF: For (1), observe first that the W -action on R^h arises from the Γ -action on R_+^h and the fact that $R_+^h \rightarrow R^h$ is a torsor under $\text{Stab}_{\mathcal{D}}$, so that the normal subgroup W of Γ acts on each connected component of R_+^h . The effectivity of the action follows from the fact that Γ acts freely on the complement of the discriminant in R_+^h .

Since $R \rightarrow A$ is an isomorphism over the complement of the discriminant, it follows that $\deg((R_{P,\mathcal{D}} \times_S S^h)^1 \rightarrow A) = \#\Gamma$. So $\deg(R^h \rightarrow A) = \#W$, and then $R^h//W \rightarrow \tilde{A}$ is an isomorphism by Galois theory.

For (2) we start by copying the proof of Lemma 3.3 of [Ar74].

Consider the formal deformation functor for X'_s . This has a hull $\text{Def}_{X'_s}$, which is formally smooth since, from the short exact sequence

$$0 \rightarrow T_{X'_s}(-\log F) \rightarrow T_{X'_s} \rightarrow \oplus N_{F_j/X'_s} \rightarrow 0$$

and our assumption that $H^2(X'_s, T_{X'_s}(-\log F))$ vanishes, the obstruction space $H^2(X'_s, T_{X'_s})$ also vanishes. Let \widehat{R}_P denote the completion of $R_P \times_S S^h$ at any one of its closed points x . Since $R_P \times_S S^h \rightarrow R^h$ is an isomorphism in an étale neighbourhood of x , \widehat{R}_P is identified with the completion of R^h . There are morphisms

$$\text{Def}_{X'_s} \xleftarrow{\hat{\beta}} \widehat{R}_P \longrightarrow \widehat{S}$$

where \widehat{S} is the completion of S at s and $\hat{\beta}$ is provided by the semi-universal property of a hull. By Lemma 3.3 of [Ar74] $\hat{\beta}$ is formally smooth and (2) is proved.

In $\text{Def}_{X'_s}$ there is a divisor \widehat{D}'_i , formally smooth over Λ , which is the locus where the exceptional (-2) -curve E_i survives. The Zariski tangent space to \widehat{D}'_i is $H^1(X'_s, T_{X'_s}(-\log E_i))$ and the Zariski tangent space to the locus $\cap \widehat{D}'_i$ where each E_i survives is $H^1(X'_s, T_{X'_s}(-\log \sum E_i))$.

Put $\widehat{D}_i = \hat{\beta}^{-1}(\widehat{D}'_i)$.

Lemma 2.12 (1) *The natural homomorphism*

$$H^1(X'_s, T_{X'_s}(-\log \sum E_i)) \rightarrow H^1(X'_s, T_{X'_s})$$

is injective and its image is of codimension l .

(2) $\widehat{D}_1, \dots, \widehat{D}_l$ are transverse divisors in \widehat{R}_P .

PROOF: Take the cohomology of the exact sequence

$$0 \rightarrow T_{X'_s}(-\log E) \rightarrow T_{X'_s} \rightarrow \oplus N_{E_i/X'_s} \rightarrow 0.$$

The lemma follows from the facts that $\mathcal{N}_{E_i/X'_s} \cong \mathcal{O}_{\mathbb{P}^1}(-2)$ and, since, by assumption, the space $H^2(X'_s, T_{X'_s}(-\log F))$ is zero, $H^2(X'_s, T_{X'_s}(-\log E))$ is also zero. \square

The irreducible curves are simple roots in Q . They define a chamber \mathcal{D} . We shall identify $R_{P,\mathcal{D},1}^0$ with R^h .

Let σ_i denote the reflexion in the simple root α_i .

In R_P there is an effective divisor $D_{P,i}$ defined by

$$D_{P,i}(T) = \{(\widetilde{X}_T, \widetilde{\phi}) | \widetilde{\phi}(\alpha_i) \text{ is the class of an effective divisor}\}.$$

Then $\widehat{D}_i = D_{P,i} \times_{R_P} \widehat{R}_P$. So $D_{P,i}$ is smooth over the coefficient ring Λ and the divisors $D_{P,1}, \dots, D_{P,l}$ are transverse. Let $D_{+,i}$ denote the image of $D_{P,i} \times_S S^h$ in R_+^h and D_i^h its image in R^h . By construction, D_i^h is the locus in R^h that parametrizes the deformations of X'_s where the simple root r_i survives as an effective curve.

We have observed that $p : R_P \times_S S^h \rightarrow R_+^h$ is étale and that R_+^h is the largest separated quotient of $R_P \times_S S^h$, so that p is Γ -equivariant. Moreover, p is an isomorphism over the complement of the discriminant in S^h (or A).

Therefore, given $r \in R_+^h$ and $\gamma \in \Gamma - \{1\}$, $r \in \text{Fix}_\gamma$ if and only if there is a henselian trait $V = \{0, v\}$ and two morphisms $a, b : V \rightarrow R_P \times_S S^h$ such that $a \neq b$, $a(v) = b(v)$ and $a(0) = b(0) \circ \gamma$.

We show next that the generic point $r_{+,i}$ of $D_{+,i}$ lies in the fixed locus Fix_{+, σ_i} of σ_i acting on R_+^h . Since Γ acts freely on $R_P \times_S S^h$, the point $r_{+,i}$ lies in Fix_{+, σ_i} if, by the remark just made, there are two morphisms $a, b : V \rightarrow R_P \times_S S^h$ such that $a \neq b$, $p \circ a = p \circ b$, $r = p \circ a(0) = p \circ b(0)$ and $p \circ a(v) \neq p \circ b(v)$.

To find these morphisms, take any such V and a morphism $f : V \rightarrow R_+^h$ such that $f(0) = r_{+,i}$ and otherwise f is in general position; then the closed fibre of $X_V \rightarrow V$ has a single A_1 singularity and the generic fibre is smooth. It is well known (“the existence of flops”) that if a smoothing of an A_1 singularity possesses a resolution, then it has two such, and they are not isomorphic. Therefore $f : V \rightarrow R_+^h$ has two liftings $a, b : V \rightarrow R_P \times_S S^h$ as described above, so that $r_{+,i} \in \text{Fix}_{+, \sigma_i}$. It follows that D_i^h is contained in the fixed point locus $\text{Fix}_{\sigma_i}^h$ of σ_i acting on R^h .

To complete the proof of (3), we need to show that $\text{Fix}_{\sigma_i}^h \subset D_i^h$.

Suppose that $\eta = \eta_1 = [E_1]$ is a simple root. Suppose also that $x \in (R_{P,\mathcal{D}} \times_S S^h)^1$ with $x \mapsto t \in S$, where $\mathcal{D} = \sum_{i=1}^l \mathbb{R}_{\geq 0} \varpi_i$. Assume that η is not the class of a (-2) -curve on the minimal resolution X'_t . That is, E_1 does not survive as an effective class when X'_s deforms to X'_t .

Now \mathcal{D} is contained in the nef cone $\text{NE}(X'_t)$, by the definition of $(R_{P,\mathcal{D}} \times_S S^h)^1$.

Lemma 2.13 $\sigma_\eta(\mathcal{D}) \subseteq \text{NE}(X'_t)$.

PROOF: $\sigma_\eta(\mathcal{D})$ is spanned by $\varpi_1 + \eta_1, \varpi_2, \dots, \varpi_l$. Any irreducible (-2) -curve F_t on X'_t specializes an effective cycle F_s on X'_s ; then $[F_s]$ is a positive root ϕ and $\phi \neq \eta_1$. So $\phi \cdot \varpi_j \geq 0$ for all j , while $\phi \cdot (\varpi_1 + \eta_1) \geq 0 - 1 = -1$ since $\phi \neq \eta_1$. Suppose $\phi \cdot (\varpi_1 + \eta_1) < 0$; then $\phi \cdot \varpi_1 = 0$ and $\phi \cdot \eta_1 = -1$. But $\phi = \sum_{j \geq 1} n_j \eta_j$ with $n_j \geq 0$, so that $\phi = \sum_{j \geq 2} n_j \eta_j$ and $\phi \cdot \eta_1 = -1$, which is absurd. \square

Therefore $\sigma_\eta(x) \in (R_{P,\mathcal{D}} \times_S S^h)^1$ also; since Γ acts freely on $\cup_{\mathcal{D}} (R_{P,\mathcal{D}} \times_S S^h)^1 = R_P \times_S S^h$ it follows that $\sigma_\eta(x) \neq x$. However, the morphism $p_{\mathcal{D}} : (R_{P,\mathcal{D}} \times_S S^h)^1 \rightarrow R_+^h$ is an isomorphism, so $\sigma_\eta(p(x)) \neq p(x)$. Hence $\text{Fix}_{\sigma_i}^h \subseteq D_i^h$, so that $\text{Fix}_{\sigma_i}^h = D_i^h$.

So part (3) of the theorem is proved for every simple reflexion.

Every reflexion σ_r in a positive root r is conjugate in W to a simple reflexion, so the locus $\text{Fix}_{\sigma_r}^h$ is a smooth divisor \tilde{D}_r^h . In characteristic zero we know, by Proposition 2.4 (i) of [Wa79], that $\tilde{D}_r^h = D_r^h$; since D_r^h is also a smooth divisor, it follows that $D_r^h = \text{Fix}_{\sigma_r}^h$, and (3) is proved.

Finally, if Def_{X_s} is formally smooth over Λ then $S^h = \tilde{A}$, and now Theorem 2.11 is proved. \square

Corollary 2.14 *Suppose given a normal local henselian scheme $(T, 0)$ and a family $g : Y \rightarrow T$ of surfaces such that the closed fibre Y_0 has only du Val singularities and the generic fibre is smooth. Then $Y \rightarrow T$ has a resolution if and only if the Galois action on the cohomology $H^2(Y_{\bar{\eta}}, \mathbb{Z}_\ell(1))$, of the generic fibre $Y_{\bar{\eta}}$ is trivial.*

PROOF: The question is local on Y , so we may assume that $Y \rightarrow T$ is pulled back via a morphism $T \rightarrow S$ whose image does not lie in the discriminant locus of $f : X \rightarrow S$. On the one hand, the W -covering $R_{\mathcal{D}}^0 \rightarrow S^0$ is exactly the covering defined by the monodromy action on H^2 of the geometric generic fibre of $X \rightarrow S$, while on the other $Y \rightarrow T$ has a resolution if and only if the map $T \rightarrow S$ factors through $R_{\mathcal{D}}$. Since T is normal, we are done. \square

Remark: (1) Note that for families that map to the discriminant locus in S , it might be necessary to take an inseparable cover; for example, this happens for the family $xy + z^2 + t = 0$ of A_1 singularities in characteristic 2.

(2) Suppose that $X \rightarrow C$ is a morphism from the germ of a smooth threefold to the germ of a smooth curve, that the closed fiber X_s has a du Val singularity and that $\text{char } k = 0$. Then the monodromy (the image of a generator of the local

π_1 , the fundamental group of the punctured curve) is a Coxeter element of W [Dm75]. However, in positive or mixed characteristic the local π_1 is a local Galois group and is not cyclic and it is not clear how to describe the image of this Galois group in W . For example, if the residue characteristic is 3 and the type of the singularity is A_2 then the image of Galois equals W .

3 Some topology of the situation

Suppose in this section that $X \rightarrow S$ is versal at s , that $H^2(X'_s, T_{X'_s}(-\log E))$ and $H^2(X'_s, \mathcal{O}_{X'_s})$ are both trivial, that X_s has only RDPs and that S is a local henselian scheme.

We have seen that the non-separated scheme R_P is obtained by glueing copies $R_{P,\mathcal{D}}^1$ of the (separated) scheme R_+^h along open subschemes. Moreover,

- (1) W acts on each connected component $R_{P,\mathcal{D},g}^0$ of $R_{P,\mathcal{D}}^1$, where g runs over $\text{Stab}_{\mathcal{D}}$, and the connected components $R_{P,\mathcal{D},g}^0$ are all isomorphic, say to $R_{P,\mathcal{D}}^0$,
- (2) the image of $R_{P,\mathcal{D},g}^0$ in R_+^h is a connected component R_g^h of R_+^h ,
- (3) R_g^h is isomorphic to R^h and is the maximal separated quotient of a connected component R_P^0 of R_P and
- (4) W acts on R^h .

Now that we know, thanks to Theorem 2.11, that R^h contains a hyperplane arrangement consisting of the divisors D_r^h parametrized by the positive roots r , we can describe this glueing more precisely, as an analogue of the construction of the real algebraic prevariety $\mathcal{Z}(\mathcal{A})$ of [Pr07].

Proposition 3.1 *R_P^0 is the non-separated scheme obtained by glueing copies $R_{\mathcal{D}}^h$ of R^h , one copy for each chamber \mathcal{D} in the Euclidean vector space $P_{\mathbb{R}}$, as follows.*

For chambers \mathcal{D}, \mathcal{E} in $P_{\mathbb{R}}$, the intersection $R_{\mathcal{D}}^h \cap R_{\mathcal{E}}^h$ is given by

$$R_{\mathcal{D}}^h \cap R_{\mathcal{E}}^h = R^h - \cup_r D_r^h$$

where r runs over those positive roots r such that in $P_{\mathbb{R}}$ the wall H_r separates \mathcal{D} and \mathcal{E} .

PROOF: It's enough to prove the analogous statement for the possibly disconnected schemes $(R_{P,\mathcal{D}})^1$.

Observe that $(R_{P,\mathcal{D}})^1 \cap (R_{P,\mathcal{E}})^1$ consists of those points x in R_P that map to a point $t \in S$ such that on the minimal resolution X'_t of X_t the chambers \mathcal{D} and \mathcal{E} both lie in $\text{NE}(X'_t)$. This is equivalent to saying that \mathcal{D} and \mathcal{E} both lie on the positive side of the wall H_r if the positive root r corresponds to an effective cycle on X'_t , and there is nothing left to prove. \square

Corollary 3.2 W acts freely on R_P^0 and $R = R_P^0/W$.

Now suppose that $k = \mathbb{R}$ and that X_s has a du Val singularity. According to Brieskorn [Br70] and Slodowy [Sl80] we can write $S = \mathfrak{t}/W$ and $R^h = \mathfrak{t}$, where \mathfrak{t} is a Cartan subalgebra of the Lie algebra of the relevant split simple algebraic group. So R^h is a complexified hyperplane arrangement. Let $S_0 \subseteq S$ be the complement of the discriminant and \tilde{S}_0 the inverse image of S_0 in R^h .

Corollary 3.3 $S_0(\mathbb{C})$ is weakly homotopy equivalent to $R(\mathbb{R})$.

PROOF: By Corollary 3.1, $R_P^0(\mathbb{R})$ is nothing but the non-separated manifold $\mathcal{Z}(\mathcal{A})(\mathbb{R})$. The main result of [Pr07] is that there is a W -equivariant map $\tilde{S}_0(\mathbb{C}) \rightarrow \mathcal{Z}(\mathcal{A})(\mathbb{R})$ that is a weak homotopy equivalence. Since W acts freely on both sides, taking quotients by W gives a weak homotopy equivalence

$$S_0(\mathbb{C}) \rightarrow \mathcal{Z}(\mathcal{A})(\mathbb{R})//W = R_P^0(\mathbb{R})//W = R(\mathbb{R}).$$

□

Recall that to the Coxeter system (W, Σ) , where Σ is the set of simple reflexions in W , there is associated a generalized braid group π_W .

Corollary 3.4 $S_0(\mathbb{C})$ is a $K(\pi_W, 1)$.

PROOF: This follows from Deligne's result [D72] that $R(\mathbb{R})$ is a $K(\pi_W, 1)$. □

4 Polynomial rings of W -invariants over \mathbb{Z} via RDPs

The ring of invariants for the action of a Weyl group W on the polynomial ring $\mathbb{Z}[P]$ generated by the corresponding weight lattice P is not always a polynomial ring. The point of this section is to exploit our results about the actions of Weyl groups W on deformation spaces from Section 2 to prove Theorem 4.9 below, which states that, nevertheless, the action of W on the polynomial ring $\mathbb{Z}[M]$ generated by the corresponding root lattice M can be extended to a larger polynomial ring where the ring of invariants is polynomial.

Consider the polynomials f in $\mathbb{Z}[x, y, z]$ given in the table below. In each

Type	f	Π_{fund}	Π_{extra}
A_n	$xy + z^{n+1}$	$(2, \dots, n+1)$	(1)
D_{2n}	$x^2 + z^2y + zy^n$	$(2, 4, \dots, 4n-2, 2n)$	$(1^{(2)}, 3, 5, \dots, 2n-1)$
D_{2n+1}	$x^2 + z^2y + y^n x$	$(2, 4, 6, \dots, 4n-2, 4n, 2n+1)$	$(1, 3, 5, \dots, 2n-1)$
E_6	$x^2 + xz^2 + y^3$	$(2, 5, 6, 8, 9, 12)$	$(1, 2, 3, 4, 6)$
E_7	$x^2 + z^3y + y^3$	$(2, 6, 8, 10, 12, 14, 18)$	$(1, 3, 4, 5, 9)$
E_8	$x^2 + y^3 + z^5$	$(2, 8, 12, 14, 18, 20, 24, 30)$	$(3, 4, 5, 6, 9, 10, 15)$

case the surface X_0 defined by $f = 0$ in $\mathbb{A}_{\mathbb{Z}}^3 = \mathbf{Spec} \mathbb{Z}[x, y, z]$ has an effective action of $\mathbb{G}_{m, \mathbb{Z}} = \mathbf{Spec} \mathbb{Z}[\lambda^{\pm}]$ and has an RDP of the indicated type at every field-valued point of the origin 0 in $\mathbb{A}_{\mathbb{Z}}^3$. Further calculation shows that, in each case, there is, for some N , a linear action of $\mathbb{G}_{m, \mathbb{Z}}$ on $\mathbb{A}_{\mathbb{Z}}^N$ and a $\mathbb{G}_{m, \mathbb{Z}}$ -equivariant family $\mathcal{X} \rightarrow \mathbb{A}_{\mathbb{Z}}^N$ that is versal (taking \mathbb{Z} to be the coefficient ring) at every field-valued point of the origin $0_S \cong \mathbf{Spec} \mathbb{Z}$ in $\mathbb{A}_{\mathbb{Z}}^N$ and that

- (1) no weight¹ of the $\mathbb{G}_{m, \mathbb{Z}}$ -action on $\mathbb{A}_{\mathbb{Z}}^N$ is zero and
- (2) the set Π of positive weights of this action can be written, counting multiplicities, as $\Pi = \Pi_{fund} \cup \Pi_{extra}$, where Π_{fund} and Π_{extra} are as tabulated.

Here, Π_{fund} is the set of fundamental degrees (that is, exponents plus 1) of the corresponding root system (see the Planches of [GrLie4-6]) and Π_{extra} is some set of extra weights. The negative weights of the \mathbb{G}_m -action on $\mathbb{A}_{\mathbb{Z}}^N$ are omitted.

Write $N = N^+ + N^-$ where $N^+ = \#\Pi$ is the number of positive weights, N^- the number of negative weights, r is the rank of the corresponding root lattice (that is, r is the suffix in the notation A_n etc.) and $\rho = N^+ - r = \#\Pi_{extra}$.

Because having RDPs and versality are both open conditions, there is a $\mathbb{G}_{m, \mathbb{Z}}$ -invariant open subscheme S of $\mathbb{A}_{\mathbb{Z}}^N$ such that S contains 0_S , the induced family $X \rightarrow S$ is everywhere versal and all its geometric members are affine surfaces with RDPs.

Since none of the weights of the $\mathbb{G}_{m, \mathbb{Z}}$ -action on $\mathbb{A}_{\mathbb{Z}}^N$ is zero, the origin 0_S is the fixed locus of the $\mathbb{G}_{m, \mathbb{Z}}$ -action on $\mathbb{A}_{\mathbb{Z}}^N$, and so of the action on S .

Fix a prime ℓ . Then, over $S[1/\ell] = S \otimes \mathbb{Z}[1/\ell]$ there is a non-separated scheme $R_P = R_P^{(\ell)}$ given by the construction of Section 2.

Note that $\mathbb{G}_{m, \mathbb{Z}[1/\ell]} \times \Gamma$ acts on $R_P^{(\ell)}$ and the morphism $R_P^{(\ell)} \rightarrow S[1/\ell]$ is $\mathbb{G}_{m, \mathbb{Z}[1/\ell]}$ -equivariant.

Let $\pi^{(\ell)} : Q^{(\ell)} \rightarrow S[1/\ell]$ denote the normalization of $S[1/\ell]$ in $R_P^{(\ell)}$. The next lemma is well known.

Lemma 4.1 *Suppose that G is a smooth affine group scheme over some affine normal base $\mathbf{Spec} B$ and that $X \rightarrow Y$ is a dominant quasi-finite G -equivariant morphism of affine normal G -schemes over B .*

Then there is a unique induced G action on the normalization N of Y in X .

PROOF: The normalization N is the affine scheme $N = \mathbf{Spec} \mathcal{O}_N$ where $\mathcal{O}_N = \{f \in \mathcal{O}_X \mid f \text{ is integral over } \mathcal{O}_Y\}$. Let $f \in \mathcal{O}_N$ and suppose that $\sum_0^r a_i f^i = 0$ with $a_i \in \mathcal{O}_Y$ and $a_r = 1$. Let $\mu_Z : \mathcal{O}_Z \rightarrow \mathcal{O}_Z \otimes_B \mathcal{O}_G$ be the co-actions, for $Z = X, Y$. Then $\sum \mu_Y(a_i) \mu_X(f)^i = 0$, so that $\mu_X(f)$ lies in $\mathcal{O}_X \otimes \mathcal{O}_G$ and is integral over

¹In this section the word ‘‘weight’’ has two meanings: one arises in the consideration of a linear action of $\mathbb{G}_{m, \mathbb{Z}}$ on an affine space $\mathbb{A}_{\mathbb{Z}}^N$ and the other arises in the context of lattices. Of course these uses are closely related but it seems helpful to mention this point explicitly.

$\mathcal{O}_Y \otimes \mathcal{O}_G$. That is, $\mu_X(f)$ is in the normalization of $\mathcal{O}_Y \otimes \mathcal{O}_G$ in $\mathcal{O}_X \otimes \mathcal{O}_G$. But this normalization is $\mathcal{O}_N \otimes \mathcal{O}_G$, since G is smooth over B , and therefore μ_X restricts to a co-action on \mathcal{O}_N . \square

Proposition 4.2 (1) $Q^{(\ell)}$ is smooth over $\mathbb{Z}[1/\ell]$.

(2) $Q^{(\ell)}$ is the separated quotient of $R_P^{(\ell)}$.

(3) $\mathbb{G}_{m,\mathbb{Z}[1/\ell]} \times \Gamma$ acts on $Q^{(\ell)}$ and $\pi^{(\ell)} : Q^{(\ell)} \rightarrow S[1/\ell]$ is $\mathbb{G}_{m,\mathbb{Z}[1/\ell]}$ -equivariant.

(4) Γ acts effectively on $Q^{(\ell)}$ and the morphism $\pi^{(\ell)} : Q^{(\ell)} \rightarrow S[1/\ell]$ identifies $S[1/\ell]$ with $Q^{(\ell)}/\Gamma$.

PROOF: According to Lemmas 2.9 and 2.10, $R_+^{(\ell),h}$ is, for all henselizations S^h of $S[1/\ell]$, the normalization of both S^h and $R_P^{(\ell),h}$ in $R_P^{(\ell)} \times_{S[1/\ell]} S^h$. Now $R_+^{(\ell),h} \rightarrow R^{(\ell),h}$ is a $\text{Stab}_{\mathcal{D}}$ -torsor, so étale, and $R^{(\ell),h}$ is smooth, and so $Q^{(\ell)} \times_{S[1/\ell]} S^h$ is smooth. (1) follows.

(2) can be checked after passing to S^h , where it follows from Lemma 2.9

For (3), it is clear that Γ acts. The existence of the $\mathbb{G}_{m,\mathbb{Z}[1/\ell]}$ -action follows from the definition of $Q^{(\ell)}$ as the normalization of $S[1/\ell]$ in $R_P^{(\ell)}$.

(4) follows from the facts that $\pi^{(\ell)}$ is finite, its degree is $\#\Gamma$ and $S[1/\ell]$ is smooth, so normal. \square

Now suppose that ℓ' is a second prime. Then over $S[1/\ell\ell']$ the schemes $R_P^{(\ell)}$ and $R_P^{(\ell')}$ are isomorphic, by Corollary 2.2. Therefore $Q^{(\ell)}$ and $Q^{(\ell')}$ are isomorphic over $S[1/\ell\ell']$ and so can be glued to give $\pi : Q \rightarrow S$.

Proposition 4.3 (1) Q is smooth over \mathbb{Z} .

(2) Q is the separated quotient of R_P .

(3) $\mathbb{G}_{m,\mathbb{Z}} \times \Gamma$ acts on Q and $\pi : Q \rightarrow S$ is $\mathbb{G}_{m,\mathbb{Z}}$ -equivariant.

(4) Γ acts effectively on Q and the morphism $\pi : Q \rightarrow S$ identifies S with Q/Γ .

PROOF: This is an immediate consequence of Proposition 4.2. \square

Since the weights of the $\mathbb{G}_{m,\mathbb{Z}}$ -action on $\mathbb{A}_{\mathbb{Z}}^N$ are never zero the fixed locus of the action is 0_S .

Since S is a $\mathbb{G}_{m,\mathbb{Z}}$ -equivariant neighbourhood of 0_S in $\mathbb{A}_{\mathbb{Z}}^N$, it follows that S contains S^+ and S^- , where

$$S^{\pm} = \{x \in \mathbb{A}_{\mathbb{Z}}^N \mid \lim_{\lambda^{\pm} \rightarrow 0} \lambda(x) \subset 0\}.$$

Then S^+ (resp., S^-) is defined as a subscheme of S by the vanishing of the co-ordinates of negative (resp., positive) weight, so $S^{\pm} \cong \mathbb{A}_{\mathbb{Z}}^{N^{\pm}}$ and S^+ and S^- intersect transversely in the origin 0_S .

Lemma 4.4 $\pi^{-1}(0_S)_{red} \cong \mathbf{Spec} \mathbb{Z} \times \text{Stab}_{\mathcal{D}}$.

PROOF: Calculation shows that, in each case, the minimal resolution $X'_0 \rightarrow X_0$ is obtained by successively blowing-up along copies of $\mathbf{Spec} \mathbb{Z}$, so that X'_0 is

smooth over \mathbb{Z} . Then for any chamber \mathcal{D} , the set of markings $\phi : L \rightarrow \mathrm{NS}(X'_0)$ such that $\phi(\langle \gamma, \mathcal{D} \rangle)$ lies in the nef cone (and then equals the nef cone) is a torsor under $\mathrm{Stab}_{\mathcal{D}}$. So X'_0 defines a $\mathbf{Spec} \mathbb{Z} \times \mathrm{Stab}_{\mathcal{D}}$ -point of $R_{P, \mathcal{D}}$. Similarly, for any field k and for every morphism $f : \mathbf{Spec} k \rightarrow 0_S$, $X_0 \otimes k$ has a unique minimal resolution. Therefore the set of lifts of f to $\pi^{-1}(0_S)$ is a torsor under $\mathrm{Stab}_{\mathcal{D}}$, and therefore $\pi^{-1}(0_S)_{red} \rightarrow \mathbf{Spec} \mathbb{Z} \times \mathrm{Stab}_{\mathcal{D}}$ is an isomorphism. \square

Define $Q^\pm = \pi^{-1}(S^\pm)$ and $0_Q = \pi^{-1}(0_S)_{red}$. Choose a connected component \tilde{Q} of Q , and write $\tilde{Q}^\pm = \tilde{Q} \cap Q^\pm$ and $0_{\tilde{Q}} = \tilde{Q} \cap 0_Q$. Then $\mathbb{G}_{m, \mathbb{Z}} \times W$ acts on \tilde{Q} and on \tilde{Q}^\pm . Observe that

$$\tilde{Q}^\pm = \{x \in \tilde{Q} \mid \lim_{\lambda^\pm \rightarrow 0} \lambda(x) \subset 0_{\tilde{Q}}\},$$

so that the closure of each $\mathbb{G}_{m, \mathbb{Z}}$ -orbit in \tilde{Q}^\pm meets $0_{\tilde{Q}}$.

Lemma 4.5 *The fixed locus of $\mathbb{G}_{m, \mathbb{Z}}$ acting on \tilde{Q} is $0_{\tilde{Q}}$.*

PROOF: 0_S is the fixed locus of $\mathbb{G}_{m, \mathbb{Z}}$ acting on S and the fixed locus of $\mathbb{G}_{m, \mathbb{Z}}$ on Q is smooth over $\mathbf{Spec} \mathbb{Z}$. \square

The next lemma is a version of Theorem 2.5 of [KR82], but in mixed characteristic.

Lemma 4.6 *Suppose that $\mathbb{G}_{m, \mathbb{Z}}$ acts on a smooth affine \mathbb{Z} -scheme $X = \mathbf{Spec} A$ such that the fixed locus of the $\mathbb{G}_{m, \mathbb{Z}}$ -action is isomorphic to $\mathbf{Spec} \mathbb{Z}$ and meets the closure of every geometric orbit. Then X is $\mathbb{G}_{m, \mathbb{Z}}$ -equivariantly isomorphic to some affine space $\mathbb{A}_{\mathbb{Z}}^d$ on which $\mathbb{G}_{m, \mathbb{Z}}$ acts linearly.*

PROOF: The hypotheses imply that A is a graded ring, say $A = \bigoplus_{n \in \mathbb{Z}} A_n$. Since, over every algebraically closed field, the fixed locus $F = \mathrm{Fix}(\mathbb{G}_{m, \mathbb{Z}}|_X)$ meets every orbit, it follows that *either* $A_n = 0$ for all $n < 0$ *or* $A_n = 0$ for all $n > 0$. Without loss of generality we assume that $A_n = 0$ for all $n < 0$.

Then $A_0 = \mathbb{Z}$ and the ideal I of F is $I = \bigoplus_{n \geq 1} A_n$. Now the argument from [KR82] goes through to show that X is $\mathbb{G}_{m, \mathbb{Z}}$ -equivariantly isomorphic to some affine space $\mathbb{A}_{\mathbb{Z}}^d$ with a linear $\mathbb{G}_{m, \mathbb{Z}}$ -action, since their Corollary 1.4 is stated and proved over any base. \square

Lemma 4.7 $\tilde{Q}^+ \cap \tilde{Q}^- = 0_{\tilde{Q}}$ as schemes and \tilde{Q}^\pm is isomorphic to $\mathbb{A}_{\mathbb{Z}}^{N^\pm}$.

PROOF: Certainly the Krull dimension of \tilde{Q}^\pm is $\dim \tilde{Q}^\pm = \dim S^\pm = N^\pm + 1$.

From the description of \tilde{Q}^\pm as limit loci it follows that the tangent \mathbb{Z} -module $T_{0_{\tilde{Q}}} \tilde{Q}^\pm$ is the part of the representation $T_{0_{\tilde{Q}}} \tilde{Q}$ of $\mathbb{G}_{m, \mathbb{Z}}$ where the weights are all of the corresponding sign. No weight of this tangent action is zero, and so \tilde{Q}^+, \tilde{Q}^- intersect transversely in $0_{\tilde{Q}}$ and are therefore smooth along $0_{\tilde{Q}}$ of the dimensions already indicated.

Since \tilde{Q}^\pm is smooth along $0_{\tilde{Q}}$ and on each one of them $0_{\tilde{Q}}$ meets the closure of any given $\mathbb{G}_{m, \mathbb{Z}}$ -orbit it follows from Lemma 4.6 that they are affine spaces over \mathbb{Z} . \square

Since W acts on the henselization Q_t^h of Q at each field-valued point t of Φ as a Coxeter system $(W, \Sigma = \{\sigma_1, \dots, \sigma_8\})$ (that is, the fixed locus of each $\sigma \in \Sigma$ acting on Q_t^h is a smooth divisor and these divisors are transverse), it follows that (W, Σ) also acts as a Coxeter system on \tilde{Q}^+ .

Lemma 4.8 *For any field k the action of W on $\tilde{Q}^+ \otimes k$ is effective.*

PROOF: The construction so far has been made for deformations where the coefficient ring is \mathbb{Z} . When repeated with \mathbb{Z} replaced by k as coefficient ring, that is, when we consider only deformations over k of $X_0 \otimes k$, the spaces R_P, Q etc., are replaced by $R_P \otimes k, Q \otimes k$ etc. Then the result follows from the effectivity of the action of $\text{Stab}_{\mathcal{D}}$ given by Proposition 4.3. \square

Let M denote a root lattice of type ADE and r its rank. In the polynomial ring $\mathbb{Z}[M]$ regard the elements of M as being of degree 1 and so write $\mathbb{Z}[M] = \mathbb{Z}[1^r]$. Recall the set Π_{extra} of extra weights.

Theorem 4.9 *The action of W on the root lattice M extends to a graded action of W on a polynomial ring $\mathcal{O} = \mathbb{Z}[M][\Pi_{extra}] = \mathbb{Z}[1^r \cup \Pi_{extra}]$ on N^+ variables over \mathbb{Z} such that*

- (1) \mathcal{O}^W is a polynomial \mathbb{Z} -algebra $\mathbb{Z}[\Pi_{fund} \cup \Pi_{extra}]$ and
- (2) for all normal domains A , W acts effectively on $\mathcal{O} \otimes A$ and $(\mathcal{O} \otimes A)^W = \mathcal{O}^W \otimes A$.

PROOF: We first prove three lemmas. The theorem will follow from them at once.

Lemma 4.10 *$S^+ = \tilde{Q}^+ // W$ and $S^+ \otimes_{\mathbb{Z}} A = (\tilde{Q}^+ \otimes_{\mathbb{Z}} A) // W$ for all normal domains A .*

PROOF: Since W acts effectively on \tilde{Q}^+ the commutative diagram

$$\begin{array}{ccc} \tilde{Q}^+ & \longrightarrow & Q \\ \downarrow & & \downarrow \pi \\ \tilde{Q}^+ // W & \longrightarrow & Q // \Gamma = S \end{array}$$

is Cartesian in a neighbourhood of the generic point of $\tilde{Q}^+ // W$ and so the morphism $\tilde{Q}^+ // W \rightarrow S$ identifies $\tilde{Q}^+ // W$ with the normalization of its image, which is S^+ .

The same argument applies after tensoring with A . \square

Write $\mathcal{O}_{\tilde{Q}^+} = \mathbb{Z}[x_1, \dots, x_{N^+}]$. This is a positively graded polynomial \mathbb{Z} -algebra with a graded W -action, where $\deg x_i \geq 1$ for all i . For each $\sigma_i \in \Sigma$ and for every henselization Q_t^h at a point t of Φ , the fixed locus $\text{Fix}(\sigma_i|_{Q_t^h})$ contains $(Q^-)_t^h$, as already remarked. So $\text{Fix}(\sigma_i|_Q)$ is a smooth divisor that contains Q^- and the divisors $\text{Fix}(\sigma_i|_Q)$, for $i = 1, \dots, r$, are transverse. Set $D_{\sigma_i} = \text{Fix}(\sigma_i|_{\tilde{Q}^+})$;

then $D_{\sigma_1}, \dots, D_{\sigma_r}$ are smooth transverse divisors in \tilde{Q}^+ . Say $D_\sigma = (f_\sigma)_0$; then each f_σ is unique up to an element of $\mathcal{O}_{\tilde{Q}^+}^* = \pm 1$, so that for each σ there is a character χ_σ of $\mathbb{G}_{m, \mathbb{Z}}$ such that $\lambda(f_\sigma) = \chi_\sigma(\lambda)f_\sigma$. That is, each f_σ is homogeneous. Since the divisors D_{σ_i} are transverse, we can then assume $x_i = f_{\sigma_i}$ for $i = 1, \dots, r$. The reflexions in W are conjugate, so all of x_1, \dots, x_r have equal degrees. Put $M_1 = \sum_1^r \mathbb{Z}x_i$.

Lemma 4.11 *W acts on the subring $\mathbb{Z}[x_1, \dots, x_r]$ of $\mathcal{O}_{\tilde{Q}^+}$ via its standard action on the root lattice M .*

PROOF: The fixed locus $Fix(W|_{\tilde{Q}^+})$ is given by $Fix(W|_{\tilde{Q}^+}) = \cap_1^r D_{\sigma_i} = \cap_\sigma D_\sigma$, where in the second intersection σ runs over all reflexions in W . So for every reflexion σ , f_σ lies in the ideal $M_1 \cdot \mathcal{O}_{\tilde{Q}^+}$; since $\deg f_\sigma = \deg x_1$, it follows that $f_\sigma = \sum \alpha_i x_i$ and every α_i lies in \mathbb{Z} . Since $Fix(w\sigma w^{-1}) = w(Fix(\sigma))$ it follows that M_1 is a representation of W . Since the x_i are algebraically independent they are certainly linearly independent, so that $\text{rank } M_1 = r$.

Moreover, σ_i preserves and acts trivially on the hyperplane $(x_i)_0$ in \tilde{Q}^+ , so that for every $g \in \mathcal{O}_{\tilde{Q}^+}$ $\sigma_i(g) - g$ lies in the ideal (x_i) . In particular, $\sigma_i(x_j) = x_j + b_{ji}x_i$ for some $b_{ji} \in \mathbb{Z}$ if $i \neq j$ and $\sigma_i(x_i) = -x_i$.

If $i \neq j$ and $(\sigma_i \sigma_j)^2 = 1$ then $b_{ij} = b_{ji} = 0$, while if $(\sigma_i \sigma_j)^3 = 1$ then $b_{ij} b_{ji} = 1$, so that $b_{ij} = b_{ji} = \pm 1$. Since the Coxeter diagram is a tree we can choose $b_{ij} = 1$ for all such i, j , and then deduce that the representations M_1 and M of W are isomorphic. \square

Lemma 4.12 *$\deg x_i = 1$ and $\mathcal{O}_{\tilde{Q}^+} = \mathbb{Z}[M][y_1, \dots, y_\rho]$ where the degrees of the extra generators y_1, \dots, y_ρ run over Π_{extra} .*

PROOF: The degrees can be read by tensoring with \mathbb{Q} . \square

The theorem now follows from Lemmas 4.10, 4.11 and 4.12 by taking $\mathcal{O} = \mathcal{O}_{\tilde{Q}^+}$. \square

5 Chow rings of complete flag varieties

It is well known that the structure of rings of invariants under Weyl groups is connected with the topology of the corresponding simply connected split reductive group. For example, if M is either the root or the weight lattice of a split semisimple algebraic group G over any base, then $\mathbb{Q}[M]^W$ is isomorphic to $A^*(G/B) \otimes \mathbb{Q}$, where A^* denotes the Chow ring. (Recall that, for flag varieties, $H^*((G/B)(\mathbb{C}), \mathbb{Z})$, which we shall abbreviate to $H^*(G/B, \mathbb{Z})$, is naturally isomorphic to the Chow ring, up to a doubling of degrees. Therefore we do not need to be concerned here with the base over which G is defined.)

Given an \mathbb{N} -graded ring $A = \bigoplus_{n \in \mathbb{N}} A_n$, set $A_+ = \bigoplus_{n > 0} A_n$. If a group Γ acts on A set $A_\Gamma = A/I$, where I is the ideal in A generated by A_Γ^+ . This is the ring of *co-invariants*.

Demazure [Dm73] constructed a finite graded \mathbb{Z} -algebra $H = H(\mathbb{Z}[M])$ and a graded W -homomorphism $c : \mathbb{Z}[M] \rightarrow H$, all purely in terms of M and W . He then showed that H is isomorphic to the integral cohomology ring $H^*(G/B, \mathbb{Z}) = A^*(G/B)$ and that c induces an injective homomorphism $\bar{c} : \mathbb{Z}[M]_W \hookrightarrow H$ whose image is the subring generated by the first Chern classes of line bundles on G/B that can be G -linearized.

As is well known, c (or \bar{c}) is not surjective for many groups; this is related to the fact that the ring $\mathbb{Z}[M]^W$ of invariants is not a polynomial ring, even when M is the weight lattice. From now on take M to be the root lattice. In Section 4 we have constructed a polynomial $\mathbb{Z}[M]$ -algebra \mathcal{O} such that \mathcal{O}^W is a polynomial \mathbb{Z} -algebra; what we prove here is that c extends to a surjective homomorphism $c : \mathcal{O} \rightarrow H$ and that $\bar{c} : \mathcal{O}_W \rightarrow H$ is an isomorphism.

Theorem 5.1 *Suppose that M is the root lattice belonging to a finite Coxeter system (W, Σ) of type ADE. Suppose that \mathcal{O} is a graded polynomial $\mathbb{Z}[M]$ -algebra on ρ further generators and that W acts on \mathcal{O} compatibly with its action on $\mathbb{Z}[M]$. Let $\pi : \mathbf{Spec} \mathcal{O} \rightarrow \mathbf{Spec} \mathbb{Z}[M]$ be the projection and assume that*

- (1) *for every $\sigma \in \Sigma$, the fixed locus $Fix(\sigma|_{\mathbf{Spec} \mathcal{O}})$ equals the inverse image $\pi^{-1}(Fix(\sigma|_{\mathbf{Spec} \mathbb{Z}[M]}))$,*
- (2) *W acts effectively on $\mathbf{Spec}(\mathcal{O} \otimes R)$ for every normal domain R and*
- (3) *\mathcal{O}^W is a polynomial \mathbb{Z} -algebra.*

Then there is a surjective W -equivariant homomorphism $c : \mathcal{O} \rightarrow H$ that induces an isomorphism $\bar{c} : \mathcal{O}_W \rightarrow H$.

PROOF: We begin by recapitulating results from [Dm73] and extending some of them slightly to cover our situation. These extensions are easy.

Lemma 5.2 *\mathcal{O}_W is flat over \mathbb{Z} .*

PROOF: $I = \mathcal{O} \cdot \mathcal{O}_+^W$ defines the subscheme $\pi^{-1}(0)$ of $\mathbf{Spec} \mathcal{O}$. By Lemma 4.4 this is 1-dimensional and is quasi-finite over $\mathbf{Spec} \mathbb{Z}$; since I is generated by $r + \rho$ elements these generators form a regular sequence, and the result follows. \square

Say N is the number of reflexions in W . This is also the number of positive roots in M . The fixed locus $Fix(\sigma|_{\mathbf{Spec} \mathcal{O}})$ equals the zero locus $(x_\sigma)_0$ of some element $x_\sigma \in M \subset \mathcal{O}_1$. Define $d = \prod_\sigma x_\sigma \in \mathcal{O}_N$ and $J = \sum \det(w)w$.

Since any reflexion σ acts trivially on the divisor $(x_\sigma)_0$ in $\mathbf{Spec} \mathcal{O}$, $u - \sigma(u)$ lies in the ideal (x_σ) , so that there is an \mathcal{O}^W -linear divided difference operator $\Delta_\sigma : \mathcal{O} \rightarrow \mathcal{O}$ of degree -1 defined, up to ± 1 , by

$$\Delta_\sigma(u) = (u - \sigma(u))/x_\sigma.$$

If $\sigma = \sigma_i \in \Sigma$ then write $\Delta_{\sigma_i} = \Delta_i$.

The formulae (3) – (6) of [Dm73] for the maps Δ_σ are valid.

let \mathcal{D} denote the \mathcal{O}^W -algebra of \mathcal{O}^W -linear endomorphisms of \mathcal{O} generated by \mathcal{O} and the Δ_σ . It is a left \mathcal{O} -module and is graded, since each Δ_σ is homogeneous of degree -1 .

Lemma 5.3 (= Lemme 2 of [Dm73]) For all $\Delta \in \mathcal{D}$ there exist $(\Delta'_i, \Delta''_i)_{i=1, \dots, n}$ such that $\Delta(uv) = \sum \Delta'_i(u)\Delta''_i(v)$.

Let $\epsilon : \mathcal{O} \rightarrow \mathbb{Z}$ be the augmentation map. Then $\epsilon\mathcal{D}$ is a \mathbb{Z} -module of linear maps $\mathcal{O} \rightarrow \mathbb{Z}$ that kill I , so is finitely generated. Say $H(\mathcal{O})$ is its \mathbb{Z} -dual. There is a dual \mathbb{Z} -linear map $c = c_{\mathcal{O}} : \mathcal{O} \rightarrow H(\mathcal{O})$. By Lemma 5.3 $H(\mathcal{O})$ is a graded co-algebra.

Proposition 5.4 $H(\mathcal{O})$ has a unique structure as a graded commutative ring for which c is a graded ring homomorphism. Moreover, W acts on $H(\mathcal{O})$ in such a way that c is W -equivariant.

PROOF: Prop. 2 of [Dm73] and the remarks following. \square

According to Th. 1 of [Dm73], there is a well defined operator D_w for each $w \in W$ such that $D_\sigma = \Delta_\sigma$ for each reflexion σ . If w_0 is the longest element of W then $D_{w_0} = J/d$ ([Dm73], Prop. 3 (b)).

Proposition 5.5 (= Cor. 4 of [Dm73]) (1) $\{\epsilon D_w\}_{w \in W}$ is a \mathbb{Z} -basis of $\epsilon\mathcal{D}$.

(2) Write $z_w = \epsilon D_w$. Then $\{z_w\}_{w \in W}$ is the unique \mathbb{Z} -basis of $H(\mathcal{O})$ such that c is given by $c(u) = \sum_{w \in W} \epsilon D_w(u)z_w$.

(3) $H(\mathcal{O})_i$ is based by $\{z_w | \ell(w) = i\}$.

Proposition 5.6 (1) (Poincaré duality) The multiplication $H(\mathcal{O})_i \times H(\mathcal{O})_{N-i} \rightarrow H(\mathcal{O})_N = \mathbb{Z} \cdot z_{w_0}$ is a perfect pairing of \mathbb{Z} -modules.

(2) The ring $H(\mathcal{O})$ is the same for all polynomial $\mathbb{Z}[M]$ -algebras \mathcal{O} that satisfy Hypothesis 1 of Theorem 5.1. That is, there is a W -equivariant commutative diagram

$$\begin{array}{ccc} \mathbb{Z}[M] & \xrightarrow{c_{\mathbb{Z}[M]}} & H(\mathbb{Z}[M]) \\ \downarrow & & \downarrow \cong \\ \mathcal{O} & \xrightarrow{c_{\mathcal{O}}} & H(\mathcal{O}) \end{array}$$

where $H(\mathbb{Z}[M]) \rightarrow H(\mathcal{O})$ is an isomorphism.

PROOF: Cor., p. 293 of [Dm73], and Remarque 2) following. \square

Let H denote the common value of the rings $H(\mathcal{O})$ and $H(\mathbb{Z}[M])$. As explained in [Dm73], H is isomorphic (after a doubling of degrees) as a graded \mathbb{Z} -algebra with W -action to the integral cohomology ring $H^*(G/B, \mathbb{Z})$ where G is the split simple algebraic group of the given type A, D or E . So \bar{c} is an injective graded W -equivariant homomorphism

$$\bar{c} : \mathcal{O}_W \hookrightarrow H^*(G/B, \mathbb{Z})$$

of finite flat \mathbb{Z} -algebras.

The next three results are taken from or inspired by [Bro09].

Lemma 5.7 *There is an \mathcal{O}^W -linear splitting $\nu : \mathcal{O} \rightarrow \mathcal{O}^W$ of the inclusion $\mathcal{O}^W \hookrightarrow \mathcal{O}$ whose kernel is a free \mathcal{O}^W -module.*

PROOF: Consider first the induced homomorphism $j : \mathbb{Z} = \mathcal{O}^W/\mathcal{O}_+^W \rightarrow \mathcal{O}_W$. Via the augmentation map $\mathcal{O} \rightarrow \mathbb{Z}$ with kernel \mathcal{O}_+ , j has an \mathcal{O}^W -linear splitting. So there is a \mathbb{Z} -basis $\{1, x_1, \dots, x_{\#W-1}\}$ of \mathcal{O}_W where each x_i is homogeneous and lies in \mathcal{O}_{W+} . Since $\mathcal{O}^W \rightarrow \mathcal{O}$ is finite, any lifting of this \mathbb{Z} -basis to a subset Φ consisting of homogeneous elements of \mathcal{O} is, by the graded version of Nakayama's Lemma, a generating set of \mathcal{O} as an \mathcal{O}^W -module. Now $\mathcal{O}^W \rightarrow \mathcal{O}$ is also flat, since both rings are regular of the same dimension, and therefore Φ is an \mathcal{O}^W -basis of \mathcal{O} . In particular, there is an \mathcal{O}^W -basis of \mathcal{O} that includes the element 1. The lemma follows. \square

Lemma 5.8 *The different \mathfrak{D} of the morphism $\mathbf{Spec} \mathcal{O} \rightarrow \mathbf{Spec} \mathcal{O}^W$ is defined by the principal ideal (d) .*

PROOF: By assumption, W acts effectively on $\mathbf{Spec}(\mathcal{O} \otimes k)$ for all fields k , so that, in particular, \mathfrak{D} does not contain the mod 2 fibre $\mathbf{Spec}(\mathcal{O} \otimes \mathbb{F}_2)$. Now on one hand \mathfrak{D} is an effective W -invariant Cartier divisor on $\mathbf{Spec} \mathcal{O}$, since $\mathbf{Spec} \mathcal{O} \rightarrow \mathbf{Spec} \mathcal{O}^W$ is a finite flat and separable morphism of regular schemes, and on the other hand the ideal $\mathcal{I}_{\mathfrak{D}}$ of \mathfrak{D} coincides with (d) over $\mathbf{Spec} \mathbb{Z}[1/2]$, since the only elements of W that act on $\mathbf{Spec} \mathcal{O}$ with fixed points in codimension one are the reflexions σ , the fixed locus of each of which is the corresponding wall $(x_\sigma)_0$. Therefore $\mathcal{I}_{\mathfrak{D}} = (d)$ over $\mathbf{Spec} \mathbb{Z}$. \square

Proposition 5.9 *There exists $a \in \mathcal{O}_N$ such that $J(a) = d$.*

PROOF: Write $S = \mathcal{O}$, $R = \mathcal{O}^W$. Since $R \rightarrow S$ is finite and flat, the relative dualizing module $\omega_{S/R}$ is the graded S -module

$$\omega_{S/R} = \mathrm{Hom}_R(S, R).$$

Since R and S are smooth \mathbb{Z} -algebras, Lemma 5.8 gives, by the well known canonical isomorphism $\mathfrak{D}^{-1} \xrightarrow{\cong} \omega_{S/R}$, an isomorphism

$$\phi : S.d^{-1} \xrightarrow{\cong} \omega_{S/R}$$

of graded S -modules defined by the R -bilinear pairing

$$S.d^{-1} \times S \rightarrow R : (a/d, s) \mapsto \mathrm{Tr}(as/d) = \sum_{w \in W} w(as/d).$$

Choose $a \in S$ such that $\phi(a/d) = \nu$, where ν is the splitting provided by Lemma 5.7; since $\nu(1) = 1$, we get $\mathrm{Tr}(a/d) = 1$. Since ϕ is a graded isomorphism we can choose a to be homogeneous, and then $a \in S_N$. Since $w(d) = \det w.d$ the result follows from the definition of J . \square

Since \mathcal{O}^W is a polynomial ring and \mathcal{O} is flat over \mathcal{O}^W , the ideal I is generated by a regular sequence in \mathcal{O} , so that \mathcal{O}_W is a finite flat complete intersection \mathbb{Z} -algebra. The ring H is a finite flat Gorenstein \mathbb{Z} -algebra because multiplication gives a perfect pairing into H_N , by Proposition 5.6.

Now we can prove the theorem. It is enough to show that the injective homomorphism $\bar{c} : \mathcal{O}_W \rightarrow H$ is surjective.

By Proposition 5.9, the fact that $D_{w_0} = J/d$ and the fact that, by the definition of H , H_N is generated as a \mathbb{Z} -module by D_{w_0} , the map $\bar{c}_N : (\mathcal{O}_W)_N \rightarrow H_N$ is surjective, and so is an isomorphism, since both sides are torsion-free \mathbb{Z} -modules of rank one. Therefore $\bar{c} : \mathcal{O}_W \rightarrow H$ is a graded homomorphism of finite flat graded Gorenstein \mathbb{Z} -algebras such that, for every field k , the induced homomorphism $\bar{c} \otimes 1_k : (\mathcal{O}_W) \otimes k \rightarrow H \otimes k$ of finite local Gorenstein k -algebras induces an isomorphism of socles. Therefore $\bar{c} \otimes 1_k$ is injective. Since both algebras have the same dimension over k (namely, $\#W$), $\bar{c} \otimes 1_k$ is an isomorphism, and we are done. \square

Corollary 5.10 *There is an action of W on a polynomial $\mathbb{Z}[M]$ -algebra $\mathcal{O} = \mathbb{Z}[M][\Pi_{extra}]$ such that \mathcal{O}^W is a polynomial \mathbb{Z} -algebra $\mathbb{Z}[\Pi_{fund} \cup \Pi_{extra}]$ and $H^*(G/B, \mathbb{Z})$ is isomorphic to \mathcal{O}_W .*

PROOF: According to Theorem 4.9, the hypotheses of Theorem 5.1 are satisfied, and the Corollary follows. \square

6 Partial resolutions of RDPs and partial flag varieties

A *partial resolution* of an RDP over a field k is a quasi-projective surface Y with RDPs that is a partial resolution of an affine surface X that is defined over k and has RDPs. That is, there are proper birational morphisms

$$X' \rightarrow Y \rightarrow X$$

where X is affine with RDPs and X' is the minimal resolution of both X and Y .

If $X \rightarrow S$ is a flat family of affine surfaces where each geometric fibre has RDPs then a partial resolution of X over S consists of $Y \rightarrow X$ where $Y \rightarrow S$ is flat and for each field-valued point s of S the morphism $Y_s \rightarrow X_s$ is a partial resolution.

Fix a ring Λ as in Section 2.

Lemma 6.1 *The local deformation space of a partial resolution Y of an RDP over a field k is formally smooth over Λ .*

PROOF: $H^2(Y, \mathcal{F}) = 0$ for all coherent sheaves \mathcal{F} on Y . \square

Now take $\Lambda = \mathbb{Z}$ and suppose that X_0 is one of the RDPs over \mathbb{Z} considered in Section 4. Let r denote the suffix in the type of the defining equation of X_0 . The minimal resolution of X_0 is defined over \mathbb{Z} and is smooth over \mathbb{Z} , and the exceptional locus is a tree of r copies of $\mathbb{P}_{\mathbb{Z}}^1$ arranged in the tree that is the Dynkin diagram. Then there are 2^r partial resolutions Y of X_0 over \mathbb{Z} ; they correspond, via taking the curves contracted by $X' \rightarrow Y$, to subsets Θ of the set of simple roots. So, for example, $Y = X'$ if $\Theta = \emptyset$.

Suppose also that S is the base of the deformation considered there, that S^h is the henselization of S at a point s of 0_S and that T^h is the henselian base of a versal deformation of Y_s . So, by the previous lemma, S^h and T^h are both smooth, in the henselian sense.

Suppose that $W_{\Theta} \subseteq W$ is the subgroup of W generated by the reflexions in the members of Θ . By Theorem 2.11 1, $S^h \cong R^h // W$ and $T^h \cong R_1^h // W_{\Theta}$ where each of R^h and R_1^h is the base of a versal deformation of X' . Both R^h and R_1^h are smooth since $H^2(X', \mathcal{F}) = 0$ for all coherent sheaves \mathcal{F} on X' .

For the rest of this section we use the notation of Section 4.

Theorem 6.2 (1) $R^h // W_{\Theta}$ is smooth.

(2) $\tilde{Q} // W_{\Theta}$ is smooth over $\mathbf{Spec} \mathbb{Z}$.

(3) $\tilde{Q}^+ // W_{\Theta}$ is the spectrum of a polynomial ring over \mathbb{Z} .

PROOF: The preceding discussion, combined with Theorem 2.11 1, shows that $R_1^h // W_{\Theta}$ is smooth. Then (1) follows from the fact that, since any two versal deformation spaces of the same object are smoothly equivalent, R_1^h is smoothly equivalent to R^h . (2) follows from (1) and the fact that at every point lying over the origin 0_S of S the henselization of Q is smoothly equivalent to R^h . (3) is then proved using a $\mathbb{G}_{m, \mathbb{Z}}$ -action, as in Section 4. \square

Fix a Borel subgroup B of G and let P denote the parabolic subgroup of G that contains B and corresponds to Θ (so that, for example, $P = B$ if $\Theta = \emptyset$.) Put $\mathcal{O} = \mathcal{O}_{\tilde{Q}^+}$.

Corollary 6.3 (1) $\mathcal{O}^{W_{\Theta}}$ is a polynomial \mathbb{Z} -algebra.

(2) $\mathcal{O}^{W_{\Theta}} / \mathcal{O}_+^W \cdot \mathcal{O}^{W_{\Theta}}$ is isomorphic to $H^*(G/P, \mathbb{Z})$.

PROOF: (1) is just Theorem 6.2 2 above.

For (2), write $C = \mathcal{O}$, $B = \mathcal{O}^{W_{\Theta}}$ and $A = \mathcal{O}^W$. Then, since $H^*(G/P, \mathbb{Z}) = H^*(G/B, \mathbb{Z})^{W_{\Theta}}$ ([BGG73], Theorem 5.5), there is a commutative diagram of rings

$$\begin{array}{ccccc}
 C & \longrightarrow & C/A_+ \cdot C & \xrightarrow[\cong]{\bar{c}} & H^*(G/B, \mathbb{Z}) \\
 \uparrow & & \uparrow & & \uparrow \\
 B & \longrightarrow & B/A_+ \cdot B & \xrightarrow{\bar{b}} & H^*(G/P, \mathbb{Z}) \\
 \uparrow & & \uparrow & & \uparrow \\
 A & \longrightarrow & A/A_+ & \xrightarrow[\cong]{\bar{a}} & \mathbb{Z}
 \end{array}$$

Lemma 6.4 *If B_1 is a B -algebra that is torsion-free as a \mathbb{Z} -module, then $B_1 = (C \otimes_B B_1)^{W_\Theta}$.*

PROOF: By Lemma 5.7 there is a B -linear splitting $\nu : C \rightarrow B$ whose kernel N is a free B -module. So $C = B \oplus N$, and then $C \otimes_B B_1 = B_1 \oplus (N \otimes_B B_1)$. Therefore the inclusion $B_1 \hookrightarrow C \otimes_B B_1$ is cotorsion-free as a \mathbb{Z} -module, and then $B_1 \hookrightarrow (C \otimes_B B_1)^{W_\Theta}$ is also cotorsion-free as a \mathbb{Z} -module. Now

$$B_1 \otimes_{\mathbb{Z}} \mathbb{Q} = (C \otimes_B B_1 \otimes_{\mathbb{Z}} \mathbb{Q})^{W_\Theta} = (C \otimes_B B_1)^{W_\Theta} \otimes \mathbb{Q},$$

since finite groups are linearly reductive over \mathbb{Q} , and so $B_1 = (C \otimes_B B_1)^{W_\Theta}$. \square

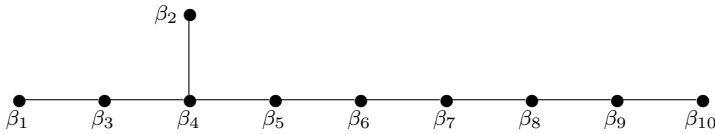
Take $B_1 = B/A_+.B$. Then

$$\begin{aligned} B/A_+.B &= (C \otimes_B (B/A_+.B))^{W_\Theta} = (C/A_+.C)^{W_\Theta} \\ &= H^*(G/B, \mathbb{Z})^{W_\Theta} = H^*(G/P, \mathbb{Z}). \end{aligned}$$

\square

7 Moduli of Enriques surfaces

Suppose that k is an algebraically closed field of characteristic 2 and that Y is a smooth Enriques surface over k . Then $\text{NS}(Y)$ is isomorphic to the even unimodular lattice $E = E_{10}(-1)$. Denote by $O(E)$ the orthogonal group of E and $O^+(E)$ the index 2 subgroup of $O(E)$ consisting of elements that preserve the two cones of positive vectors in $E_{\mathbb{R}}$. It is known [CD89] that $O^+(E)$ is the Weyl group $W(E)$, the group generated by reflexions in the roots of E . Fix, once and for all, a chamber \mathcal{D}_0 defined by the roots (that is, the (-2) -vectors) in the positive cone of $E \otimes \mathbb{R}$. (We shall not always be scrupulous in distinguishing between \mathcal{D} and its closure.) This defines a root basis $\beta_1, \dots, \beta_{10}$ of E that in turn defines a diagram of type $T_{2,3,7}$ as in the diagram below. We recover \mathcal{D}_0 as $\mathcal{D}_0 = \sum \mathbb{R}_{\geq 0} \varpi_i$, where $\varpi_1, \dots, \varpi_{10}$ are the fundamental dominant weights defined by the root basis. That is, $\varpi_i \cdot \beta_j = \delta_{ij}$. We label the simple roots $\beta_1, \dots, \beta_{10}$ according to the diagram



According to [L15] and [EHS], the weight ϖ_1 defines a Cossec–Verra polarization λ on Y ; in turn this defines a birational contraction $Y \rightarrow Z_0$, where Z_0 has du Val singularities, such that λ descends to an ample class λ on Z_0 . The forgetful map $\text{Def}_{Z_0, \lambda} \rightarrow \text{Def}_{Z_0}$ of infinitesimal deformation functors is an isomorphism, as is the corresponding morphism of henselian functors, and [Li10] these functors are formally smooth over either $\mathbf{Spec} \mathbf{W}(k)[[f, g]]/(fg - 2)$ or $\mathbf{Spec} \mathbf{W}(k)$ according to whether Pic_Y^τ is isomorphic to α_2 or not.

Suppose that $Z \rightarrow S$ is a versal deformation of (Z_0, λ) and that S is henselian; then consider the stack $\text{Res}_{\mathcal{D}}$ over S that is defined much as before: the objects over an S -scheme T consist of a resolution $\pi : \widetilde{Z}_T \rightarrow Z_T$ together with a choice of chamber \mathcal{D} in the nef cone of $\widetilde{Z}_T \rightarrow T$ such that $\pi^*\lambda$ is the vector corresponding to ϖ_1 under the unique isomorphism $\mathcal{D}_0 \rightarrow \mathcal{D}$. As before, $\text{Res}_{\mathcal{D}}$ is represented by a finite local S -scheme $R_{\mathcal{D}}$ that carries an action of a Weyl group W such that the geometric quotient $R_{\mathcal{D}}//W$ is S . The Weyl group is that associated to the configuration of singularities on Z_0 . The root lattice corresponding to this configuration embeds into the root lattice of type D_9 , since D_9 is the orthogonal complement of ϖ_1 in E_{10} .

The infinitesimal functors Def_Y and $\text{Def}_{Y, \mathcal{D}}$ are also isomorphic, so there is, as before, a diagram

$$\begin{array}{ccc} \text{Def}_Y & \xleftarrow{\alpha} & \widehat{R}_{\mathcal{D}} \\ & & \downarrow \\ & & \widehat{S} \end{array}$$

where α is formally smooth.

We can summarize this discussion in the following result.

Theorem 7.1 *A versal deformation space Def_Y of a smooth Enriques surface Y is smoothly equivalent to a regular complete local scheme $R_{\mathcal{D}}$ that carries an action of a Weyl group W such that $R_{\mathcal{D}}//W$ is regular. Moreover, W belongs to a root sublattice of the root lattice D_9 and $R_{\mathcal{D}}//W$ is formally smooth over either \mathbf{W} or $\mathbf{W}[[f, g]]/(fg - 2)$.*

PROOF: The only thing that has not been proved is that Def_Y is regular. Except for surfaces that are both classical and exceptional (that is, $\text{Pic}_Y^\tau \cong \mathbb{Z}/2$ and Y has vector fields) this is proved in [EHS]. More precisely, if Y is an α_2 -surface, then a hull for Def_Y is given by $\mathbf{W}[[f, x_2, \dots, x_{12}]]/(fg - 2)$, where g is divisible by neither 2 nor f , and otherwise it is a power series ring over \mathbf{W} . The remaining cases are dealt with in [Sc19]. \square

Remark: This picture describing the local moduli of Enriques surfaces can now be described in terms of the local picture above.

Suppose that S_1 is the base of a versal deformation $Z_1 \rightarrow S_1$ of a sufficiently small affine étale neighbourhood of $\text{Sing } Z_0$. Then there is a W -covering $R_{1, \mathcal{D}} \rightarrow S_1$, where $R_{1, \mathcal{D}}$ represents the stack of \mathcal{D} -polarized resolutions of $Z_1 \rightarrow S_1$, and a commutative square

$$\begin{array}{ccc} R_{\mathcal{D}} & \longrightarrow & R_{1, \mathcal{D}} \\ \downarrow & & \downarrow \\ S & \longrightarrow & S_1. \end{array}$$

Since both vertical arrows are W -coverings, the square is Cartesian (which was not obvious *a priori*).

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