



King's Research Portal

DOI:

[10.1007/s00526-020-01784-8](https://doi.org/10.1007/s00526-020-01784-8)

Document Version

Peer reviewed version

[Link to publication record in King's Research Portal](#)

Citation for published version (APA):

Bourni, T., Tinaglia, G., & Langford, M. (2020). Convex ancient solutions to curve shortening flow. *CALCULUS OF VARIATIONS AND PARTIAL DIFFERENTIAL EQUATIONS*, 59(4), Article 133.

<https://doi.org/10.1007/s00526-020-01784-8>

Citing this paper

Please note that where the full-text provided on King's Research Portal is the Author Accepted Manuscript or Post-Print version this may differ from the final Published version. If citing, it is advised that you check and use the publisher's definitive version for pagination, volume/issue, and date of publication details. And where the final published version is provided on the Research Portal, if citing you are again advised to check the publisher's website for any subsequent corrections.

General rights

Copyright and moral rights for the publications made accessible in the Research Portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognize and abide by the legal requirements associated with these rights.

- Users may download and print one copy of any publication from the Research Portal for the purpose of private study or research.
- You may not further distribute the material or use it for any profit-making activity or commercial gain
- You may freely distribute the URL identifying the publication in the Research Portal

Take down policy

If you believe that this document breaches copyright please contact librarypure@kcl.ac.uk providing details, and we will remove access to the work immediately and investigate your claim.

CONVEX ANCIENT SOLUTIONS TO CURVE SHORTENING FLOW

THEODORA BOURNI, MAT LANGFORD, AND GIUSEPPE TINAGLIA

ABSTRACT. We show that the only convex ancient solutions to curve shortening flow are the stationary lines, shrinking circles, Grim Reapers and Angenent ovals, completing the classification initiated by Daskalopoulos, Hamilton and Šešum and X.-J. Wang.

1. INTRODUCTION

A smooth one-parameter family $\{\Gamma_t\}_{t \in I}$ of connected, immersed planar curves $\Gamma_t \subset \mathbb{R}^2$ evolves by curve shortening flow if

$$\partial_t \gamma(\theta, t) = \vec{\kappa}(\theta, t) \text{ for each } (\theta, t) \in \Gamma \times I$$

for some smooth family $\gamma : \Gamma \times I \rightarrow \mathbb{R}^2$ of immersions $\gamma(\cdot, t) : \Gamma \rightarrow \mathbb{R}^2$ of Γ_t , where $\vec{\kappa}(\cdot, t)$ is the curvature vector of $\gamma(\cdot, t)$. The solution $\{\Gamma_t\}_{t \in I}$ is called *ancient* if I contains the interval $(-\infty, t_0)$ for some $t_0 \in \mathbb{R}$, which we may, without loss of generality, take to be zero. We refer to a solution as *compact* if $\Gamma \cong S^1$, *convex* if the timeslices Γ_t each bound convex domains (in which case the immersions $\gamma(\cdot, t)$ are proper embeddings), *locally uniformly convex* if the curvature κ is always positive and *maximal* if it cannot be extended forwards in time.

The following families of curves constitute maximal convex ancient solutions to curve shortening flow.

- The *stationary line* $\{L_t\}_{t \in (-\infty, \infty)}$, where $L_t := \{(x, 0) : x \in \mathbb{R}\}$.
- The *shrinking circle* $\{S^1_{\sqrt{-2t}}\}_{t \in (-\infty, 0)}$.
- The *Grim Reaper* $\{G_t\}_{t \in (-\infty, \infty)}$, where $G_t := \{(x, y) : \cos x = e^{t-y}\}$ [17].
- The *Angenent oval* $\{A_t\}_{t \in (-\infty, 0)}$, where $A_t := \{(x, y) : \cos x = e^t \cosh y\}$ [3].

We will prove that the aforementioned examples are the only ones possible (modulo spacetime translation, spatial rotation and parabolic rescaling).

Theorem 1.1. *The only convex ancient solutions to curve shortening flow are the stationary lines, shrinking circles, Grim Reapers and Angenent ovals.*

Theorem 1.1 completes the classification of convex ancient solutions to curve shortening flow initiated by Daskalopoulos, Hamilton and Šešum [10] and X.-J. Wang [18]. Indeed, Daskalopoulos, Hamilton and Šešum showed that the shrinking circles and the Angenent ovals are the only *compact* examples [10]. Their arguments are based on the analysis of a certain Lyapunov functional. On

the other hand, Wang's results imply, in particular, that a convex ancient solution $\{\Gamma_t\}_{t \in (-\infty, 0)}$ must either be entire (i.e. sweep out the whole plane, in the sense that $\cup_{t < 0} \Omega_t = \mathbb{R}^2$, where Ω_t is the convex body bounded by Γ_t) or else lie in a strip region (the region bounded by two parallel lines) [18, Corollary 2.1]. He also proved that the only entire examples are the shrinking circles [18, Theorem 1.1]. His arguments are based primarily on the concavity of the arrival time function (whose level t set is the curve Γ_t) [18, Lemmas 4.1 and 4.4].

In fact, we provide a new, self-contained proof of the full classification result of Theorem 1.1: In Section 2, we present a simple geometric argument which shows that the Grim Reapers are the only noncompact examples, other than the stationary lines, which lie in strip regions. The argument is based on novel techniques that were developed by the authors to construct and classify new examples of ancient solutions to mean curvature flow which lie in slab regions [4, 5]. In Section 3, we use the same method to provide a simple new proof that the Angenent ovals are the only compact examples which lie in strip regions. In the fourth and final section, we prove that a convex ancient solution which is not a shrinking circle necessarily lies in a strip region, mostly following Wang's original argument.

2. NONCOMPACT SOLUTIONS IN STRIPS

Consider first a convex ancient solution $\{\Gamma_t\}_{t \in (-\infty, 0)} \subset \mathbb{R}^2$ which lies in the strip $\Pi := \{(x, y) : |x| < \pi/2\}$ and in no smaller strip but is *not compact*. By applying the strong maximum principle to the evolution equation for the curvature [13], we find that $\{\Gamma_t\}_{t \in (-\infty, 0)}$ is either a stationary line or else locally uniformly convex. We henceforth assume the latter. Denote by θ the turning angle of the solution (the angle made by the x -axis and its tangent vector with respect to a counterclockwise parametrization). Since the solution is convex and lies in the strip Π , we can arrange (by reflecting across the x -axis if necessary) that the Gauss image $\theta(\Gamma_t)$ is the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$ for all $t < 0$. Denote by $\gamma : (-\frac{\pi}{2}, \frac{\pi}{2}) \times (-\infty, 0) \rightarrow \mathbb{R}^2$ the turning angle parametrization and set $p(t) = \gamma(0, t)$. By translating vertically, we can arrange that $y(p(0)) = 0$.

We begin with some basic asymptotics.

Lemma 2.1. *The translated family $\{\Gamma_t^s\}_{t \in (-\infty, -s)}$ defined by $\Gamma_t^s := \Gamma_{t+s} - p(s)$ converges locally uniformly in the smooth topology as $s \rightarrow -\infty$ to the Grim Reaper $\{rG_{r-2t}\}_{t \in (-\infty, \infty)}$, where $r := \lim_{s \rightarrow -\infty} \kappa^{-1}(0, s)$.*

Proof. Since the solution is convex and locally uniformly convex, $\sup_{\Gamma_t} \kappa < \infty$ for each t , so the differential Harnack inequality [2, 7, 14] implies that the curvature is non-decreasing in time with respect to the turning angle parametrization. In particular, the limit $\kappa_\infty := \lim_{s \rightarrow -\infty} \kappa(0, s)$ exists. Since each translated solution contains the origin at time zero, and the curvature is uniformly bounded on compact subsets of spacetime, it follows from standard bootstrapping results that

a subsequence of each of the families converges locally uniformly in the smooth topology to a weakly convex eternal limit flow lying in a strip of the same width. By the strong maximum principle, the limit must be locally uniformly convex, since its normal at the origin at time zero is parallel to the strip (which rules out a stationary line as the limit). That is, κ_∞ is positive. Since the curvature of the limit is constant in time with respect to the turning angle parametrization, the rigidity case of the differential Harnack inequality implies that it moves by translation. Since the Grim Reapers are the only translating solutions to curve shortening flow other than the stationary lines [17], we conclude that the limit is the Grim Reaper with bulk velocity $v := \kappa_\infty e_2$. The claim follows since the limit is independent of the subsequence. \square

Lemma 2.2. *The solution sweeps out all of Π ; that is, $\Pi = \cup_{t < 0} \Omega_t$, where Ω_t is the convex region bounded by Γ_t .*

Proof. Since the solution is locally uniformly convex, $\Omega_{t_2} \subset \Omega_{t_1}$ for all $t_1 \leq t_2 < 0$. By hypothesis, given any $\delta \in (0, 1)$, there exist points $p_1, p_2 \in \Omega := \cup_{t < 0} \Omega_t$ such that

$$x(p_1) = \frac{\pi}{2} - \delta \quad \text{and} \quad x(p_2) = -\frac{\pi}{2} + \delta.$$

By convexity, the rays $\{(x(p_i), y(p_i) + s) : s > 0\}$ are also contained in Ω . Since $y(p(t)) \rightarrow -\infty$ as $t \rightarrow -\infty$, we conclude that Ω contains the lines $\{(\frac{\pi}{2} - \delta, s) : s \in \mathbb{R}\}$ and $\{(-\frac{\pi}{2} + \delta, s) : s \in \mathbb{R}\}$ and hence, by convexity, also the strip $\Pi_\delta := \{(x, y) \in \mathbb{R}^2 : |x| \leq \frac{\pi}{2} - \delta\}$. So $\Pi = \cup_{\delta \in (0, 1)} \Pi_\delta \subset \Omega \subset \Pi$, which implies the claim. \square

By the differential Harnack inequality and Lemma 2.1, the maximal displacement satisfies

$$(1) \quad \ell(t) := \max_{p \in \Gamma_t} -y(p) = -y(p(t)) \geq -r^{-1}t = -\kappa_\infty t.$$

Let $A(t)$ and $B(t)$ be the two points on Γ_t such that

$$y(A(t)) = y(B(t)) = 0$$

arranged so that

$$x(A(t)) > x(B(t)).$$

Let $\theta_A(t)$ and $\theta_B(t)$ be the corresponding turning angles, so that $\gamma(\theta_A(t), t) = A(t)$ and $\gamma(\theta_B(t), t) = B(t)$. Then the area $\text{Area}(t)$ enclosed by Γ_t and the x -axis satisfies [12, 17]

$$-\text{Area}'(t) = \int_{B(t)}^{A(t)} \kappa ds = \int_{\theta_B(t)}^{\theta_A(t)} d\theta = \theta_A(t) - \theta_B(t) \leq \pi.$$

Since $\text{Area}(0) = 0$, integrating from $-t$ to 0 yields

$$\text{Area}(t) \leq -\pi t.$$

Using the displacement estimate (1) and the fact that the solution approaches the boundary of the strip Π , we will prove that the enclosed area grows too quickly as $t \rightarrow -\infty$ if $r < 1$ (see Figure 1).

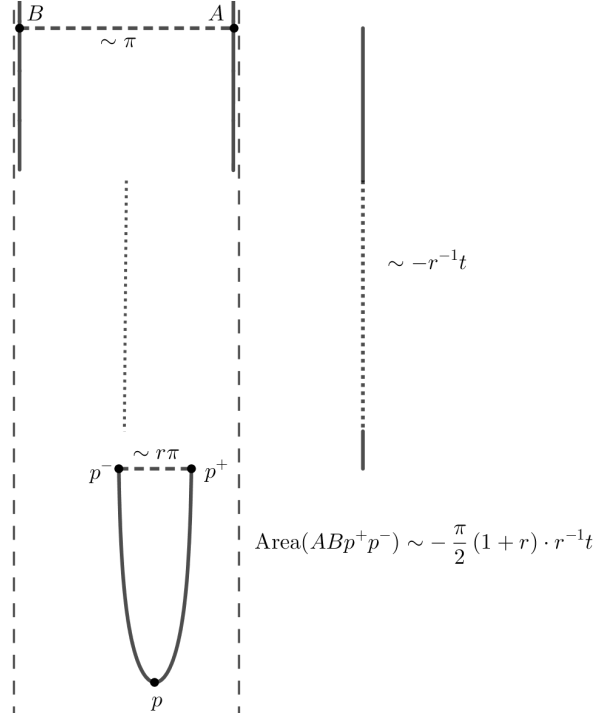


FIGURE 1. The enclosed area grows too quickly if $r < 1$.

Lemma 2.3. *The width of the asymptotic Grim Reaper is maximal: $r = 1$.*

Proof. By Lemma 2.2, for any $\delta \in (0, 1)$ we can find $t_\delta < 0$ such that

$$\pi \geq x(A(t)) - x(B(t)) \geq \pi - \delta$$

for every $t < t_\delta$. By Lemma 2.1, choosing t_δ smaller if necessary, we can also find two points $p^\pm(t)$ and a constant C_δ with the following properties:

$$\begin{aligned} y(p^+(t)) &= y(p^-(t)) \\ \pi r - \delta &< x(p^+(t)) - x(p^-(t)) < \pi r \\ 0 &< y(p^\pm(t)) - y(p(t)) < C_\delta. \end{aligned}$$

Since the enclosed area is bounded below by that of the trapezium with vertices A , B , p^- and p^+ , we find that

$$-\pi t \geq \text{Area}(t) \geq \frac{1}{2} (\pi r - \delta + \pi - \delta) (-r^{-1}t - C_\delta)$$

and hence

$$-(\pi(1-r) - 2\delta)t \leq (\pi(1+r) - 2\delta)rC_\delta$$

for all $t < t_\delta$. Taking $t \rightarrow -\infty$, we conclude that $(1-r)\pi \leq 2\delta$ for any $\delta > 0$. Taking $\delta \rightarrow 0$ then yields the claim. \square

The Alexandrov reflection principle [8, 9] can now be employed to prove that the solution is reflection symmetric about the y -axis (cf. [6]).

Lemma 2.4. *Let $\{\Gamma_t\}_{t \in (-\infty, 0)}$ be a convex ancient solution which is not compact and is contained in the strip $\Pi := \{(x, y) : |x| < \pi/2\}$ and in no smaller strip. Then Γ_t is reflection symmetric about the y -axis for all $t < 0$.*

Proof. Set

$$H_\alpha := \{(x, y) \in \mathbb{R}^2 : x < \alpha\}$$

and denote by R_α the reflection about ∂H_α . It is a consequence of the convexity of $\{\Gamma_t\}_{t \in (-\infty, 0)}$ and the convergence of its ‘tip’ to the Grim Reaper that, given any $\alpha \in (0, \pi/2)$, there exists a time t_α such that

$$(R_\alpha \cdot \Gamma_t) \cap (\Gamma_t \cap H_\alpha) = \emptyset$$

for all $t < t_\alpha$ (cf. [4, Claim 6.2.1]). By the Alexandrov reflection principle [8, 9], this is true for all $t < 0$. Taking $\alpha \searrow 0$ proves the lemma. \square

We can now prove that the solution is the Grim Reaper.

Theorem 2.5. *Let $\{\Gamma_t\}_{t \in (-\infty, 0)}$ be a convex ancient solution which is contained in the strip $\Pi := \{(x, y) : |x| < \pi/2\}$ and in no smaller strip and is not compact. Then $\{\Gamma_t\}_{t \in (-\infty, 0)}$ is a Grim Reaper.*

Proof. Since, by the Harnack inequality, the curvature of $\{\Gamma_t\}_{t \in (-\infty, 0)}$ is non-decreasing in t with respect to the turning angle parametrization, the maximal vertical displacement ℓ of $\{\Gamma_t\}_{t \in (-\infty, 0)}$ satisfies

$$\frac{d}{dt}(\ell(t) + t) \leq 0.$$

So the limit

$$C := \lim_{t \rightarrow -\infty} (\ell + t)$$

exists in $(0, \infty]$.

Claim 2.6. *The asymptotic displacement is equal to that of the Grim Reaper:*

$$C = 0.$$

Proof of the claim. Suppose, contrary to the claim, that $C > 0$ (the case $C < 0$ is ruled out similarly). Then, since the tips of both solutions reach the origin at time zero, we can find $t_0 \leq 0$ such that

$$(2) \quad \ell(t) > -t \text{ for all } t < t_0 \text{ but } \ell(t_0) = -t_0.$$

Define the halfspaces H_α and the reflection R_α about ∂H_α as in Lemma 2.4. Given any $\alpha \in (0, \pi/2)$, set

$$\tilde{G}_t = (R_\alpha \cdot G_t) \cap \{(x, y) \in \mathbb{R}^2 : x < 0\},$$

where $\{G_t\}_{t \in (-\infty, \infty)}$ is the Grim Reaper, and

$$\tilde{\Gamma}_t = \Gamma \cap \{(x, y) \in \mathbb{R}^2 : x < 0\}.$$

Then, by (2), $\partial \tilde{G}_t \cap \partial \tilde{\Gamma}_t = \emptyset$ for all $t < t_0$. Moreover, by convexity and the convergence of the tip to the Grim Reaper, there exists $t_\alpha < t_0$ depending on α such that $\tilde{G}_t \cap \tilde{\Gamma}_t = \emptyset$ for all $t < t_\alpha$. It then follows by the strong maximum principle that $\tilde{G}_t \cap \tilde{\Gamma}_t = \emptyset$ for all $t < t_0$. Letting $\alpha \searrow 0$, we find that Γ_t lies below G_t for all $t < t_0$. This contradicts the strong maximum principle, since the tips of both solutions coincide at time $t = t_0$. \square

Now consider, for any $\tau > 0$, the solution $\{\Gamma_t^\tau\}_{t \in (-\infty, 0)}$ defined by $\Gamma_t^\tau = \Gamma_{t+\tau}$. Since

$$C_\tau := \lim_{t \rightarrow -\infty} (\ell(t + \tau) + t) > 0,$$

we may argue as above to conclude that Γ_t^τ lies above G_t for all $t < 0$. Taking $\tau \rightarrow 0$, we find that Γ_t lies above G_t for all $t < 0$. Since the two curves reach the origin at time zero, they intersect for all $t < 0$ by the avoidance principle. The strong maximum principle then implies that the two coincide for all t . \square

3. COMPACT SOLUTIONS IN STRIPS

Combined with the theorems of Hamilton, Daskalopoulos and Šešum [10] and Wang [18, Corollary 2.1], Theorem 2.5 already implies Theorem 1.1. In this section, we present a different proof, along the same lines as the noncompact case, that the Angenent oval is the only compact example which lies in a strip. In the following section, we prove that the shrinking circle is the only example which does not lie in a strip, mostly following [18].

So assume that $\{\Gamma_t\}_{t \in (-\infty, 0)} \subset \mathbb{R}^2$ is a *compact*, convex ancient solution contained the strip region $\Pi := \{(x, y) : |x| < \pi/2\}$ and in no smaller one. By the Gage–Hamilton theorem [12], we may assume that it shrinks to a single point $p \in \Pi$ at time zero. After a vertical translation, we can arrange that $y(p) = 0$.

Denote by $\theta(p, t)$ the angle the tangent vector to Γ_t at $p \in \Gamma_t$ with respect to a counter-clockwise parametrization makes with the positive x -axis and let $\gamma : S^1 \times (-\infty, 0) \rightarrow \mathbb{R}^2$ be the corresponding family of turning angle parametrizations for $\{\Gamma_t\}_{t \in (-\infty, 0)}$. Set

$$p^-(t) := \gamma(0, t) \quad \text{and} \quad p^+(t) := \gamma(\pi, t).$$

Since Γ_t is convex, $p^+(t)$ and $p^-(t)$ are its maximal vertical displacements. I.e.

$$\ell_\pm(t) := \max_{p \in \Gamma_t} \pm y(p) = y(p^\pm(t)).$$

Lemma 3.1. *The pair of translated families $\{\Gamma_{s,t}^\pm\}_{t \in (-\infty, -s)}$ defined by $\Gamma_{s,t}^\pm := \Gamma_{t+s} - p^\pm(s)$ converge locally uniformly in the smooth topology as $s \rightarrow -\infty$ to the Grim Reapers $\{-r_+ G_{r_+^{-2}t}\}_{t \in (-\infty, \infty)}$ and $\{r_- G_{r_-^{-2}t}\}_{t \in (-\infty, \infty)}$ respectively, where $r_- := \lim_{s \rightarrow -\infty} \kappa^{-1}(0, s)$, $r_+ := \lim_{s \rightarrow -\infty} \kappa^{-1}(\pi, s)$ and κ is the curvature of γ .*

Moreover, $\{\Gamma_t\}_{t \in (-\infty, 0)}$ sweeps out all of Π .

Proof. The proof is similar to those of Lemmas 2.1 and 2.2. \square

By the Harnack inequality, the maximal vertical displacements satisfy

$$(3) \quad \ell_\pm(t) \geq -t\kappa_\pm = -r_\pm^{-1}t.$$

The area $\text{Area}(t)$ enclosed by Γ_t satisfies [12, 17]

$$(4) \quad \text{Area}(t) = -2\pi t.$$

Using the displacement estimate (3) and Lemma 3.1, we will prove that the enclosed area grows too quickly as $t \rightarrow -\infty$ unless $r_\pm = 1$.

Lemma 3.2. *The widths of the asymptotic Grim Reapers are maximal: $r_\pm = 1$.*

Proof. Let $A(t)$ and $B(t)$ be the points of intersection of Γ_t and the x -axis, with $x(B(t)) \leq x(A(t))$. For every $\delta \in (0, 1)$ we can find t_δ such that, for all $t < t_\delta$,

$$x(A(t)) - x(B(t)) \geq \pi - \delta.$$

By Lemma 3.1, choosing t_δ smaller if necessary, we can find a constant C_δ and points $p_1^+(t)$, $p_2^+(t)$, $p_1^-(t)$ and $p_2^-(t)$ on Γ_t such that, for all $t < t_\delta$,

$$\begin{aligned} y(p_1^\pm(t)) &= y(p_2^\pm(t)) \\ |y(p^\pm(t)) - y(p_{1,2}^\pm(t))| &\leq C_\delta \\ \pi r_\pm - \delta &\leq x(p_1^\pm(t)) - x(p_2^\pm(t)) < \pi r_\pm. \end{aligned}$$

Estimating the area from below by that of the two trapezia $ABp_1^\pm p_2^\pm$, we find

$$\begin{aligned} -2\pi t &\geq \text{Area}(t) \\ &\geq \frac{1}{2} (\pi(r_+ + 1) - 2\delta) (-r_+^{-1}t - C_\delta) + \frac{1}{2} (\pi(r_- + 1) - 2\delta) (-r_-^{-1}t - C_\delta). \end{aligned}$$

That is,

$$-((r_+^{-1} + r_-^{-1})(\pi - 2\delta) - 2\pi)t \leq (\pi(r_+ + r_- + 2) - 4\delta)C_\delta.$$

Taking $t \rightarrow -\infty$ and then $\delta \rightarrow 0$ yields $r_+^{-1} + r_-^{-1} \leq 2$ and hence $r_+ = r_- = 1$. \square

We can now prove that the solution is the Angenent oval.

Theorem 3.3. *Let $\{\Gamma_t\}_{t \in (-\infty, 0)}$ be a compact, convex ancient solution which is contained in the strip $\Pi := \{(x, y) : |x| < \pi/2\}$ and in no smaller strip. Then $\{\Gamma_t\}_{t \in (-\infty, 0)}$ is an Angenent oval.*

Proof. The claim follows as in the noncompact case: using Alexandrov reflection, we first show that the solution is symmetric with respect to reflections across the y -axis and then we compare with the Angenent oval. \square

4. COMPLETING THE CLASSIFICATION

As we have mentioned, Theorem 1.1 already follows from Theorems 2.5 and 3.3 and the results of X.-J Wang mentioned in the introduction. We present here a different proof of the required results (Lemmas 4.2 and 4.4), which mostly follows Wang's original arguments (particularly [18, Lemmas 2.1, 2.2, 4.1 and 4.4]).

Let $\{\Gamma_t\}_{t \in (-\infty, 0)}$ be a convex ancient solution to curve shortening flow, not necessarily compact. After a spacetime translation, we may arrange that the solution reaches the origin at time zero. We can also arrange that the origin is a regular point in the noncompact case and a singular point in the compact case [12]. Denote by $\gamma(\cdot, t) : N_t \rightarrow \mathbb{R}^2$ the turning angle parametrization for Γ_t , where $N_t \subset \mathbb{R}/2\pi\mathbb{Z}$. That is, $\gamma(\theta, t)$ is the point on Γ_t such that $\gamma_\theta(\theta, t) = (\cos \theta, \sin \theta)$. For compact solutions $N_t = \mathbb{R}/2\pi\mathbb{Z}$ for all t and Γ_t bounds a disk, while for noncompact solutions $N_t = (\theta_1(t), \theta_2(t))$ with $\theta_2(t) - \theta_1(t) \leq \pi$ and Γ_t is a graph over some line.

We begin by classifying the possible blow-downs of $\{\Gamma_t\}_{t \in (-\infty, 0)}$. Our main tool is the monotonicity formula. Recall that the Gaussian area Θ of $\{\Gamma_t\}_{t \in (-\infty, 0)}$ is defined, for $t < 0$, by

$$\Theta(t) := \int_{\Gamma_t} \Phi(p, t) d\mathcal{H}^1(p),$$

where

$$\Phi(p, t) := (-4\pi t)^{-\frac{n}{2}} e^{\frac{|p|^2}{4t}}.$$

Lemma 4.1. *There exists a constant $C = C(n) < \infty$ such that*

$$\sup_{k > 0} \frac{1}{k^{n/2}} \int_{M^n} e^{\frac{-|y|^2}{k}} d\mathcal{H}^n(y) < C$$

for any convex hypersurface M^n of \mathbb{R}^{n+1} .

Proof. Let $A_i = B_{(i+1)\sqrt{k}}(0) \setminus B_{i\sqrt{k}}(0)$ for $i \in \mathbb{N}$. If $M^n \subset \mathbb{R}^{n+1}$ is convex, then

$$|M^n \cap A_i| \leq |\partial B_{(i+1)\sqrt{k}}(0)| = c_n (i+1)^n k^{n/2}$$

and hence

$$\begin{aligned} \frac{1}{k^{n/2}} \int_{M^n} e^{\frac{-|y|^2}{k}} d\mathcal{H}^n(y) &= \frac{1}{k^{n/2}} \sum_{i=0}^{\infty} \int_{M^n \cap A_i} e^{\frac{-|y|^2}{k}} d\mathcal{H}^n(y) \\ &\leq \frac{1}{k^{n/2}} \sum_{i=0}^{\infty} c_n (i+1)^n k^{n/2} e^{-i^2} = c_n \sum_{i=0}^{\infty} (i+1)^n e^{-i^2}, \end{aligned}$$

which proves the claim. \square

Lemma 4.2. *The family of rescaled solutions $\{\lambda\Gamma_{\lambda^{-2}t}\}_{t \in (-\infty, 0)}$ converges locally uniformly in the smooth topology, as $\lambda \rightarrow 0$, to either*

- (a) *the shrinking circle,*
- (b) *a stationary line of multiplicity one passing through the origin, or*
- (c) *a stationary line of multiplicity two passing through the origin.*

In case (a), $\{\Gamma_t\}_{t \in (-\infty, 0)}$ is the shrinking circle. In case (b), $\{\Gamma_t\}_{t \in (-\infty, 0)}$ is a stationary line passing through the origin.

Proof. Since the shrinking circle reaches the origin at time zero, it must intersect the solution at all negative times by the avoidance principle. Thus,

$$\min_{p \in \Gamma_t} |p| \leq \sqrt{-2t}.$$

Since the speed, κ , of the solution is non-decreasing by the Harnack inequality, it is bounded in any compact subset of $\mathbb{R}^2 \times (-\infty, 0)$ after the rescaling. Therefore, given any sequence $\lambda_i \searrow 0$, we can find a subsequence along which $\{\lambda_i \Gamma_{\lambda_i^{-2}t}\}_{t \in (-\infty, 0)}$ converges locally uniformly in the smooth topology to a non-empty limit flow. We claim that the limit is always a shrinking solution. By Lemma 4.1, $\Theta(t)$ is uniformly bounded, so the monotonicity formula [15] holds. That is,

$$(5) \quad \frac{d}{dt} \Theta(t) = - \int_{\Gamma_t} \left| \vec{\kappa}(p) + \frac{p^\perp}{-2t} \right|^2 \Phi(p, t) d\mathcal{H}^1(p).$$

It follows that $\Theta(t)$ converges to some limit as $t \rightarrow -\infty$. But then

$$\begin{aligned} - \int_a^b \int_{\lambda\Gamma_{\lambda^{-2}t}} \left| \vec{\kappa}(p) + \frac{p^\perp}{-2t} \right|^2 \Phi(p, t) d\mathcal{H}^1(p) &= \Theta_\lambda(b) - \Theta_\lambda(a) \\ &= \Theta(\lambda^{-2}b) - \Theta(\lambda^{-2}a) \rightarrow 0 \end{aligned}$$

as $\lambda \rightarrow 0$ for any $a < b < 0$, where Θ_λ is the Gaussian area of the rescaled flow $\{\lambda\Gamma_{\lambda^{-2}t}\}_{t \in (-\infty, 0)}$. We conclude that the integrand vanishes identically in the limit and hence any limit of $\{\lambda\Gamma_{\lambda^{-2}t}\}_{t \in (-\infty, 0)}$ along a sequence of scales $\lambda_i \rightarrow 0$ is a shrinking solution to curve shortening flow. But the only convex examples which can arise are the shrinking circle and the stationary lines of multiplicity either one or two [1, 11]. By (5),

$$\lim_{\lambda \rightarrow 0} \Theta_\lambda(t) \geq \lim_{\lambda \rightarrow \infty} \Theta_\lambda(t).$$

If the shrinking circle arises as a backwards limit (i.e. as $\lambda \rightarrow 0$), then the solution is compact and hence, by the Gage–Hamilton theorem [12], the forwards limit (i.e. as $\lambda \rightarrow \infty$) is also the shrinking circle. It follows from (5) that Θ is constant and we conclude that the solution is the shrinking circle. Else, the backwards limits are all stationary lines. Note that the backwards limit is unique in this case since, by convexity, the limiting convex region bounded by each subsequential limit is contained in the limiting convex region bounded by any other. If the backwards

limit has multiplicity one, then the forwards limit cannot be a shrinking circle (by the monotonicity formula, since the latter has larger Gaussian area). So the solution is noncompact and, since the spacetime origin is a regular point, the forwards limit is also a stationary line. We conclude from (5) that Θ is constant and hence that the solution is a stationary line. This completes the proof. \square

It remains to show that when case (c) of Lemma 4.2 holds $\{\Gamma_t\}_{t \in (-\infty, 0)}$ lies in a strip. A more general version of this statement is shown by X.-J. Wang [18, Corollary 2.1]. We present here a modified version of Wang's argument adapted to our setting (cf. [18, Lemmas 2.1 and 2.2]). A key ingredient is concavity of the arrival time.

Set $\Omega = \cup_{t < 0} \Omega_t$, where Ω_t is the convex domain bounded by Γ_t and let $u : \Omega \rightarrow \mathbb{R}$ be the *arrival time* of $\{\Gamma_t\}_{t \in (-\infty, 0)}$. That is, $u(p)$ is the unique time $t < 0$ such that $p \in \Gamma_t$. Then $\Gamma_t = \{x \in \Omega : u(x) = t\}$, and u satisfies the *level set flow*

$$-|Du| \operatorname{div} \left(\frac{Du}{|Du|} \right) = 1.$$

Moreover, by hypothesis, $u \leq 0$ and $u(0) = 0$.

Proposition 4.3 (cf. [16, Appendix A]). *The arrival time of a convex, locally uniformly convex ancient solution to curve shortening flow is locally concave.*

Proof. Let $\gamma : S^1 \times (-\infty, 0) \rightarrow \mathbb{R}^2$ be a turning angle parametrization for the solution. Since

$$u(\gamma(\theta, t)) = t,$$

we find that

$$Du \cdot \tau = 0 \quad \text{and hence} \quad Du \cdot \nu = \frac{1}{\kappa}.$$

It follows that

$$\begin{aligned} D^2u(\tau, \tau) &= -1, \\ D^2u(\tau, \nu) &= 0 \end{aligned}$$

and

$$D^2u(\nu, \nu) = -\frac{\kappa_t}{\kappa^3}.$$

The latter is nonpositive by the differential Harnack inequality [14]. \square

Lemma 4.4 ([18, Corollary 2.1]). *In case (c) of Lemma 4.2, $\{\Gamma_t\}_{t \in (-\infty, 0)}$ is contained in a strip region.*

Proof. After a rotation, we can assume that $\{\lambda \Gamma_{\lambda^{-2}t}\}_{t \in (-\infty, 0)}$ converges as $\lambda \rightarrow 0$ to the y -axis with multiplicity 2. In the non-compact case, after reflecting along the x -axis if necessary, $0 \in N_t \rightarrow (-\frac{\pi}{2}, \frac{\pi}{2})$ as $t \rightarrow -\infty$. For each t , we define the points $p_t^- = \gamma(0, t) \in \Gamma_t \cap \{y < 0\}$, $p_t^+ = \gamma(\pi, t) \in \Gamma_t \cap \{y > 0\}$ and $q_t^\pm \in \Gamma_t \cap \{(x, 0) \mid \pm x > 0\}$, omitting the point p_t^+ in case Γ_t is non-compact. The existence of these points is a consequence of the fact that $0 \in \Omega_t$ for all $t < 0$.

Since the blow-down is the y -axis with multiplicity two, for any $\varepsilon > 0$ we can find $t_\varepsilon < 0$ such that

$$(6) \quad |p_t^\pm \cdot e_2| \geq \frac{\sqrt{-t}}{\varepsilon} \quad \text{and} \quad |q_t^\pm| \leq \varepsilon \sqrt{-t} \quad \text{for all } t < t_\varepsilon.$$

Define also $a_t^\pm := y(p_t^\pm)$, with $a_t^+ := \infty$ in the non-compact case. Then Γ_t is the union of two graphs over the y -axis,

$$\Gamma_t = \{(-v^+(y, t), y) : y \in [a_t^-, a_t^+]\} \cup \{(-v^-(y, t), y) : y \in [a_t^-, a_t^+]\},$$

with $v^+(\cdot, t) : [a_t^-, a_t^+] \rightarrow \mathbb{R}$ convex, $v^- : [a_t^-, a_t^+] \rightarrow \mathbb{R}$ concave, $(v^\pm(0, t), 0) = -p_t^\pm$ and $v^\pm(a_t^\pm, t) = 0$.

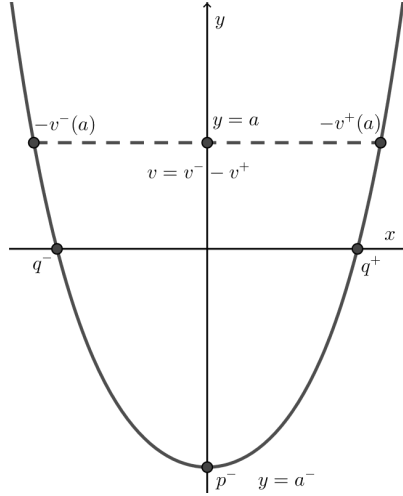


FIGURE 2. Writing the solution as a union of two graphs over the y -axis

Since their graphs move by curve shortening flow, the functions v^\pm satisfy

$$(7) \quad \frac{\partial v^\pm}{\partial t} = \frac{v_{yy}^\pm}{1 + (v_y^\pm)^2} = -\kappa^\pm \sqrt{1 + (v_y^\pm)^2},$$

where κ^\pm are the curvatures of the respective graphs.

Claim 4.5. [18, Claim 1 in Lemma 2.1 and Lemma 2.2] *There exists $t_0 < 0$ such that*

$$|a_t^\pm|(v^-(0, t) - v^+(0, t)) \geq -\frac{\pi}{4}t \quad \text{for all } t \leq t_0.$$

Proof. Fix $t < t_\varepsilon$ and assume, without loss of generality, that $|a_t^-| \leq |a_t^+|$ in the compact case. By convexity/concavity of the graphs, the two segments connecting p_t^\pm to q_t^- lie below the graph of v^- and the two segments connecting p_t^\pm to p_t^+

lie above the graph of v^+ . Thus, comparing their slopes with the slope of the tangents to $v^\pm(\cdot, t)$ at 0, we obtain

$$\begin{aligned} \frac{v^-(0, t) - v^+(0, t)}{|a_t^-|} &\geq \frac{\partial v^-}{\partial y}(0, t) - \frac{\partial v^+}{\partial y}(0, t) \\ &\geq \frac{-v^-(0, t) + v^+(0, t)}{|a_t^+|} \geq \frac{-v^-(0, t) + v^+(0, t)}{|a_t^-|} \end{aligned}$$

and hence, using (6), we find that

$$(8) \quad \left| \frac{\partial v^-}{\partial y}(0, t) - \frac{\partial v^+}{\partial y}(0, t) \right| \leq \frac{v^-(0, t) - v^+(0, t)}{|a_t^-|} \leq 2\varepsilon^2.$$

It follows that

$$\int_{\Gamma_t \cap \{(x, y) : x > 0\}} \kappa \, ds \geq \pi - 2\varepsilon^2,$$

where κ is the curvature and s an arc-length parameter of Γ_t . Since Γ_t moves by curve shortening flow,

$$\frac{d}{dt} \mathcal{H}^2(\Omega_t \cap \{(x, y) : y < 0\}) = - \int_{\Gamma_t \cap \{(x, y) : x > 0\}} \kappa \, ds \leq -\pi + 2\varepsilon^2,$$

where Ω_t is the convex region bounded by Γ_t . Integrating this between $t < 4t_\varepsilon$ and t_ε yields, upon choosing $\varepsilon = \frac{\sqrt{\pi}}{3\sqrt{2}}$,

$$\mathcal{H}^2(\Omega_t \cap \{(x, y) : y < 0\}) \geq -\frac{3}{4}(\pi - 2\varepsilon^2)t \geq -\frac{2\pi}{3}t.$$

By convexity, the region $\Omega_t \cap \{(x, y) : y < 0\}$ lies between the tangent lines to $v^\pm(\cdot, t)$ at 0. By (8), these tangent lines intersect the line $y = a_t^-$ at two points with distance at most $2(v^-(0, t) - v^+(0, t))$. It follows that

$$\frac{3}{2}(v^-(0, t) - v^+(0, t))|a_t^-| \geq \frac{\pi}{4}(-t),$$

which finishes the proof of Claim 4.5. \square

Set $v := v^- - v^+$. Note that $v(\cdot, t)$ is positive and concave with $v(a_t^\pm, t) = 0$. We need to show that $v(\cdot, t)$ stays bounded as $t \rightarrow -\infty$. It suffices to show that $v_t(0)$ stays bounded as $t \rightarrow -\infty$ since, by (6), $|a_t^\pm| \rightarrow \infty$ as $t \rightarrow -\infty$. Set $t_k = 2^k t_\varepsilon$ for some to-be-determined $\varepsilon > 0$ and write $a_k^\pm := a_{t_k}^\pm$, $v_k := v(\cdot, t_k)$ and $\Gamma_k := \Gamma_{t_k}$.

Claim 4.6. [18, Claim 1 in Lemma 2.1 and Lemma 2.2] *There exists $\varepsilon > 0$ such that*

$$(9) \quad v_k(0) \leq v_{k-1}(0) + \sqrt{-t_\varepsilon} 2^{-\frac{k}{4}}$$

for all $k \geq 1$.

Proof. By concavity of the arrival time, for each y the functions $t \mapsto v^-(y, t)$ and $t \mapsto -v^+(y, t)$ are concave and hence so is $t \mapsto v(y, t)$. It follows that

$$(10) \quad \frac{d}{dt} \frac{v(y, t)}{-t} \geq 0 \text{ for every } y \in (a_t^-, a_t^+).$$

In particular,

$$\frac{v_k(0)}{-t_k} \leq \frac{v_0(0)}{-t_0} \text{ and hence } v_k(0) \leq 2^k v_0(0) \leq 2^{k+1} \varepsilon \sqrt{-t_\varepsilon},$$

where in the last inequality we used (6). Given $k_0 \geq 8$ (to be determined momentarily) we chose $\varepsilon = \varepsilon(k_0)$ small enough that $2^{k_0+1} \varepsilon \leq 2^{-\frac{k_0}{4}}$, hence

$$(11) \quad v_k(0) \leq 2^{-\frac{k_0}{4}} \sqrt{-t_\varepsilon} \leq \frac{1}{4} \sqrt{-t_\varepsilon} \text{ for all } k \leq k_0.$$

In particular, (9) holds for each $k \leq k_0$. So suppose that (9) holds up to some $k \geq k_0$. Then

$$v_k(0) \leq v_{k_0}(0) + \sqrt{-t_\varepsilon} \sum_{i=k_0+1}^k 2^{-\frac{i}{4}},$$

where the second term on the right hand side is taken to be zero if $k = k_0$. Since $\sum_{i=1}^{\infty} 2^{-\frac{i}{4}} < \infty$, we can choose k_0 so that $\sum_{i=k_0+1}^{\infty} 2^{-\frac{i}{4}} < 1/4$. Applying (11), we then obtain

$$(12) \quad v_k(0) \leq \frac{1}{2} \sqrt{-t_\varepsilon}.$$

By (10),

$$\frac{v_{k+1}(0)}{-t_{k+1}} \leq \frac{v_k(0)}{-t_k} \text{ and hence } v_{k+1}(0) \leq 2v_k(0) \leq \sqrt{-t_\varepsilon}.$$

Since $t \mapsto v(0, t)$ is decreasing, we conclude that

$$v(0, t) \leq \sqrt{-t_\varepsilon} \text{ for every } t \leq t_{k+1}.$$

Using this estimate in Claim 4.5 along with the monotonicity of the intervals $[a_k^-, a_k^+]$, we obtain

$$(13) \quad |a_{k+1}^\pm| \geq |a_k^\pm| \geq \frac{-\pi t_k}{4v_k(0)} \geq \frac{-\pi}{4\sqrt{-t_\varepsilon}} t_k \text{ and } |a_{k-1}^\pm| \geq \frac{-\pi}{4\sqrt{-t_\varepsilon}} t_{k-1} = \frac{-\pi}{8\sqrt{-t_\varepsilon}} t_k.$$

Define now

$$L_k := \left\{ y \in \mathbb{R} : \frac{\pi}{8\sqrt{-t_\varepsilon}} t_k < y < \frac{\pi}{8\sqrt{-t_\varepsilon}} (-t_k) \right\}.$$

By the concavity of $y \mapsto v(y, t)$ we find, using also (13) and (12),

$$\frac{v_k(\pm|y|)}{\pm|y| + |a_k^\mp|} \leq \frac{v_k(0)}{|a_k^\mp|} \text{ and hence } v_k(y) \leq 2v_k(0) \leq \sqrt{-t_\varepsilon} \text{ for all } y \in L_k.$$

By the concavity of $t \mapsto v(y, t)$ for $y \in L_k$ and the above estimate, we find, for $t \in [t_{k+1}, t_k]$,

$$v(y, t) \leq \frac{t}{t_k} v_k(y) \leq 2\sqrt{-t_\varepsilon}.$$

These estimates, (13) and the concavity of $y \mapsto v(y, t)$ and $t \mapsto v(y, t)$ yield, for all $(y, t) \in L_k \times [t_{k+1}, t_k]$,

$$(14) \quad |\partial_y v(y, t)| \leq \max \left\{ \frac{v(y, t)}{a_t^+ - |y|}, \frac{v(y, t)}{|a_t^-| - |y|} \right\} \leq \frac{16t_\varepsilon}{\pi} \frac{1}{t_k} \leq \frac{8t_\varepsilon}{t_k}$$

and

$$(15) \quad 0 \geq \partial_t v(y, t) \geq \frac{v(y, t)}{t} \geq \frac{v_k(y)}{t_k} \geq \frac{2\sqrt{-t_\varepsilon}}{t_k}.$$

Next we want to bound $\partial_y^2 v(y, t)$ in $L_k \times [t_{k+1}, t_k]$ in terms of t_k . Let $f : \mathbb{N} \rightarrow \mathbb{R}$ be a positive function, to be determined later and define

$$\chi = \{(y, t) \in L_k \times [t_{k+1}, t_k] : -\partial_y^2 v(y, t) \geq f(k)\}.$$

By concavity of v we find, for any $t \in (t_{k+1}, t_k)$,

$$\begin{aligned} \mathcal{H}^1(\{y \in L^k : (y, t) \in \chi\}) f(k) &\leq \int_{L_k} -\partial_y^2 v(y, t) dy \\ &= -\partial_y v \left(\frac{\pi}{8\sqrt{-t_\varepsilon}}(-t_k), t \right) + \partial_y v \left(\frac{\pi}{8\sqrt{-t_\varepsilon}} t_k, t \right) \leq 2 \sup_{L_k} |\partial_y v(\cdot, t)| \leq \frac{16t_\varepsilon}{t_k}, \end{aligned}$$

where at the last inequality we used (14). Integrating between t_{k+1} and t_k yields

$$\mathcal{H}^2(\chi) \leq (t_k - t_{k+1}) \frac{16t_\varepsilon}{t_k f(k)} = -\frac{16t_\varepsilon}{f(k)}.$$

Consider now another positive function $g : \mathbb{N} \rightarrow \mathbb{R}$, which will be also determined later. Since

$$\int_{L_k} \mathcal{H}^1(\chi \cap \{y = z\}) dz = \mathcal{H}^2(\chi) \leq -\frac{16t_\varepsilon}{f(k)},$$

there exists $\hat{L}_k \subset L_k$ with $\mathcal{H}^1(\hat{L}_k) \leq -\frac{16t_\varepsilon}{f(k)g(k)}$ such that

$$(16) \quad \mathcal{H}^1(\chi \cap \{y = z\}) \leq g(k) \text{ for all } z \in L_k \setminus \hat{L}_k.$$

Now for any $y \in L_k \setminus \hat{L}_k$ we have

$$(17) \quad \begin{aligned} v(y, t_{k+1}) - v(y, t_k) &= \int_{t_{k+1}}^{t_k} -\partial_t v(y, t) dt \\ &= \int_{\chi \cap \{y=z\}} -\partial_t v(y, t) dt + \int_{[t_{k+1}, t_k] \setminus (\chi \cap \{y=z\})} -\partial_t v(y, t) dt. \end{aligned}$$

Using (15) and (16) to bound the first integral on the right hand side of (17) and the graphical curve shortening flow equation (7) to bound the second, we find

$$(18) \quad v_{k+1}(y) - v_k(y) \leq \frac{2\sqrt{-t_\varepsilon}}{-t_k} g(k) - f(k)t_k.$$

We now choose $f(k) = \frac{2^{-k(1+\beta)}}{\sqrt{-t_\varepsilon}}$ and $g(k) = 2^{k(1-\beta)}(-t_\varepsilon)$ for some $\beta \in (0, 1)$. Then

$$\mathcal{H}^1(L_k \setminus \hat{L}_k) \geq \frac{-\pi t_k}{4\sqrt{-t_\varepsilon}} - 16\sqrt{-t_\varepsilon}2^{2k\beta} = \frac{\pi}{4}2^k\sqrt{-t_\varepsilon} - 16\sqrt{-t_\varepsilon}2^{2k\beta},$$

which is positive for $k \geq k_0$ sufficiently large. Without loss of generality, we can assume that $\partial_y v(0, t_k) \leq 0$. Moreover, by convexity of the solution and the preceding measure estimate, it suffices to consider points $y \in L_k \setminus \hat{L}_k \cap \{0 < y < 16\sqrt{-t_\varepsilon}2^{2k\beta}\}$. Then $v(0, t_k) \geq v(y, t_k)$ and, by the concavity of $y \mapsto v(y, t)$,

$$\frac{v_{k+1}(0)}{a_{k+1}^+} \leq \frac{v_{k+1}(y)}{a_{k+1}^+ - y}.$$

Since

$$\frac{a_{k+1}^+}{a_{k+1}^+ - y} \leq \frac{a_{k+1}^+}{a_{k+1}^+ - 16\sqrt{-t_\varepsilon}2^{2k\beta}} = 1 + \frac{16\sqrt{-t_\varepsilon}2^{2k\beta}}{a_{k+1}^+ - 16\sqrt{-t_\varepsilon}2^{2k\beta}} \leq 1 + 17 \cdot 2^{k(2\beta-1)},$$

for $k \geq k_0$ sufficiently large, where in the last step we used (13), we obtain

$$v_{k+1}(0) \leq (1 + 17 \cdot 2^{k(2\beta-1)})v_{k+1}(y).$$

Applying (18), estimating $v_k(y) \leq v_k(0)$ and recalling (12), we conclude that

$$\begin{aligned} v_{k+1}(0) &\leq (1 + 17 \cdot 2^{k(2\beta-1)})(v_k(0) + 3\sqrt{-t_\varepsilon}2^{-k\beta}) \\ &\leq v_k(0) + \sqrt{-t_\varepsilon}(9 \cdot 2^{k(2\beta-1)} + 3 \cdot 2^{-k\beta} + 51 \cdot 2^{k(\beta-1)}) \end{aligned}$$

Choosing now $\beta \in (1/4, 1/3)$, we obtain for k sufficiently large

$$v_{k+1}(0) \leq v_k(0) + 63\sqrt{-t_\varepsilon}2^{-k\beta} \leq v_k(0) + \sqrt{-t_\varepsilon}2^{-\frac{k}{4}}.$$

which finishes the proof of the claim. \square

Taking $k \rightarrow \infty$, we conclude that v is bounded uniformly in time, which completes the proof of Lemma 4.4. \square

REFERENCES

- [1] ABRESCH, U., AND LANGER, J. The normalized curve shortening flow and homothetic solutions. *J. Differential Geom.* 23, 2 (1986), 175–196.
- [2] ANDREWS, B. Harnack inequalities for evolving hypersurfaces. *Math. Z.* 217, 2 (1994), 179–197.
- [3] ANGENENT, S. B. Shrinking doughnuts. In *Nonlinear diffusion equations and their equilibrium states, 3. Proceedings from the third conference, held August 20-29, 1989 in Gregynog, Wales, United Kingdom*. Boston, MA etc.: Birkhäuser, 1992, pp. 21–38.

- [4] BOURNI, T., LANGFORD, M., AND TINAGLIA, G. Collapsing ancient solutions of mean curvature flow. To appear in *J. Differential Geom.*
- [5] BOURNI, T., LANGFORD, M., AND TINAGLIA, G. On the existence of translating solutions of mean curvature flow in slab regions. To appear in *Analysis & PDE*.
- [6] BRYAN, P., AND LOUIE, J. Classification of convex ancient solutions to curve shortening flow on the sphere. *J. Geom. Anal.* *26*, 2 (2016), 858–872.
- [7] CHOW, B. On Harnack’s inequality and entropy for the Gaussian curvature flow. *Comm. Pure Appl. Math.* *44*, 4 (1991), 469–483.
- [8] CHOW, B. Geometric aspects of Aleksandrov reflection and gradient estimates for parabolic equations. *Comm. Anal. Geom.* *5*, 2 (1997), 389–409.
- [9] CHOW, B., AND GULLIVER, R. Aleksandrov reflection and geometric evolution of hypersurfaces. *Comm. Anal. Geom.* *9*, 2 (2001), 261–280.
- [10] DASKALOPOULOS, P., HAMILTON, R., AND SESUM, N. Classification of compact ancient solutions to the curve shortening flow. *J. Differential Geom.* *84*, 3 (2010), 455–464.
- [11] EPSTEIN, C. L., AND WEINSTEIN, M. I. A stable manifold theorem for the curve shortening equation. *Comm. Pure Appl. Math.* *40*, 1 (1987), 119–139.
- [12] GAGE, M., AND HAMILTON, R. The heat equation shrinking convex plane curves. *J. Differ. Geom.* *23* (1986), 69–96.
- [13] GAGE, M. E. Curve shortening makes convex curves circular. *Invent. Math.* *76*, 2 (1984), 357–364.
- [14] HAMILTON, R. S. Harnack estimate for the mean curvature flow. *J. Differential Geom.* *41*, 1 (1995), 215–226.
- [15] HUISKEN, G. Asymptotic behavior for singularities of the mean curvature flow. *J. Differential Geom.* *31*, 1 (1990), 285–299.
- [16] KOHN, R. V., AND SERFATY, S. A deterministic-control-based approach to motion by curvature. *Comm. Pure Appl. Math.* *59*, 3 (2006), 344–407.
- [17] MULLINS, W. W. Two-dimensional motion of idealized grain boundaries. *J. Appl. Phys.* *27* (1956), 900–904.
- [18] WANG, X.-J. Convex solutions to the mean curvature flow. *Ann. of Math. (2)* *173*, 3 (2011), 1185–1239.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TENNESSEE KNOXVILLE, KNOXVILLE TN, 37996-1320

E-mail address: `tbourni@utk.edu`

E-mail address: `mlangford@utk.edu`

DEPARTMENT OF MATHEMATICS, KING’S COLLEGE LONDON, LONDON, WC2R 2LS, U.K.

E-mail address: `giuseppe.tinaglia@kcl.ac.uk`

SCHOOL OF MATHEMATICAL AND PHYSICAL SCIENCES, THE UNIVERSITY OF NEWCASTLE, NEWCASTLE, NSW, AUSTRALIA, 2308

E-mail address: `mathew.langford@newcastle.edu.au`