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# Localization and universality of eigenvectors in directed random graphs

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Although the spectral properties of random graphs have been a long-standing focus of network theory, the properties of right eigenvectors of directed graphs have so far eluded an exact analytic treatment. We present a general theory for the statistics of the right eigenvector components in directed random graphs with a prescribed degree distribution and with randomly weighted links. We obtain exact analytic expressions for the inverse participation ratio and show that right eigenvectors of directed random graphs with a small average degree are localized. Remarkably, the critical mean degree for the localization transition is independent of the degree fluctuations. We also show that the dense limit of the distribution of the right eigenvectors is solely determined by the degree fluctuations, which generalizes standard results from random matrix theory. We put forward a classification scheme for the universality of the eigenvector statistics in the dense limit, which is supported by an exact calculation of the full eigenvector distributions. More generally, this paper provides a theoretical formalism to study the eigenvector statistics of sparse non-Hermitian random matrices.

*Introduction.* Complex systems, such as neural networks [1–3], ecosystems [4], and the World Wide Web [5, 6], consist of components that interact along the edges of large directed networks. Therefore, a problem of fundamental importance is how network structure affects the properties of complex systems.

Much insight in the dynamics of a complex system is gained from the eigenvalues and eigenvectors of the adjacency matrix representing its interaction network. For example, the dynamics in the vicinity of a stationary state is governed by the eigenvalues and eigenvectors of the adjacency matrix [7, 8], which is important in the study of disease spreading [9–12], synchronization of coupled oscillators [13, 14], and stability of biological systems, such as, neural networks [15, 16], ecosystems [17, 18], and gene regulatory networks [19, 20]. Eigenvalues and eigenvectors of sparse matrices are also important to evaluate spectral algorithms for ranking nodes [21–23], measuring node centrality [24, 25], detecting communities [26–28], and for recovering signals [29].

Properties of eigenvectors of random graphs have been mainly studied for undirected graphs [30–43], where an important feature is the *delocalization-localization transition*. Localized eigenvectors occupy a few sites, whereas delocalized eigenvectors are extended over the whole system. In general, the delocalization-localization transition implies a qualitative change in the properties of a system. Examples are the metal-insulator phase transition in solid state physics [30, 37], the transition from an algorithmically successful to a failure phase in spectral algorithms [29, 44], and the transition from a phase governed by a collective mode to a phase governed by a localized mode in dynamical systems [10]. Besides that, eigenvector localization also impacts the efficiency of net-

work centrality measures [25, 45] and the propagation of perturbations in ecosystems [46].

The statistical properties and the localization of eigenvectors of directed random graphs have so far eluded a mathematical analysis. Notable exceptions are models defined on one-dimensional chains, such as the Hatano-Nelson model [47, 48] and the Feinberg-Zee model [49] for the (de)pinning of vortex lines in superconductors. Recently these models have been extended to consider localization in one-dimensional biological systems [16, 50].

In this Letter, we develop an exact theory for the statistical properties of the right (or left) eigenvectors of directed random graphs with a prescribed degree distribution and random couplings. We derive exact analytic expressions for the inverse participation ratio and for the critical point of the localization-delocalization transition. Surprisingly, when the moments of the degree distribution are finite, the critical point of the localization-delocalization transition is independent of the degree distribution. Moreover, the right eigenvectors are localized if the degree distribution has diverging moments. We also show that in the dense limit the statistics of the components of right eigenvectors are only determined by degree fluctuations. In this limit, we obtain distinct universality classes that depend on an exponent that quantifies the degree fluctuations.

*Model set-up.* We consider random matrices  $\mathbf{A}$  of dimension  $n \times n$  with elements

$$A_{ij} = J_{ij}C_{ij}, \quad i, j \in \{1, 2, \dots, n\}, \quad (1)$$

where  $C_{ij} \in \{0, 1\}$  are the entries of the adjacency matrix  $\mathbf{C}$  of a directed random graph with a prescribed degree distribution

$$p_{K^{\text{in}}, K^{\text{out}}}(k, \ell) = p_{K^{\text{in}}}(k)p_{K^{\text{out}}}(\ell) \quad (2)$$

of indegrees  $K^{\text{in}}$  and outdegrees  $K^{\text{out}}$ . We set  $C_{ij} = 1$  when there exists a directed link pointing from  $i$  to  $j$ , such that the outdegree (indegree) of the  $i$ -th node is given by  $K_i^{\text{out}} = \sum_{j=1}^n C_{ij}$  ( $K_i^{\text{in}} = \sum_{j=1}^n C_{ji}$ ). The  $J_{ij}$  are real-valued independent and identically distributed random variables drawn from a distribution  $p_J(x)$ .

Directed random graphs with a prescribed degree distribution [51–56] have been used to model the World Wide Web [5, 6] and neural networks [1, 3, 57]. In this model, the indegrees and outdegrees are drawn independently from Eq. (2) subject to the constraint  $\sum_{j=1}^n K_j^{\text{in}} = \sum_{j=1}^n K_j^{\text{out}}$ , and subsequently nodes are randomly connected according to the given degree sequences. Since the degree distributions are specified at the outset, this model provides the ideal setting to explore the influence of network topology on the spectral properties of  $\mathbf{A}$ .

In what follows, brackets  $\langle \cdot \rangle$  denote the average with respect to the distribution of  $\mathbf{A}$ . In particular, we use

$$c = \langle K^{\text{out}} \rangle \quad (3)$$

for the mean outdegree, and we denote the variance of a random variable  $X$  by  $\text{var}(X) = \langle X^2 \rangle - \langle X \rangle^2$ .

*Spectra of infinitely large matrices  $\mathbf{A}$ .* The spectrum of  $\mathbf{A}$  has been studied in Refs. [58–61]. For  $n \rightarrow \infty$  and  $c > 1$ , directed random graphs have a giant strongly connected component [62] and the spectral distribution  $\rho_{\mathbf{A}}(\lambda) = n^{-1} \sum_{j=1}^n \delta[\lambda - \lambda_j(\mathbf{A})]$  of the eigenvalues  $\{\lambda_j(\mathbf{A})\}_{j=1}^n$  is supported on a disk of radius  $|\lambda_b| = \sqrt{c \langle J^2 \rangle}$  centered at the origin of the complex plane. In addition, if

$$c > c_{\text{gap}} = \frac{\langle J^2 \rangle}{\langle J \rangle^2}, \quad (4)$$

then there exists an eigenvalue outlier located at  $\lambda_{\text{isol}} = c \langle J \rangle$  that is separated from the boundary  $\lambda_b$  by a finite gap. Figure 1 shows the eigenvalues for an example of a directed random graph, where one clearly identifies the outlier  $\lambda_{\text{isol}}$  and the boundary  $\lambda_b$  of  $\rho_{\mathbf{A}}(\lambda)$  for  $n \rightarrow \infty$ .

*Distribution of the right eigenvector components.* A right eigenvector  $\vec{R}(\lambda)$  associated to an eigenvalue  $\lambda$  of  $\mathbf{A}$  satisfies

$$\mathbf{A} \vec{R}(\lambda) = \lambda \vec{R}(\lambda). \quad (5)$$

In this paper, we study localization of  $\vec{R}(\lambda)$  with the distribution

$$p_R(r|\lambda) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \delta[r - R_i(\lambda)] \quad (6)$$

of the entries of  $\vec{R}$  and we also study universality classes in the dense limit  $c \rightarrow \infty$ .

If  $\lambda$  is an outlier ( $\lambda = \lambda_{\text{isol}}$ ) or  $\lambda$  is located at the boundary of the spectrum ( $\lambda = \lambda_b$ ), then  $p_R(r|\lambda)$  fulfills

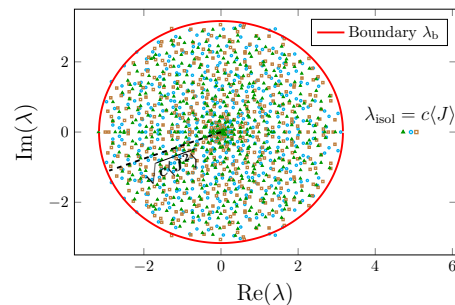


FIG. 1. Eigenvalues of three realizations (circles, triangles, and squares) of the adjacency matrix  $\mathbf{A}$  of directed random graphs with  $n = 500$  (see Eq. (1)). The indegrees and outdegrees follow a Poisson distribution with average  $c = 5$ . The weights  $J_{ij}$  are drawn from a Gaussian distribution  $p_J$  with mean and variance equal to one.

the equation [59–61]

$$p_R(r|\lambda) = \sum_{k=0}^{\infty} p_{K^{\text{out}}}(k) \int \left( \prod_{j=1}^k dx_j d^2 r_j p_J(x_j) p_R(r_j|\lambda) \right) \times \delta \left( r - \frac{1}{\lambda} \sum_{j=1}^k x_j r_j \right), \quad (7)$$

with  $d^2 r \equiv d\text{Re}r d\text{Im}r$ . Equation (7) is exact for infinitely large directed random graphs with a prescribed degree distribution [61].

*Inverse participation ratio.* The localization of  $\vec{R}(\lambda)$  can be characterized in terms of the inverse participation ratio (IPR) [36, 63, 64]

$$\mathcal{I}(\lambda) \equiv \lim_{n \rightarrow \infty} \frac{n \sum_{i=1}^n |R_i(\lambda)|^4}{\left( \sum_{i=1}^n |R_i(\lambda)|^2 \right)^2} = \frac{\langle |R(\lambda)|^4 \rangle}{\langle |R(\lambda)|^2 \rangle^2}, \quad (8)$$

where we have used that  $\mathcal{I}$  is self-averaging [65]. The IPR is finite if  $\vec{R}(\lambda)$  is delocalized, whereas  $\mathcal{I}(\lambda)$  diverges if  $\vec{R}(\lambda)$  is localized on a finite number of nodes.

From Eq. (7), we derive in the Supplemental Material [65] exact expressions for the IPR when  $\lambda = \lambda_{\text{isol}}$  or  $\lambda = \lambda_b$ . We find that

$$\mathcal{I}(\lambda_b) = \frac{(\gamma + 1) [\langle (K^{\text{out}})^2 \rangle - c]}{c(c - \langle J^4 \rangle / \langle J^2 \rangle^2)}, \quad (9)$$

where  $\gamma = 2$  when  $\lambda_b \in \mathbb{R}$  and  $\gamma = 1$  when  $\lambda_b \notin \mathbb{R}$ . From Eq. (9), it follows that  $\mathcal{I}(\lambda_b) \geq \gamma + 1$  and, consequently,  $\{R_i(\lambda_b)\}_{i=1}^n$  are non-Gaussian random variables if either  $p_{K^{\text{out}}}$  or  $p_J$  has nonzero variance. Analogously, the IPR

at  $\lambda = \lambda_{\text{isol}}$  reads

$$\begin{aligned} \mathcal{I}(\lambda_{\text{isol}}) &= \frac{3\beta_1 \langle J^2 \rangle^2}{(c^4 \langle J^4 \rangle - c \langle J^4 \rangle)} + \frac{\beta_3 (c^2 \langle J^2 \rangle - c \langle J^2 \rangle)^2}{\beta_1^2 (c^4 \langle J^4 \rangle - c \langle J^4 \rangle)} \\ &+ \frac{12\beta_1 \langle J^3 \rangle \langle J^2 \rangle (c^2 \langle J^2 \rangle - c \langle J^2 \rangle)}{(c^4 \langle J^4 \rangle - c \langle J^4 \rangle) (c^3 \langle J^3 \rangle - c \langle J^3 \rangle)} \\ &+ \frac{4\beta_2 \langle J^3 \rangle (c^2 \langle J^2 \rangle - c \langle J^2 \rangle)^2}{\beta_1 (c^4 \langle J^4 \rangle - c \langle J^4 \rangle) (c^3 \langle J^3 \rangle - c \langle J^3 \rangle)} \\ &+ \frac{6\beta_2 \langle J^2 \rangle (c^2 \langle J^2 \rangle - c \langle J^2 \rangle)}{\beta_1 (c^4 \langle J^4 \rangle - c \langle J^4 \rangle)}, \end{aligned} \quad (10)$$

where

$$\beta_\ell \equiv \sum_{k=\ell+1}^{\infty} p_{K^{\text{out}}}(k) \frac{k!}{(k-\ell-1)!}, \quad \ell = 1, 2, 3. \quad (11)$$

Figure 2 illustrates Eqs. (9) and (10) as a function of  $c$  for a Gaussian distribution  $p_J$  and three different outdegree distributions: Poisson, exponential, and Borel distributions (see Supplemental Material [65]). All moments of these degree distributions are finite and each  $p_{K^{\text{out}}}$  is parametrized only by  $c$ . Figure 2 shows that the IPR is finite if  $c$  is large enough and it diverges for small  $c$ , which proves the existence of a delocalization-localization phase transition in directed random graphs.

*The localization phase transition.* There are two mechanisms for localization, one which is governed by fluctuations of  $J_{ij}$ , and a second one that is governed by degree fluctuations.

The first mechanism is illustrated in Fig. 2 and it holds for an arbitrary  $p_{K^{\text{out}}}$  with finite moments. In this case, from Eqs. (9) and (10) it follows that right eigenvectors associated to  $\lambda = \lambda_b$  and  $\lambda = \lambda_{\text{isol}}$  are localized when  $c$  is smaller than

$$c_b = \frac{\langle J^4 \rangle}{\langle J^2 \rangle^2} \quad \text{and} \quad c_{\text{isol}}^3 = \frac{\langle J^4 \rangle}{\langle J \rangle^4}, \quad (12)$$

respectively. Thus, the critical points for the localization transitions only depend on the lower moments of  $p_J$  and they are independent of  $p_{K^{\text{out}}}$ . When the  $J_{ij}$  are constant, then  $c_b = c_{\text{isol}} = 1$  such that the delocalization-localization transition is governed by the percolation transition for the strongly connected component [62]. On the other hand, when there is disorder in  $J_{ij}$ , then  $c_b > 1$  and  $c_{\text{isol}} > 1$ .

In Fig. 3, we present the phase diagram obtained when  $p_J$  is a Gaussian distribution with mean  $\mu$  and variance  $\sigma^2$ . In this case,  $c_{\text{gap}}$ ,  $c_b$  and  $c_{\text{isol}}$  only depend on the ratio  $\sigma/\mu$ . A few generic aspects of eigenvector localization in directed random graphs, which also hold for non-Gaussian  $p_J$ , are illustrated in Fig. 3. First, the eigenvector  $\vec{R}(\lambda_{\text{isol}})$  is delocalized when  $\langle J^2 \rangle^3 > \langle J^4 \rangle \langle J \rangle^2$  because  $c_{\text{gap}} > c_{\text{isol}}$ . Second, the transition lines fulfill  $c_{\text{gap}} < c_{\text{isol}} < c_b$  for  $\langle J^2 \rangle^3 < \langle J^4 \rangle \langle J \rangle^2$ . Lastly, we observe

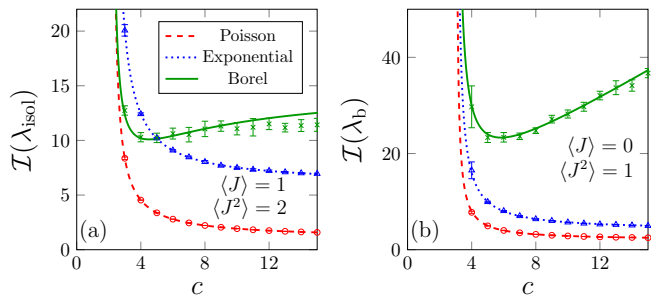


FIG. 2. The inverse participation ratio  $\mathcal{I}(\lambda)$  of the right eigenvectors associated to the outlier eigenvalue  $\lambda_{\text{isol}}$  [Panel (a)] and to an eigenvalue  $\lambda_b \notin \mathbb{R}$  at the boundary of the spectrum [Panel (b)]. Equations (9) and (10) (different line styles) are shown as a function of the average degree  $c$  for different outdegree distributions: Poisson, exponential, and Borel (see Supplemental Material [65]). The weights  $J_{ij}$  are drawn from a Gaussian distribution  $p_J$  with first and second moments indicated on each panel. The different symbols are results obtained from the numerical solution of Eq. (7), while direct diagonalization results for  $\mathcal{I}(\lambda)$  are presented in the Supplemental Material [65]. The results for the Borel distribution are rescaled as  $\mathcal{I}(\lambda_{\text{isol}}) \rightarrow \mathcal{I}(\lambda_{\text{isol}})/c$  in panel (a).

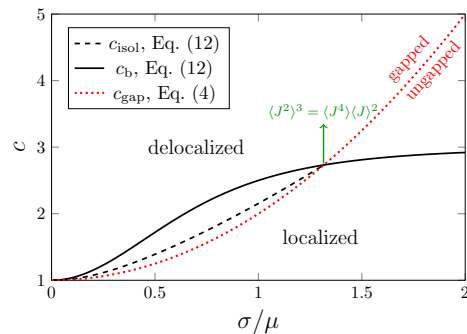


FIG. 3. Phase diagram for localization of right eigenvectors associated to the outlier  $\lambda_{\text{isol}}$  and to eigenvalue  $\lambda_b$  at the boundary of the spectrum. The distribution  $p_J$  is Gaussian with mean  $\mu$  and standard deviation  $\sigma$ .

that the critical transitions  $c_{\text{gap}}$ ,  $c_{\text{isol}}$  and  $c_{\text{gap}}$  intersect in a common point because of the identity  $c_{\text{isol}}^3 = c_b c_{\text{gap}}^2$ .

The second mechanism for localization is due to large degree fluctuations. From Eqs. (9) and (10), it follows that  $\mathcal{I}(\lambda_b) \rightarrow \infty$  if  $\langle (K^{\text{out}})^2 \rangle \rightarrow \infty$  and  $\mathcal{I}(\lambda_{\text{isol}}) \rightarrow \infty$  if  $\langle (K^{\text{out}})^4 \rangle \rightarrow \infty$ , independently of the distribution  $p_J$ . Hence, localization of  $\vec{R}(\lambda_b)$  and  $\vec{R}(\lambda_{\text{isol}})$  also occurs in graphs with power-law degree distributions. In the sequel, we show that degree-based localization persists in the dense limit.

*Localization and universality in the dense limit.* Let us explore the localization and universality of eigenvectors in the dense limit  $c \rightarrow \infty$ . In Fig. 2, we observe that  $\mathcal{I}(\lambda)$  flows to different asymptotic values for  $c \gg 1$ .

In order to identify the universality classes in the limit  $c \rightarrow \infty$ , we analyze the moments of the distribution  $p_R$ . We characterize the dense limit of  $p_R(r|\lambda_{\text{isol}})$  using the relative variance

$$\mathcal{R}_c = \frac{\text{var}[R(\lambda_{\text{isol}})]}{\langle R(\lambda_{\text{isol}}) \rangle^2}, \quad (13)$$

while we choose to characterize the dense limit of  $p_R(r|\lambda_b)$  through the kurtosis

$$\mathcal{K}_c = \frac{\langle (\text{Re}R(\lambda_b))^4 \rangle}{\langle (\text{Re}R(\lambda_b))^2 \rangle^2} = \frac{(4-\gamma)}{2} \mathcal{I}(\lambda_b), \quad (14)$$

where the second equality in Eq. (14) follows from the fact that odd moments of  $p_R(r|\lambda_b)$  are zero [65]. Setting  $c \rightarrow \infty$  in Eqs. (13) and (14), we obtain [65]

$$\mathcal{R}_\infty = \lim_{c \rightarrow \infty} \frac{\text{var}[K^{\text{out}}]}{c^2}, \quad (15)$$

$$\mathcal{K}_\infty = 3 \left( 1 + \lim_{c \rightarrow \infty} \frac{\text{var}[K^{\text{out}}]}{c^2} \right), \quad (16)$$

which shows that the dense limit of  $p_R$  is determined by the degree distribution. We see that, in general,  $p_R(r|\lambda_b)$  and  $p_R(r|\lambda_{\text{isol}})$  are not Gaussian in the dense limit.

With the purpose of classifying the universal behavior of  $p_R$  for  $c \rightarrow \infty$ , let us consider degree distributions that satisfy

$$\text{var}[K^{\text{out}}] = Bc^\alpha \quad (c \gg 1), \quad (17)$$

where  $\alpha$  and  $B$  depend on the specific choice of  $p_{K^{\text{out}}}(k)$ . Equation (17) holds for most degree distributions, including those addressed in Fig. 2. Plugging this *ansatz* for  $\text{var}[K^{\text{out}}]$  in Eqs. (15) and (16), we obtain three universality classes for  $\lim_{c \rightarrow \infty} p_R(r|\lambda)$ , which are determined by the exponent  $\alpha$  that controls the degree fluctuations. The results for the universality classes are summarized in table I. We find that for  $\alpha \leq 2$  the eigenvectors  $\vec{R}(\lambda_b)$  and  $\vec{R}(\lambda_{\text{isol}})$  are delocalized in the limit  $c \rightarrow \infty$ , whereas for  $\alpha > 2$  these eigenvectors are localized due to large degree fluctuations.

	$\alpha < 2$	$\alpha = 2$	$\alpha > 2$
$\mathcal{R}_\infty$	0	$B$	$\infty$
$\mathcal{K}_\infty$	3	$3(1+B)$	$\infty$
Example	Poisson	Exponential	Borel

TABLE I. The relative variance  $\mathcal{R}_c$  of  $\vec{R}(\lambda_{\text{isol}})$  and the kurtosis  $\mathcal{K}_c$  of  $\vec{R}(\lambda_b)$  in the dense limit  $c \rightarrow \infty$  (see Eqs. (15) and (16)), together with an example of the outdegree distribution  $p_{K^{\text{out}}}$  in each regime of  $\alpha$  (see Eq. (17)).

*The eigenvector distributions in the dense limit.* The results in Table I indicate that  $p_R(r|\lambda)$  is universal in the dense limit. Below we present explicit expressions for  $p_R(r|\lambda)$  when  $c \rightarrow \infty$ . Henceforth we set  $\langle |R|^2 \rangle = 1$  without losing generality.

The characteristic function of  $p_R(r|\lambda)$  is given by [65]

$$g_R(u, v|\lambda) = \sum_{k=0}^{\infty} p_{K^{\text{out}}}(k) e^{k \ln F(u, v|\lambda)}, \quad (18)$$

where

$$F(u, v|\lambda) = \int dx p_J(x) \int d^2r p_R(r|\lambda) e^{-\frac{xzr}{2\lambda} + \frac{xz^*r^*}{2\lambda^*}}, \quad (19)$$

and  $z = u + iv$ . The symbol  $(\dots)^*$  denotes complex-conjugation. If  $\lambda \in \mathbb{R}$ , the eigenvector components are real and  $F(u, v|\lambda)$  does not depend on  $v$ .

Setting  $\lambda = \lambda_{\text{isol}}$  or  $\lambda = \lambda_b$  in Eq. (19), we can expand  $F(u, v|\lambda)$  for  $c \gg 1$  up to order  $O(1/c)$  if  $\alpha \leq 2$  (see table I). This approach does not work for  $\alpha > 2$ , because the moments of  $p_R$  can diverge in this regime. Thus, performing this expansion for  $\alpha \leq 2$  and substituting the resulting expression for  $F(u, v|\lambda)$  in Eq. (18), we obtain [65]

$$g_R(u, v|\lambda_b) = \sum_{k=0}^{\infty} p_{K^{\text{out}}}(k) \exp \left[ -\frac{\gamma k}{4c} (u^2 + (2-\gamma)v^2) \right], \quad (20)$$

$$g_R(u, v|\lambda_{\text{isol}}) = \sum_{k=0}^{\infty} p_{K^{\text{out}}}(k) \exp \left( -\frac{iuk}{c\sqrt{Bc^{\alpha-2} + 1}} \right). \quad (21)$$

Remarkably, the characteristic functions in the dense limit are fully specified by degree fluctuations and are independent of  $p_J$ .

For degree distributions where  $\lim_{c \rightarrow \infty} \text{var}[K^{\text{out}}]/c^2 = 0$  ( $\alpha < 2$ ), it is reasonable to set  $p_{K^{\text{out}}}(k) = \delta_{k,c}$  in Eqs. (20) and (21), leading to [65]

$$p_R(r|\lambda_b) = \frac{1}{\pi} e^{-|r|^2} \quad (\lambda_b \notin \mathbb{R}), \quad (22)$$

$$p_R(r|\lambda_{\text{isol}}) = \delta[\text{Im}(r)] \delta[\text{Re}(r) - 1]. \quad (23)$$

Equation (22) yields the well-known Porter-Thomas distribution for the eigenvector components of Gaussian random matrices [66, 67]. Thus, standard results from random matrix theory are recovered when  $\alpha < 2$ .

If  $p_{K^{\text{out}}}$  is an exponential distribution, for which  $\alpha = 2$ , we obtain in the limit  $c \rightarrow \infty$  [65]

$$p_R(r|\lambda_b) = \frac{2}{\pi} K_0(2|r|) \quad (\lambda_b \notin \mathbb{R}), \quad (24)$$

$$p_R(r|\lambda_{\text{isol}}) = \sqrt{2} \delta[\text{Im}(r)] \Theta[\text{Re}(r)] e^{-\sqrt{2}\text{Re}(r)}, \quad (25)$$

where  $\Theta(x)$  is the Heaviside step function and  $K_0(x)$  is a modified Bessel function of the second kind [68]. Figure 4 illustrates the shape of the distributions  $p_R$  given

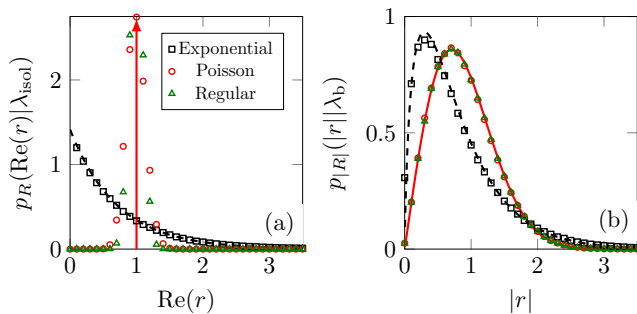


FIG. 4. The dense limit  $c \rightarrow \infty$  of the distribution  $p_R(\text{Re}(r)|\lambda_{\text{iso}})$  of the real part of the eigenvector components at  $\lambda_{\text{iso}}$  [Panel (a)], and of the distribution  $p_{|R|}(|r||\lambda_b)$  of the norm of the eigenvector components at  $\lambda_b \notin \mathbb{R}$  [Panel (b)]. The solid red lines and the dashed black lines are, respectively, the analytic results for regular/Poisson and exponential degree distributions (see Eqs. (22-25)), while the symbols are data obtained from the numerical solutions of Eq. (7) with  $c = 100$ . The numerical data for regular/Poisson graphs in panel (a) is a Gaussian distribution with variance of  $O(1/c)$ , approaching the Dirac delta distribution (vertical arrow) for  $c \rightarrow \infty$ .

by Eqs. (22-25), and compares them with numerical solutions of Eq. (7) for  $c = 100$ . In the Supplemental Material [65], we also derive the analytic expressions for  $\lim_{c \rightarrow \infty} p_R(r|\lambda_b \in \mathbb{R})$  when  $\alpha \leq 2$ .

*Conclusions.* We have shed light on the relationship between graph topology and the localization of right eigenvectors in directed random graphs. If the moments of the outdegree distribution  $p_{K^{\text{out}}}$  are finite, then right eigenvectors at the edge of the spectrum are localized below a critical mean outdegree. It is striking that the critical points for the localization transitions are universal, in the sense they only depend on the lower moments of the distribution  $p_J$  of the edge weights, regardless of the network topology. Therefore, localization in directed random graphs is fundamentally different from localization in undirected graphs, for which degree fluctuations are important [34–36, 39, 69–71]. Indeed, eigenvectors in the tail of the spectrum of undirected random graphs are localized for any  $p_J$  if the degree distribution has an unbounded support. Degree-based localization is also possible for directed random graphs, but then  $p_{K^{\text{out}}}$  has divergent moments.

We have also studied localization and universality of the eigenvectors in the dense limit. In this limit, the distribution  $p_R$  of the right eigenvector components is only determined by the graph topology, independently of the distribution  $p_J$ . If the outdegree fluctuations are small enough, then eigenvectors are delocalized and  $p_R$  is given by the same universal distribution as in the case of Gaussian random matrices [66, 67]. On the other hand, if the outdegree fluctuations are large enough, then eigenvectors are localized and the distribution  $p_R$  depends on

$p_{K^{\text{out}}}$ . More generally, these results indicate that Gaussian random matrix theory describes well the spectral properties of dense graphs only when the degree fluctuations are sufficiently small.

The techniques developed in the present paper can be used to study localization phenomena in non-Hermitian quantum systems [47–49, 72, 73], neural networks [16, 50], ecosystems [17, 18], and real-world networks [42, 74]. The relation between the dynamical properties of these systems and the localization properties of eigenvectors is an interesting topic of future research.

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