One for all, all for one—von Neumann, Wald, Rawls, and Pareto I. Theory*

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Abstract

Applications of the maximin criterion extend beyond economics to statistics, computer science, politics, and operations research. However, the maximin criterion—be it von Neumann's, Wald's, or Rawls'—draws fierce criticism due to its extremely pessimistic stance. I propose a novel concept, dubbed the *optimin criterion*, which is based on (Pareto) optimizing the worst-case payoffs of tacit agreements. The optimin criterion generalizes and unifies results in various fields: It not only coincides with (i) Wald's statistical decision-making criterion when Nature is antagonistic, (ii) the core in cooperative games when the core is nonempty, though it exists even if the core is empty, but it also generalizes (iii) Nash equilibrium in n-person constant-sum games, (iv) stable matchings in matching models, and (v) competitive equilibrium in the Arrow-Debreu economy. Moreover, every Nash equilibrium satisfies the optimin criterion in an auxiliary game.

Keywords: maximin criterion, noncooperative games, cooperative games, Nash equilibrium

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1 Introduction and motivating examples

The maximin criterion is perhaps one of the few economic concepts that has wide-ranging applications in fields outside of economics, including computer science, decision sciences, operations research, philosophy, political science, and statistics. However, not only does the maximin criterion—be it von Neumann's, Wald's, or Rawls'—draw fierce criticism due to its extremely pessimistic stance, it also fails to perform well in predicting behavior in the non-zero-sum situations in which most economic and social interactions take place.

In this paper, I propose a novel concept that I dub the *optimin criterion*, which builds on the maximin criterion. This paper contributes primarily to the literature on noncooperative solution concepts. Although the optimin criterion, which I formally define in the following page, is mainly devised for noncooperative games, it generalizes and unifies results in various fields. The optimin criterion coincides with (i) von Neumann's minimax equilibria in zero-sum games and Wald's statistical decision-making criterion when Nature is antagonistic, and (ii) the core in cooperative games when the core is nonempty, though it exists even if the core is empty. In addition, the optimin criterion generalizes (iii) Nash equilibria in n-person constant-sum games, (iv) stable matching in matching models, and (v) competitive economic equilibrium in the Arrow-Debreu economy. Moreover, every Nash equilibrium satisfies the optimin criterion in an auxiliary game. Finally, when restricted to pure strategies, an optimin point always exists in finite games. Finally, the optimin criterion is consistent with the direction of non-Nash deviations in games in which cooperation has been extensively studied, including the finitely repeated prisoner's dilemma, the centipede game, the traveler's dilemma, and the finitely repeated public goods game.

The maximin approach makes perfect sense in zero-sum games, for which von Neumann (1928) first proved his renowned minimax theorem. A maximin strategist chooses an action to maximize the minimum utility the player might receive under any conceivable deviation by the other player, "even assuming that his opponent is guided by the desire to inflict a loss rather than to achieve a gain" (von Neumann and Morgenstern, 1944, p. 555). As is the case under strategic games, the maximin criterion is extremely pessimistic as a decision-making rule. For example, Rawls (1971) proposed the so-called maximin criterion, which prescribes maximizing the situation of the worst-off individuals, no matter how costly it is for society. Rawls proposed a thought experiment in which people in the "original position" choose the principles of social and political justice "behind the veil of ignorance," that is, under complete uncertainty about our position in society, in order to reach fair agreements (or social contracts). His proposal contributes to the contractarian or social contract tradition of Locke, Rousseau, and Kant, replacing it with a "hypothetical contract" agreed upon behind the veil of ignorance.

Under the optimin criterion, players act cautiously under a hypothetical contract or an agreement that their behavior satisfies some "reasonable" rationality assumptions. For illustrative purposes, I define reasonable behavior as follows: Players do not harm themselves for the sake of harming others.[1] Accordingly, in n-person games, an optimin point is an agreement in which cautious players (Pareto) optimize their worst-case payoffs under a hypothetical contract.

Motivation: Why might the maximin criterion be implausible in games? Consider the following 2×2 game in which the unique Nash equilibrium predicts the outcome to be (U, L). However, the maximin strategy of player 1 (she) is D because it guarantees a payoff of 1, irrespective of her opponent's choice, whereas U guarantees only 0 because player 2 (he) may choose R.

	L	R
U	$2, 2$	$0, 1$
D	$1, 2$	$1, 1$

But playing R is not plausible for player 2. Put differently, player 2 must necessarily harm himself (e.g., by 'shooting himself in the leg') to minimize player 1's utility when player 1 plays U. This example shows that we need to reconsider the maximin criterion in non-zero-sum games. Just like the Nash equilibrium, the optimin criterion singles out (U, L) as the unique solution in this game, albeit for a different reason. I next formally define the optimin criterion.

Formal definition: Let $(\Delta X_i, u_i)_{i \in N}$ be an n-person noncooperative game in mixed extension and $p \in \Delta X$ be a strategy profile. An agreement $\bar{p}$ is said to satisfy the *optimin criterion*, or called an *optimin point*, if $\bar{p}$ is Pareto optimal with respect to value function $v : \Delta X \to \mathbb{R}^n$, whose i'th component, $i \in N$, is defined as $v_i(p) = \min\{u_i(p), \inf_{p'_{-i} \in B_{-i}(p)} u_i(p_i, p'_{-i})\}$, where $B_i(p) = \{p'_i \in \Delta X_i | u_i(p'_i, p_{-i}) > u_i(p_i, p_{-i})\}$ is the better-response correspondence and $B_{-i}(p) = \times_{j \in N \setminus \{i\}} (B_j(p) \cup \{p_j\})$. Equivalently, in the primitives of the game, $\bar{p}$ solves the following multi-objective optimization problem.

$$\bar{p} \in \arg\max_{\hat{p} \in \Delta X} \left(\inf_{p'_{-1} \in B_{-1}(\hat{p})} u_1(\hat{p}_1, p'_{-1}), \inf_{p'_{-2} \in B_{-2}(\hat{p})} u_2(\hat{p}_2, p'_{-2}), \ldots, \inf_{p'_{-n} \in B_{-n}(\hat{p})} u_n(\hat{p}_n, p'_{-n}) \right).$$

[1]Because I make this definition over the utility function, even "sadists" are covered by the definition.

Intuition: Suppose that before playing a game players can make a tacit agreement to play a strategy profile $p \in \Delta X$. Because the agreement is nonbinding, each player has the option of either honoring or breaking the agreement. The optimin criterion, which is based on two steps, identifies potential agreements that players can cautiously enter into. First, player i's value $v_i(p)$ (i.e., the worst-case payoff) of following a tacit agreement p is defined as the minimum utility the player receives (i) from the agreement, or (ii) under the other players' unilateral profitable deviations from the agreement, which rules out implausible deviations. Second, the worst-case payoffs of players are Pareto optimized. The rationale behind this is the assumption that players each evaluate a potential agreement by its worst-case payoffs and would rather agree on profile p than p' if the worst-case payoffs at agreement p are greater than or equal to the worst-case payoffs at p'—in other words, p Pareto dominates p'. Accordingly, an agreement satisfies the optimin criterion if it is not possible to increase a player's worst-case payoff without decreasing any other player's worst-case payoff.

Illustrative example: To illustrate the optimin criterion, consider the game in Figure 1 (left) in which attention is restricted to pure strategies for simplicity. The maximin strategy concept does not have a predictive power in this game because every action is a maximin strategy, guaranteeing a payoff of 0. Consider a tacit agreement to play (Top, Left). Even if player 2 (he) has a profitable deviation to 'Center', player 1 (she), who follows her part of the agreement, would still receive 100 at profile (Top, Center). True, player 2 could also deviate to 'Right'—a viable deviation under the maximin strategy concept—but this would be implausible because he must inflict a huge loss on himself by doing so. Thus, the worst-case payoff or the "value" associated with (Top, Left) is 100 for each player whether (i) the opponent honors the tacit agreement, or (ii) betrays the agreement by making a unilateral profitable deviation. To give another example, consider a potential agreement on (Middle, Center), which also seems attractive. The minimum or worst-case payoff associated with this agreement is 0 for player 1 because player 2 could profitably deviate to 'Right', in which case player 1, who honors the agreement, would receive 0 (and vice versa).[2] It turns out that (Top, Left) is the unique optimin point because it (Pareto) optimizes the worst-case payoffs illustrated in Figure 1 (right).

The profile (Top, Left) can be justified as a solution as follows. Suppose that before the game, players set up a binding contract in which, if they agree to play a particular strategy profile, no one will deviate from it in a nonprofitable manner. After the agreement has

[2] For the calculation of worst-case payoffs in each case, see section 3.

	Left	Center	Right
Top	100, 100	100, 105	0, 0
Middle	105, 100	95, 95	0, 210
Bottom	0, 0	210, 0	5, 5

	Left	Center	Right
Top	(100, 100)	(100, 0)	(0, 0)
Middle	(0, 100)	(0, 0)	(0, 5)
Bottom	(0, 0)	(5, 0)	(5, 5)

Figure 1: An illustrative game (left) and its worst-case payoffs (right). The unique optimin point is the agreement (Top, Left), whereas every strategy is a maximin strategy in this game. The unique Nash equilibrium is (Bottom, Right).

been made, they will make their choices simultaneously and independently. Under this contract, notice that a player may still make a unilateral profitable deviation from (Top, Left), and therefore this cannot be an equilibrium solution; however, by sticking to the agreement, a player actually guarantees 100—the highest amount that can be guaranteed in the game. If cautious players have the option of agreeing to this contract, then they clearly have incentives to do so—the amount they can guarantee increases from 0 to 100. Moreover, if they can reach this agreement under a formal contract, then they can also reach the same agreement through a hypothetical contract or tacit agreement under which everyone observes the following simple rule: Do not harm yourself for the sake of harming the other. Players may imagine that they are in the "original position" (just as in Rawls, 1971) and reach the agreement via a thought experiment. This would rationalize (Top, Left) as the unique solution that guarantees the highest payoffs under such a hypothetical contract.

Cooperative games: The optimin criterion can also be applied to cooperative games in characteristic function form. When the core is nonempty, an allocation is in the core if and only if it satisfies the optimin criterion. But as I show in subsection 4.1.1, optimin points exist even when the core is empty. As an example, consider the following cooperative game in characteristic function form in which $N = \{1, 2, 3\}$, $u(\{1\}) = 35$, $u(\{2\}) = 30$, $u(\{3\}) = 25$, $u(\{1, 2\}) = 90$, $u(\{1, 3\}) = 80$, $u(\{2, 3\}) = 70$, and $u(N) = 110$, where $u(S)$ denotes the worth of coalition S. The core of this game is empty, whereas the points that satisfy the optimin criterion can be characterized by the following set, which is illustrated in Figure 2.[3]

$$\{x \in \mathbb{R}^3 | x_1 = 40, x_2 + x_3 = 70, x_2 \geqslant 30, x_3 \geqslant 25\}.$$

The Shapley value is (44.166, 36.666, 29.166), and the nucleolus is (46.666, 36.666, 26.666). Notice that at each of these solutions, player 2 and player 3 can profitably break away

[3]For the formal definition and calculations, see subsection 4.1.1.

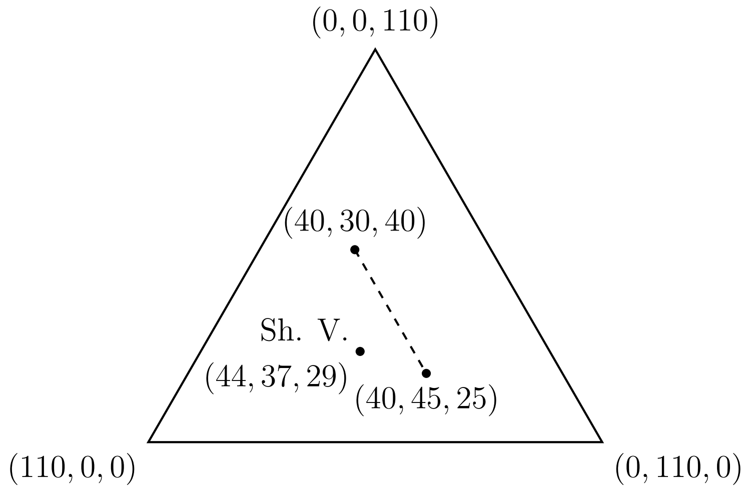

Figure 2: A game with an empty core. The set of optimin points is shown by the dashed line.

from the grand coalition to receive a joint payoff of 70.[4] As a result, the worst-case payoff of player 1 would be equal to her individual payoff, $u(\{1\}) = 35$, under both the Shapley value and the nucleolus. Notably, under the optimin criterion, player 1 receives less than both the Shapley value and the nucleolus. But this is compensated for by the fact that coalition $\{2, 3\}$ does not have any incentive to deviate from an optimin point, so player 1 can safely enjoy the (worst-case) payoff of 40 as opposed to the worst-case payoff of 35 under the Shapley value or the nucleolus.

The Arrow critique: Arrow (1973) strongly disagrees with Rawls' maximin principle, mainly from a welfare economics perspective. He draws attention to the first fundamental theorem of welfare economics, which asserts that under some mild conditions, every competitive equilibrium is Pareto optimal, given that all economic agents are rational (i.e., maximize individual utilities). Clearly, in a competitive economic system such as an Arrow-Debreu economy, the maximin criterion would not lead to the competitive equilibrium. By contrast, I will show in subsection 4.2 that every competitive economic equilibrium must satisfy the optimin principle. This result is nontrivial, in part because optimin points are logically distinct from the Nash equilibrium and Pareto optimality.

[4]This is not surprising because the Shapley value is generally regarded as an a priori assessment of the game.

2 Relevant literature

This paper contributes mainly to the literature on noncooperative solution concepts with a view to explaining the puzzling cooperative behavior in noncooperative games. The closest concept to the optimin criterion, as I have discussed, is that of the maximin criterion, which has been proposed in different contexts by a number of researchers, including Borel (1921), von Neumann (1928), Wald (1950), and Rawls (1971). There are also axiomatizations of the maximin criterion proposed by including Milnor (1954) and Gilboa and Schmeidler (1989). In their seminal work, Gilboa and Schmeidler (1989) show that maxmin expected utility can be characterized by a set of intuitive axioms for cautious decision makers with a set C of (multi-prior) subjective beliefs. Just like one can vary C in the Gilboa-Schmeidler maxmin model and rationalize different choices, one can vary the interdependent constraint B_i in the optimin model and rationalize different choices in an interactive setting. For another related axiomatization, see Puppe and Schlag (2009), who show that the axioms of Milnor (1954) that characterize the maximin decision rule are consistent with ignoring some states in which all payoffs are "small." Under reasonable conditions, optimin criterion coincides with Wald's maximin criterion and the maxmin expected utility. In more general strategic and nonstrategic contexts, the optimin criterion is yet to be axiomatized. The literature on solution concepts which incorporates various levels of cautiousness in games includes Selten (1975) and more recently Perea et al. (2006), Renou and Schlag (2010), and Iskakov et al. (2018). Prominent equilibrium concepts under ambiguity include Dow and Werlang (1994), Lo (1996), Klibanoff (1996), Marinacci (2000), and more recently Azrieli and Teper (2011), Bade (2011), Riedel and Sass (2014), and Battigalli et al. (2015). For an overview of the ambiguity models in games, see Mukerji and Tallon (2004) and Beauchêne (2014).

The optimin criterion differs from the aforementioned models and solution concepts in three main dimensions: (i) Conceptual/cognitive background, (ii) scope of application, and (iii) predictions. First, the optimin criterion is a non-equilibrium concept in which players evaluate tacit agreements by their worst-case payoffs. As such, it provides a novel extension of maximin reasoning from two-person zero-sum games to non-zero-sum n-person games. Second, although the optimin criterion is mainly devised for noncooperative games, it can also be applied to cooperative games, matching, individual decision making, statistical decision theory, and competitive economy. Third, unlike other solution concepts, the optimin criterion is consistent with the direction of non-Nash deviations in games in which cooperation has been extensively studied, including the finitely repeated prisoner's dilemma, the centipede game, the traveler's dilemma, and the finitely repeated public goods game.

Limitations: All in all, it seems unlikely that human behavior can be captured by a single reasoning process. For example, the '11-21'-type games introduced by Arad and Rubinstein (2012) seem to naturally invoke level-k reasoning (Stahl, 1993), whereas in complex games, such as Blotto games, players—who are unable to calculate optimal strategies—look at the characteristics of strategies rather than the strategies themselves (for a formalization of this type of reasoning, see Arad and Rubinstein, 2019). Moreover, in two-person zero-sum games, the optimin criterion does not tell us anything new because it coincides with the maximin strategies and the Nash equilibrium in those games. It remains to be seen the extent to which the optimin reasoning can complement other reasoning processes.

3 Optimin criterion in noncooperative games

3.1 Definitions

Let $(\Delta X_i, u_i)_{i \in N}$ be an n-person noncooperative game in mixed extension, where $N = \{1, ..., n\}$ is the finite set of players, ΔX_i the set of all probability distributions over the finite action set X_i, and $u_i : \Delta X \to \mathbb{R}$ the von Neumann-Morgenstern expected utility function of player $i \in N$. An agreement $p \in \Delta X$ denotes a strategy profile.[5]

Definition 1. The better-response correspondences $B_i(p)$ and $B_{-i}(p)$ are defined as

$$B_i(p) = \{p'_i \in \Delta X_i | u_i(p'_i, p_{-i}) > u_i(p_i, p_{-i})\}, \text{ and } B_{-i}(p) = \bigtimes_{j \in N \setminus \{i\}} (B_j(p) \cup \{p_j\}).$$

I next define the worst-case payoffs of an agreement p.

Definition 2. Given an agreement $p \in \Delta X$ and $i \in N$, the i'th component of the (optimin) *value function* $v : \Delta X \to \mathbb{R}^n$ is defined as

$$v_i(p) = \inf_{p'_{-i} \in B_{-i}(p)} u_i(p_i, p'_{-i}).$$

[5]For a detailed discussion of the mixed-strategy concept, see Luce and Raiffa (1957, p. 74). For a more recent discussion, see Rubinstein (1991). As is standard in game theory, I assume that what matters is the consequence of strategies (consequentialist approach) so that I can define the utility functions over the strategy profiles.

In other words, player i's value from an agreement p is defined as the minimum payoff the player receives (i) from the agreement, or (ii) under the better-response correspondence of other players. The next step is to make comparisons among the evaluations of agreements. Because the value function is a multi-variable function, I use a well-known multi-criteria maximization technique called Pareto-optimality. An agreement is an optimin point if its value is Pareto optimal.

Definition 3. An agreement $\bar{p} \in \Delta X$ is said to satisfy the *optimin criterion* or called an *optimin point* if for every player $i \neq j$ with $i \in N$ and every $p' \in \Delta X$, $v_i(p') > v_i(\bar{p})$ implies that there is some j with $v_j(p') < v_j(\bar{p})$. Equivalently, in the primitives of the game, $\bar{p}$ solves the following multi-objective optimization problem.

$$\bar{p} \in \arg\max_{\hat{p} \in \Delta X} \left(\inf_{p'_{-1} \in B_{-1}(\hat{p})} u_1(\hat{p}_1, p'_{-1}), \inf_{p'_{-2} \in B_{-2}(\hat{p})} u_2(\hat{p}_2, p'_{-2}), \ldots, \inf_{p'_{-n} \in B_{-n}(\hat{p})} u_n(\hat{p}_n, p'_{-n}) \right).$$

It can be stated more compactly as follows: $\bar{p} \in \arg\max_{q \in \Delta X} \inf_{p'_{-i} \in B_{-i}(q)} u_i(q_i, p'_{-i})$ for every i simultaneously. In Ismail (2014), I called $\bar{p}$ a 'maximin equilibrium' if it is an optimin point or for every i, $\bar{p}_i \in \arg\max_{q_i \in \Delta X_i} \inf_{p'_{-i} \in B_{-i}(q_i, \bar{p}_{-i})} u_i(q_i, p'_{-i})$, which could be thought of as the equilibrium counterpart of the optimin.

3.2 The intuition and rationale behind the definitions

First, Definition 1 is a standard way to define the better-response correspondence. Second, the value function in Definition 2 assigns for each player a unique value or worst-case payoff to every potential agreement p, which is a profile of strategies. To illustrate the rationale behind this definition, suppose that before playing a noncooperative game players can make a tacit (nonbinding) agreement to play a strategy profile p. Then, each player chooses his or her strategy simultaneously and independently, so each player has the option of either honoring or breaking the agreement because it is nonbinding. The optimin value assumes that players evaluate such an agreement cautiously, ruling out implausible deviations from the agreement—i.e., the deviations which do not strictly improve the payoff of the deviator, holding the others' strategies fixed. Accordingly, a player's value or worst-case payoff of following a tacit agreement is defined as the minimum utility the player receives (i) from the agreement, or (ii) under the other players' unilateral profitable deviations from the agreement, which is a rough formalization of the following simple rule I put forward earlier: Do not harm yourself for the sake of harming others.[6]

[6]Note that if p is a Nash equilibrium, then $B_i(p) = \varnothing$ because it is a self-enforcing agreement—each player best-responds holding the others' strategies fixed. But when agreement p is not a Nash equilibrium,

Notably, the value function includes only unilateral and noncooperative deviations—i.e., coalitional or correlated profitable deviations are not considered, which I consider in subsection 4.1.1. Instead of better-response deviations, other type of deviations such as best-response deviations might also be considered. All in all, definitions 1–3 would remain well-defined if correlated or best-response deviations are included.[7]

To illustrate the optimin value I return to the illustrative game in Figure 3 (left), restricting attention to pure strategies for simplicity. Figure 3 (right) shows that the value of (Top, Left) is (100, 100). This is because even if, for example, player 2 profitably deviates to 'Center', player 1 (who follows the tacit agreement) would still receive a payoff of 100. Notice that this is the only profitable deviation from agreement (Top, Left), because a deviation to 'Right' would be implausible. To give another example, the value of (Bottom, Center) is $(5, 0)$ because (i) player 1 has no profitable deviation from it, and (ii) player 2 may profitably deviate to 'Right', in which case player 1 would receive a payoff of 5. The value or worst-case payoff of (Top, Center) is 100 for player 1 because player 2 has no profitable deviation from it. By contrast, for player 2 the worst-case payoff of (Top, Center) is 0 because player 1 can profitably deviate to Bottom, which decreases player 2's payoff to 0.

Finally, Definition 3 suggests that the value function or the worst-case payoffs of players be Pareto optimized. The rationale behind this is the assumption that the greater the optimin value of an agreement for all players, the higher its likelihood to be (tacitly) agreed upon. Put differently, players—who each evaluate a potential agreement by its worst-case payoff—would rather agree on profile p than p' if the worst-case payoffs at agreement p are greater than or equal to the worst-case payoffs at p' (in other words, p

there are profitable deviations from p, in which case the "cost" of such deviations to nondeviators is measured by the worst-case payoffs under these deviations. Moreover, unlike in maximin strategies, the (optimin) value function does not assign a unique value to every strategy. Instead, the evaluation of uncertainty is attached to the strategy profile, as in the value notion of Nash (1951). Nash (1951, p. 291) defines the value of a game to a player as the payoff that the player receives from a Nash equilibrium when all of the Nash equilibria payoffs are the same for that player. The optimin value assigns a value to each agreement, including the Nash equilibria. In particular, when an agreement is a Nash equilibrium, the value of a player at this profile is equal to her Nash equilibrium payoff because it is self-enforcing.

[7] The optimin point is an application of the evaluation and comparison method I propose for solving games. For a more general application, we may interpret $B_j(p)$ as being the belief of some player i about player j's (or a coalition's) potential deviations, and apply the evaluation step accordingly. As mentioned earlier, just like one can vary C in the Gilboa-Schmeidler maxmin model and rationalize different choices, one can vary B_j and rationalize different choices in a game setting. Maximin strategy corresponds to the case in which a player's belief about her opponent's deviations is the whole strategy set of the opponent. That is, player i does not take individual rationality of the opponent into account. The optimin principle can be incorporated with stronger or weaker individual rationality assumptions, even with different ones for different players, by following the same method we follow in this section.

	Left	Center	Right
Top	$100, 100$	$100, 105$	$0, 0$
Middle	$105, 100$	$95, 95$	$0, 210$
Bottom	$0, 0$	$210, 0$	$5, 5$

	Left	Center	Right
Top	$(100, 100)$	$(100, 0)$	$(0, 0)$
Middle	$(0, 100)$	$(0, 0)$	$(0, 5)$
Bottom	$(0, 0)$	$(5, 0)$	$(5, 5)$

Figure 3: An illustrative game (left) and its value function (right). The unique optimin point is the agreement (Top, Left), whereas every strategy is a maximin strategy.

Pareto dominates p'). So, the most likely tacit agreements lie on the Pareto frontier of the value function.[8] In other words, an agreement satisfies the optimin criterion if it is not possible to increase a player's worst-case payoff without decreasing another player's worst-case payoff.[9]

To illustrate the optimin criterion, I return to the game and its value function in Figure 3. Notice that the value function is Pareto optimized at the agreement (Top, Left), so it is the unique optimin point. To compare this solution with the maximin strategy and Nash equilibrim, notice that every strategy is a maximin strategy, which guarantees a payoff of 0, whereas the unique Nash equilibrium is (Bottom, Right).

As mentioned in the introduction, the optimin point (Top, Left) can be justified as a solution as follows. Suppose that before the game, Alice and Bob set up a binding contract in which, if they agree to play a particular profile, no one will make a nonprofitable deviation. After the agreement has been made, they will make their choices simultaneously and independently. Under the contract, a player may still make a unilateral profitable deviation from (Top, Left), so it cannot be an equilibrium solution; but by sticking to the agreement, a player actually guarantees 100—the highest amount that can be guaranteed in this game. If cautious players have the option of agreeing to this contract, then they clearly have incentives to do so—the amount they can guarantee increases from 0 to 100. Moreover, if they can reach this agreement under a formal contract, then they can also reach the same agreement through a hypothetical contract or tacit agreement under which everyone observes the following simple rule: Do not harm yourself for the sake of harming the other. As mentioned earlier, players may imagine that they are in the "original position" and reach the agreement via a thought experiment rather than signing

[8] Notice that Pareto optimality applies only to the value function of players, *not* to the utility function. Furthermore, there is also no logical relationship between Pareto optimality and the optimin point. In the battle of the sexes game, for example, the two optimin points are (Football, Football) and (Opera, Opera), which are Pareto optimal. However, the optimin point may be Pareto dominated. In the prisoner's dilemma, the unique optimin point is (Defect, Defect), which is Pareto dominated.

[9] In an earlier version, I also used another maximization principle in the comparison step, namely the Nash equilibrium, by applying it to the game defined by the value function.

a binding contract. This would rationalize (Top, Left) as a solution that guarantees the highest payoffs under such a hypothetical contract.

3.3 Existence, properties, and applications

The following lemma presents a property of the value function which will be used in the following existence result.

Lemma 1. *The value function of a player is upper semi-continuous.*

Proof. In several steps, I will show that the value function v_i of player i in a game $\Gamma = (\Delta X_1, \Delta X_2, u_1, u_2)$ is upper semi-continuous.[10]

First, we decompose the value function as

$$v_i(p) = \min\{\inf_{p'_j \in B_j(p)} u_i(p_i, p'_j),\ u_i(p)\},$$

where $B_j(p)$ is the (strict) better response correspondence of player j with respect to p, representing the set of profitable deviations, which is defined as

$$B_j(p) = \{p'_j \in \Delta X_j | u_j(p_i, p'_j) > u_j(p)\}.$$

I next show that the correspondence $B_j : \Delta X_i \times \Delta X_j \twoheadrightarrow \Delta X_j$ is lower hemi-continuous. For this, it is enough to show the graph of B_j defined as follows is open.

$$Gr(B_j) = \{(q, p_j) \in \Delta X_i \times \Delta X_j \times \Delta X_j |\ p_j \in B_j(q)\}.$$

$Gr(B_j)$ is open in $\Delta X_i \times \Delta X_j \times \Delta X_j$ if and only if its complement is closed. Let $[(p_j, q_i, q_j)^k]_{k=1}^{\infty}$ be a sequence in $[Gr(B_j)]^c = (\Delta X_i \times \Delta X_j \times \Delta X_j)\backslash Gr(B_j)$, converging to (p_j, q_i, q_j) where $p_j^k \notin B_j(q^k)$ for all k. That is, we have $u_j(p_j^k, q_i^k) \leqslant u_j(q^k)$ for all k. Continuity of u_j implies that $u_j(p_j, q_i) \leqslant u_j(q)$, which means $p_j \notin B_j(q)$. Hence $[Gr(B_j)]^c$ is closed, implying that B_j is lower hemi-continuous.

Next, we define $\hat{u}_i : \Delta X_i \times \Delta X_j \times \Delta X_j \to \mathbb{R}$ by $\hat{u}_i(q_i, q_j, p_j) = u_i(p_j, q_i)$ for all $(q_i, q_j, p_j) \in \Delta X_i \times \Delta X_j \times \Delta X_j$. Since u_i is continuous, $\hat{u}_i$ is also continuous. In addition, we define $\bar{u}_i : Gr(B_j) \to \mathbb{R}$ as the restriction of $\hat{u}_i$ to $Gr(B_j)$, that is $\bar{u}_i = \hat{u}_{i_{|Gr(B_j)}}$. The continuity of $\hat{u}_i$ implies the continuity of its restriction $\bar{u}_i$, which in turn implies $\bar{u}_i$ is upper semi-continuous.

[10] The extension of the arguments to n-person case is completely analogous as long as n is finite, as is assumed.

By Theorem 1 of Berge (1959, p. 115), lower hemi-continuity of B_j and lower semi-continuity of $-\bar{u}_i : Gr(B_j) \to \mathbb{R}$ imply that the function $-\bar{v}_i : \Delta X_i \times \Delta X_j \to \mathbb{R}$—defined as $-\bar{v}_i(q) = \sup_{p_j \in B_j(q)} -\bar{u}_i(p_j, q)$—is lower semi-continuous.[11] This implies that the function $\bar{v}_i(q) = \inf_{p_j \in B_j(q)} \bar{u}_i(p_j, q)$ is upper semi-continuous.[12]

As a result, the value function $v_i(q) = \min\{\bar{v}_i(q), u_i(q)\}$ of player i is upper semi-continuous because the minimum of two upper semi-continuous functions is also upper semi-continuous. □

The following theorem shows that the optimin point exists in mixed strategies.

Theorem 1. *Every mixed extension of a finite game has an optimin point.*

Proof. Define $v_i^{max} = \arg\max_{q \in \Delta X_i \times \Delta X_j} v_i(q)$, which is a nonempty compact set because $\Delta X_i \times \Delta X_j$ is compact, and v_i is upper semi-continuous by Lemma 1. Since v_i^{max} is compact and v_j is also upper semi-continuous, the set $v_{ij}^{max} = \arg\max_{q \in v_i^{max}} v_j(q)$ is nonempty and compact. Clearly, the profiles in v_{ij}^{max} are Pareto optimal with respect to the value function, implying that v_{ij}^{max} is a nonempty compact subset of the set of optimin points in the game. Analogously, the set v_{ji}^{max} is also a nonempty compact subset of the set of optimin points. (Note that these arguments can be applied to games with any finite number of players.) □

Notice that we have used neither the convexity of the strategy sets nor the concavity of the utility functions in the proof of Lemma 1 or Theorem 1. Thus, the latter result can be stated more generally as follows: Any game with continuous utility functions and compact strategy spaces possesses an optimin point.

Harsanyi and Selten (1988, p. 70) argued that invariance with respect to positive linear transformations of the utilities is a fundamental requirement for a solution concept. This requirement is satisfied by the optimin point as the next proposition shows.

Proposition 1. *Optimin points are invariant to positive linear transformation of the utilities.*

Proof. Let Γ and $\hat{\Gamma}$ be two games such that that $\hat{u}_i = \alpha u_i + \beta$ for some $\alpha > 0$ and some constant β. First, we have that $\hat{v}_i = \alpha v_i + \beta$ because strict better response correspondence does not change, and we can take α and β out of the infimum in the definition of v_i. Second, a profile p is a Pareto optimal profile with respect to v if and only if it is Pareto

[11] I follow the terminology, especially the definition of upper hemi-continuity, presented in Aliprantis and Border (1994, p. 569).

[12] I use the fact that a function f is lower semi-continuous if and only if $-f$ is upper semi-continuous.

optimal with respect to $\hat{v}$ because each v_i is a positive linear transformation of $\hat{v}_i$. As a result, the set of optimin points of Γ and $\hat{\Gamma}$ are the same. □

The following proposition shows that without the individual rationality assumption, the optimin criterion would coincide with the maximin criterion.

Proposition 2. *Suppose that in the definition of the value function $B_{-i}(p)$ is replaced with ΔX_{-i}. Then, the optimin criterion solution reduces to a profile of maximin strategies.*

Proof. We take the infimum over ΔX_{-i} instead of taking it over $B_{-i}(p)$ in the value function definition. Hence, we have that $\bar{v}_i(p) = \inf_{p'_{-i} \in \Delta X_{-i}} u_i(p_i, p'_{-i})$. Then, $\bar{p}$ is a Pareto dominant profile of the value function $\bar{v}$ where $\bar{p}_i \in \arg\max_{p'_i \in \Delta X_i} \inf_{p'_{-i} \in \Delta X_{-i}} u_i(p'_i, p'_{-i})$. It is clear that $\bar{p}_i$ is a maximin strategy of player i. □

The next proposition shows that the optimin criterion in constant-sum games generalizes the Nash equilibrium.

Proposition 3. *Every Nash equilibrium in an n-person constant-sum game is an optimin point.*

Proof. The utility vector of a Nash equilibrium is the same as its value because there is no unilateral profitable deviation from it. In addition, it always holds that $v_i(p) \leqslant u_i(p)$ for every p and every player i. Because every strategy profile is Pareto optimal in n-person constant-sum games, the value of a Nash equilibrium must be Pareto optimal. Therefore, a Nash equilibrium must be an optimin point in any n-person constant-sum game. □

For every n-person game, we can define a fictitious $(n + 1)$-person game in which, all else being equal, the $(n+1)$st player has only one strategy and his or her payoffs are such that the new game is of constant-sum.

Proposition 4. *Every Nash equilibrium in a general n-person game satisfies the optimin criterion in the $(n + 1)$-person fictitious game.*

Proof. Note that a Nash equilibrium in an n-person game is also a Nash equilibrium in the $(n+1)$-person fictitious game (given the action of the fictitious player). By Proposition 3, a Nash equilibrium in the fictitious game must be an optimin point because it is a constant-sum game. □

The following proposition shows the existence of an optimin point in pure strategies when the game is restricted to pure strategies.

Proposition 5. *Every finite game restricted to pure strategies has an optimin point in pure strategies.*

The proof of this proposition is straightforward because there are finitely many pure strategies in finite games, so a Pareto optimal point of the value function exists, which corresponds to an optimin point of the original game. This proposition is useful in part because it guarantees the existence of pure optimin points when attention is restricted to pure strategies, as finding mixed strategy equilibria can be tedious in many games.

3.3.1 Applications: Explaining non-Nash deviations towards cooperation in non-cooperative games

Cooperation among individuals has been the subject of many experimental investigations in economics and other (social) sciences. A well-established and systematic finding is that individuals are more cooperative than the Nash equilibrium suggests. This is puzzling, especially because players earn more payoffs by cooperating than they would by playing noncooperative equilibrium. Another consistent finding is that we cannot disregard "selfish" noncooperative behavior because, while for some economically relevant parameters in a given game the play converges towards cooperation, for some other parameters the play converges to strictly noncooperative (equilibrium-like) behavior.

Most common games in which cooperation has been studied include the finitely repeated prisoner's dilemma, the traveler's dilemma, the centipede game, and the finitely repeated public goods game. The optimin criterion can selectively explain the direction of non-Nash deviations in these games. It is selective in the sense that when cooperation satisfies the optimin criterion—i.e., when the worst-case payoffs under cooperation are greater than under defection—noncooperative behavior typically does not satisfy it, and vice versa.

The predictions and applications of the optimin criterion in well-studied non-cooperative games have been presented in a companion "applications" paper to ensure that the current paper does not become excessively lengthy. Below, I give a non-technical summary of the optimin predictions in well-known economic games, which is self-contained and accesible to anyone who is familiar with those games.[13]

The traveler's dilemma is a two-person game (illustrated in Figure 4) in which each player picks a number between 2 and 100; the one who chooses the smaller number, n, receives n plus a reward $r > 1$, and the other receives n minus r; if they both choose n,

[13] For the technical details of the derivations, please see https://arxiv.org/abs/1912.00211.

	100	99	$\cdots$	3	2
100	$100,\ 100$	$99-r,\ 100+r$	$\cdots$	$3-r,\ 3+r$	$2-r,\ 2+r$
99	$100+r,\ 99-r$	$99,\ 99$	$\cdots$	$3-r,\ 3+r$	$2-r,\ 2+r$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
3	$3+r,\ 3-r$	$3+r,\ 3-r$	$\cdots$	$3,\ 3$	$2-r,\ 2+r$
2	$2+r,\ 2-r$	$2+r,\ 2-r$	$\cdots$	$2+r,\ 2-r$	$2,\ 2$

Figure 4: Traveler's dilemma with reward/punishment parameter r. The unique optimin point is to play the highest (lowest) number when r is small (big). The unique Nash equilibrium is to choose the lowest number regardless of r.

then they each receive n (Basu, 1994). Consistent experimental findings show that the behavior of subjects crucially depends on the reward/punishment parameter: When r is "small," as in the original game, the subjects' behavior converges towards the highest number, whereas when r is "large," their behavior converges towards the lowest number. (See, e.g., Goeree and Holt, 2001, Capra et al., 1999, and Rubinstein, 2007.) The Nash equilibrium is insensitive to r, which predicts that the lowest number will always be chosen. By contrast, the optimin point is responsive to r: The unique optimin point coincides with the Nash equilibrium when r is large, but when r is small, only the highest pair of numbers satisfies the optimin criterion. The reason is that, as the reward parameter increases, the worst-case payoffs of cooperation decrease, and at some point, the worst-case payoffs for the highest number (100) become smaller than the worst-case payoffs for the lowest number (2).

The centipede game is a two-person extensive-form game of perfect information where each player can choose to either continue (cooperate) or stop (defect) at each node (Rosenthal, 1981). One of the most common and replicated finding is that, on average, subjects show the most cooperative behavior in increasing-sum centipedes and the most noncooperative behavior in constant-sum centipedes. (See, e.g., McKelvey and Palfrey, 1992, Krockow et al., 2016, and the references therein.) Yet, the unique subgame perfect equilibrium is always choosing to stop at every decision node whether there are gains from cooperation or not. The optimin criterion can explain the direction of these non-Nash deviations. The unique optimin point leads to cooperation in increasing-sum centipedes whenever the number of decision nodes is greater than or equal to four, whereas the optimin criterion uniquely coincides with the equilibrium prediction in constant-sum centipedes, suggesting that the player should play stop immediately. Moreover, as the number of decision nodes increases, the worst-case payoffs between cooperation and defection become larger in increasing-sum centipedes, but this gap decreases as the number of decision nodes decreases. Eventually the worst-case payoffs for defection become greater than the worst-case payoffs for cooperation as the game progresses. This provides an explanation as to why cooperation may decrease as the game proceeds.

The finitely repeated n-person public goods game is a repeated game in which players simultaneously choose to contribute something to a public pot in the stage game. Not contributing anything (i.e., free-riding) is a dominant strategy for every player, but if everyone contributes (i.e., if they cooperate), then everyone will be better off. Experimental research indicates that cooperation (i) is significantly greater in games with high marginal per capita return (MPCR) compared to games with low MPCR, (ii) decreases as the game progresses, (iii) restarts if the finitely repeated game is played again, and (iv) is magnified by pre-play communication. (See, e.g., Isaac et al., 1984, Lugovskyy et al., 2017, and the references therein). While the unique subgame perfect equilibrium predicts 0 contribution in every round irrespective of parameters such as MPCR, the optimin criterion gives an explanation for these experimental findings. Comparative statics on exogenous parameters of the game shows the following regularities. First, for high (low) values of MPCR, cooperative (free-riding) behavior satisfies the optimin criterion. Second, as the game progresses, the worst-case payoffs of free-riding get closer to, and eventually becomes greater than, the worst-case payoffs of cooperation. But if the finitely repeated game is played again, the value of cooperation at the beginning of the game is again greater than the value of free-riding, which can explain the "restart" effect. Finally, pre-play communication facilitates players agreeing to cooperative behavior—though these are certainly tacit agreements because they are nonbinding.

The finitely repeated prisoner's dilemma is a well-known two-person repeated game in which it is a dominant strategy to defect in the stage game. As is well-known, the unique subgame perfect equilibrium prescribes defection in every round. Experiments suggest that (i) initial cooperation increases as the number of rounds increases, and (ii) cooperation decays as the end of the game approaches. (See, e.g., Axelrod, 1980, Embrey et al., 2017, and the references therein). The optimin criterion gives an explanation for these regularities. Although in the one-shot game the unique optimin point coincides with the unique Nash equilibrium, cooperation generally satisfies the optimin criterion in the repeated prisoner's dilemma because, even if a player tries to take advantage of cooperative behavior, the worst-case payoff of the cooperator is greater than the subgame perfect equilibrium payoff. As the number of rounds increases, the worst-case payoffs of cooperation increase. However, these worst-case payoffs gradually decrease as the game progresses, and they eventually become less than the worst-case payoffs of defection.

4 Applications beyond noncooperative games

4.1 Cooperative games

4.1.1 Games in characteristic function form

In their groundbreaking book, von Neumann and Morgenstern (1944) introduced cooperative games in which the so-called characteristic function assigns a unique number to each coalition or subset of players. In this section, I assume transferable utility—i.e., the utility of a coalition can be redistributed among its members. The concepts introduced in this section can be analogously extended to games with nontransferable utility.

Let (N, u) be an n-person cooperative game in characteristic function form, where $N = \{1, ..., n\}$ is the finite set of players and $u : 2^N \to \mathbb{R}$ with $u(\varnothing) = 0$ is the characteristic function which is cohesive.[14] Notation $S \subseteq N$ denotes a coalition, $(x_i)_{i \in S}$ denotes a payoff allocation for coalition S where $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^N$, and $x(S) = \Sigma_{i \in S} x_i$. Characteristic function U is called *cohesive* if u satisfies the following property: $u(N) \geqslant \sum_{k=1}^{K} u(S_k)$ for every partition $\{S_1, ..., S_K\}$ of N. This assumption restricts the attention to games in which the grand coalition N forms.

A vector $x \in \mathbb{R}^N$ is called *feasible* if $x(N) \leqslant u(N)$. A feasible $x \in \mathbb{R}^N$ is called an *imputation* if $x_i \geqslant u(\{i\})$ for all $i \in N$ and $x(N) = u(N)$. The set of all imputations is denoted by $I(v)$. A vector $y \in \mathbb{R}^N$ is said to *dominate* $z \in \mathbb{R}^N$ via coalition S if for all $i \in S$ and $z(S) < y(S) \leqslant u(S)$, in which case S has a *profitable deviation* from z. An allocation z is dominated by another allocation y if y dominates z. The set of all imputations that are not dominated by another imputation is called the *core*, which may be empty.

Definition 4. The *value function* of a cooperative game (N, u) is a mapping $V : X \to \mathbb{R}^N$. For any feasible payoff vector $x \in X \subset \mathbb{R}^N$, the i'th component of V, $V_i : X \to \mathbb{R}$, is defined as

$$V_i(x) = \min\{x_i, \min_{S \in D_{-i}(x)} (x_i - \frac{x(N \backslash S) - u(N \backslash S)}{|N \backslash S|})\},$$

where $D_{-i}(x) = \{S \subseteq N \backslash \{i\} \mid x \text{ is dominated via coalition } S\}$—the set of all coalitions excluding player i that dominate the payoff vector x.

[14] Note that characteristic function is usually called value function denoted by v in the literature; for a reference textbook, see, e.g., Peters (2015). To avoid confusion with the value function I define earlier in this paper, I call the characteristic function utility function denoted by u.

The value of a payoff distribution x to a player i is the minimum payoff the player receives between (i) her payoff and (ii) the worst-case utility she would get if coalition S profitably deviates from x. The intuition is that if x is dominated via S, then player i looks at her worst-case payoff when coalition S indeed deviates.[15]

Of course, it is possible, and perhaps desirable, to consider more "clever" profitable coalitional deviations than we assume when calculating the worst-case payoffs of a distribution. For example, Harsanyi (1975) criticized the core as it is based on "myopic" deviations because a deviating coalition does not consider the possibility of another coalition deviating further. Harsanyi's (1975) observation has led to a large literature on solution concepts with farsighted individuals. First, it is possibility to define the value or the worst-case payoff of a player under profitable farsighted deviations rather than just one-off deviations (For a comprehensive survey and references, see, e.g., Ray and Vohra, 2015). Second, there may be cases in which a payoff distribution, x, is dominated via some S. Then, the worst-case payoff of player i in that situation would depend on the worth of nondeviating coalition $N\backslash S$ which includes i, and how this worth is distributed among players in $N\backslash S$. Of course, distributing worth of a coalition among its members defines another game, which may be solved recursively. Third, there may be two different subsets of $N\backslash\{i\}$ that have a profitable deviation from x, in which case it would be sensible to consider only "maximal" or "best-response" deviations—i.e., deviations that give the largest payoff to a deviating coalition.

While these are all potential research directions, the purpose of this section is to illustrate how "evaluate and compare" method and its specific application, the optimin criterion, I introduced in this paper can be applied to cooperative games. As before, the evaluation step gives a value to each payoff distribution based on deviations that are deemed "reasonable," and the comparison step makes comparison among these evaluations.

Definition 5. A feasible payoff distribution $x \in \mathbb{R}^N$ is said to satisfy the optimin criterion or called an *optimin point* if for every player $i \neq j$ and every feasible $x' \in \mathbb{R}^N$, $V_i(x') > V_i(x)$ implies that there is some j with $V_j(x') < V_j(x)$.

As before, if the value of a feasible payoff vector is Pareto optimal, then it is called an

[15]For convenience, I defined the worst-case payoff of a player i from some x when a coalition S profitably deviates from x by $\min_{S \in D_{-i}(x)}(x_i - \frac{x(N\backslash S) - u(N\backslash S)}{|N\backslash S|})$. The intuition is that if the total utility of the non-deviating coalition $N\backslash S$ is less than the total payoff this coalition was supposed to receive before the deviation, then the losses are shared equally. It could be defined more generally—it is sufficient that when a coalition S profitably deviates from x, it must hold that $\sum_{i \in N\backslash S} V_i(x) = u(N\backslash S)$.

optimin point.[16] I next present two useful results before giving an illustrative example.

Theorem 2. *There exists an optimin point in every cooperative game in characteristic function form.*

Proof. We can apply the same steps as in Lemma 1 and Theorem 1 given the following facts. First, note that the "better-response correspondence"—i.e., the set of dominating payoff allocations given an allocation—is lower hemi-continuous because it is a finite set since there are finitely many coalitions. Second, the utility function of players are continuous because the domain of the function is simply finite. Then, applying the same steps as in Lemma 1 and Theorem 1 show that the set of optimin points in a cooperative game is nonempty. □

Theorem 3. *Suppose that the core is nonempty. Then, a feasible payoff distribution x is in the core if and only if x is an optimin point.*

Proof. Suppose that x is in the core, which is nonempty by assumption. By definition, there is no individual or coalitional profitable deviation from an element x in the core. Thus, the value of x is equal to its payoff vector, which is Pareto optimal. As a result, x is an optimin point. Conversely, to reach a contradiction suppose that x satisfies the optimin criterion but is not in the core. It implies that there is some coalition S who has a profitable deviation from x—i.e., $x(S) < u(S)$. Then, the total value of the players in the non-deviating coalition $N\backslash S$ must be less than the sum of their payoffs—i.e., $\sum_{j\in N\backslash S} V_j(x) \leqslant x(N\backslash S)$, by definition of the value function. Moreover, cohesiveness of the characteristic function implies that both the deviating coalition S and the non-deviating coalition $N\backslash S$ would each receive (and guarantee in terms of value) in total weakly more payoff in case the grand coalition forms. The last two statements imply that $\sum_{i\in N} V_i(x) < u(N)$, which is possible to achieve as the core is nonempty. The difference $(u(N) - \sum_{i\in N} V_i(x))$ then can be reallocated in a way that makes at least a player better off without reducing the value of another player. Therefore, x cannot be an optimin point as is assumed. As a result, we obtain that if x is an optimin point, which we know it exists by Theorem 2, it must be in the core, which is nonempty by assumption. □

Corollary 1. *The nucleolus satisfies the optimin criterion whenever the core is nonempty.*

Proof. When the core is nonempty the nucleolus is in the core. Thus, by Theorem 3 nucleolus satisfies the optimin criterion. □

[16] One could also define "optimin core" as the core of the cooperative game with respect to the worst-case payoffs function.

As Theorem 3 illustrates the set of optimin points is equivalent to the core whenever the core is nonempty. Corollary 1 shows that when the nucleolus is in the core it satisfies the optimin criterion. But when the core is empty, this result no longer holds as the following example illustrates. The game is adapted from Kahan and Rapoport (1984, p. 61) to compare the core, the Shapley value (Shapley, 1953), the nucleolus (Schmeidler, 1969), and the optimin criterion.

Example: A game with an empty core. Suppose that $N = \{1, 2, 3\}$ and $u(\{1\}) = 35$, $u(\{2\}) = 30$, $u(\{3\}) = 25$, $u(\{1, 2\}) = 90$, $u(\{1, 3\}) = 80$, $u(\{2, 3\}) = 70$, and $u(N) = 110$.

Solution of the game: First note that the core of this game is empty because $x_1 + x_2 + x_3 = 110$, $x_1 + x_2 \geqslant 90$, $x_1 + x_3 \geqslant 80$, and $x_2 + x_3 \geqslant 70$ imply that $x_1 \geqslant 50$, $x_2 \geqslant 40$, and $x_3 \geqslant 30$, which lead to a contradiction. The Shapley value of this game can be calculated by taking the average of marginal contributions, which is $(44.166, 36.666, 29.166)$ as is illustrated in Figure 5. The nucleolus of the game is $(46.666, 36.666, 26.666)$. Next, I show that the set of points that satisfy the optimin criterion can be characterized by

$$\{x \in \mathbb{R}^3 | x_1 = 40, x_2 + x_3 = 70, x_2 \geqslant 30, x_3 \geqslant 25\}.$$

First, suppose that x is an optimin point with the property that any of the two players can profitably deviate. Then the value of $x_i = u(\{i\})$ or less. Suppose that $x_1 + x_3 = 80$ so that $\{1, 3\}$ would not deviate, but then 2 would get only 30, which is no greater than it can get individually. If $x_1 + x_2 = 90$ so that $\{1, 2\}$ would not deviate, then $x_3 = 20$, which is less than what 3 can get individually. Finally, suppose that $x_2 + x_3 = 70$ so that coalition $\{2, 3\}$ does not have an incentive to deviate, hence $v_1(x) = 40$, which is actually greater than what 1 can get individually (35). The values of 2 and 3 are $v_2(x) = 30$ and $v_3(x) = 25$ because at distribution $x = (40, x_2, x_3)$ with $x_2 + x_3 = 70$ and $x_2 \geqslant 30, x_3 \geqslant 25$ player 1 can profitably form a coaliton with either 2 or 3. As a result, the worst-case payoff that 2 or 3 can expect is their individual payoff.

To compare the optimin criterion with the Shapley value, notice that every two-player coalition would like to deviate from the payoff distribution suggested by the Shapley value.[17] For example, $\{2, 3\}$ would profitably deviate from the Shapley value distribution and get 70 together, as a result of which 1's payoff would decrease to 35. Compared to the Shapley value (44) and the nucleolus (47), player 1's payoff is lower under the

[17]This is not surprising because Shapley (1953) himself put it: "... the value is best regarded as an a priori assessment of the situation, based on either ignorance or disregard of the social organization of the players."

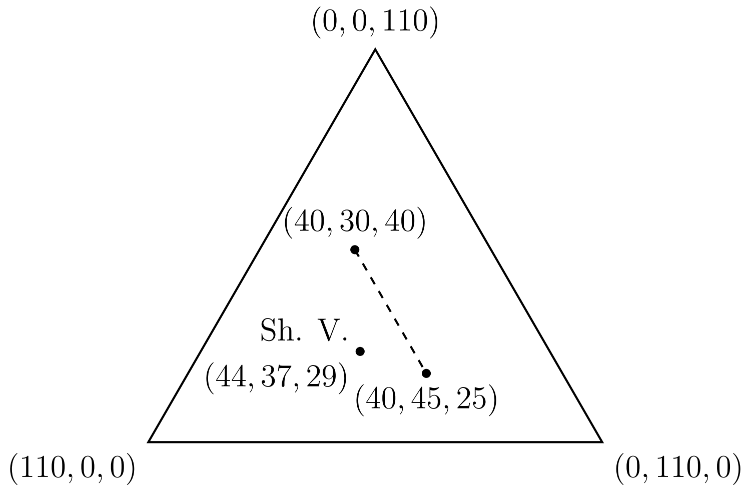

Figure 5: A game with an empty core. The set of optimin points are shown by the dashed line.

optimin criterion (40) in a way that gives 2 and 3 just enough payoff to prevent them from deviating (because they receive 70 in total). Thus, having a lower payoff gives player 1 the safety to enjoy the (worst-case) payoff of 40.

Let's modify Example 7 so that $u(N) = 110 + c$ with $c > 0$, everything else being equal. As c increases, the set of optimin points follows a pattern similar to before

$$\{x \in \mathbb{R}^3 | x_1 = 40 + c, x_2 + x_3 = 70, x_2 \geqslant 30, x_3 \geqslant 25\},$$

up to $c = 10$, in which case the optimin point becomes unique, which is $(50, 40, 30)$ as is illustrated in Figure 6. This is because if $x_1 > 50$, then $\{2, 3\}$ would jointly deviate from N to receive 70, in which case player 1's value would decrease to 35. By similar arguments, one can show that $(50, 40, 30)$ is indeed the unique optimin point. It turns out that $(50, 40, 30)$ is also the unique element in the core of the game when $u(N) = 120$, and the Shapley value is $(47.5, 40, 32.5)$. When $c > 10$, the core gets larger, hence the set of the optimin points.

4.1.2 Matching markets

Gale and Shapley (1962) published in the *American Mathematical Monthly*, a paper that is generally considered to have initiated matching theory. Another remarkable point about this paper is that it contained almost no explicit mathematics such as formulas. In this paper, the authors introduced a "two-sided" matching model in which there are two

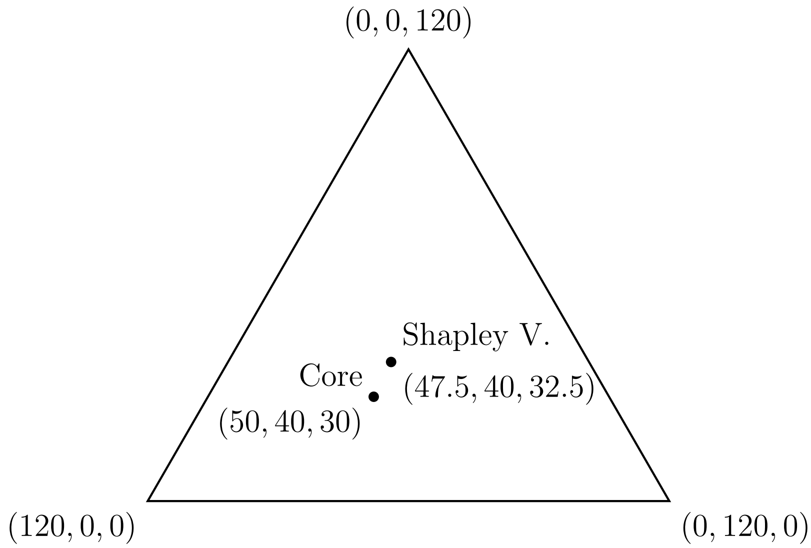

Figure 6: The core, the nucleolus, and the unique optimin point coincide at $(50, 40, 30)$ when $u(N) = 120$. The Shapley value is $(47.5, 40, 32.5)$.

sets of invididuals (or objects) that need to be paired or matched. Two-sided matching problems include the marriage market, school choice, medical labor markets; one-sided matching problems include housing markets and kidney exchange. The literature on matching markets has grown considerably since the publication of Gale and Shapley (1962) and another seminal paper by Shapley and Scarf (1974). For related literature, see the references in surveys, such as Roth and Sotomayor (1992), and Sönmez and Ünver (2011).

Let $(A, B, (\succ_i)_{i \in A \cup B})$ be a *marriage problem* where A and B are two disjoint finite sets, in which each individual i in a set A or B ranks those potential partners in the other set. For convenience, I assume that there are n individuals in each set and preferences are strict. Preference of i is captured by $\succ_i$, which is over $C \cup \{i\}$ where C is A or B and $i \notin C$. Notation $i \succ_i j$ means that individual i would not like to marry j.

Matching in a marriage problem is a function, $\mu : A \cup B \to A \cup B$ that satisfies the following properties:

1. $\mu(a) \notin B$ implies that $\mu(a) = a$ for all a in A;
2. $\mu(b) \notin A$ implies that $\mu(b) = b$ for all b in B;
3. $\mu(a) = b$ if and only if $\mu(b) = a$ for all a in A and b in B.

A matching μ is called *individually rational* if there is no individual i such that $i \succ_i$

$\mu(i)$. A matching is called *stable* if it is individually rational and there are no $a \in A$ and $b \in B$ who are not married to each other yet prefer each other to their current partners.

The optimin principle's application to a matching problem is similar to its application to cooperative games. Let μ be a matching and $I \subset A \cup B$ be a group of players in which each player in I is either single or matched with another partner in I. Then, I is said to be a *profitable deviation* from a matching μ if every player in I prefers their new partner to their current partner. As before, the first step is to evaluate a matching. The *value* of a matching μ to an individual i is the worst-case outcome under (i) the matching μ or (ii) any profitable deviation by an individual or group, I. Let $\succ_{v_i}$ denote the preferences of i based on the value of a matching—i.e., if the value of matching μ is greater than or equal to the value of matching μ', then $\mu \succeq_{v_i} \mu'$. The second step would be to make comparisons among the evaluations of various matchings. Accordingly, a matching is said to satisfy the optimin criterion if its value is Pareto optimal—no individual can improve his or her worst-case outcome without decreasing someone else's worst-case outcome.

Definition 6. A matching μ is said to satisfy the *optimin criterion* if for every player $i \neq j$ and every μ', $\mu' \succ_{v_i} \mu$ implies that there is some j with $\mu \succ_{v_j} \mu'$.

Proposition 6 shows that every stable matching in the marriage problem is an optimin point.

Proposition 6. *Every stable matching satisfies the optimin criterion.*

Proof. Because there is neither a unilateral nor a group profitable deviation from a stable matching (see, e.g., Roth and Sotomayor, 1992), each individual's value of a stable matching is equal to the matching's "payoff" to the individual. (This is similar to the fact that the value of a Nash equilibrium is exactly its payoff vector.) It is left to show that the value of a stable matching is Pareto optimal, which is true because every stable matching is Pareto optimal (see, e.g., Abdulkadiroğlu and Sönmez, 2013). □

By similar arguments, one could show that the result would extend to college admission problems (many-to-one matching). However, there are problems such as the roommate problem, in which the existence of stable matchings is not guaranteed. In such situations, a matching that satisfies the optimin criterion would always exist as long as there are finitely many individuals or objects to be matched. I omit this existence proof as it is essentially the same as the proof of the existence of pure optimin points in strategic games (Proposition 5).

Shapley and Scarf (1974) proposed a housing market (one-sided matching) model in which a set of houses is to be assigned to a set of individuals who have initial endowments. (For formal model see, e.g., Sönmez and Ünver, 2011). Gale's Top Trading Cycles (TTC) algorithm gives a rather strong solution to this problem: It chooses unique matching in the core of the housing market, which is Pareto efficient and individually rational (Roth and Postlewaite, 1977). Definition of the optimin principle in one-sided matching models would be similar to its definition in two-sided markets. The aforementioned properties of the TTC algorithm show that its outcome satisfies the optimin criterion. Moreover, the outcome of the TTC algorithm turns out to be the unique competitive equilibrium allocation. Indeed, in the next subsection (4.2), I show that every competitive equilibrium satisfies the optimin criterion.

4.2 The Arrow critique and Arrow-Debreu economy

Arrow's (1973) disagreement with Rawls' maximin principle mainly stems from its predictions in welfare economics. For example, the first fundamental theorem of welfare economics asserts that under some mild conditions every competitive equilibrium is Pareto optimal provided that economic agents are utility-maximizers. In contrast to the maximin principle, I show below that every competitive economic equilibrium must satisfy the optimin principle.

Arrow and Debreu (1954) define an abstract economy as a game situation in which—unlike a standard game—the strategy sets of players depend on the strategies chosen by other players. In an abstract economy, the players are consumers, producers, and the market participants who choose prices. The strategy set of a player depends on the choices of the others because players have constraints such as the budget constraint of consumers, which depend on the price and, in turn, depend on the choices of the players.

Let $\mathcal{E} = (Y_i, u_i, F_i)_{i \in N}$ be an *abstract economy* where N denotes the set of players, Y_i the strategy set, u_i the utility function, and $F_i : Y_{-i} \twoheadrightarrow Y_i$ the feasibility correspondence of player $i \in N$, which gives player i's set of feasible strategies given the other players' strategies.

The value function in an abstract economy is defined completely analogous to the value function in games: For every profile, the value function gives the minimum utility a player would receive under the other players' unilateral profitable deviations. The following two lemmata will be used in the proof of the existence theorem.

Lemma 2. *The value function of each player is upper semi-continuous.*

The proof follows essentially the same steps as the proof of Lemma 1, therefore I do not reproduce it. (Because utility functions are continuous, the strict better reply correspondences have open graphs, so Berge's aforementioned theorem applies.)

Lemma 3. *The correspondence $\mathcal{F}(y) = \{y \in Y | y_i \in F_i(y_{-i})$ for all $i\}$ is compact for all y.*

Proof. We first show that $\mathcal{F}(y)$ is closed where $y \in Y$. Take a sequence y^k converging to $\bar{y}$ such that such that $y^k \in \mathcal{F}(y)$ for all k. This implies that $y^k \in Gr(F_i) = \{y \in Y | y_i \in F_i(y_{-i}\}$ for all i. Since $Gr(F_i)$ is closed by our supposition and y^k converges to $\bar{y}$, we have $\bar{y} \in Gr(F_i)$ for every i. This implies that $\bar{y} \in \mathcal{F}(y)$. Thus, $\mathcal{F}(y)$ is compact because it is a closed subset of a compact set. □

The following theorem shows the existence of an optimin point under some topological assumptions on the primitives of an abstract economy.

Theorem 4. *Let $\mathcal{E} = (Y_i, u_i, F_i)_{i \in N}$ be an abstract economy. If Y_i is compact, u_i is continuous, and F_i has a closed graph, then the economy $\mathcal{E}$ has an optimin point.*

The proof of Theorem 1 can be applied to prove this theorem with an addition that one maximizes the value function with respect to $\mathcal{F}$ (i.e., $v_i^{max} = \arg\max_{y \in \mathcal{F}(y)} v_i(y)$), which is shown to be compact for every y by Lemma 3.

Corollary 2. *Competitive economic equilibrium satisfies the optimin criterion.*

This follows from Theorem 4 and the second fundamental theorem of welfare economics: A competitive equilibrium in an Arrow-Debreu model (i.e., a Nash equilibrium of the abstract economy) is Pareto optimal. Therefore, it is an optimin point. However, even if a competitive equilibrium does not exist, an optimin point may exist in an abstract economy by Theorem 4. This is because, I do not assume convexity of strategy sets, quasi-concavity of utility functions, or continuity and convexity of feasibility correspondences in this theorem.

Surely, I only show the mere possibility that an optimin point exists when a competitive equilibrium does not. In addition, I did not say anything about the conditions on consumer preferences and producer technologies in this case, which I leave for a future project.

4.3 The optimin criterion in decisions

As mentioned in the introduction, the maximin criterion is too pessimistic and may lead to extreme conclusions which do not coincide with common sense. Harsanyi (1975) argues that acting on such pessimism may prevent individuals from performing even daily tasks such as crossing a street. In the case of providing drugs to the terminally ill patients, Harsanyi defends a utilitarian solution: Spending resources, for example, on the higher education of a mathematical genius, is preferred to a treatment that would extend the life of a terminally ill patient. I believe that such a solution does not conform with the morals of many people, as it does not sound just or fair. However, in such situations, the maximin principle has its own problems: It does not address how to effectively allocate limited resources in a society. As mentioned previously, extra resources can be used more efficiently by investing in medical R&D and technology, thereby saving even more human lives in the future.

With this in mind, I next present a rather general definition of the optimin criterion in nonstrategic decision-making situations. I define a decision problem as an abstract economy or a generalized game, which a decision maker plays against Nature. This is not a new approach. Just like Wald's (1950) statistical decision theory (see subsection 4.4), I model Nature as a player, though he assumes that Nature is antagonistic—it tries to minimize the decision maker's payoffs, whereas in a decision problem as defined below Nature may or may not be modeled as an antagonist. Given a decision problem, a profile of acts is said to satisfy the optimin criterion if it is an optimin point of the abstract economy. Note that this definition is analogous to previous definitions of the optimin criterion in various contexts. To formalize the concept, let's fix some notation.

Let $\mathcal{D} = (\Delta X_i, F_i, \mathcal{U}_i)_{i\in\{1,2\}}$ denote a *decision problem*, which is defined as a two-player abstract economy in which player 1 is called the decision maker (DM), and player 2 is called Nature. $X_i = \{x_i^1, x_i^2, ..., x_i^m\}$ denotes the finite set of pure *acts* of $i \in \{1,2\}$, ΔX_i the set of all probability distributions over X_i, $p_i \in \Delta X_i$ a mixed act, and $p \in \Delta X_1 \times \Delta X_2$ a mixed act profile. $F_i : \Delta X_{-i} \twoheadrightarrow \Delta X_i$ denotes the feasibility correspondence of $i \in \{1,2\}$, which gives i's set of feasible acts given the other's acts. F_i is an extension of pure feasibility correspondence $F_i' : X_{-i} \twoheadrightarrow X_i$, where for all $x_i \in X_{-i}$, $F_i'(x_{-i}) \subseteq X_i$. The mixed feasibility correspondence is then defined as follows:

$$F_i(p_{-i}) = \{p_i \in \Delta X_i \mid \forall x_i^k \in supp(p_i), supp(p_{-i}) \subseteq F_i'(x_i^k)\}.$$

Put differently, given the DM's mixed act, p_1, a mixed act of Nature, p_2, is feasible if for all pure acts of Nature, x_2^k with $k \in \{1,2,...,m\}$, in the support of p_2, DM's acts in the support of p_1 must be feasible—i.e., $supp(p_1) \subseteq F_1'(x_2^k)$. The feasibility correspondence captures that Nature's acts may depend on the DM's choices, and vice versa. The intuition behind this is that Nature may pick a different "strategy" (e.g., a state of nature) depending on whether the DM goes to work by metro or by bike. Note that I do not assume that the DM tries to define the states of Nature "perfectly." Indeed, there may be better (or more fine grained) ways to define the states of Nature.[18] Von Neumann-Morgenstern expected utility function of i is defined as $\mathcal{U}_i : Z \to \mathbb{R}$, where $Z = \{(p_i, p_{-i}) \in \Delta X_i \times \Delta X_{-i} \mid p_{-i} \in F_{-i}(p_i)\}$. If Nature is antagonistic, then $\mathcal{U}_2 = -\mathcal{U}_1$.

Definition 7. For of $i \in \{1, 2\}$, the *optimism constraint*, $OC_i(p)$ at p is a subset of $F_i(p_{-i})$:

$$OC_i(p) = \{p'_i \in F_i(p_{-i}) | -i \text{ deems } p'_i \text{ possible}\}.$$

In other words, for a given an act profile, p, the DM is optimistic that Nature's choice will be in $OC_2(p)$. This is neither the first model nor the only way to model optimistic pessimism in decision making situations. For example, Hurwicz (1951) captures optimism of a DM with a parameter to take a convex combination of the worst and the best outcomes associated with acts.

The optimism of the DM may be captured by a hypothetical or written contract, which may represent the beliefs, moral ideals, or norms of the DM. Because different individuals and societies have different beliefs, morals, and norms, the optimin criterion based on the optimism constraint would be responsive to such characteristics. The optimism constraint assumes that the optimism of the DM may depend on his or her own act. For example, going to a game at a stadium (or watching a game on television) to support one's favorite team may make the DM more optimistic about the outcome of the game—even if there is no evidence for it.

Definition 8. The i'th component of the *value function*, $\mathcal{V} : \Delta X \to \mathbb{R}^2$, is defined as

$$\mathcal{V}_i(p) = \inf_{p'_{-i} \in OC_{-i}(p)} \mathcal{U}_i(p_i, p'_{-i}).$$

Put differently, the *value* of a choice is defined as the minimum utility the decision-maker would receive under a hypothetical or formal contract, the optimism constraint, which captures the optimism of the DM. The value function captures the pessimism of the DM about his or her choice given the OC.

As before, the next step is to make comparisons among the evaluations of acts. The value function is then optimized by applying Pareto optimality: An act profile satisfies the optimin criterion if its value is Pareto optimal under the optimism constraint.

[18]See, e.g., the discussion in section 12 in Gilboa (2009).

Definition 9. An act, p_i, as part of a tacit agreement p is said to satisfy the *optimin criterion* in a decision problem, $\mathcal{D} = (\Delta X_i, F_i, \mathcal{U}_i)_{i\in\{1,2\}}$, if p is an optimin point of the decision problem under the optimism constraint—i.e., the value of p is Pareto optimal.

The decision problem is modeled as an abstract economy played between a decision maker and Nature. A tacit agreement, p, between the DM and Nature satisfies the optimin criterion if its value is Pareto optimal. Under this criterion, the DM is optimistically pessimistic about his or her choice.

As an example, consider a person deciding on a housing mortgage, and suppose that she has recently achieved tenure at work. In many countries, a tenured employee may still be fired in some extreme situations. Thus, if she makes a decision based on the maximin principle, she should not actually buy a house because she might lose her job in the worst-case scenario. On the other hand, if getting a mortgage makes her more optimistic that she will work reasonably well and therefore will not be subject to the conditions of being fired under extreme situations, then the optimin principle would suggest that she buy the house. In the hypothetical contract she makes with herself, she has a job guarantee under reasonable situations. She could be optimistic about her job security but pessimistic about the possibility of promotion. Note that her preferences can be represented by the maxmin expected utility, which is not a coincidence as I will show in Proposition 7 below.

Posing the problem of spending resources in a society in terms of the optimin principle gives us a different perspective on the issue: We should (Pareto) optimize the situation of the worst-off individuals under a reasonable (perhaps hypothetical) social contract—e.g., the funding of drug treatments should be sustainable so that everyone who needs them could be treated without assessing their backgrounds. This hypothetical or written contract could represent the beliefs and moral ideals of the society, through the policymaker, as well as the sustainability of the funding. In fact, policymakers have already found a solution in such situations: The UK's National Health Service provides newly developed drugs as long as they are "effective" and sustainable, among other considerations, without making any discrimination based on the social background or race of those who receives them. The optimin principle seems to justify this simple rule of thumb of the policymakers in this decision-making situation.

The following proposition illustrates when the optimin criterion preferences can be represented by the maxmin expected utility (Gilboa and Schmeidler, 1989).

Proposition 7. *Let $\mathcal{D} = (\Delta X_i, F_i, \mathcal{U}_i)_{i\in\{1,2\}}$ be a decision problem and assume that the following are satisfied. $\mathcal{D}$ is a game rather than an abstract economy with an antagonistic Nature, and for every act profile $p = (p_1, p_2)$,*

1. *$OC_2(p) = OC_2(p')$ for all p'.*

2. *$OC_2(p)$ is convex and closed.*

3. *$\mathcal{V}(p)$ Pareto dominates $\mathcal{V}(p')$ if and only if $\mathcal{V}_1(p) > \mathcal{V}_1(p')$ where $p' \in \Delta X$.*

Then, the preferences of the DM over the acts can be represented by the Gilboa-Schmeidler maxmin expected utility. In addition, if for all x, $OC_2(x) = X_2$, then optimin criterion (in pure acts) can be axiomatized by Milnor's (1954) maximin criterion axioms.

If the conditions in Proposition 7 are satisfied, then it is clear that acts of the DM can be ranked based on the minimum payoff given the unique set of multi-prior beliefs where set C in Gilboa and Schmeidler (1989) corresponds to set $OC_2(p)$ above. I next turn to Wald's statistical decision theory, which has been recently gaining a momentum as an alternative to hypothesis testing (see, e.g., Manski, 2019).

4.4 Wald's theory of statistics

Wald's (1950) theory is based on the idea that a statistician should use a maximin strategy to minimize the maximum risk in a carefully constructed game against Nature. The statistician faces a decision problem under uncertainty and assumes that Nature wants to maximize the risk, which makes the game between the statistician and Nature a zero-sum game. This approach views statistical decision-making as a game against Nature. Formally, a *statistical game* is denoted by a tuple $S = (Y_1, Y_2, u_1, u_2)$ where Y_1 and Y_2 denote the set (which is not necessarily finite) of strategies of the statistician and the Nature, respectively.

To illustrate, consider the following simplified version of Bulmer's (1979, p. 416) game. Suppose that a possibly unfair coin has a probability of either 1/4 or 1/2 of coming up heads. The task is to choose between (i) $p = 1/4$ or (ii) $p = 1/2$ upon tossing the coin once.

What is the "optimal" decision in this problem? Figure 7 illustrates four actions of the experimenter: (1) never choose $p = 1/4$; (2) always choose $p = 1/4$; (3) choose $p = 1/4$ if it comes up heads; (4) choose $p = 1/4$ if it comes up tails. Nature has two (pure) actions, (i) $p = 1/4$ or (ii) $p = 1/2$. Payofs are simply the expected probability of guessing right in each case.

In this game against Nature, the experimenter's optimal maximin strategy is to play $(\frac{1}{5}, 0, 0, \frac{4}{5})$ and Nature's optimal strategy is to play $(\frac{2}{5}, \frac{3}{5})$. The experimenter's probability of guessing it right is $\frac{3}{5}$, which is significantly higher than a random guess. There is no

Say $p = 1/4$	$p = 1/4$	$p = 1/2$
never	0	1
always	1	0
if it comes up heads	1/4	1/2
if it comes up tails	3/4	1/4

Figure 7: A statistical game against Nature, which picks $p = 1/4$ or $p = 1/2$.

other strategy that can guarantee a higher probability of being correct. As Bulmer (1979, p. 416) shows, if one is allowed to toss the coin twice, then the probability of being correct increases to $\frac{9}{14}$.

In what follows, I will show that the optimin criterion coincides with Wald's maximin criterion (with a pessimist Nature) and von Neumann's (1928) maximin strategies in zero-sum games. The following theorem shows that a strategy profile is an optimin point if and only if it is a pair of maximin strategies in zero-sum games.

Theorem 5. *Let S be a statistical game. A profile $(y_1^*, y_2^*) \in Y_1 \times Y_2$ is an optimin point if and only if $y_1^* \in \arg\max_{y_1} \inf_{y_2} u_1(y_1, y_2)$ and $y_2^* \in \arg\max_{y_2} \inf_{y_1} u_2(y_1, y_2)$.*

Proof. '$\Rightarrow$' First, we show that $v_i(y_i, y_j) = \inf_{y'_j \in Y_j} u_i(y_i, y'_j)$ for each $i \neq j$. Suppose that there exists $\bar{y}_j \in Y_j$ such that $\bar{y}_j \in \arg\min_{y'_j \in Y_j} u_i(y_i, y'_j)$. Then, we have that $v_i(y_i, y_j) = \min_{y'_j \in Y_j} u_i(y_i, y'_j) = u_i(y_i, \bar{y}_j)$. Suppose, otherwise, that for all $y'_j \in Y_j$ there exists $y''_j \in Y_j$ such that $u_i(y_i, y''_j) < u_i(y_i, y'_j)$. This implies that

$$v_i(y_i, y_j) = \inf_{y'_j : u_i(y_i, y'_j) < u_i(y_i, y_j)} u_i(y_i, y'_j) = \inf_{y'_j \in Y_j} u_i(y_i, y'_j).$$

Next, we show that the value of an optimin point (y_1^*, y_2^*) must be Pareto dominant in a zero-sum game. By contraposition, suppose that its value is not Pareto dominant, that is, there is another optimin point $(\hat{y}_1, \hat{y}_2)$ such that $v_i(y_1^*, y_2^*) > v_i(\hat{y}_1, \hat{y}_2)$ and $v_j(y_1^*, y_2^*) < v_j(\hat{y}_1, \hat{y}_2)$ for $i \neq j$. Then, we have $v_1(y_1^*, y_2^*) = v_1(y_1^*, \hat{y}_2)$ and $v_2(\hat{y}_1, \hat{y}_2) = v_2(y_1^*, \hat{y}_2)$. This implies the value of $(y_i^*, \hat{y}_j)$ Pareto dominates the value of (y_1^*, y_2^*), which is a contradiction to our supposition that (y_1^*, y_2^*) is an optimin point. Since the value of (y_1^*, y_2^*) is Pareto dominant, each strategy is a maximin strategy of the respective players.
'$\Leftarrow$' Suppose that for each i we have $y_i^* \in \arg\max_{y_i} \inf_{y_j} u_i(y_i, y_j)$. This implies that for all $(y'_1, y'_2) \in Y_1 \times Y_2$ and for each i we have $v_i(y_1^*, y_2^*) \geqslant v_i(y'_1, y'_2)$. Hence the value of (y_1^*, y_2^*) is Pareto dominant, which implies that it is an optimin point. $\square$

5 Conclusions

The maximin criterion has far-reaching applications in statistics, politics, philosophy, operations research, and engineering, as well as economics. In this paper, I have proposed a novel concept, dubbed the *optimin criterion*, which (i) addresses criticisms of the maximin criterion, (ii) extends the maximin criterion to n-person non-zero-sum games, and (iii) is equivalent to the core in cooperative games whenever core exists. Finally, the maximin criterion is consistent with well-established non-Nash experimental deviations towards cooperation in noncooperative games.

In addition to games, natural research directions for the optimin criterion are the ones in which the maximin criterion has been extensively applied. These include the study of ambiguity or Knightian uncertainty (see, e.g., Wakker, 2010), fair division (see, e.g., Brams and Taylor, 1996), and other aforementioned fields. For example, it is natural to ask what axioms would rationalize the optimin criterion under various frameworks including games and decisions. When I consider the reduced-normal form of an extensive-form game, optimin point solutions seem to have forward induction reasoning rather than backward induction reasoning except in (strictly) competitive games. Therefore, extending the definition of the optimin criterion to extensive-form games is another research direction opened up by this paper. This direction would lead to the exploration of the optimin criterion's relationship to forward and backward induction reasoning. Finally, under reasonable conditions, optimin criterion coincides with Wald's maximin criterion and the maxmin expected utility. In more general strategic and nonstrategic contexts, the optimin criterion is yet to be axiomatized.

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