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Gravity, Dual Gravity and A_1^{+++}

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Abstract

We construct the non-linear realisation of the semi-direct product of the very extended algebra A_1^{+++} and its vector representation. This theory has an infinite number of fields that depend on a spacetime with an infinite number of coordinates. Discarding all except the lowest level field and coordinates the dynamics is just Einstein's equation for the graviton field. We show that the gravity field is related to the dual graviton field by a duality relation and we also derive the equation of motion for the dual gravity field.

1. Introduction

Some time ago it was conjectured that the non-linear realisation of the semi-direct product of E_{11} and it's vector representation (l_1), denoted $E_{11} \otimes_s l_1$, leads to the low energy effective action for the theory of strings and branes [1,2]. This theory contains an infinite number of fields associated with E_{11} that live on a space-time that contains an infinite number of coordinates. The field equations follow from the symmetries of the non-linear realisation. If one takes the decomposition of E_{11} into its GL(11) subalgebra then one finds a theory whose lowest level fields are those of eleven dimensional supergravity and the level zero coordinates are those of the eleven dimensional spacetime we are familiar with. The essentially unique equation of motion that follow in this decomposition were found relatively recently [3,4] and, in the restriction just mentioned, they were precisely those of eleven dimensional supergravity. To be more precise they were the equations of motion for the graviton ${h_a}^b$ and the three form field $A_{a_1 a_2 a_3}$.

At the next two levels one finds a six form field $A_{a_1 \ldots a_6}$ and a field $h_{a_1 \ldots a_8, b}$. The former field was the well known dual of the three form while the latter fields was proposed to be the dual of the usual gravity field [1], the dual graviton. The field $h_{a_1 a_2, b}$ had been previously investigated in five dimensions and proposed as a candidate for the dual graviton [5], while the field $h_{a_1 \ldots a_{D-3}, b}$ had been proposed in D dimensions [6] as a candidate for the dual graviton. It was shown in reference [1] that this field did indeed describe the degrees of freedom of gravity in D dimensions at the linearised level. as well as the references they contain. Previous work on the dual graviton in the context of E_{11} can be found in references [7] and [8] and a review of E theory can be found in references [10], [11] and [12].

E_{11} is a very extended algebra Kac-Moody algebra which can be found by adding three nodes to the Dynkin diagram of E_8 [13]. In terms of this construction one can write $E_{11} = E_8^{+++}$. Indeed this is a general procedure and one can add three nodes in this way to any semi-simple finite dimensional Lie Algebra, that is, the Lie algebras in the list of Cartan which was actually found by Killing. For each of these algebras one can carry out a corresponding non-linear realisation. It was realised that for $K_{27} \equiv D_{24}^{+++} \otimes_s l_1$, where l_1 is the vector (first fundamental) representation, one finds the fields of the effective action of the twenty six dimensional bosonic string [1]. It is inevitable that the equations of motion of the lowest level fields are those of this effective action. It was also proposed that the very extended A_{D-3} algebra, denoted by A_{D-3}^{+++} describes gravity in D dimensions [14]. These theories contain at lowest level the graviton field and at the next level the dual graviton [15].

In this paper we will consider the case of four dimensional gravity and so the non-linear realisation of algebra $A_1^{+++} \otimes_s l_1$ where this denotes the semi-direct product of A_1^{+++} and it first fundamental (vector) representation l_1. The algebra $A_1^{+++} \otimes_s l_1$ was worked out at low levels in reference [16] and the invariant tangent space metric and an invariant gauge fixing was found in reference [17]. In this paper we calculate the low level equations of motion for the non-linear realisation of $A_1^{+++} \otimes_s l_1$ at low levels. If we restrict the theory to just contain the gravity ${h_a}^b$ and dual gravity ${\tilde{h}_a}{}^b$ fields and the level zero coordinates $x^\mu, \mu = 0, 1, 2, 3$ then we find that the gravity field does obey Einstein's equation and a duality relation that relates the gravity field to the dual gravity field. We also derive the much sort after fully non-linear equation of motion for the dual gravity field. This equation involves the usual graviton field as well as the dual graviton and as a result it avoids the no go theoems of reference [9]. We also comment in the paper on the dual graviton equation derived in reference [8].

The idea that gravity could be found as a result of a non-linear realisation dates back to an old paper of Aleksandr Borisov and Victor Ogievetsky [18] who proposed that gravity was the non-linear realisation of $GL(4) \otimes_s l_1$ where l_1 is the familiar vector representation. This non-linear realisation lead to equations of motion that were far from unique but they proposed that one could take the simultaneous non-linear realisation of this algebra with the conformal algebra. This did lead uniquely to Einstein's equation as must have been the case as it was shown that the simultaneous action of GL(4) and the conformal group on the vector representation lead to general coordinate transformations [19]. In early days of E_{11} when only some of the symmetries were being used to find the equations of motion it was proposed to also use the conformal group but this was found not to be helpful. The uniqueness of the equations of motion in the $E_{11} \otimes_s l_1$ non-linear realisation, including that of gravity, was found to be a consequence of the higher level symmetries and in particular the local symmetries of the Cartan involution invariant subalgebra of E_{11}, denoted $I_c(E_{11})$, beyond those at level zero [3,4]. In this paper we will find that the equations of motion of gravity and dual gravity are essentially unique once we use the higher level symmetries in $A_1^{+++} \otimes_s l_1$.

2 The Kac-Moody algebra A_1^{+++}

We now establish the basic properties of the Kac-Moody algebra A_1^{+++} and it's l_1 representation [16] at low levels. The Dynkin diagram for the Kac-Moody algebra A_1^{+++} is

$$\begin{array}{ccccccc} \bullet & - & \bullet & - & \bullet & = & \otimes \\ 1 & & 2 & & 3 & & 4 \end{array}$$

which corresponds to the Cartan matrix

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -2 \\ 0 & 0 & -2 & 2 \end{pmatrix} . \qquad (2.1)$$

Like all Kac-Moody algebras that are not finite dimensional or affine there is no known listing of the generators that A_1^{+++}. Deleting node four in the above Dynkin diagram we find the residual algebra of $GL(4)$ and one can investigate the A_1^{+++} algebra once it has been decomposed into this later algebra. The generators that one finds are classified by a level which is the number of up minus down $GL(4)$ indices on that generator all divided by two. The decomposition of A_1^{+++} in terms of this subalgebra was given at low levels [16].

The positive level generators to low levels are given by

$$K^a{}_b(16); \quad R^{(ab)}(10); \quad R^{a_1a_2,(b_1b_2)}(45); \quad R^{a_1a_2,b_1b_2,(c_1c_2)}(126), \quad R^{a_1a_2a_3,b_1b_2,c}(64);$$

$$R^{a_1a_2,b_1b_2,c_1c_2,(d_1d_2)}, \quad R^{a_1a_2a_3,b_1b_2,(c_1c_2c_3)}, \quad R_{(1)}^{a_1a_2a_3,b_1b_2,c_1c_2,d}, \quad R_{(2)}^{a_1a_2a_3,b_1b_2,c_1c_2,d},$$

$$R^{a_1a_2a_3,b_1b_2b_3,(c_1c_2)}, \quad R^{a_1a_2,(b_1b_2)}, \quad R^{a_1a_2a_3,d}, \ldots \tag{2.2}$$

where the generators at levels zero, one, two, ... are separated by a semi-colon and the numbers in brackets for the first few generators are the dimensions of the representations. All the upper indices are assumed to be anti-symmetric except for the indices which appear with ( ) brackets and these are symmetric. In what follows we will drop these brackets, for example $R^{[a_1a_2],(b_1b_2)}$ will just be written as $R^{a_1a_2,b_1b_2}$. The subscript indicate that a generator has multiplicity greater than one and the (1) and (2) distinguishing the different generators. These generators possess the $GL(4)$ irreducibility properties

$$R^{[a_1a_2,b_1]b_2} = 0, \quad R^{[a_1a_2,b_1]b_2,c_1c_2} = 0, \quad R^{[a_1a_2,|b_1b_2|,c_1]c_2} = 0,$$

$$R^{[a_1a_2a_3,b_1]b_2,c} = 0, \quad R^{a_1a_2a_3,[b_1b_2,c]} = 0, \ldots \tag{2.3}$$

The negative level generators $R_{ab},\ R_{ab,cd}, \ldots$ possess analogous symmetry and irreducibility properties to their positive level counterparts.

The generators belong to representations of $GL\,(4)$ and so the commutators of ${K^a}_b$ with the positive generators are

$$[{K^a}_b,\, {K^c}_d] = {\delta^c}_b\, {K^a}_d - \delta^a_d\, {K^c}_b,$$

$$[{K^a}_b,\, R^{c_1c_2}] = 2\,\delta_b^{(c_1}\, R^{|a|c_2)}, \quad [{K^a}_b,\, R_{c_1c_2}] = -\,2\,\delta^a_{(c_1}\, R_{|b|c_2)},$$

$$\left[{K^a}_b,\, R^{cd,ef}\right] = \delta^c_b\, R^{ad,ef} + \delta^d_b\, R^{ca,ef} + \delta^e_b\, R^{cd,af} + \delta^f_b\, R^{cd,ea},$$

$$[{K^a}_b,\, R_{cd,ef}] = -\,\delta^a_c\, R_{bd,ef} - \delta^a_d\, R_{cb,ef} - \delta^a_e\, R_{cd,bf} - \delta^a_f\, R_{cd,eb}. \tag{2.4}$$

$$[{K^a}_b, R^{c_1c_2,d_1d_2,e_1e_2}] = {\delta^{c_1}}_b R^{ac_2,d_1d_2,e_1e_2} + \ldots + {\delta^{e_2}}_b R^{c_1c_2,d_1d_2,e_1a},$$

$$[{K^a}_b, R_{c_1c_2,d_1d_2,e_1e_2}] = -{\delta^a}_{c_1} R_{bc_2,d_1d_2,e_1e_2} - \ldots - {\delta^a}_{e_2} R_{c_1c_2,d_1d_2,e_1b},$$

$$[{K^a}_b, R^{c_1c_2c_3,d_1d_2,e}] = {\delta^{c_1}}_b R^{ac_2c_3,d_1d_2,e} + {\delta^{c_2}}_b R^{c_1ac_3,d_1d_2,e} + \ldots + {\delta^e}_b R^{c_1c_2c_3,d_1d_2,a},$$

$$[{K^a}_b, R_{c_1c_2c_3,d_1d_2,e}] = -{\delta^a}_{c_1} R_{bc_2c_3,d_1d_2,e} - \ldots - {\delta^a}_e R_{c_1c_2c_3,d_1d_2,b},$$

The commutators of the level 2 (-2) must give on the right-hand side the unique level 2 (-2) generators and so these commutators must be of the form

$$\left[R^{ab},\, R^{cd}\right] = R^{ac,bd} + R^{bd,ac}, \quad [R_{ab},\, R_{cd}] = R_{ac,bd} + R_{bd,ac}. \tag{2.5}$$

where the normalisation of the level 2 (-2) generators are fixed by these relations. The commutators between the positive and negative level generators are given by

$$\left[R^{ab},\, R_{cd}\right] = 2\,\delta^{(a}_{(c}\, {K^{b)}}_{d)} - \delta^{(ab)}_{cd} \sum_e {K^e}_e,$$

$$\left[R^{ab,cd},\, R_{ef}\right] = \delta^{(bd)}_{ef}\, R^{ac} + \delta^{(bc)}_{ef}\, R^{ad} - \delta^{(ac)}_{ef}\, R^{bd} - \delta^{(ad)}_{ef}\, R^{bc},$$

$$\left[R_{ab,cd},\, R^{ef}\right] = \delta^{ef}_{bd}\, R_{ac} + \delta^{(ef)}_{bc}\, R_{ad} - \delta^{(ef)}_{ac}\, R_{bd} - \delta^{ef}_{ad}\, R_{bc}. \tag{2.6}$$

where $\delta^{(ab)}_{cd} = \delta^{(a}_{c}\delta^{b)}_{d}$.

The Cartan involution acts on the generators of A_1^{+++} as follows

$$I_c\left(K^a{}_b\right) = -\,K^b{}_a, \quad I_c\left(R_{ab}\right) = -\,R^{ab}, \quad I_c\left(R^{ab,cd}\right) = R_{ab,cd}, \ldots . \tag{2.7}$$

The Cartan-involution invariant generators are given by

$$J_{ab} = \eta_{ac}K^c{}_b - \eta_{bc}K^c{}_a,$$

$$S_{ab} = R^{cd}\eta_{ca}\eta_{db} - R_{ab}, \quad S_{a_1a_2,b_1b_2} = R^{c_1c_2,d_1d_2}\eta_{c_1a_1}\eta_{c_2a_2}\eta_{d_1b_1}\eta_{d_2b_2} - R_{a_1a_2,b_1b_2}, \ldots \tag{2.8}$$

They generate the Cartan involution-invariant subalgebra denoted by $I_c(A_1^{+++})$ whose low level commutators are

$$\begin{aligned}
[J_{a_1a_2}, J_{b_1b_2}] &= \eta_{a_2b_1}J_{a_1b_2} - \eta_{a_2b_2}J_{a_1b_1} - \eta_{a_1b_1}J_{a_2b_2} + \eta_{a_1b_2}J_{a_2b_1}\\
[J_{a_1a_2}, S_{b_1b_2}] &= \eta_{a_2b_1}S_{a_1b_2} + \eta_{a_2b_2}S_{a_1b_1} - \eta_{a_1b_1}S_{a_2b_2} - \eta_{a_1b_2}S_{a_2b_1}\\
[S_{a_1a_2}, S_{b_1b_2}] &= 2S_{(a_1|(b_1,b_2)|a_2)} - 2\eta_{(b_1|(a_1}J_{a_2)|b_2)}, \quad \ldots
\end{aligned} \tag{2.9}$$

The first fundamental representation, also called the vector representation, is denoted by l_1. This representation has, at low levels, the generators

$$P_a \ ; \quad Z^a \ ; \quad Z^{(a_1a_2a_3)}, \quad Z^{a_1a_2,b}, \quad Z^{a_1a_2,(b_1b_2b_3)}, \quad Z^{a_1a_2,b_1b_2,c}_{(1)}, \quad Z^{a_1a_2,b_1b_2,c}_{(2)},$$

$$Z^{a_1a_2a_3,(b_1b_2)}, \quad Z^{a_1a_2a_3,b_1b_2}, \quad Z^{a_1a_2,b_1b_2,(c_1c_2c_3)}_{(1)}, \quad Z^{a_1a_2,b_1b_2,(c_1c_2c_3)}_{(2)}, \quad \ldots \tag{2.10}$$

where, as before, the upper indices with no brackets are anti-symmetric, while those with ( ) brackets are symmetric. The subscripts denote the different generators when the multiplicity is greater than one. These generators satisfy the irreducibility conditions

$$Z^{[a_1a_2,b]} = 0, \quad \ldots \tag{2.11}$$

The semi-direct product of the A_1^{+++} with the generators in l_1 representation is denoted by $A_1^{+++} \otimes_s l_1$. The commutators of the A_1^{+++} generators with those of the vector representation have the form

$$\begin{gathered}
[K^a{}_b,\, P_c] = -\,\delta^a_c\, P_b + \frac{1}{2}\,\delta^a_b\, P_c, \quad [K^a{}_b,\, Z^c] = \delta^c_b\, Z^a + \frac{1}{2}\,\delta^a_b\, Z^c,\\
[K^a{}_b,\, Z^{cde}] = \delta^c_b\, Z^{ade} + \delta^d_b\, Z^{cae} + \delta^e_b\, Z^{cda} + \frac{1}{2}\delta^a_b Z^{cde},\\
[K^a{}_b,\, Z^{cd,e}] = \delta^c_b\, Z^{ad,e} + \delta^d_b\, Z^{ca,e} + \delta^e_b\, Z^{cd,a} + \frac{1}{2}\delta^a_b Z^{cd,e}. \qquad (2.12)\\
[R^{ab},\, P_c] = \delta^{(a}_c\, Z^{b)}, \quad [R^{ab},\, Z^c] = Z^{abc} + Z^{c(a,b)}.\\
[R^{ab,cd},\, P_e] = -\,\delta^{[a}_e\, Z^{b]cd} + \frac{1}{4}\left(\delta^a_e\, Z^{b(c,d)} - \delta^b_e\, Z^{a(c,d)}\right) - \frac{3}{8}\left(\delta^c_e\, Z^{ab,d} + \delta^d_e\, Z^{ab,c}\right)
\end{gathered}$$

The commutators with the negative level A_1^{+++} generators are given by

$$[R_{ab},\, P_c] = 0, \quad [R_{ab},\, Z^c] = 2\, \delta^{\,c}_{(a}\, P_{b)},$$

$$[R_{ab},\, Z^{cde}] = \frac{2}{3}\, \left(\delta^{cd}_{(ab)}\, Z^e + \delta^{de}_{(ab)}\, Z^c + \delta^{ec}_{(ab)}\, Z^d\right),$$

$$[R_{ab},\, Z^{cd,e}] = \frac{4}{3}\, \left(\delta^{de}_{(ab)}\, Z^c - \delta^{ce}_{(ab)}\, Z^d\right). \tag{2.13}$$

We also have an algebra formed from the $I_c(A_1^{+++})$ generators and the l_1 generators

$$[J_{a_1a_2}, P_b] = 2P_{[a_1}\eta_{a_2]b}, \quad [S^{ab}, P_c] = \delta^{(a}{}_c Z^{b)},$$

$$[J^{a_1a_2}, Z^b] = -2\eta^{b[a_1}Z^{a_2]},$$

$$[S^{a_1a_2}, Z^b] = (Z^{a_1a_2b} + Z^{b(a_1,a_2)}) - 2\eta^{b(a_1}P^{a_2)}. \tag{2.14}$$

3. Non-linear realisations of $A_1^{+++} \otimes_s l_1$

The construction of the non-linear realisation of $E_{11} \otimes_s l_1$ was discussed in detail in the previous papers on E_{11}. The reader may like to look at reference [10] and the review of reference [11]. The general features of this construction apply to the non-linear realisation of $A_1^{+++} \otimes_s l_1$ which we now briefly summarise. It starts with the group element group element $g \in A_1^{+++} \otimes_s l_1$ that can be written as

$$g = g_l g_A \tag{3.1}$$

In this equation g_A is a group element of A_1^{+++} which can be written in the form $g_A = \Pi_{\underline{\alpha}} e^{A_{\underline{\alpha}} R^{\underline{\alpha}}}$ where the $R^{\underline{\alpha}}$ are the generators of A_1^{+++} given in equations (2.2) as well as their negative level counter parts. The group element g_l is formed from the generators of the vector (l_1) representation and so has the form $\Pi_A e^{z^A L_A}$ where z^A are the coordinates of the generalised space-time. The fields $A_{\underline{\alpha}}$ depend on the coordinates z^A.

The above group elements can, up to level three, be written in the form

$$g_A = \dots\, e^{A_{a_1a_1a_3,b_1b_2,c}R^{a_1a_2a_3,b_1b_2,c}}$$

$$e^{A_{a_1a_2,b_1b_2,c_1c_2}\, R^{a_1a_2,b_1b_2,c_1c_2}}\; e^{A_{a_1a_2,b_1b_2}\, R^{a_1a_2,b_1b_2}}\; e^{A_{a_1a_2}\, R^{a_1a_2}}\; e^{h_a{}^b\, K^a{}_b}\dots . \tag{3.2}$$

where … at the beginning of the equation corresponds to the presence of the higher positive level generators and the … at the end of the equation corresponds to the presence of the negative level generators While the group element g_l can be taken to be of the form

$$g_l = e^{x^a\, P_a}\, e^{y_a\, Z^a}\, e^{x_{abc}\, Z^{abc}}\, e^{x_{ab,c}\, Z^{ab,c}} \dots \tag{3.3}$$

In the above group elements we have introduced the fields

$$h_a{}^b;\;\; A_{(a_1a_2)};\;\; A_{a_1a_2,(b_1b_2)};\;\; A_{a_1a_2,b_1b_2,(c_1c_2)}\;\; A_{a_1a_2a_3,b_1b_2,c},$$

$$A_{a_1a_2,b_1b_2,c_1c_2,(d_1d_2)},\ A_{a_1a_2a_3,b_1b_2,c_1c_2,d}, \ldots \tag{3.4}$$

where as a block of indices is antisymmetric in its indices, except if if it is contained between () in which case it is symmetrised. We will in what follows drop these latter brackets but the reader should recall that the indices are symmetrised. The fields obey the GL(4) irreducibility conditions, for example

$$A_{[a_1a_2,b_1]b_2} = 0,\ A_{[a_1a_2,b_1]b_2,c_1c_2} = 0,$$

$$A_{[a_1a_2|,b_1b_2,|c_1]c_2} = 0,\ A_{[a_1a_2a_3,b_1]b_2,c} = 0,\ A_{a_1a_2,[b_1b_2,c_1]c_2} = 0, \tag{3.5}$$

These fields have 45, 126 and 64 components respectively. In arriving at this count we took account of the fact that $A_{[a_1a_2,b_1b_2]} = 0$ as well as similar conditions for the other two fields.

We have also introduced the generalized coordinates of the space-time

$$x^a;\ y_a;\ x_{abc},\ x_{ab,c}; x_{a_1a_2,b_1b_2b_3},\ x_{a_1a_2,b_1b_2,c},\ x_{a_1a_2a_3,b_1b_2},\ x_{a_1a_2a_3,(b_1b_2)},\ \ldots \tag{3.6}$$

which possess the same symmetries as their corresponding generators in the vector representation, for example $x_{abc} = x_{(a_1a_2a_3)}$. The fields and coordinates obey the same irreducibility as their corresponding generators.

The field ${h_a}^b$ is the usual graviton, the field A_{ab} is the dual graviton and the field $A_{ab,cd}$ is the dual dual-graviton etc. The coordinates x^a are the usual coordinates of space-time while the coordinates y_a are the coordinates associated with the dual graviton. expand.

The non-linear realisation is, by definition, invariant under the transformations

$$g \to g_0 g, \quad g_0 \in A_1^{+++} \otimes_s l_1, \quad \text{as well as} \quad g \to gh, \quad h \in I_c(A_1^{+++}) \tag{3.7}$$

The group element $g_0 \in A_1^{+++}$ is a rigid transformation, that is, it is a constant. The group element h belongs to the Cartan involution invariant subalgebra $I_c(A_1^{+++})$ of A_1^{+++} and it is a local transformation meaning that it depends on the coordinates of the space-time.

As the generators in g_l form a representation of A_1^{+++} the above transformations for $g_0 \in A_1^{+++}$ can be written as

$$g_l \to g_0 g_l g_0^{-1}, \quad g_A \to g_0 g_A \quad \text{and} \quad g_A \to g_A h \tag{3.8}$$

Using these transformations we can set to zero all parts of the group element g_A which depend on the negative level generators.

The dynamics of the non-linear realisation is just a set of equations of motion, that are invariant under the transformations of equation (3.7). We will construct the dynamics of the $A_1^{+++} \otimes_s l_1$ non-linear realisation from the Cartan forms which are given by

$$\mathcal{V} \equiv g^{-1}dg = \mathcal{V}_A + \mathcal{V}_l, \tag{3.9}$$

where

$$\mathcal{V}_A = g_A^{-1}dg_A \equiv dz^\Pi G_{\Pi,\underline{\alpha}} R^{\underline{\alpha}}, \ \text{and}\ \mathcal{V}_l = g_A^{-1}(g_l^{-1}dg_l)g_A = g_A^{-1}dz\cdot l g_A \equiv dz^\Pi {E_\Pi}^A l_A \tag{3.10}$$

Clearly $\mathcal{V}_A$ belongs to the A_1^{+++} algebra and it is the Cartan form of A_1^{+++} while $\mathcal{V}_l$ is in the space of generators of the l_1 representation. The object $E_\Pi{}^A = (\Pi_\alpha e^{A_{\underline{\alpha}} D^{\underline{\alpha}}})_\Pi{}^A$ is the vielbein on the spacetime introduced in the non-linear realisation.

Both $\mathcal{V}_A$ and $\mathcal{V}_l$, when viewed as forms, are invariant under rigid transformations, but under the local $I_c(A_1^{+++})$ transformations of equation (1.3) they change as

$$\mathcal{V}_A \to h^{-1}\mathcal{V}_A h + h^{-1}dh \quad \text{and} \quad \mathcal{V}_l \to h^{-1}\mathcal{V}_l h \tag{3.11}$$

The Cartan form of $I_c(A_1^{+++})$ can be written as

$$\mathcal{V}_A = G_a{}^b K^a{}_b + \overline{G}_{a_1a_2} R^{a_1a_2} + G_{a_1a_2,b_1b_2} R^{a_1a_2,b_1b_2} + \ldots \tag{3.12}$$

Substituting the group element of equation (3.2) we find that the Cartan forms are given by

$$G_a{}^b = (e^{-1}de)_a{}^b$$
$$\overline{G}_{a_1a_2} = e_{a_1}{}^{\mu_1} e_{a_1}{}^{\mu_2} dA_{\mu_1\mu_2},$$
$$G_{a_1a_2,b_1b_2} = e_{a_1}{}^{\mu_1} e_{a_2}{}^{\mu_2} e_{b_1}{}^{\nu_1} e_{b_2}{}^{\nu_2} (dA_{\mu_1\mu_2,\nu_1\nu_2} - A_{[\mu_1|(\nu_1} dA_{\nu_2)|\mu_2]}) \tag{3.13}$$

One can easily verify that $G_{a_1a_2,b_1b_2}$ really does satisfy the irreducibility condition $G_{[a_1a_2,b_1]b_2} = 0$. The presence of the $(\det e)^{\frac{1}{2}}$ factors arises from the unexpected terms with coefficient one half in equation (2.12).

The generalised vielbein and it's inverse up to level one [16] are given by

$$E_\Pi{}^A = (\det e)^{-\frac{1}{2}} \begin{pmatrix} e_\mu{}^a & -e_\mu{}^b A_{ba} \\ 0 & (e^{-1})_a{}^\mu \end{pmatrix}, \quad (E^{-1})_A{}^\Pi = (\det e)^{\frac{1}{2}} \begin{pmatrix} (e^{-1})_a{}^\mu & A_{ab} e_\mu{}^b \\ 0 & e_\mu{}^a \end{pmatrix}, \tag{3.14}$$

The Cartan form transforms under the local $I_c(A_1^{+++})$ transformation as expressed in equation (3.11). The Cartan involution invariant subalgebra at level zero is the Cartan involution invariant subalgebra of $GL(4)$ which is $SO(1,3)$ and the Cartan forms transform under this symmetry as their indices suggest. At the next level they transform under the group element $h = I - \Lambda_{a_1a_2} S^{a_1a_2} \in I_c(A_1^{+++})$ as

$$\delta\mathcal{V}_A = [\Lambda_{a_1a_2} S^{a_1a_2}, \mathcal{V}_A] - S^{a_1a_2} d\Lambda_{a_1a_2} \tag{3.15}$$

These variations are given explicitly by

$$\delta G_a{}^b = 2\Lambda^{cb}\overline{G}_{ca} - \delta_a{}^b \Lambda^{c_1c_2}\overline{G}_{c_1c_2}, \quad \delta\overline{G}_{a_1a_2} = -2\Lambda_{(a_1}{}^b G_{a_2)b} - 4G_{(a_1|b_1|,a_2)b_2}\Lambda^{b_1b_2} - d\Lambda_{a_1a_2}$$
$$\delta G_{a_1a_2,b_1b_2} = 2\Lambda_{[a_1|(b_1|} G_{|a_2]|b_2)}, \tag{3.16}$$

As in the E_{11} case [4], we must require that the local transformations of equation (3.15) preserve the gauge choice. Demanding that the transformed Cartan form has no negative level parts we find that the $\Lambda_{a_1a_2}$ parameter is restricted by

$$d\Lambda_{a_1a_2} - 2\Lambda_{(a_1}{}^b G_{|b|a_2)} = 0. \tag{3.17}$$

This equation implies that the parameter $\Lambda^{\mu\nu}$, that is, the one with upper world indices, is a constant. Using equation (3.17) in equation (3.16) we find that we can re-express $\delta\overline{G}_{a_1a_2}$ as

$$\delta\overline{G}_{a_1a_2} = -4\Lambda_{(a_1}{}^{b}G_{(a_2)b)} - 4G_{(a_1|b_1|,a_2)b_2}\Lambda^{b_1b_2}. \tag{3.18}$$

While the Cartan forms when written as forms are invariant under the above transformations once we consider them as components, that is, we remove the forms dz^{Π} they are no longer invariant under the rigid transformation $g_0 \in A_1^{+++} \otimes_s l_1$. To get an object that is invariant under these rigid transformations we consider the objects

$$G_{A,\bullet} = (E^{-1})_A{}^{\Pi}G_{\Pi,\bullet} \tag{3.19}$$

where $\bullet$ is any A_1^{+++} index. However, these A indices transform under the local $h \in I_c(A_1^{+++})$ transformations given by equation (3.11) and as a result on their first (l_1) index the Cartan forms of equation (3.19) transform as

$$\delta G_{a,\bullet} = -\Lambda_{ab}\hat{G}^{b}, \quad \delta\hat{G}^{a,}{}_{\bullet} = 2\Lambda^{ab}G_{b,\bullet}, \tag{3.20}$$

where the hat indicates a derivative with respect to the level one coordinate y_a. Thus the Cartan forms transform under the simultaneous effect of equations (3.16), (3.18) and (3.20).

The non-linear realisation results in an invariant set of equations which are constructed from the fields of the theory of equation (3.4) which depend on the generalised space-time coordinates of equation (3.6).

4. Derivation of the Duality Equations

We will now construct equations that are first order in derivatives using the Cartan forms of equation (3.13) which are invariant under the rigid $g_0 \in A_1^{+++} \otimes_s l_1$. As a result we do not need to take further account of these transformations. The Cartan forms do, however, transform under the local $h \in I_c(A_1^{+++})$ transformations and so it invariance under these transformations that we will require. At level zero the $I_c(A_1^{+++})$ transformations are just local Lorentz transformations $SO(1,3)$. While the transformations at the level one are given in equations (3.16) , (3.18) and (3.20). As for the case of E_{11} we demand that these first order equations will only be invariant under the above transformations of the non-linear realisation but modulo certain gauge transformations. This some what subtle point is explained in detail in references [20,4,12].

The level one transformations of equations (3.16) and (3.18) transform Cartan forms of a given level into Cartan forms that have a level increased or decreased by one. Hence the variation of the Cartan form associated with our usual formulation of gravity, that is, the one constructed from the gravity field $h_a{}^b$, will led to the Cartan form associated with the field $\tilde{A}_{a_1a_2}$ associated with dual graviton. Thus we expect a duality relation that relates the gravity field to the dual gravity field. We will start by considering the well known spin connection which in terms of the gravity Cartan form is given by

$$(\det e)^{1/2}\omega_{a,b_1b_2} = (-G_{b_1,(b_2a)} + G_{b_2,(b_1a)} + G_{a,[b_1b_2]}) \tag{4.2}$$

While one could proceed by writing down the most general equation constructed from the Cartan forms and test its invariance it is easier, and equivalent, to start from the spin connection and see what terms one must add by demanding $I_c(A_1^{+++})$ invariance.

Simply using Lorentz symmetry we find that the equation should be of the generic form

$$E_{a,b_1b_2} \equiv (\det e)^{1/2}\omega_{a,b_1b_2} + \frac{\tilde{e}_1}{2}\varepsilon_{b_1b_2}{}^{c_1c_2}\overline{G}_{c_1,c_2a} \doteq 0 \tag{4.3}$$

where $\tilde{e}_1$ is a constant. The factor of $(\det e)^{\frac{1}{2}}$ correspond to the same factors in equation (3.13), that is, such factors appear in the Cartan forms.

We observe that the spin connection has to transform under local $I_c(A_1^{+++})$ transformations into not just the dual gravity Cartan form but the one which has its first two indices anti-symmetrised, namely $\overline{G}_{[c_1,c_2]a}$ as it is this object that occurs in equation (4.3). While the spin connection does not do this we can add to it terms that involve derivatives with respect to higher level coordinates, the so called l_1 terms, such that it does. The required object is

$$(\det e)^{1/2}\Omega_{a,b_1b_2} = (\det e)^{1/2}\omega_{a,b_1b_2} - \frac{1}{2}\eta_{b_2a}\hat{\overline{G}}{}^{e,}{}_{b_1e} + \frac{1}{2}\eta_{b_1a}\hat{\overline{G}}{}^{e,}{}_{b_2e} \tag{4.4}$$

Its variation is given by

$$\begin{aligned}\delta[(\det e)^{1/2}\Omega_{a,b_1b_2}] = {}& 2\Lambda^e{}_a\overline{G}_{[b_2,b_1]e} + 2\Lambda^e{}_{b_2}\overline{G}_{[a,b_1]e} + 2\Lambda^e{}_{b_1}\overline{G}_{[b_2,a]e} + 2\eta_{b_2a}\Lambda^{e_1e_2}\overline{G}_{[b_1,e_1]e_2} \\ & -2\eta_{b_1a}\Lambda^{e_1e_2}\overline{G}_{[b_2,e_1]e_2}.\end{aligned} \tag{4.5}$$

In arriving at this result we have used equation (3.20).

As for the case of E_{11}, we will only compute the equations of motion and duality relations to lowest level in the derivatives of the coordinates, meaning that they contain only derivatives with respect to the usual coordinates x^a of spacetime. As a result we only keep terms in the local $I_c(A_1^{+++})$ variations that have no derivatives with respect to the higher level coordinates. However, terms in the equation that is being varied that are linear in derivatives with respect to the level one coordinates y_a will, according to equation (3.20), vary into terms that have ordinary derivatives. Such terms will contain as one of its factors the Cartan forms $\hat{G}_{a,\bullet}$. As a result we will require such terms in the equations we are varying. We will refer to such terms as l_1 terms.

To summarise we will find the equation that is the result of the variation only up to derivatives with respect to the level zero coordinates but to do this we will be required to find the equations that are being varied up to derivatives with respect to the level one coordinates. Indeed by varying equations one can find the terms that they contain that have derivatives with respect to the level one coordinates. We will refer to this as the l_1 extension of the equation.

Taking all this into account we vary the object E_{a,b_1b_2} of equation (4.3) but as a help along the way we may use the object of equation (4.4) instead of the usual spin connection. Adding further l_1 terms one finds that the l_1 extended object duality relation between the gravity and dual gravity fields is given by

$$\mathcal{E}_{a,b_1b_2} \equiv (\det e)^{1/2}\Omega_{a,b_1b_2} + \frac{1}{2}\varepsilon_{b_1b_2}{}^{c_1c_2}\overline{G}_{c_1,c_2a} + \frac{1}{2}\varepsilon_{b_1b_2}{}^{c_1c_2}(\hat{G}_{c_2,[c_1a]} + \frac{1}{2}\hat{G}_{a,[c_1c_2]})$$

$$-\frac{1}{2}\varepsilon_{b_1b_2}{}^{c_1c_2}(\hat{G}^{e,}{}_{c_1a,c_2e}+\frac{1}{2}\hat{G}^{e,}{}_{c_1c_2,ae})-\frac{1}{4}\eta_{ab_1}\overline{G}^{e,}{}_{b_2e}+\frac{1}{4}\eta_{ab_2}\Lambda^{e_1e_2}\overline{G}^{e,}{}_{b_1e}\doteq 0 \tag{4.6}$$

and it varies under a local $I_c(A_1^{+++})$ transformations as follows

$$\delta\mathcal{E}_{a,b_1b_2}=\frac{1}{2}\varepsilon_{b_1b_2}{}^{c_1c_2}\Lambda_a{}^eE_{e,c_1c_2}+\varepsilon_{b_1b_2}{}^{c_1c_2}\Lambda_{c_2}{}^eE_{e,c_1a}-\frac{1}{2}\eta_{ab_1}\Lambda^{e_1e_2}\overline{E}_{b_2,e_1e_2}$$

$$+\frac{1}{2}\eta_{ab_2}\Lambda^{e_1e_2}\overline{E}_{b_1,e_1e_2}+\Lambda_{b_1}{}^e\overline{E}_{b_2,ae}-\Lambda_{b_2}{}^e\overline{E}_{b_1,ae}+e_a{}^\mu\partial_\mu\tilde{\Lambda}_{b_1b_2} \tag{4.7}$$

In the process of carry out this calculation one finds that the variation of $\mathcal{E}_{a,b_1b_2}=0$ leads to a trivial dynamics unless $\tilde{e}_1=1$ which is the value we now adopt.

Setting the variation of $\mathcal{E}_{a,b_1b_2}\doteq 0$ we find the gravity-dual gravity relation $E_{a,bc}\doteq 0$, from which we started, as well as a dual graviton- dual dual graviton duality relation which is given by

$$\overline{E}_{a,b_1b_2}\equiv\overline{G}_{a,b_1b_2}+\varepsilon_a{}^{e_1e_2e_3}G_{e_1,e_2e_3,b_1b_2}\doteq 0 \tag{4.8}$$

In the variation of equation (4.7) we also find the local Lorentz transformations

$$e_a{}^\mu\partial_\mu\tilde{\Lambda}_{b_1b_2}=-\varepsilon_{b_1b_2}{}^{c_1c_2}(\Lambda_{c_2}{}^eG_{a,(c_1e)}-G_{a,e_1c_1,c_2e_2}\Lambda^{e_1e_2})-2\Lambda_{e[b_1|}\overline{G}_{a,|b_2]e}. \tag{4.9}$$

As we have noted some of the equations we find only hold modulo certain local transformations and in the case of the gravity-dual gravity duality relation these include local Lorentz transformations. The symbol $\doteq$ indicates that the equations only hold modulo the local transformations.

5. The gravity and dual gravity equations of motion

In this section we will use the symmetries of the non-linear realisation to find the equations of motion for the graviton and the dual graviton which are second order in derivatives. Since the level one local transformations with parameter Λ^{ab} change the level of the Cartan form on which it acts by plus or minus one, the variation of the gravity equation must led to the dual gravity equation. We begin with the the usual Ricci tensor

$$(\det e)R_a{}^b=(\det e)\{e_a{}^\mu\partial_\mu(\omega_{\nu,}{}^{bd})e_d{}^\nu-\partial_\nu(\omega_\mu{}^{bd})e_d{}^\nu e_a{}^\mu+\omega_{a,}{}^b{}_c\omega_{d,}{}^{cd}-\omega_{d,}{}^b{}_c\omega_{a,}{}^{cd}\} \tag{5.1}$$

In order to carry out its local $I_c(A_1^{+++})$ variation we must express the Ricci tensor in terms of the Cartan forms of section three, the result is

$$(\det e)R_a{}^b=(\det e)^{\frac{1}{2}}e_a{}^\mu\partial_\mu[(\det e)^{\frac{1}{2}}\omega_{d,}{}^{bd}]-(\det e)^{\frac{1}{2}}e_d{}^\nu\partial_\nu[(\det e)^{\frac{1}{2}}\omega_{a,}{}^{bd}]$$

$$+(\det e)\omega^{c,bd}\omega_{d,ca}+G_{c,d}{}^c(\det e)^{\frac{1}{2}}\omega_{a,}{}^{bd}-\frac{1}{2}G_{a,c}{}^c(\det e)^{\frac{1}{2}}\omega_{d,}{}^{bd}-\frac{1}{2}G_{d,c}{}^c(\det e)^{\frac{1}{2}}\omega_{a,}{}^{bd} \tag{5.2}$$

where the expression for the spin connection in terms of the gravity Cartan forms is given in equation (4.2).

We begin by considering the Ricci tensor as we expect our equation of motion will turn out to be that this object will vanish. As such we define

$$E_a{}^b\equiv(\det e)R_a{}^b \tag{5.3}$$

and consider its variation under local $I_c(A_1^{+++})$ transformations. As we explained above in carrying out the variation we must find the l_1 extension of the equations we are varying. We denote this l_1 extended object by ${\mathcal{E}_a}^b$. A help towards the result is achieved if one replaced the usual spin connection by its l_1 extension of equation (4.4) which has the variation of equation (4.5). After a somewhat lengthy calculation one finds that

$$\begin{aligned}
\mathcal{E}'_{ab} \equiv (\det e)\mathcal{R}_{ab} = (\det e)\{{e_a}^\mu\partial_\mu({\Omega_{\nu,}}^{bd}){e_d}^\nu - \partial_\nu({\Omega_\mu}^{bd}){e_d}^\nu {e_a}^\mu + {\Omega_{a,}}^b{}_c{\Omega_{d,}}^{cd} - {\Omega_{d,}}^b{}_c{\Omega_{a,}}^{cd}\} \\
+(\det e)^{\frac{1}{2}}{\hat{e}_a}{}^\nu\partial_\nu{\overline{G}_{[c,b]}}{}^c + {\hat{G}_{a,}}{}^c{}_d{\overline{G}_{[c,b]}}{}^d - {\hat{G}_{a,}}{}^d{}_b{\overline{G}_{[c,d]}}{}^c + {\hat{G}_{a,}}{}^d{}_c{\overline{G}_{[b,d]}}{}^c - \frac{1}{2}{\hat{G}_{a,}}{}^c{}_c{\overline{G}_{[b,d]}}{}^d \\
+(\det e)^{\frac{1}{2}}{\hat{e}_b}{}^\nu\partial_\nu{\overline{G}_{[c,a]}}{}^c + {\hat{G}_{b,}}{}^c{}_d{\overline{G}_{[c,a]}}{}^d - {\hat{G}_{b,}}{}^d{}_a{\overline{G}_{[c,d]}}{}^c + {\hat{G}_{b,}}{}^d{}_c{\overline{G}_{[a,d]}}{}^c - \frac{1}{2}{\hat{G}_{b,}}{}^c{}_c{\overline{G}_{[a,d]}}{}^d \\
-\eta_{ab}((\det e)^{\frac{1}{2}}\hat{e}^e{}_\nu\partial_\nu{\overline{G}_{[c,e]}}{}^c + \hat{G}^{e,c}{}_d{\overline{G}_{[c,e]}}{}^d - \hat{G}^{e,d}{}_e{\overline{G}_{[c,d]}}{}^c + \hat{G}^{e,d}{}_c{\overline{G}_{[e,d]}}{}^c - \frac{1}{2}\hat{G}^{e,c}{}_c{\overline{G}_{[e,d]}}{}^d) \\
-(\hat{G}^{e,c}{}_a\overline{G}_{[b,c]e} + \hat{G}^{e,c}{}_b\overline{G}_{[a,c]e} + \eta_{ab}\hat{G}^{e,}{}_{[f_1f_2]}\overline{G}^{[f_1,f_2]}{}_e) + \hat{\partial}^e\overline{G}_{[a,b]e} + \hat{G}^{e,}{}_e{}^c\overline{G}_{[a,b]c} \\
+\frac{1}{2}\hat{G}^{c,}{}_e{}^e\overline{G}_{[a,b]c} - \frac{1}{2}(\det e)^{\frac{1}{2}}\Omega_{d,b}{}^d\hat{\overline{G}}{}^{e,}{}_{ae} - \frac{1}{2}(\det e)^{\frac{1}{2}}\Omega_{a,b}{}^d\hat{\overline{G}}{}^{e,}{}_{ed} \qquad (5.4)
\end{aligned}$$

has the variation

$$\begin{aligned}
\delta\mathcal{E}'_{ab} = -4\Lambda_{ea}\overline{E}'{}_b{}^e - 4\Lambda_{eb}\overline{E}'{}_a{}^e + 4\eta_{ab}\Lambda^{e_1}{}_{e_2}\overline{E}'{}_{e_1}{}^{e_2} \\
+\Lambda^{e_1e_2}\varepsilon_{bc}{}^{f_1f_2}(E_{e_2,a}{}^e(\det e)^{\frac{1}{2}}\omega_{e_1,f_1f_2} - E_{e_1,f_1f_2}(\det e)^{\frac{1}{2}}\omega_{e_2,a}{}^c) \\
+\Lambda^{e_1e_2}\varepsilon_{ac}{}^{f_1f_2}(E_{e_2,b}{}^e(\det e)^{\frac{1}{2}}\omega_{e_1,f_1f_2} - E_{e_1,f_1f_2}(\det e)^{\frac{1}{2}}\omega_{e_2,b}{}^c) \qquad (5.5)
\end{aligned}$$

where we defined

$$\begin{aligned}
\overline{E}'{}_a{}^b \equiv (\det e)^{\frac{1}{2}}e^{\nu[c}\partial_\nu\overline{G}_{[c,a]}{}^{b]} + G^{[c,b]}{}_d\overline{G}_{[c,a]}{}^d - G^{[c|,d}{}_c\overline{G}_{[d,a]}{}^{|b]} \\
-G^{[d|,c}{}_a\overline{G}_{[d,c]}{}^{|b]} + \frac{1}{2}G^{[d|}{}_{,c}{}^c\overline{G}_{[d,a]}{}^{|b]}. \qquad (5.6)
\end{aligned}$$

Converted to world volume indices $\overline{E}'{}_a{}^b$ takes the form

$$\overline{E}'{}^\mu{}_\tau = \partial_{[\nu}((\det e)^{\frac{1}{2}}\overline{G}^{[\nu,\mu]}{}_{\tau]}). \qquad (5.7)$$

We observe that the variation of $\mathcal{E}'{}_{ab}$ contains our previously discussed the first order gravity-dual gravity relation $E_{a,b}{}^c$, found in the previous section, as well as the new object $\overline{E}'{}_a{}^b$. We note that these occur in a different ways in relation to the parameter Λ_{ab}. As a result, we may take the equations of motion to be $E_{ab} = 0$ and $\overline{E}'{}_a{}^b = 0$ as these are an invariant set of equations up to the level computed. The first equation is just Einstein's equation for gravity, as one might be expect, while the second equation would be that for the dual graviton.This conclusion would however, be premature. It over looks the fact that the l_1 extension of the Einstein equation could contain terms $\hat{G}_{b,\bullet}X$ where X is any function of the Cartan forms with derivatives that are with respect to the usual spacetime coordinates. These terms would lead in the variation of the Einstein equation $\mathcal{E}'_{ab}$ to terms of the form $\Lambda^{be}\overline{G}_{e,\bullet}X$ where $\bullet$ is a E_{11} index. Looking at the variation of equation (5.5) we see that such a term would result in an addition to the dual graviton equation $\overline{E}_{ab}$ of a term of the form index $G_{b,\bullet}X$. We note this is a term which contains a spacetime derivative with an index that corresponds to the second index on $\overline{E}'_a{}^b$. The primes on $\mathcal{E}'_a{}^b$ and $\overline{E}'_a{}^b$ are to indicate that we have not so far taken account of this possibility and so these objects are not the final results. We will now take account of this possibility and find which terms can be added in the way suggested.

The dual graviton equation is by definition the equation of motion for dual graviton and as a result it should have the same symmetries as the dual gravity field, that is, it should be symmetric in its two indices. While the effect of exchanging the a and b indices is obvious for the $G\bar{G}$ terms in equation (5.6) it is not so obvious for the first term. To clarify this we rewrite the first term in equation (5.6) as

$$(\det e)^{\frac{1}{2}}e^{\nu[c}\partial_\nu\overline{G}_{[c,a]}{}^{b]} = \frac{1}{4}(\det e)^{\frac{1}{2}}e^{\nu c}\partial_\nu\overline{G}_{c,a}{}^{b} - \frac{1}{4}(\det e)^{\frac{1}{2}}(e^{\nu c}\partial_\nu\overline{G}_{a,c}{}^{b} + e^{\nu c}\partial_\nu\overline{G}^{b,}{}_{ac})$$

$$+\frac{1}{8}(\det e)^{\frac{1}{2}}(e^{\nu b}\partial_\nu\overline{G}_{a,c}{}^{c} + e_a{}^{\nu}\partial_\nu\overline{G}^{b,c}{}_{c})$$

$$-\frac{1}{4}(\det e)^{\frac{1}{2}}(e^{\nu b}\partial_\nu\overline{G}_{c,a}{}^{c} - e^{\nu}{}_{c}\partial_\nu\overline{G}^{b,}{}_{a}{}^{c}) + \frac{1}{8}(\det e)^{\frac{1}{2}}(e^{\nu b}\partial_\nu\overline{G}_{a,c}{}^{c} - e_a{}^{\nu}\partial_\nu\overline{G}^{b,c}{}_{c}) \qquad (5.8)$$

The first three terms are obviously symmetric under the interchange of a and b. While the effect of this interchange on the last two terms is not so clear we can further rewrite them using the Maurer Cartan equations.

The form $\mathcal{V} = g^{-1}dg$ it obeys the Maurer Cartan equation $d\mathcal{V} = -\mathcal{V}\wedge\mathcal{V}$, or equivalently $\partial_\mu\mathcal{V}_\nu - \partial_\nu\mathcal{V}_\mu + \mathcal{V}_\mu\mathcal{V}_\nu - \mathcal{V}_\nu\mathcal{V}_\mu = 0$. Using the form of $\mathcal{V}$ of equation (3.12) we find, amongst other equations, that

$$(\det e)^{\frac{1}{2}}e_c{}^{\mu}\partial_\mu\overline{G}_{d,ab} - (\det e)^{\frac{1}{2}}e_d{}^{\mu}\partial_\mu\overline{G}_{c,ab} + G_{c,d}{}^{e}\overline{G}_{e,ab} - G_{d,c}{}^{e}\overline{G}_{e,ab} - \frac{1}{2}G_{c,e}{}^{e}\overline{G}_{d,ab}$$

$$+\frac{1}{2}G_{d,e}{}^{e}\overline{G}_{c,ab} + G_{c,a}{}^{e}\overline{G}_{d,eb} + G_{c,b}{}^{e}\overline{G}_{d,ae} - G_{d,a}{}^{e}\overline{G}_{c,eb} - G_{d,b}{}^{e}\overline{G}_{c,ae} = 0 \qquad (5.9)$$

Using this last equation we can rewrite the last two terms of equation (5.8) as

$$-\frac{1}{4}(\det e)^{\frac{1}{2}}[e^{\nu b}\partial_\nu\overline{G}_{c,a}{}^{c} - e^{\nu}{}_{c}\partial_\nu\overline{G}^{b,}{}_{a}{}^{c}] = +\frac{1}{4}(G^{b,}{}_{c}{}^{e}\overline{G}_{e,a}{}^{c} - G_{c,}{}^{be}\overline{G}_{e,a}{}^{c} - \frac{1}{2}G^{b,e}{}_{e}\overline{G}_{c,a}{}^{c}$$

$$+\frac{1}{2}G_{c,e}{}^{e}\overline{G}^{b,}{}_{a}{}^{c} + G^{b,}{}_{a}{}^{e}\overline{G}_{c,e}{}^{c} + G^{b,ce}\overline{G}_{c,ae} - G_{c,a}{}^{e}\overline{G}^{b,}{}_{e}{}^{c} - G_{c,}{}^{ce}\overline{G}^{b,}{}_{ae}) \qquad (5.10)$$

and

$$\frac{1}{8}(\det e)^{\frac{1}{2}}(e^{\nu b}\partial_\nu\overline{G}_{a,c}{}^{c} - e_a{}^{\nu}\partial_\nu\overline{G}^{b,c}{}_{c}) = -\frac{1}{8}(G^{b,}{}_{a}{}^{e}\overline{G}_{e,c}{}^{c} - G_{a,}{}^{be}\overline{G}_{e,}{}^{c}{}_{c} - \frac{1}{2}G^{b,e}{}_{e}\overline{G}_{a,c}{}^{c}$$

$$+\frac{1}{2}G_{a,e}{}^{e}\overline{G}^{b,c}{}_{c}+G^{b,}{}_{c}{}^{e}\overline{G}_{a,}{}^{c}{}_{e}-G_{a,c}{}^{e}\overline{G}^{b,}{}_{e}{}^{c}-G_{a,c}{}^{e}\overline{G}^{b,c}{}_{e}) \tag{5.11}$$

Using equations (5.10) and (5.11) and explicitly writing out the anti-symmetrisations of the $G\bar{G}$ terms we find that the dual graviton expression $\bar{E}'{}_{ab}$ of equation (5.6) can written as

$$\begin{aligned}\overline{E}'{}_{a}{}^{b}=\frac{1}{4}(\det e)^{\frac{1}{2}}(&e^{\nu c}\partial_\nu\overline{G}_{c,a}{}^{b}-e^{\nu c}\partial_\nu\overline{G}_{a,c}{}^{b}-e^{\nu c}\partial_\nu\overline{G}^{b,}{}_{ac}+\frac{1}{2}e^{\nu b}\partial_\nu\overline{G}_{a,c}{}^{c}+\frac{1}{2}e_{a}{}^{\nu}\partial_\nu\overline{G}^{b,c}{}_{c})\\
&+\frac{1}{8}[G^{c,b}{}_{e}(2\overline{G}_{c,a}{}^{e}-2\overline{G}^{e,}{}_{ac}-2\overline{G}_{a,c}{}^{e})+G^{b,}{}_{c}{}^{e}(-2\overline{G}_{a,}{}^{c}{}_{e}+2\overline{G}_{e,a}{}^{c}+2\overline{G}^{c,}{}_{ae})\\
&+G^{e,c}{}_{e}(-2\overline{G}_{c,a}{}^{b}+2\overline{G}_{a,c}{}^{b})+G^{d,c}{}_{a}(-2\overline{G}_{d,c}{}^{b}+2\overline{G}_{c,d}{}^{b})+G^{b,c}{}_{a}(2\overline{G}_{d,c}{}^{d}-2\overline{G}_{c,d}{}^{d})\\
&+G^{d,c}{}_{c}(\overline{G}_{d,a}{}^{b}-\overline{G}_{a,d}{}^{b}+\overline{G}^{b,}{}_{ad})+G^{b,c}{}_{c}(-\overline{G}_{d,a}{}^{d}+\overline{G}_{a,d}{}^{d}+\frac{1}{2}\overline{G}_{a,d}{}^{d}-\overline{G}_{d,a}{}^{d})+G_{a,}{}^{be}\overline{G}_{e,c}{}^{c}\\
&+G^{b,}{}_{a}{}^{e}(-\overline{G}_{e,c}{}^{c}+2\overline{G}_{c,e}{}^{c})+G_{a,c}{}^{e}(\overline{G}^{b,}{}_{e}{}^{c}+\overline{G}^{b,c}{}_{e})+G_{c,a}{}^{e}(-2\overline{G}^{b,}{}_{e}{}^{c})\\
&+G_{c,}{}^{ce}(-2\overline{G}^{b,}{}_{ae})+G_{a,e}{}^{e}(-\frac{1}{2}\overline{G}^{b,c}{}_{c})]\end{aligned} \tag{5.12}$$

Clearly this expression for $\bar{E}'{}_{ab}$ of equation (5.12) is not symmetric under $a \leftrightarrow b$ and so setting it to zero can not lead to the dual graviton equation. However, we can exploit the above ambiguity to add terms to the l_1 extension of the Einstein expression and so to the dual graviton expression of equation (5.12). The terms of equation (5.12) can be divided in to three types

(a) terms which contain a $G_{b,\bullet}$ factor ,
(b) terms which contain a $G_{a,\bullet}$ factor,
(c) the remaining terms.

The type (a) terms can all be removed by adding terms to $\mathcal{E}'{}_{a}{}^{b}$ as explained above. These terms occur in equation (5.12) as the number 3, 6, 8, 10 , 12, 13, 14 terms as well as the last expression in term 7. The type (b) terms by definition contain a $G_{a,\bullet}$ factor and they occur in equation (5.12) as the terms number 2 (only last expression), 4 (only last expression), 7 (only middle expression) and term 9. These terms are given by

$$+\frac{1}{8}(-2G^{c,b}{}_{e}\overline{G}_{a,c}{}^{e}+2G^{e,c}{}_{e}\overline{G}_{a,c}{}^{b}-G^{d,c}{}_{c}\overline{G}_{a,d}{}^{b}+G_{a,}{}^{be}\overline{G}_{e,c}{}^{c}) \tag{5.13}$$

For each of these terms we can swop the a and b indices and add the resulting term to the dual graviton equation as it contains a $G_{b,\bullet}$ factor. Put another way, we can in effect symmetrise type (b) terms by hand. The effect is that we add the terms

$$+\frac{1}{8}(-2G^{c,}{}_{ae}\overline{G}^{b,}{}_{c}{}^{e}+2G^{e,c}{}_{e}\overline{G}^{b,}{}_{ca}-G^{d,c}{}_{c}\overline{G}^{b,}{}_{da}+G^{b,}{}_{a}{}^{e}\overline{G}_{e,c}{}^{c}) \tag{5.14}$$

to the dual graviton equation.

The terms of the type (c) are given by

$$+\frac{1}{2}(G^{c,be}\overline{G}_{[c,e]a}-G^{c,d}{}_{a}\overline{G}_{[c,d]}{}^{b})-\frac{1}{4}G^{e,c}{}_{e}\overline{G}_{c,a}{}^{b}+\frac{1}{8}G^{d,c}{}_{c}\overline{G}_{d,a}{}^{b} \tag{5.15}$$

The last two terms are symmetric in $a \leftrightarrow b$ while the first two terms can be written as

$$+\frac{1}{2}(G^{c,bd}\overline{G}_{[c,d]a}+G^{c,}{}_{a}{}^{d}\overline{G}_{[c,d]}{}^{b})-\frac{1}{4}\varepsilon^{cde_1e_2}\overline{G}_{[e_1,e_2]a}\overline{G}_{[c,d]}{}^{b}+\frac{1}{2}(E_{a,}{}^{cd}-\frac{1}{2}G_{a,}{}^{[c,d]})\overline{G}_{[c,d]}{}^{b} \tag{5.16}$$

The first and second terms in this expression are symmetric, while the third term is the gravity-dual gravity duality relation and the fourth term can be viewed as a modulo transformation to which this duality relation holds. While the first two terms contribute to the dual gravity equation of motion, the last two terms can be reinterpreted as terms that explicitly occur in the variation of the l_1 extended gravity equation of motion $\mathcal{E}_{ab}$, see equation (5.18) below.

After carrying out all the above steps we add the above terms to $\bar{E}'{}_a{}^b$ to find that the dual graviton equation which is given by

$$\begin{aligned}
\overline{E}_a{}^b \equiv \frac{1}{4}(\det e)^{\frac{1}{2}}(e^{\nu c}\partial_\nu\overline{G}_{c,a}{}^{b} - e^{\nu c}\partial_\nu\overline{G}_{a,}{}^{b}{}_{c} - e^{\nu c}\partial_\nu\overline{G}^{b,}{}_{ac} + \frac{1}{2}e^{\nu b}\partial_\nu\overline{G}_{a,c}{}^{c} + \frac{1}{2}e_a{}^{\nu}\partial_\nu\overline{G}^{b,c}{}_{c})\\
-\frac{1}{4}G^{e,c}{}_{e}\overline{G}_{c,a}{}^{b} + \frac{1}{8}G^{d,c}{}_{c}\overline{G}_{d,a}{}^{b}\\
-\frac{1}{4}\varepsilon^{cde_1e_2}\overline{G}_{[e_1,e_2]a}\overline{G}_{[c,d]}{}^{b} + \frac{1}{2}(G^{c,bd}\overline{G}_{[c,d]a} + G^{c,}{}_{a}{}^{d}\overline{G}_{[c,d]}{}^{b})\\
-\frac{1}{4}(G^{c,}{}_{ae}\overline{G}^{b,}{}_{c}{}^{e} + G^{c,b}{}_{e}\overline{G}_{a,c}{}^{e}) + \frac{1}{4}G^{e,c}{}_{e}(\overline{G}^{b,}{}_{ca} + \overline{G}_{a,c}{}^{b})\\
-\frac{1}{8}G^{d,c}{}_{c}(\overline{G}^{b,}{}_{da} + \overline{G}_{a,d}{}^{b}) + \frac{1}{8}(G^{b,}{}_{a}{}^{e} + G_{a,}{}^{be})\overline{G}_{e,c}{}^{c} = 0
\end{aligned} \tag{5.17}$$

It is indeed symmetric under the interchange of a and b. That one can use the ambiguity to find an expression that is symmetric under the interchange of a and b is very non-trivial. The corresponding l_1 extension of the Einstein equation, denoted $\mathcal{E}_a{}^b$, is given in appendix A. The variation of $\mathcal{E}_a{}^b$ obeys the equation

$$\begin{aligned}
\delta\mathcal{E}_{ab} = -4\Lambda_{ea}\overline{E}_b{}^e - 4\Lambda_{eb}\overline{E}_a{}^e + 4\eta_{ab}\Lambda^{e_1}{}_{e_2}\overline{E}_{e_1}{}^{e_2}\\
+\Lambda^{e_1e_2}\varepsilon_{bc}{}^{f_1f_2}(E_{e_2,a}{}^{e}(\det e)^{\frac{1}{2}}\omega_{e_1,f_1f_2} - E_{e_1,f_1f_2}(\det e)^{\frac{1}{2}}\omega_{e_2,a}{}^{c})\\
+\Lambda^{e_1e_2}\varepsilon_{ac}{}^{f_1f_2}(E_{e_2,b}{}^{e}(\det e)^{\frac{1}{2}}\omega_{e_1,f_1f_2} - E_{e_1,f_1f_2}(\det e)^{\frac{1}{2}}\omega_{e_2,b}{}^{c})\\
-2\Lambda_{ea}(E_{b,}{}^{cd} - \frac{1}{2}G_{b,}{}^{[c,d]})\overline{G}_{[c,d]}{}^{e} - 2\Lambda_{eb}(E_{a,}{}^{cd} - \frac{1}{2}G_{a,}{}^{[c,d]})\overline{G}_{[c,d]}{}^{e} +\\
+2\eta_{ab}\Lambda^{e_1}{}_{e_2}(E_{e_1,}{}^{cd} - \frac{1}{2}G_{e_1,}{}^{[c,d]})\overline{G}_{[c,d]}{}^{e_2}
\end{aligned} \tag{5.18}$$

It is equation (5.5) with the primes removed and an extra term involving the gravity-dual gravity duality relation. The equations $E_a{}^b = 0$ and $\bar{E}_a{}^b = 0$, together with the gravity-dual gravity duality relation, form a set of equations that are transformed into each other and we can take them to be our equations of motion.

The above process has one further ambiguity associated with terms that are both of type (a) and type (b), that is, they are of the form $G_{a,\bullet}G_{b,\bullet}$. Clearly one can either remove them or symmetrise them. The net effect is that we can add the terms to the dual graviton equation that are of the form

$$+c_1(G_{b,c}{}^{e}\overline{G}_{a,ce} + G_{a,c}{}^{e}\overline{G}_{b,ce}) \tag{5.19}$$

$$+c_2(G_{b,c}{}^{c}\overline{G}_{a,d}{}^{d} + G_{a,c}{}^{c}\overline{G}_{b,d}{}^{d}) \tag{5.20}$$

where c_1 and c_2 are constants.

One very stringent check of the above dual graviton equation (5.17) is that it is Lorentz invariant. Under the transformations

$$\delta\bar{G}_{a,bc} = \Lambda_a{}^{e}\bar{G}_{e,bc} + \Lambda_b{}^{e}\bar{G}_{a,ec} + \Lambda_c{}^{e}\bar{G}_{a,be},$$

$$\delta G_{a,bc} = \Lambda_a{}^{e}\bar{G}_{e,bc} + \Lambda_b{}^{e}\bar{G}_{a,ec} + \Lambda_c{}^{e}\bar{G}_{a,be} + e_a{}^{\mu}\partial_\mu\Lambda^{cb} \tag{5.21}$$

One does not need the possible additional terms of equations (5.19) and (5.20). The former expression is not invariant and so we can not add it to the dual graviton equation, however, the latter terms is invariant and so it is still a possible addition. It would be interesting to discuss the other expected symmetries of the dual graviton equation of motion. Of particular interest are diffeomorphism and the gauge symmetry. The corresponding transformations are discussed in section seven. As we observe there the transformation of the dual graviton is not as a general relativity tensor. We hope to return to this point in a future publication.

It is instructive to express the dual graviton equation in terms of objects carrying world indices. Defining $\overline{F}_{\mu,\nu_1\nu_2} = \partial_\mu A_{\nu_1\nu_2}$ the dual graviton equation (5.17) in world indices reads as

$$\begin{aligned}
\overline{E}_{\mu\nu} = g^{\rho\sigma}\partial_{[\sigma|}\overline{F}_{[\rho,\nu]|\mu]} + \frac{1}{4}g^{\rho\sigma}G_{\tau,\rho}{}^{\tau}(\overline{G}_{\nu,\mu`\sigma} + \overline{G}_{\mu,\sigma\nu} - \overline{G}_{\sigma,\mu\nu})\\
+\frac{1}{4}g^{\rho\sigma}G_{\rho,\tau}{}^{\tau}(-\overline{G}_{\nu,\mu\sigma} - \overline{G}_{\mu,\nu\sigma} + \overline{G}_{\sigma,\mu\nu}) + \frac{1}{4}g^{\rho\sigma}G_{\rho,\sigma}{}^{\tau}(-\overline{G}_{\tau,\mu\nu} + \overline{G}_{\mu,\nu\tau} + \overline{G}_{\nu,\mu\tau})\\
-\frac{1}{4}g^{\rho\sigma}G_{\nu,\rho}{}^{\tau}\overline{G}_{\mu,\tau\sigma} - g^{\rho\sigma}\frac{1}{4}G_{\mu,\rho}{}^{\tau}\overline{G}_{\nu,\tau\sigma} + \frac{1}{16}g^{\rho\sigma}G_{\nu,\tau}{}^{\tau}\overline{G}_{\mu,\rho\sigma} + \frac{1}{16}g^{\rho\sigma}G_{\mu,\tau}{}^{\tau}\overline{G}_{\nu,\rho\sigma}\\
-\frac{1}{4}(\det e)^{-1}\varepsilon^{\tau_1\tau_2\tau_3\tau_4}\overline{G}_{[\tau_1,\tau_2]\mu}\overline{G}_{[\tau_3,\tau_4]\nu}
\end{aligned} \tag{5.22}$$

In reference [8] the equations for the dual graviton in eleven dimensions was discussed. In particular the gravity-dual gravity duality relation and the dual graviton equation of motion are derived. While the former duality relation is directly derived from E_{11} transformations and is correct, the derivation of the latter equation of motion relies on some additional steps that have not been used in other E_{11} papers. In particular it relied on diffeomorphism symmetry and the non-linear form of certain modulo transformations. As is apparent from this paper the result for the dual graviton equation in reference [8] is likely to be incorrect as these additional steps were not correctly applied. However, it should be straightforward to apply the techniques used in this paper to derive the dual graviton equation of motion in eleven dimensions.

Another way to find the dual gravity equation is to carry out its variation under the non-linear symmetries. However, this is a very complicated task. We illustrate how it goes in the next section at the linearised level.

6. Derivation of the Linearised Equations of Motion and Variations

In this section we will carry out the variation of the dual gravity equation of motion under the local $I_c(A_1^{+++})$ transformations, but only at the linearised level. The dual graviton transforms under the $I_c(A_1^{+++})$ transformations of equation (3.18) into terms involving the graviton and dual dual-graviton, and so the resulting variation may be expected to involve the second order in derivatives gravity and dual gravity equations as well as derivatives of the previously derived first order duality relations. At the linearised level the dual graviton equation is given by

$$\overline{E}_a{}^b{}_{(lin)} \equiv \partial^{[c} G_{[c,a],}{}^{b]} = 0 \tag{6.1}$$

The l_1 extension of the linearised dual graviton equation, $\mathcal{E}_a{}^b{}_{(lin)}$ transforms under the $I_c(A_1^{+++})$ transformation Λ^{ab} as

$$\begin{aligned}\delta\overline{\mathcal{E}}_a{}^b = \frac{1}{2}\Lambda_a{}^c R_c{}^b + \frac{1}{2}R_a{}^c \Lambda^b{}_c + 3\partial^b E_{dac_1,}{}^d{}_{c_2}\Lambda^{c_1c_2} + \partial^b E_{c_2,ac_1}\Lambda^{c_1c_2} \\ -\frac{3}{2}\partial^d(E_{dac_1,bc_2} + E_{dbc_1,ac_2})\Lambda^{c_1c_2} + \frac{3}{2}\partial^d E_{abc_1,dc_2}\Lambda^{c_1c_2} \\ -\frac{1}{4}\varepsilon_{abc_1}{}^e\partial^d\overline{E}_{d,e}{}^{c_2}\Lambda^{c_1}{}_{c_2} + \varepsilon_{abc_1}{}^e\overline{E}_e{}^{c_2}\Lambda^{c_1}{}_{c_2}\end{aligned} \tag{6.2}$$

where

$$\begin{aligned}\overline{\mathcal{E}}_a{}^b{}_{(lin)} = \overline{E}_a{}^b{}_{(lin)} + \frac{1}{2}(\hat{\partial}_a(G^s)^{[d,}{}_d{}^{b]} + \hat{\partial}^b(G^s)_{[d,a]}{}^d) + \frac{1}{4}\hat{\partial}^c(G^{b,}{}_{(ac)} - (G^s)_{a,}{}^b{}_c + 2G^{b,}{}_{[ac]}) \\ +\frac{1}{4}\hat{\partial}^c(\varepsilon_{abc}{}^e\overline{G}_{[e,d]}{}^d - 2G^{[d,}{}_{da,}{}^{b]}{}_c - 2G^{d,}{}_{[d}{}^{b,}{}_{a]c} - G^{b,}{}_{ad,}{}^d{}_c)\end{aligned} \tag{6.3}$$

In these equation we define $G^s_{a,bc} = G_{a,(bc)}$ and

$$E_{a_1a_2a_3,bc} = \frac{1}{3!}\varepsilon_{a_1a_2a_3}{}^e\overline{E}_{e,bc}. \tag{6.4}$$

where $\overline{E}_{e,bc}$ is defined in equation (4.8). In the equations in this section all Cartan forms and duality relations are to be taken to only contain their linearised expressions. Setting $\overline{\mathcal{E}}_a{}^b = 0$ we find, at the linearised level, the gravity equation of motion $E_a{}^b = R_a{}^b = 0$ and the dual gravity- dual dual gravity duality relation $E_{a,b_1b_2}\doteq 0$ which can also be written in the form of equation (6.4). Thus the equations shows that the variation of the dual gravity equation of motion transforms into quantities that we already know to vanish.

We have carried out some parts of the above calculation at the non-linear level. It is a much more difficult calculation. One of the most difficult aspects is that one must take account of the transformations that the duality relations hold subject to. We note in particular that the dual dual graviton equation of motion involves three derivatives and the duality relations this field satisfies even with two derivatives will only hold modulo certain transformations.

7. Gauge transformations for A_1^{+++}

The equations of motion that resulted from the non-linear realisation of $E_{11} \otimes_s l_1$ were essentially uniquely determined by its symmetries which were given in equation (3.7). These transformations do not include the usual gauge transformations. However, the resulting equations when restricted to contain just derivatives with respect to the usual coordinates of spacetime are invariant under all the usual gauge symmetries, that is, diffeomorphisms and the standard gauge transformations of the form fields. It was proposed in [21] that the theory was invariant under a set of gauge transformations whose parameters were in a one to one correspondence with the l_1 representation. Indeed they were contained in the parameter Λ^A and the transformation of the fields of the theory could be given in terms of the variation of the vierbein in the formula [16]

$$E^{-1}{}_A{}^\Pi \delta E_\Pi{}^B = (D^{\underline{\alpha}})_A{}^B C_{\underline{\alpha\beta}} (D^{\underline{\beta}})_C{}^D D_D \Lambda^C \tag{7.1}$$

where $\Lambda^A = \Lambda^\Pi E_\Pi{}^A$, $C_{\underline{\alpha\beta}}$ is the Cartan-Killing metric of E_{11}, D_A is a suitable covariant derivative and the $D^{\underline{\alpha}}$ are the l_1 representation matrices which appear in the $E_{11} \otimes_s l_1$ algebra in the commutator

$$[R^{\underline{\alpha}}, l_A] = -(D^{\underline{\alpha}})_A{}^B l_B \tag{7.2}$$

This formula does indeed lead to the usual diffeomorphisms and form gauge transformations. These proposed gauge transformations have not played a central role in the construction of the eleven [4], seven [22] and five dimensional [3] theories from the non-linear realisation. However, it is expected that they will play a more important role when a more systematic construction of the dynamics is given at all levels. These first order in derivatives relations only hold modulo certain transformations and these are very closely linked to the above local transformations.

In this section we wish to find the analogous gauge transformations for the non-linear realisation $A_1^{+++} \otimes_s l_1$. Indeed we can simply apply the same formula as for the $E_{11} \otimes_s l_1$ case but now for the Kac-Moody algebra $A_1^{+++} \otimes_s l_1$. The gauge parameter Λ^C contains the components

$$\xi^a,\ \hat{\xi}_a,\ \Lambda_{a_1 a_2, a_3},\ \Lambda_{a_1 a_2 a_3},\ \ldots \tag{7.3}$$

We expect the first component to be the parameter for the usual diffeomorphisms and the second component to be the corresponding analogue for the dual graviton.

At low levels the first few $D^{\underline{\alpha}}$ matrices in the $A_1^{+++} \otimes_s l_1$. algebra are found to be given by

$$(D^a{}_b) = \begin{pmatrix} \delta^a{}_c \delta^b{}_d - \frac{1}{2}\delta^a{}_b \delta^d{}_c & 0 \\ 0 & -\delta^c{}_b \delta^a{}_d - \frac{1}{2}\delta^a{}_b \delta^c{}_d \end{pmatrix},$$

$$(D^{ab}) = \begin{pmatrix} 0 & -\delta^{(a}{}_c\delta^{b)}{}_d \\ 0 & 0 \end{pmatrix}, \quad (D_{ab}) = \begin{pmatrix} 0 & 0 \\ -2\delta^c{}_{(a}\delta^d{}_{b)} & 0 \end{pmatrix}, \quad \ldots \tag{7.4}$$

The Cartan-Killing metric of A_1^{+++} can be found, as usual, by noting that it is invariant under E_{11} transformations and so obeys the equation

$$g([R^{\underline{\alpha}}, R^{\underline{\beta}}], R^{\underline{\gamma}}) = g(R^{\underline{\alpha}}, [R^{\underline{\beta}}, R^{\underline{\gamma}}]) \tag{7.5}$$

It is straight forward using the level zero $GL(4)$ and level one invariances to prove that it must, at low levels, take the form

$$g_{\underline{\alpha},\underline{\beta}} = \begin{pmatrix} \delta^c{}_b\delta^a{}_d - \frac{1}{2}\delta^a{}_b\delta^c{}_d & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{7.6}$$

Using the above facts we find that equation (7.1) implies that

$$E^{-1}{}_a{}^\Pi \delta E_\Pi{}^b = D_a\xi^b - \frac{1}{2}\delta^b_a D_e\xi^e - \tilde{D}^b\tilde{\xi}_a - \frac{1}{2}\delta^b_a\tilde{D}^e\tilde{\xi}_e$$

$$E^{-1}{}_a{}^\Pi \delta E_{\Pi\tilde{b}} = 2D_{(a}\tilde{\xi}_{b)}, \quad E^{-1\tilde{a}\Pi}\delta E_\Pi{}^b = 2\tilde{D}^{(a}\xi^{b)} \tag{7.7}$$

Evaluating the left hand sides of these equations using the vielbein of equation (3.14) we find that

$$E^{-1}{}_a{}^\Pi \delta E_\Pi{}^b = e_a{}^\mu \delta e_\mu{}^b - \frac{1}{2}\delta^b_a e_c{}^\mu \delta e_\mu{}^c, \; E^{-1}{}_a{}^\Pi \delta E_{\Pi\tilde{b}} = -e_a{}^\mu e_b{}^\nu \delta A_{\mu\nu}, \; E^{-1\tilde{a}\Pi}\delta E_\Pi{}^b = 0 \tag{7.8}$$

The appearance of the zero in the last equation is a consequence of the fact that we used the local $I_c(E_{11})$ transformation of the non-linear realisation to set to zero all the negative level fields in the group element. It is this step that resulted in the upper triangular form of the vielbein as well as its inverse as shown in equation (3.14). and so the above zero.

We observe that our gauge transformations of equation (7.1) does not preserved our gauge choice using the local $I_c(A_1^{+++})$ transformations. The solution is to carry out a simultaneous correctional local $I_c(A_1^{+++})$ transformation so as to preserve the gauge choice. The local $I_c(A_1^{+++})$ transformations with parameter $\Lambda^{\underline{\alpha}}$ act as

$$E^{-1}{}_A{}^\Pi \delta E_\Pi{}^B = (D^{\underline{\alpha}} - D^{-\underline{\alpha}})\Lambda^{\underline{\alpha}} \tag{7.9}$$

Taking the local transformation constructed from plus and minus one generators this transformation takes the form

$$E^{-1}{}_A{}^\Pi \delta E_\Pi{}^B = (D^{ab} - D_{ab})_A{}^B \Lambda_{ab} \tag{7.10}$$

and we find that

$$E^{-1}{}_a{}^\Pi \delta E_\Pi{}^b = 0, \; E^{-1}{}_a{}^\Pi \delta E_{\Pi\tilde{b}} = -\Lambda_{a\tilde{b}}, \; E^{-1\tilde{a}\Pi}\delta E_\Pi{}^b = 2\Lambda^{\tilde{a}b}, \; E^{-1\tilde{a}\Pi}\delta E_{\Pi\tilde{b}} = 0, \tag{7.11}$$

Carrying out a simultaneous $I_c(A_1^{+++})$ with the parameter $\Lambda^{ab} = -\tilde{D}^{(a}\Lambda^{b)}$ we find that $E^{-1\tilde{a}\Pi}\delta E_\Pi{}^b = 0$, as it should, and that the fields transform as

$$e^{-1}{}_a{}^\mu \delta e_\mu{}^b = D_a\xi^b - \tilde{D}^b\tilde{\xi}_a - \delta_a{}^b\tilde{D}^c\hat{\xi}_c, \qquad \delta A_{\mu\nu} = e_\mu{}^a e_\nu{}^b(\tilde{D}_{(a}\xi_{b)} - 2D_{(a}\tilde{\xi}_{b)}). \tag{7.12}$$

These transformations are not completely defined as we have not specified what are the connections the covariant derivatives utilise. We notice that the transformation of the dual graviton field is not that of a rank two symmetric tensor under general relativity. In particular it does not contain terms of the form of an ordinary spacetime derivative acting on the parameter ξ. This is to be expected as the dual graviton gives an alternative description of gravity rather than a field which couples in the normal way to gravity.

In reference [21] the linearised gauge transformations of the fields of the non-linear realisation of $E_{11} \otimes_s l_1$ were derived starting from the fact that they are constructed from the derivatives ∂_Π acting on the parameters Λ^Σ. The coefficients in front of such terms were fixed by demanding that if these quantities transform according to the $\bar{l}_1$ representation and l_1 representation respectively then the transformations must transform in the adjoint representation of E_{11}, that is, like the fields themselves. The results agree with the transformations of equation (7.12) when linearised. The reader who wants to check this ascertion may find the transformations of the coordinates under the rigid A_1^{+++} transformation $g_0 = e^{a^{a_1a_2}R_{a_1a_2}}$ useful; they are given by

$$\delta x^a = 2\tilde{x}_c a^{ca}, \quad \delta\tilde{x}_a = 0, \quad \ldots \tag{7.13}$$

This leads to the following transformations of the derivatives transform

$$\delta(\partial_a) = 0, \quad \delta(\tilde{\partial}^a) = -2a^{ac}\partial_c, \quad \ldots \tag{7.14}$$

The reader can follow the procedure given in reference [21].

8. An alternative approach to the dual graviton

Rather than start form an E_{11} view point we will now present a theory that contains the fields of gravity and dual gravity and has some of the expected symmetries. This is different to that of the non-linear realisation in that does not involve any extension of spacetime and the symmetries are proposed in an ad hoc way rather than being part of a deeper structure. It will be instructive to first recall some very well known facts about gravity. This is described by a vierbein $e_\mu{}^a$ which can be used to define a spin connection and curvature according to the equations

$$D_{[\mu}e_{\nu]}{}^a \equiv \partial_{[\mu}e_{\nu]}{}^a + \omega_{[\mu|}{}^a{}_b e_{|\nu]}{}^b = 0, \qquad D_{[\mu}\omega_{\nu]}{}^{a_1a_2} - \frac{1}{2}R_{\mu\nu}{}^{a_1a_2} = 0, \tag{8.1}$$

These equations are invariant under the local Lorentz transformations given by

$$\delta e_\nu{}^a = \Lambda^a{}_b e_\nu{}^b, \qquad \delta\omega_\mu{}^{ab} = -D_\mu\Lambda^{ab} = -(\partial_\mu\Lambda^{ab} + \omega_\mu{}^a{}_c\Lambda^{cb} + \omega_\mu{}^b{}_c\Lambda^{ac}),$$

$$\delta R_{\mu\nu}{}^{ab} = -R_{\mu\nu}{}^{ac}\Lambda_c{}^b - R_{\mu\nu}{}^{cb}\Lambda_c{}^a \tag{8.2}$$

where $\Lambda^a{}_b$ is the parameter local of the Lorentz transformation. We recall that $[D_\mu, D_\nu]T^a = R_{\mu\nu}{}^a{}_c T^c$ for any tensor T^a with obvious generalisations for tensors with more indices.

Equations (8.1) are also invariant under the transformations

$$\delta e_\nu{}^a = D_\mu \xi^a \equiv \partial_\mu \xi^a + \omega_\mu{}^a{}_b \xi^b, \quad \delta\omega_{\mu,ab} = R_{\mu c,ab}\xi^c$$

$$\delta R_{\mu\nu}{}^{ab} = \xi^c D_c R_{\mu\nu}{}^{ab} - R_{\nu c}{}^{ab} D_\mu \xi^c + R_{\mu c}{}^{ab} D_\nu \xi^c \tag{8.3}$$

which are just a combination of a usual diffeomorphism and a Local Lorentz rotation. The invariant field equations are given by

$$R_{\mu\nu}{}^{ab} e_b{}^\nu = 0 \tag{8.4}$$

.

We now introduce the field $\tilde{e}_\mu{}^a$ corresponding to the dual graviton. We define a dual spin connection $\tilde{\omega}_\mu{}^a{}_b$ and dual curvature $\tilde{R}_{\mu\nu}{}^{ab}$ as follows

$$D_{[\mu}\tilde{e}_{\nu]}{}^a + \tilde{\omega}_{[\mu|}{}^a{}_b e_{|\nu]}{}^b = \partial_{[\mu}\tilde{e}_{\nu]}{}^a + \omega_{[\mu|}{}^a{}_b \tilde{e}_{|\nu]}{}^b + \tilde{\omega}_{[\mu|}{}^a{}_b e_{|\nu]}{}^b = 0, \quad D_{[\mu}\tilde{\omega}_{\nu]}{}^{ab} - \frac{1}{2}\tilde{R}_{\mu\nu}{}^{ab} = 0, \tag{8.5}$$

where D_μ the usual covariant derivative of general relativity.

These equations together with those of equation (8.1) are invariant under the local symmetries

$$\delta\tilde{e}_\mu{}^a = \tilde{\Lambda}^a{}_b e_\nu{}^b, \quad \delta\tilde{\omega}_\mu{}^{ab} = -D_\mu\tilde{\Lambda}^{ab} = -(\partial_\mu\tilde{\Lambda}^{ab} + \omega_\mu{}^a{}_c\tilde{\Lambda}^{cb} + \omega_\mu{}^b{}_c\tilde{\Lambda}^{ac}),$$

$$\delta\tilde{R}_{\mu\nu}{}^{ab} = -R_{\mu\nu}{}^{ac}\tilde{\Lambda}_c{}^b - R_{\mu\nu}{}^{cb}\tilde{\Lambda}_c{}^a \tag{8.6}$$

where $\tilde{\Lambda}^{ab}$ is the parameter. They are also invariant under the transformations

$$\delta\tilde{e}_\mu{}^a = D_\mu\tilde{\xi}^a, \quad \delta\tilde{\omega}_\mu{}^{ab} = -R_{\mu c}{}^{ab}\tilde{\xi}^c$$

$$\delta\tilde{R}_{\mu\nu}{}^{ab} = +\tilde{\xi}^c D_c R_{\mu\nu}{}^{ab} - R_{\nu c}{}^{ab} D_\mu\tilde{\xi}^c + R_{\mu c}{}^{ab} D_\nu\tilde{\xi}^c \tag{8.7}$$

We take as our equation of motion

$$\tilde{E}_\mu{}^a \equiv \tilde{R}_{\mu\nu}{}^{ab} e_b{}^\nu - R_{\mu\nu}{}^{ab} e_d{}^\nu \tilde{e}_\tau{}^d e_b{}^\tau = 0 \tag{8.8}$$

One can verify that it is invariant under the usual diffeomorphism and local Lorentz transformations but it is also invariant under the transformations of equations (8.6) and (8.7)

We can solve the first of the equations in (7.5) for the dual spin connection in much the way that one solves for the usual spin connection using equation (8.1). One finds that

$$\tilde{\omega}_{a,bc} = -F_{b,(ca)} + F_{c,(ba)} + F_{a,[bc]} \tag{8.9}$$

where $F_{a,b}{}^c = e_a{}^\mu e_b{}^\tau(\partial_\mu\tilde{e}_\tau{}^c + \omega_{\mu,}{}^c{}_d\tilde{e}_\tau{}^d) = e_a{}^\mu e_b{}^\tau D_\mu\tilde{e}_\tau{}^c$.

If we define

$$\tilde{e}_{\mu,\nu} = \tilde{e}_{\mu}{}^{c} e_{\nu c} \tag{8.10}$$

then one can show that the dual spin connection can be written as

$$\tilde{\omega}_{\mu,bc} = e_{b}{}^{\kappa} e_{c}{}^{\lambda} 2(-\partial_{[\kappa} \tilde{e}^{S}_{\lambda]\mu} + \partial_{\mu} \tilde{e}^{A}_{\kappa\lambda} + \Gamma^{\rho}_{\mu[\kappa} \tilde{e}_{\lambda],\rho}) \tag{8.11}$$

where $\tilde{e}^{S}_{\kappa\lambda} = \tilde{e}_{(\kappa\lambda)}$, $\tilde{e}^{A}_{\kappa\lambda} = \tilde{e}_{[\kappa\lambda]}$ and $\Gamma^{\rho}_{\mu\kappa}$ is the usual Christoffel connection which obeys the relation $\partial_{\mu} e^{a}_{\nu} + \omega_{\mu,}{}^{a}{}_{b} e_{\nu} b = \Gamma^{\rho}_{\mu\nu} e_{\rho}{}^{a}$. Substituting the above expression for the dual spin connection in the equation of motion of equation (8.8) one finds the equation of motion for the dual graviton as discussed in this section.

It would be interesting to find the relationship between the formulation of dual gravity given in this section and the one that follows from the non-linear realisation of $A_1^{+++} \otimes_s l_1$ and is the main subject of this paper.

9. Discussion

In this paper we have carried out the non-linear realisation of the semi-direct product of the very extended Kac-Moody algebra A_1^{+++} with its vector representation, denoted by $A_1^{+++} \otimes_s l_1$. We found that the resulting equations of motion at lowest level describe gravity when the derivatves with respect to the higher level coordinates are discarded. At the next level we found the fully non-linear equation of motion for the dual graviton. We also find that the gravity and dual gravity fields satisfy a first order in derivative duality relation. The fields in the non-linear realisation up to level four are listed in equation (3.4). As is apparent from this list the fields at higher levels have an increasing number of indices that obey more and more complicated symmetrisation and anti-symmetrisation conditions. The third field listed in equation (3.4) is the dual dual graviton and at higher levels we find further duals of gravity. These fields have the form $A_{a_1 a_2, \ldots, d_1 d_2, (e_1 e_2)}$ and in the listing of equation (3.4) they are the second, third, fourth and sixth fields. The occurrence of such dual fields was observed in the context of E_{11} [23] and for other non-linear realisations of the semi-direct products of extended algebras and their vector representations in reference [24], although this reference did not include studies involving the very extended algebras, A_{D-3}^{+++}, associated with gravity. It would be good to know what is the physical meaning of the fourth field $A_{a_1 a_2 a_3, b_1 b_2, c}$ in the listing of equation (3.4).

The spacetime coordinates belong to the vector representation and are listed in equation (3.6). By construction these are in one to one correspondence with the generators in the vector representation. For the case of E_{11} it is clear that the multiplet of brane charges belong to the vector representation of E_{11}. As a result, it is very likely that the brane charges of the non-linear realisation of $A_1^{+++} \otimes_s l_1$ also contains all brane charges of this theory and as such are given at low levels in equation (2.10). The first entry is just the just momentum operator corresponding to translations in our usual spacetime and is associated with the gravity field. The next entry Z_a is associated with the dual graviton and it is the charge carried by the Taub-Nut solution. It would be good to know what is the physical significance of the higher charges.

When we truncate to only the lowest level field, that is, that of gravity and retain only the usual coordinate of our usual four dimensional spacetime the non-linear realisation is just Einstein's theory of gravity. However, the full non-linear realisation contains an infinite number of fields which depend on a spacetime that has an infinite number of coordinates. It is also invariant under the infinite algebra, namely $A_1^{+++} \otimes_s l_1$. As such the full nonlinear realisation of $A_1^{+++} \otimes_s l_1$ contains much more than Einstein's theory of gravity. For example, it contains an infinite number of new degrees of freedom corresponding to the infinite number of brane charges and their corresponding solutions. It has been realised that to explain features of gravity such as the entropy of black holes one needs a theory that goes beyond our usual understanding of Einstein's theory. It would be interesting to see if the additional content of the nonlinear realisation of $A_1^{+++} \otimes_s l_1$ can be used in this way and in particular if it can be used to explain black hole entropy. The brane charges correspond to weights of A_1^{+++} and as one requires large brane charges this means weights of high level. One might wonder if the calculation of the black hole entropy can be formulated as a combinatoric problem constructing such high level weights from the more fundamental weights and roots of the A_1^{+++} algebra.

A very interesting discovery in the 1950's was the existence of asymptotic (BMS) charges in gravity. This work has been considerably extended in more recent times, see reference [25] and the references it contains. Very recently it has been shown that one should also include the asymptotic charges associated with the Taub-Nut solution [26]. Could it be possible that the brane charges for the non-linear realisation of $A_1^{+++} \otimes_s l_1$ studied in this paper are closely related and even the same as the asymptotic charges that are being studied. The asymptotic charges and the charges in the vector representation appear to agree at the first two levels.

The non-linear realisations of $G^{+++} \otimes_s l_1$, where G is any Lie algebra in the Cartan List, possess an invariant tangent space metric which was constructed at low levels in reference [27] for many of these non-linear realisations including for $A_1^{+++} \otimes_s l_1$. The tangent vectors transform under $I_c(A_1^{+++})$ and arise from the l_1 representation. It we label their components by

$$T^a, \bar{T}_a, T_{a_1a_2a_2}, T_{a_1a_2,b}, \dots \tag{9.1}$$

the invariant tangent space metric is given by

$$L^2 \equiv T_aT^a + 2\bar{T}_a\bar{T}^a + 4T_{a_1a_2a_2}T^{a_1a_2a_2} + \frac{16}{3}T_{a_1a_2,b}T_{a_1a_2,b} + \dots \tag{9.2}$$

This expression provides an invariant bilinear in the brane charges l_Π which do transform in the vector representation by taking $T_A = E_A{}^\Pi l_\Pi$ where $E_A{}^\Pi$ is the inverse vierbein of equation (3.14).

It was found in reference [28] that, for the case of E_{11}, setting $L^2 = 0$ coincided at low levels with the half BPS conditions that can be derived from the supersymmetry algebra. Hence it is natural to take the condition $L^2 = 0$ to be the analogue of the half BPS conditions for the case of the non-linear realisation of $A_1^{+++} \otimes_s l_1$. This condition begins with the square of the momentum generators and the next term contains the square of the Taub-Nut charge. Such a condition has been proposed from the view point of gravitational duality relating these two charges [28,29]. As was explained in the last of these references this condition can not be derived from the usually supersymmetry algebra. It does however, follow naturally from the non-linear realisation studied in this paper. It has been know for a long time that the supersymmetry algebra does not contain all the required brane charges but that they are contained in E theory. The presence of such a relation involving the momenta and Taub-NUT charge is generic to the $G^{+++} \otimes_s l_1$ non-linear realisations including in E theory. Thus the Taub-Nut charge is just one of an infinite number of charges that is missing from the supersymmetry algebra. It would also be interesting to find for the $A_1^{+++} \otimes_s l_1$ non-linear realisation the analogue of equation (13) and the "quarter BPS" condition of equation (39) of reference [28] and also interpret the former in the sense of reference [31].

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Appendix A

In this appendix we will list the terms which are referred to, but not explicitly stated, in section five when we were constructing the symmetric dual graviton equation $\overline{E}_a{}^b$ of (5.17). In particular we will discuss in more detail the construction of the l_1 extended Einstein equation $\mathcal{E}_{ab}$ of (5.18) which lead in its variation to the terms we added to $E'_a{}^b$ of (5.12) to find the dual graviton equation of motion $\overline{E}_a{}^b$.

We recall that equation (5.12) contained terms that were derivatives of the dual graviton Cartan form and also terms that were divided into the types (a), (b) and (c). The terms of type (a) occur in equation (5.12) as the number 3, 6, 8, 10 , 12, 13, 14 terms as well as the last expression in term 7. We remove these terms by adding their negative to $\bar{E}'_a{}^b$, that is, we add the terms

$$-\frac{1}{4}G^{b,}{}_c{}^e(-\overline{G}_{a,}{}^c{}_e + \overline{G}_{e,a}{}^c + \overline{G}^{c,}{}_{ae}) + \frac{1}{4}G^{b,c}{}_a(\overline{G}_{c,d}{}^d - \overline{G}_{d,c}{}^d)$$

$$-\frac{1}{8}G^{b,c}{}_c(-2\overline{G}_{d,a}{}^d + \frac{3}{2}\overline{G}_{a,d}{}^d)$$

$$-\frac{1}{8}G^{b,}{}_a{}^e(-\overline{G}_{e,c}{}^c + 2\overline{G}_{c,e}{}^c) - \frac{1}{4}G_{a,c}{}^e\overline{G}^{b,}{}_e{}^c + \frac{1}{4}G_{c,a}{}^e\overline{G}^{b,}{}_e{}^c$$

$$+\frac{1}{4}G_{c,}{}^{ce}\overline{G}^{b,}{}_{ae} + \frac{1}{16}G_{a,e}{}^e\overline{G}^{b,c}{}_c - \frac{1}{8}G^{d,c}{}_c\overline{G}^{b,}{}_{ad} \qquad (A.1)$$

The type (b) terms by definition contain a $G_{a,\bullet}$ factor and they occur in equation (5.12) as the terms number 2 (only last expression), 4 (only last expression), 7 (only middle expression) and term 9, we list them again here for convenience

$$+\frac{1}{8}(-2G^{c,b}{}_e\overline{G}_{a,c}{}^e + 2G^{e,c}{}_e\overline{G}_{a,c}{}^b - G^{d,c}{}_c\overline{G}_{a,d}{}^b + G_{a,}{}^{be}\overline{G}_{e,c}{}^c). \qquad (A.2)$$

As explained in equation (5.14) we can obtain an expression which is symmetric in a and b by adding to $E'_a{}^b$ the terms

$$+\frac{1}{8}(-2G^{c,}{}_{ae}\overline{G}^{b,}{}_c{}^e + 2G^{e,c}{}_e\overline{G}^{b,}{}_{ca} - G^{d,c}{}_c\overline{G}^{b,}{}_{da} + G^{b,}{}_a{}^e\overline{G}_{e,c}{}^c) \qquad (A.3)$$

The net effect of this is that we find in $\bar{E}_a{}^b$ the a, b symmetric terms

$$+\frac{1}{4}(G^{c,b}{}_{e}\overline{G}_{a,c}{}^{e} + G^{c,}{}_{ae}\overline{G}^{b,}{}_{c}{}^{e}) + \frac{1}{4}(G^{e,c}{}_{e}\overline{G}_{a,c}{}^{b} + G^{e,c}{}_{e}\overline{G}^{b,}{}_{ca})$$

$$-\frac{1}{8}(G^{d,c}{}_{c}\overline{G}_{a,d}{}^{b} + G^{d,c}{}_{c}\overline{G}^{b,}{}_{da}) + \frac{1}{8}(G_{a,}{}^{be}\overline{G}_{e,c}{}^{c} + G^{b,}{}_{a}{}^{e}\overline{G}_{e,c}{}^{c}) \qquad (A.4)$$

The terms of type (c) have been listed in (5.15) and re-expressed in (5.16) and, as explained there, two of these terms are symmetric under the interchange of a and b and are part of the dual graviton equation and the final terms can be viewed as part of the variation of the gravity equation of motion.

As noted in section five some of the terms are of both (a) and (b) type, listed as they appear in $\overline{E}'_a{}^b$ they are

$$-\frac{1}{4}G^{b,}{}_{c}{}^{e}\overline{G}_{a,}{}^{c}{}_{e} + \frac{1}{4}G_{a,c}{}^{e}\overline{G}^{b,c}{}_{e} + \frac{3}{16}G^{b,c}{}_{c}\overline{G}_{a,d}{}^{d} - \frac{1}{16}G_{a,c}{}^{c}\overline{G}^{b,d}{}_{d}. \qquad (A.5)$$

Such terms can be treated as either type (a) terms, that is in effect by removal, or as type (b) terms, that is, symmetrisation. The effect of this ambiguity is that the expression for $\bar{E}_a{}^b$ can contains two terms, listed in equations (5.19) and (5.20) whose coefficients are not fixed. The first of these terms was ruled out by considerations of Lorentz symmetry and so we only take account of the second term and as a result we add to $\bar{E}'{}_a{}^b$ the term

$$c_2(G^{b,c}{}_{c}\overline{G}_{a,d}{}^{d} + G_{a,}{}^{c}{}_{c}\overline{G}^{b,}{}_{d}{}^{d}) \qquad (A.6)$$

The result of all the above consideration is that we obtain the dual graviton equation of motion by adding to $\overline{E}'{}_a{}^b$ the terms in equations (A.1), (A.3) and (A.6) as well as the term involving the gravity-dual gravity relation which arises in the type (c) terms discussed above.

The dual gravity equation of motion has been found by varying the gravity equation of motion and so the above additions to the dual gravity equation of motion arise in the variation of the gravity equation of motion by adding l_1 terms to this equation, that is adding more such terms to the $\mathcal{E}'_{ab}$ of equation (5.4). The resulting l_1 extension of the gravity equation of motion is given, using equation (3.20), to be

$$\mathcal{E}_{ab} \equiv \mathcal{E}'_{ab} + \frac{1}{2}[G^{c,}{}_{be}\hat{\overline{G}}_{a,c}{}^{e} - G^{e,c}{}_{e}\hat{\overline{G}}_{a,cb} + \frac{1}{2}G^{d,c}{}_{c}\hat{\overline{G}}_{a,db} - \frac{1}{2}\hat{G}_{a,b}{}^{e}\overline{G}_{e,c}{}^{c}$$

$$+\hat{G}_{a,c}{}^{e}(-\overline{G}_{b,}{}^{c}{}_{e} + \overline{G}_{e,b}{}^{c} + \overline{G}^{c,}{}_{be}) + \frac{1}{2}G^{d,c}{}_{c}\hat{\overline{G}}_{a,bd} - 2\hat{G}_{a,}{}^{c}{}_{b}\overline{G}_{[c,d]}{}^{d}$$

$$-G_{c,}{}^{ce}\hat{\overline{G}}_{a,be} + \frac{1}{2}\hat{G}_{a,}{}^{c}{}_{c}(-2\overline{G}_{d,b}{}^{d} + \frac{3}{2}\overline{G}_{b,d}{}^{d}) + \frac{1}{2}\hat{G}_{a,b}{}^{e}(-\overline{G}_{e,c}{}^{c} + 2\overline{G}_{c,e}{}^{c})$$

$$-G_{c,b}{}^{e}\hat{\overline{G}}_{a,e}{}^{c} + G_{b,c}{}^{e}\hat{\overline{G}}_{a,e}{}^{c} - \frac{1}{4}G_{b,e}{}^{e}\hat{\overline{G}}_{a,}{}^{c}{}_{c} - 4c_2(\hat{G}_{a,}{}^{c}{}_{c}\overline{G}_{b,d}{}^{d} + G_{b,}{}^{c}{}_{c}\hat{\overline{G}}_{a,d}{}^{d})]$$

$$+\frac{1}{2}[G^{c,}{}_{ae}\hat{\overline{G}}_{b,c}{}^{e} - G^{e,c}{}_{e}\hat{\overline{G}}_{b,ca} + \frac{1}{2}G^{d,c}{}_{c}\hat{\overline{G}}_{b,da} - \frac{1}{2}\hat{G}_{b,a}{}^{e}\overline{G}_{e,c}{}^{c}$$

$$+\hat{G}_{b,c}{}^{e}(-\overline{G}_{a,}{}^{c}{}_{e}+\overline{G}_{e,a}{}^{c}+\overline{G}^{c,}{}_{ae})+\frac{1}{2}G^{d,c}{}_{c}\hat{\overline{G}}_{b,ad}-2\hat{G}_{b,}{}^{c}{}_{a}\overline{G}_{[c,d]}{}^{d}$$

$$-G_{c,}{}^{ce}\hat{\overline{G}}_{b,ae}+\frac{1}{2}\hat{G}_{b,}{}^{c}{}_{c}(-2\overline{G}_{d,a}{}^{d}+\frac{3}{2}\overline{G}_{a,d}{}^{d})+\frac{1}{2}\hat{G}_{b,a}{}^{e}(-\overline{G}_{e,c}{}^{c}+2\overline{G}_{c,e}{}^{c})$$

$$-G_{c,a}{}^{e}\hat{\overline{G}}_{b,e}{}^{c}+G_{a,c}{}^{e}\hat{\overline{G}}_{b,e}{}^{c}-\frac{1}{4}G_{a,e}{}^{e}\hat{\overline{G}}_{b,}{}^{c}{}_{c}-4c_2(\hat{G}_{b,}{}^{c}{}_{c}\overline{G}_{a,d}{}^{d}+G_{a,}{}^{c}{}_{c}\hat{\overline{G}}_{b,d}{}^{d})]$$

$$-\frac{1}{2}\eta_{ab}[G^{c,}{}_{e_1e_2}\hat{\overline{G}}^{e_1,}{}_{c}{}^{e_2}-G^{e_1,c}{}_{e_1}\hat{\overline{G}}^{e_2,}{}_{ce_2}+\frac{1}{2}G^{d,c}{}_{c}\hat{\overline{G}}^{e,}{}_{de}-\frac{1}{2}\hat{G}^{e_1,}{}_{e_1}{}^{e_2}\overline{G}_{e_2,c}{}^{c}$$

$$+\hat{G}^{e_1,}{}_{c}{}^{e_2}(-\overline{G}_{e_1,}{}^{c}{}_{e_2}+\overline{G}_{e_2,e_1}{}^{c}+\overline{G}^{c,}{}_{e_1e_2})+\frac{1}{2}G^{d,c}{}_{c}\hat{\overline{G}}^{e,}{}_{ed}-2\hat{G}^{e,c}{}_{e}\overline{G}_{[c,d]}{}^{d}$$

$$-G_{c,}{}^{ce_1}\hat{\overline{G}}^{e_2,}{}_{e_2e_1}+\frac{1}{2}\hat{G}^{e,c}{}_{c}(-2\overline{G}_{d,e}{}^{d}+\frac{3}{2}\overline{G}_{e,d}{}^{d})+\frac{1}{2}\hat{G}^{e_1,}{}_{e_1}{}^{e_2}(-\overline{G}_{e_2,c}{}^{c}+2\overline{G}_{c,e_2}{}^{c})$$

$$-G_{c,e_1}{}^{e_2}\hat{\overline{G}}^{e_1,}{}_{e_2}{}^{c}+G_{e_1,c}{}^{e_2}\hat{\overline{G}}^{e_1,}{}_{e_2}{}^{c}-\frac{1}{4}G_{e_1,e_2}{}^{e_2}\hat{\overline{G}}^{e_1,c}{}_{c}-4c_2(\hat{G}^{e,c}{}_{c}\overline{G}_{e,d}{}^{d}+G_{e,}{}^{c}{}_{c}\hat{\overline{G}}^{e,}{}_{d}{}^{d})] \quad (A.7)$$

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