Report on the paper

Collapsing ancient solutions of mean curvature flow

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This paper deals with ancient solutions of the mean curvature flow, i.e. solutions which are defined for arbitrary negative times. Ancient solutions have attracted much attention in recent years because they are relevant in the analysis of singularities of the flow. The simplest example is given by the shrinking sphere. Few examples are also known of nonspherical convex compact ancient solutions, which become more and more eccentric as they evolve backwards in time. The oldest one is a solution of curve shortening flow in the plane, known as the Angenent oval. Higher dimensional examples have been constructed in [12,19]. While the Angenent oval is contained in a strip for all times, the higher dimensional examples of [12,19] sweep the whole space and are noncollpased in the sense of [1]. For curves in the plane, it is known that the shrinking sphere and the Angenent oval are the only compact ancient solutions. In higher dimensions, a classification of ancient solution seems much harder to obtain, although the results of [14] show that this class enjoys strong rigidity results.

This paper focuses the attention on ancient solutions which can be considered a higher dimensional analogue of the Angenent oval. The authors first prove the existence of a rotationally symmetric ancient solution which resembles a rotated Angenent oval: in particular, the solution is contained in a slab and the asymptotic profile is given by the grim reaper curve, in any direction orthogonal to the axis of rotation. In contrast with the examples of [12,19], this solution is collapsed for large negative times. The existence result is proved together with a detailed asymptotic analysis of the solution. The second part of the paper gives a uniqueness result, showing that any other compact, convex, rotationally symmetric ancient solution which is contained in the same slab must coincide with the one constructed here.

The results of the paper are new and interesting. The existence of an ancient solution looking like a rotated Angenent oval was commonly believed to be true in the community of mean curvature flow. However, the only available result in this direction was due to X.-J. Wang in [18], who proved the existence of ancient solutions contained in a slab. In the present paper, the authors use a more explicit method and provide a description of the asymptotic profile which was not given in [18]. The statement about uniqueness is probably even more interesting, since it is one of the few available results until now on the general structure of ancient solutions in higher dimensions.

In my opinion, the results of the paper are an important and original contribution to the understanding of ancient solutions of mean curvature flow. The paper analyzes the class of collapsed solutions, which had not been treated in detail before, and gives a comprehensive existence and uniqueness result in the rotationally symmetric case. The authors cannot follow the technique of the noncollpased case as in [12], and give a direct proof based on simple and explicit computations, but which often involves subtle arguments and estimations, especially in the uniqueness part. I think that the results of this paper are of great interest for the general theory of mean curvature flow, and I believe that they fully meet the quality standards of JDG.

The paper is very clearly written and well organized, and I have not detected misprints or passages where corrections are needed, except for one spot: it seems to me that statement (1) of Therem 1.1 should hold for $\lambda \to +\infty$ rather than $\lambda \to 0$. Except for this minor remark, my suggestion is that the paper can be accepted in its present form.