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Modelling Lévy space-time white noises

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Abstract

In this work, we compare Lévy space-time white noises and cylindrical Lévy processes. Lévy space-time white noises are defined as infinitely divisible independently scattered random measures and cylindrical Lévy processes are defined by means of the theory of cylindrical processes. It is shown that Lévy space-time white noises correspond to an entire sub-class of cylindrical Lévy processes, which is completely characterised by the characteristic functions of its members. We embed the Lévy space-time white noise, or the corresponding cylindrical Lévy process, in the space of general and tempered distributions. For the latter case, we show that this embedding is possible if and only if a certain integrability condition is satisfied. We establish that both embedded cylindrical processes are induced by genuine Lévy processes in the space of general or tempered distributions. We complete the picture by establishing Lévy space-time white noise as the weak derivative of Lévy additive sheets.

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1 Introduction

Gaussian random perturbations of partial differential equations are most often modelled either as a cylindrical Brownian motion or a Gaussian space-time white noise. The choice usually depends on the exploited method, in which one follows either a semi-group approach, based on the work by Da Prato and Zabczyk in [13], or a random field approach, originating from the work by Walsh in [42]. It is well known that both models essentially result in the same dynamics as established by Dalang and Quer-Sardanyons in [16].

Cylindrical Brownian motions can be naturally generalised to cylindrical Lévy processes by exploiting the theory of cylindrical measures and random variables. This was accomplished by one of us together with Applebaum in [2]. In the random field approach, Gaussian space-time white noise is generalised to Lévy space-time white noise as an infinitely divisible random measure, often represented by integrals with respect to Gaussian and Poisson random measures. Both generalisations, cylindrical Lévy processes and Lévy space-time white noises, serve as a model for random perturbations of complex dynamical systems. These applications can be found for cylindrical Lévy processes for example in the monograph S. Peszat and J. Zabczyk [34] or in Kumar and Riedle [30], and for Lévy space-time white noise in Applebaum and Wu [3], Chong [9], Chong and Kevei [10] and Dalang and Humeau [15] among many others. Another approach to model such perturbed dynamical systems, e.g. parabolic stochastic partical differential equations, is provided by the recently introduced ambit fields, presented in the monograph [7] by Barndorff-Nielsen, Benth and Veraart, and their relations to SPDE investigated in [8] by the same authors.

The main objectives of our work are the comparison of cylindrical Lévy processes and Lévy space-time white noises, as well as their embeddings in the space of general and tempered (Schwartz) distributions. It turns out that these results significantly differ from the Gaussian situation. Only the standard cylindrical Brownian motion corresponds to the Gaussian space-time white noise, see e.g. Kallianpur and Xiong [29], and Gaussian space-time white noise always can be embedded in the space of tempered distribution, see e.g. Gel'fand and Vilenkin [22].

The property of independent scattering for random measures restricts the correspondence between cylindrical processes and space-time noises in the Gaussian setting to the standard case of the identity as the covariance operator. In the non-Gaussian case, it turns out that there is an entire subclass of cylindrical Lévy processes corresponding to Lévy space-time white noises. We call this subclass *independently scattered cylindrical Lévy processes* according to its defining property. We completely characterise the sub-class of independently scattered cylindrical Lévy processes by the particular form of the characteristic function of its members.

We embed Lévy space-time white noises and, due to the aforementioned correspondence, independently scattered cylindrical Lévy processes, in the space of distributions and tempered distributions. Although the embedding in the former case is possible for all Lévy space-time white noises, the embedding to the space of tempered distribution is restricted to members of a subclass satisfying a certain integrability condition. For both embeddings, we show that the embedded cylindrical Lévy process is induced by a genuine Lévy process in the space of general or tempered distributions, i.e. the embedded cylindrical process possesses a regularised version in the sense of Itô and Nawata [24].

These embedding results enable us to compare the Lévy space-time white noise with a model of Lévy-type noise in the space of distributions, investigated in a series of papers by Dalang, Humeau, Unser and co-authors in e.g. [4, 14, 18, 19]. Their model of noise is initiated from research on developing sparse statistical models for signal and image processing. However, it turns out that these two models result in the same object only for Lévy space-time white noises which are additionally assumed to be stationary in space. Similar questions such as the embedding to the space of tempered distributions and the relation to independently scattered infinitely divisible random measures are addressed in Dalang and Humeau [14] and Fageot and Humeau [19] for the Lévy-type noise in the space of distributions.

In the restricted case of independently scattered Lévy processes which are stationary in space, we can combine our results with the work Aziznejad and Fageot [4] to determine the Besov space, in which a cylindrical Lévy process attains its values. This indicates a potential reasoning for the often observed phenomena of irregular trajectories of solutions of heat equations driven by cylindrical Lévy processes, e.g. in Brzeźniak and Zabczyk [12] and Priola and Zabczyk [35].

To complete the picture, we compare Lévy space-time white noises with Lévy additive sheets in the last part of our article. The notion of Lévy additive sheets is based on the work by Adler et al [1], its extension by Dalang and Humeau [14] and results from Pedersen [33]. We establish the relation between Lévy space-time white noise and additive sheets, which is given by the integration of the Lévy space-time white noise in space, i.e. Lévy space-time white noise can be seen as the week derivative of a Lévy additive sheet. This relation is established to be one-to-one for Lévy space-time white noise without fixed point of discontinuity in space.

Lévy space-time white noises, defined by means of random measures, do not require distinguishing the time domain. However, as we compare these with cylindrical processes, which are naturally indexed by time, we break off one coordinate as the time domain. For this purpose, we mimic Walsh's definition of a martingale measure in [42] to define Lévy space-time white noise. To differentiate our setting from the various other definitions of Lévy space-time white noises in the literature, we call our model Lévy-valued random measure.

Our article starts with two preliminary Sections 2 and 3 on Lévy-valued random measures and Lévy-valued sheets. Here, we present our precise definitions of those objects, recall some known results from the literature and add a few observations particular to our approach. In Section 4 we present our first two main results on the embedding of Lévy-valued random measures in the space of distributions and tempered distributions. The following Section 5 is devoted to the comparison of cylindrical Lévy processes and Lévy-valued random measures. Our main results here characterise exactly the subclass of cylindrical Lévy processes which correspond to Lévy-valued random measures. In the last Section 6, we complete the picture by establishing Lévy-valued random measures as the weak derivative of Lévy-valued additive sheets.

Notation: for a Borel set $\mathcal{O} \subseteq \mathbb{R}^d$ we denote the Borel σ-algebra by $\mathscr{B}(\mathcal{O})$ and define the δ-ring

$$\mathscr{B}_b(\mathcal{O}) := \{A \in \mathscr{B}(\mathcal{O})\colon A \text{ is relatively compact}\}.$$

The Lebesgue measure on $\mathscr{B}(\mathbb{R}^d)$ is denoted by leb. The closed unit ball in $\mathbb{R}^d$ is denoted by $B_{\mathbb{R}^d}$.

Throughout the paper, we fix a probability space $(\Omega, \mathcal{A}, P)$. The space of P-equivalence classes of measurable functions $f\colon \Omega \to \mathbb{R}$ is denoted by $L^0(\Omega, P)$, and of p-th integrable functions by $L^p(\Omega, P)$ for $p > 0$. These spaces are equipped with their standard metrics and (quasi-)norms.

2 Lévy-valued random measures

Our definition of Lévy-valued random measures is based on the work [36] by Rajput and Rosinski. Instead of general δ-rings, it is sufficient for us to restrict ourselves to the δ-ring $\mathscr{B}_b(\mathcal{O})$ of all relatively compact subsets of the Borel set $\mathcal{O} \in \mathscr{B}(\mathbb{R}^d)$ as the domain of the random measures.

Definition 2.1. *A map $M\colon \mathscr{B}_b(\mathcal{O}) \to L^0(\Omega, P)$ is called an* independently scattered random measure *on $\mathscr{B}_b(\mathcal{O})$ if for each collection of disjoint sets $A_1, A_2, \ldots \in \mathscr{B}_b(\mathcal{O})$ the following hold:*

(a) *the random variables $M(A_1)$, $M(A_2), \ldots$ are independent;*

(b) *if $\bigcup_{k\in\mathbb{N}} A_k \in \mathscr{B}_b(\mathcal{O})$ then $M\left(\bigcup_{k\in\mathbb{N}} A_k\right) = \sum_{k\in\mathbb{N}} M(A_k)$ P-a.s.*

An independently scattered random measure M is called infinitely divisible *if*

(c) *the random variable $M(A)$ is infinitely divisible for each $A \in \mathscr{B}_b(\mathcal{O})$.*

Analogously, an independently scattered random measure is called Gaussian *(or* Poisson*), if $M(A)$ is Gaussian (or Poisson) distributed for each $A \in \mathscr{B}_b(\mathcal{O})$.*

For an arbitrary infinitely divisible independently scattered random measure M on $\mathscr{B}_b(\mathcal{O})$ it is shown in [36] that there exist

(1) a signed measure $\gamma\colon \mathscr{B}_b(\mathcal{O}) \to \mathbb{R}$,

(2) a measure $\Sigma\colon \mathscr{B}_b(\mathcal{O}) \to \mathbb{R}_+$,

(3) a σ-finite measure $\nu\colon \mathscr{B}(\mathcal{O} \times \mathbb{R}) \to [0, \infty]$,

such that for each $A \in \mathscr{B}_b(\mathcal{O})$ the characteristics of $M(A)$ are given by $(\gamma(A), \Sigma(A), \nu_A)$, where the Lévy measure ν_A on $\mathscr{B}(\mathbb{R})$ is defined by $\nu_A(\cdot) := \nu(A \times \cdot)$. For the notion of measures on a ring see e.g. [23]. We call the triple (γ, Σ, ν) the *characteristics of* M. Furthermore, we may extend the total variation $\|\gamma\|_{\mathrm{TV}}$ of γ and Σ to σ-finite measures on $\mathscr{B}(\mathcal{O})$. In this case, the mapping

$$\lambda\colon \mathscr{B}(\mathcal{O}) \to [0, \infty], \qquad \lambda(A) = \|\gamma\|_{\mathrm{TV}}(A) + \Sigma(A) + \int_{\mathbb{R}} (|y|^2 \wedge 1)\, \nu(A, \mathrm{d}y),$$

defines a σ-finite measure, which is called the *control measure of* M. We note that $\lambda(A) < \infty$ for $A \in \mathscr{B}_b(\mathcal{O})$. The control measure λ is called *atomless* if $\lambda(\{x\}) = 0$ for all $x \in \mathcal{O}$.

We extend Definition 2.1 to include a dynamical aspect, i.e. a time variable. This extension can be thought of as a similar construction to that of Walsh in [42].

Definition 2.2. *A family* $(M(t) : t \geqslant 0)$ *of infinitely divisible random measures* $M(t)$ *on* $\mathscr{B}_b(\mathcal{O})$ *is called a* Lévy-valued random measure on $\mathscr{B}_b(\mathcal{O})$ *if, for every* $A_1, \ldots, A_n \in \mathscr{B}_b(\mathcal{O})$ *and* $n \in \mathbb{N}$, *the stochastic process*

$$\big((M(t)(A_1), \ldots, M(t)(A_n)) : t \geqslant 0\big)$$

is a Lévy process in $\mathbb{R}^n$. *We shall write* $M(t, A) := M(t)(A)$.

Let $(M(t) : t \geqslant 0)$ be a Lévy-valued random measure on $\mathscr{B}_b(\mathcal{O})$, and suppose (γ, Σ, ν) and λ are the characteristics and control measure, respectively, of the infinitely divisible random measure $M(1)$. Then, it follows from the stationary increments of the process $(M(t, A) : t \geqslant 0)$ that for each $t \geqslant 0$ the characteristics of the infinitely divisible random measure $M(t, A)$ are given by $(t\gamma, t\Sigma, t\nu)$, and the control measure of $M(t)$ is given by $t\lambda$. We shall refer to (γ, Σ, ν) as the *characteristics of* M and λ as the *control measure of* M.

Our definition above of Lévy-valued random measures assigns a special role to the time domain although this is not necessary for infinitely divisible random measures in general. However, as we will later compare Lévy-valued random measures with cylindrical Lévy processes, which are naturally carrying a time domain as generalised stochastic processes, we found it more illustrative to have the time domain distinguished. Indeed, the following theorem shows that a Lévy-valued random measure corresponds to an infinitely divisible random measure on the product space of time and space domain if the stationarity in the time domain is described by the control measure accordingly.

Proposition 2.3.

(a) *Let* $M = (M(t) : t \geqslant 0)$ *be a Lévy-valued random measure on* $\mathscr{B}_b(\mathcal{O})$*. Then, there exists a unique infinitely divisible random measure* M' *on* $\mathscr{B}_b(\mathbb{R}_+ \times \mathcal{O})$ *such that* $M'((0,t] \times A) = M(t,A)$ *for each* $t > 0$ *and* $A \in \mathscr{B}_b(\mathcal{O})$*.*

(b) *Each infinitely divisible random measure* M' *on* $\mathscr{B}_b(\mathbb{R}_+ \times \mathcal{O})$ *with control measure* $\lambda = \text{leb}\otimes\lambda_0$ *for a* σ*-finite measure* λ_0 *on* $\mathscr{B}(\mathcal{O})$ *defines by* $M(t,A) := M'((0,t] \times A)$ *for each* $t > 0$ *and* $A \in \mathscr{B}_b(\mathcal{O})$ *a Lévy-valued random measure on* $\mathscr{B}_b(\mathcal{O})$*.*

Proof. This can be proved similarly as Th. 3.2 in [38] or by using Th. 2.15 in [28]. □

Remark 2.4. Gaussian space-time white noise is usually defined equivalently to a Gaussian random measure on $\mathscr{B}_b(\mathbb{R}_+ \times \mathcal{O})$ in the sense of Definition 2.1. Typically, one assumes that the measure Σ on $\mathscr{B}_b(\mathbb{R}_+ \times \mathcal{O})$ is either the Lebesgue measure or of the form $\Sigma = \text{leb}\otimes\Sigma_0$ for a σ-finite measure Σ_0 on $\mathscr{B}_b(\mathcal{O})$; see e.g. Kallianpur and Xiong [29, De. 3.2.2]. Thus, Part (b) of Proposition 2.3 shows that our definition of a Lévy-valued random measure naturally extends the class of Gaussian space-time white noises to a Lévy-type setting.

The relation between random measures and models of Lévy-type noise utilising a Lévy-Itô decomposition is well known. We rigorously formulate this result in our setting in the following proposition; for a converse conclusion see Remark 3.6.

Proposition 2.5. *Let* ζ *be a* σ*-finite Borel measure on* $\mathscr{B}(\mathcal{O})$ *and* $(U,\mathcal{U},\nu)$ *a* σ*-finite measure space. Assume that*

(a) $\rho\colon \mathscr{B}_b(\mathcal{O}) \to \mathbb{R}$ *is a signed measure;*

(b) $W\colon \mathscr{B}_b(\mathbb{R}_+ \times \mathcal{O}) \to L^2(\Omega, P)$ *is a Gaussian random measure with characteristics* $(0, \text{leb} \otimes \zeta, 0)$*;*

(c) $N\colon \mathscr{B}_b(\mathbb{R}_+ \times \mathcal{O}) \otimes \mathcal{U} \to L^0(\Omega, P)$ *is Poisson random measure with intensity* $\text{leb}\otimes\zeta\otimes\nu$*, independent of* W*, and with compensated Poisson random measure* $\widetilde{N}$*.*

Then for any functions

(1) $b \in L^2(\mathcal{O}, \zeta)$,

(2) $c\colon \mathcal{O} \times U \to \mathbb{R}$ *with* $\int_{\mathcal{O}\times U} \left(|c(x,y)|^2 \wedge |c(x,y)|\right) (\zeta \otimes \nu)(dx, dy) < \infty$,

(3) $d : \mathcal{O} \times U \to \mathbb{R}$ *with* $\int_{\mathcal{O}\times U} \left(|d(x,y)| \wedge 1\right) (\zeta \otimes \nu)(dx, dy) < \infty$,

we define a mapping $M'\colon \mathscr{B}_b(\mathbb{R}_+\times\mathcal{O})\to L^0(\Omega,P)$ *by*

$$M'(B) = \big(\mathrm{leb}\otimes\rho\big)(B) + \int_B b(x)\,W(\mathrm{d}s,\mathrm{d}x) \\ + \int_{B\times U} c(x,y)\,\tilde{N}(\mathrm{d}s,\mathrm{d}x,\mathrm{d}y) + \int_{B\times U} d(x,y)\,N(\mathrm{d}s,\mathrm{d}x,\mathrm{d}y)\,.$$

Then we obtain a Lévy-valued random measure on $\mathscr{B}_b(\mathcal{O})$ *by the prescription*

$$M(t,A) := M'((0,t]\times A) \qquad \text{for all } A\in\mathscr{B}_b(\mathcal{O}),\ t\geqslant 0.$$

The characteristic function $\varphi_{M(t,A)}\colon \mathbb{R}\to\mathbb{C}$ *of* $M(t,A)$ *is given by*

$$\varphi_{M(t,A)}(u) = \exp\Bigg(t\bigg(iu\rho(A) - \tfrac{1}{2}u^2\int_A b^2(x)\,\zeta(\mathrm{d}x) \\ + \int_A\int_U\Big(e^{iuc(x,y)} - 1 - iuc(x,y)\Big)\,\nu(\mathrm{d}y)\,\zeta(\mathrm{d}x) + \int_A\int_U\Big(e^{iud(x,y)} - 1\Big)\,\nu(\mathrm{d}y)\,\zeta(\mathrm{d}x)\bigg)\Bigg).$$

Proof. The existence of the Gaussian integral is guaranteed by [42, Th. 2.5] and that of the Poisson integrals by [27, Le. 12.13]. The characteristic function of $M'([0,t]\times A)$, see e.g. in [39, Prop. 19.5], shows that M' is an infinitely divisible random measure, and thus applying Proposition 2.3 completes the proof. □

Example 2.6. The class of α-stable random measures is introduced for example in [40, Se. 3.3]. These can be obtained from Proposition 2.5 by defining for bounded sets B in $\mathscr{B}_b(\mathbb{R}_+\times\mathbb{R}^d)$ the random measure

$$M'(B) := \begin{cases} \int_{B\times\mathbb{R}} y\,N(\mathrm{d}s,\mathrm{d}x,\mathrm{d}y), & \text{if } \alpha\in(0,1],\\ \int_{B\times\mathbb{R}} y\,\tilde{N}(\mathrm{d}s,\mathrm{d}x,\mathrm{d}y), & \text{if } \alpha\in(1,2),\end{cases}$$

where N is a Poisson random measure on $\mathscr{B}_b(\mathbb{R}_+\times\mathbb{R}^d\times\mathbb{R})$ with intensity $\mathrm{leb}\otimes\mathrm{leb}\otimes\nu_\alpha$, and

$$\nu_\alpha(\mathrm{d}y) = \big(p\alpha y^{-\alpha-1}\mathbb{1}_{(0,\infty)}(y) + q\alpha(-y)^{-\alpha-1}\mathbb{1}_{(-\infty,0)}(y)\big)\,\mathrm{d}y$$

for some $p+q=1$ (for the case $\alpha=1$ it is required that $p=q=\frac{1}{2}$); see Balan [5] for this construction. Proposition 2.5 guarantees that, by defining $M(t,A) := M'((0,t]\times A)$ for $t\geqslant 0$ and $A\in\mathscr{B}_b(\mathbb{R}^d)$, we obtain a Lévy-valued random measure M on $\mathscr{B}_b(\mathbb{R}^d)$. Direct calculation shows that for $\alpha\neq 1$, the characteristic function of $M(t,A)$ is given by, for $t\geqslant 0$, $A\in\mathscr{B}_b(\mathbb{R}^d)$ and $u\in\mathbb{R}$,

$$\varphi_{M(t,A)}(u) = \exp\left(t\cdot\mathrm{leb}(A)\cdot\left(i\beta\frac{\alpha}{1-\alpha}u + \int_{\mathbb{R}}\big(e^{iuy} - 1 - iuy\,\mathbb{1}_{B_{\mathbb{R}}}(y)\big)\,\nu_\alpha(\mathrm{d}y)\right)\right),$$

where $\beta := p - q$, and thus we see the characteristics of M are $\left(\beta\frac{\alpha}{1-\alpha}\text{leb}, 0, \text{leb}\otimes\nu_\alpha\right)$. The control measure is given by

$$\lambda(A) = \left(\left|\beta\frac{\alpha}{1-\alpha}\right| + \frac{2}{2-\alpha}\right)\text{leb}(A) \qquad \text{for } A \in \mathscr{B}(\mathbb{R}^d).$$

For the case $\alpha = 1$, the characteristic function of $M(t, A)$ is given by

$$\varphi_{M(t,A)}(u) = \exp\left(t \cdot \text{leb}(A) \cdot \int_{\mathbb{R}} \left(e^{iuy} - 1 - iuy\, \mathbb{1}_{B_{\mathbb{R}}}(y)\right) \nu_1(\mathrm{d}y)\right)$$

with control measure $\lambda(A) = 2\text{leb}(A)$ for $A \in \mathscr{B}(\mathbb{R}^d)$.

Example 2.7. Mytnik, in [31], considers a martingale-valued measure $(M(t, A) : t \geqslant 0, A \in \mathscr{B}_b(\mathbb{R}^d))$ in the sense of Walsh [42], such that for any $A \in \mathscr{B}_b(\mathbb{R}^d)$, the process $(M(t, A) : t \geqslant 0)$ is a real-valued α-stable process ($\alpha \in (1, 2)$), with Laplace transform

$$E\left[e^{-uM(t,A)}\right] = e^{-tu^\alpha \cdot \text{leb}(A)}, \qquad t \geqslant 0, u \geqslant 0.$$

The author terms M an *α-stable measure without negative jumps.*

Example 2.8. Basse-O'Connor and Rosinski in [11, Se. 4] consider an infinitely divisible random measure M on $\mathbb{R} \times V$, for some countably-generated measure space V, which is invariant under translations over $\mathbb{R}$. By Proposition 2.3, M defines a Lévy-valued random measure on V.

3 Lévy-valued additive sheets

Just as the Brownian sheet is the generalisation of a Brownian motion to a multidimensional index set, additive sheets are defined as the corresponding generalisation of an additive process. Adler et al. [1] first defined additive random fields on $\mathbb{R}^d$, and termed them 'Lévy processes' should they be stochastically continuous. In [17], Dalang and Walsh discuss Lévy sheets in $\mathbb{R}^2$. Additive fields with stationary increments are considered by Barndorff-Nielsen and Pedersen in [6] and are called 'homogeneous Lévy sheets'. Herein we present our definition based on the deposition of Dalang and Humeau in [14], which extends [1], and results from Pedersen [33].

For $a, b \in \mathbb{R}^d$ write $a \leqslant b$ if $a_j \leqslant b_j$ for all $j = 1, \ldots, d$ and similarly $a < b$, and define boxes $(a, b] := \{s \in \mathbb{R}^d : a < t \leqslant b\}$ and $[a, b] := \{s \in \mathbb{R}^d : a \leqslant t \leqslant b\}$; $[a, b)$ and (a, b) are defined mutatis mutandi.

For a function $f\colon \mathbb{R}^d \to \mathbb{R}$, we define the increment of f over $(a,b]$ for a, $b \in \mathbb{R}^d$ with $a < b$ by

$$\Delta_a^b f := \sum_{\varepsilon_1=0}^{1} \cdots \sum_{\varepsilon_k=0}^{1} (-1)^{\varepsilon_1+\cdots+\varepsilon_k} f\big(c_1(\varepsilon_1), \ldots c_k(\varepsilon_k)\big),$$

where $c_j(0) = b_j$ and $c_j(1) = a_j$. For example, in the case $d = 2$ we have $\Delta_a^b f = f(b_1, b_2) - f(b_1, a_2) - f(a_1, b_2) + f(a_1, a_2)$.

The càdlàg property is generalised to random fields in the following way: a function $f\colon \mathbb{R}^d \to \mathbb{R}$ has limits along monotone paths (lamp) if for every $x \in \mathbb{R}^d$ and any sequence $(x_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}^d$ converging to x with either $x_{n,j} < x_j$ or $x_{n,j} \geqslant x_j$ for all $n \in \mathbb{N}$ and $j \in \{1, \ldots, d\}$ where $x = (x_1, \ldots, x_d)$ and $x_n = (x_{n,1}, \ldots, x_{n,d})$, the limit $f(x_n)$ exists as $n \to \infty$ and furthermore f is right-continuous if $f(x_n) \to f(x)$ as $n \to \infty$ for all sequences with $x \leqslant x_n$ for all $n \in \mathbb{N}$. We note that this property is a path-based property, and thus in contrast to random measures we define our sheets as mappings from $\mathbb{R}^d \times \Omega \to \mathbb{R}$.

Definition 3.1. *Let $I \subseteq \mathbb{R}^d$ with $0 \in I$. A real-valued stochastic process $(X(x) : x \in I)$ is called an* additive sheet *if the following conditions are satisfied:*

(a) *$X(x) = 0$ a.s. for all $x = (x_1, \ldots, x_d) \in I$ with $x_j = 0$ for some $j \in \{1, \ldots, d\}$;*

(b) *$\Delta_{a_1}^{b_1} X, \ldots, \Delta_{a_n}^{b_n} X$ are independent for disjoint boxes $(a_1, b_1], \ldots, (a_n, b_n] \subseteq I$;*

(c) *X is continuous in probability;*

(d) *almost all sample paths of X have limits along monotone paths and are right-continuous.*

Remark 3.2. For relaxing the requirements in Definition 3.1 we refer to [1], e.g. to capture arbitrary initial conditions or sheets which are not continuous in probability. In particular, it is shown that Conditions (a) – (c) gurantee the existence of a lamp and right-continuous modification.

If $(X(x) : x \in I)$ is an additive sheet then for fixed $x \in I$ the random variable $X(x)$ is infinitely divisible; see Adler [1, Th. 3.1]; let its characteristics be denoted by (p_x, A_x, μ_x). The additive sheet is said to be *natural* if the mapping $x \mapsto p_x$, which is necessarily continuous, is of bounded variation, or equivalently, if there exists an atomless signed measure γ with $p_x = \gamma((0,x])$ for all $x \in I$; here, we use the convention $(0,x] := \prod_{i=1}^d I_i$ where, for $x = (x_1, \ldots, x_d) \in I$, $I_i := (0, x_i]$ when $x_i > 0$ and $I_i := [x_i, 0)$ when $x_i < 0$. The notation of natural additive processes is introduced in Sato [38] for the case $d = 1$.

Similarly as for infinitely divisible random measures, we introduce a dynamical aspect in the following definition:

Definition 3.3. *A family* $(X(t,\cdot) : t \geqslant 0)$ *of natural, additive sheets* $(X(t,x) : x \in \mathbb{R}^d)$ *is called a* Lévy-valued additive sheet *if for every* $x_1, \dots, x_n \in \mathbb{R}^d$ *and* $n \in \mathbb{N}$, *the stochastic process*

$$\Big(\big(X(t,x_1), \dots, X(t,x_n)\big) : t \geqslant 0\Big)$$

is a Lévy process in $\mathbb{R}^n$.

The wording 'Lévy-valued additive sheet' is motivated by the following result:

Proposition 3.4. *A Lévy-valued additive sheet* $(X(t,\cdot) : t \geqslant 0)$ *forms a natural additive sheet* $(X(z) : z \in \mathbb{R}_+ \times \mathbb{R}^d)$.

Proof. The domain of definition and Conditions (a), (b) and (d) of Definition 3.1 are clearly met. Regarding stochastic continuity, let $(t_n, x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}_+ \times \mathbb{R}^d$ converging to $(0,x)$. For each $n \in \mathbb{N}$ the random variable $X(1,x_n)$ is infinitely divisible, say with characteristics $(p_{x_n}, V_{x_n}, \mu_{x_n})$. As $X(1,\cdot)$ is a natural, additive sheet, there exists a signed measure γ such that $p_{x_n} = \gamma((0,x_n])$. Since the Lévy process $(X(t,x_n) : t \geqslant 0)$ has stationary increments, it follows that each $X(t,x_n)$ has characteristics $(tp_{x_n}, tV_{x_n}, t\mu_{x_n})$ for every $t \geqslant 0$. Theorem 3.1 in [1] implies that there exist a measure Σ on $\mathscr{B}(\mathbb{R}^d)$ such that $V_{x_n} = \Sigma((0,x_n])$, and a measure ν on $\mathscr{B}(\mathbb{R}^d \times \mathbb{R})$ such that, for each $B \in \mathscr{B}(\mathbb{R})$, the mapping $\nu(\cdot \times B)$ is a measure on $\mathscr{B}(\mathbb{R}^d)$, and $\mu_{x_n} = \nu((0,x_n] \times \cdot)$. Therefore, the Lévy symbol of $X(t_n,x_n)$ is given by, for $u \in \mathbb{R}$,

$$\begin{aligned}\Psi_{X(t_n,x_n)}(u) = t_n\Big(& iu\gamma((0,x_n]) - \tfrac{1}{2}u^2\Sigma((0,x_n]) \\ &+ \int_{(0,x_n]\times\mathbb{R}} \big(e^{iuy} - 1 - iuy\,\mathbb{1}_{B_\mathbb{R}}(y)\big)\,\nu(\mathrm{d}x,\mathrm{d}y)\Big).\end{aligned}$$

As the set $\{x_n : n \in \mathbb{N}\}$ is bounded, there exists a bounded box $I \subseteq \mathbb{R}^d$ containing every box $(0,x_n]$, $n \in \mathbb{N}$. Thus, we obtain for each $u \in \mathbb{R}$ that

$$\big|\Psi_{X(t_n,x_n)}(u)\big| \leqslant t_n\left(u\,\|\gamma\|_{TV}(I) + \tfrac{1}{2}u^2\Sigma(I) + \int_{I\times\mathbb{R}} \big(u^2y^2 \wedge 1\big)\,\nu(\mathrm{d}x,\mathrm{d}y)\right).$$

Finiteness of the right side follows from the fact that the measures are finite on I. Therefore, it follows that $X(t_n,x_n) \to 0$ in probability as (t_n,x_n) converges to $(0,x)$. If (t_n,x_n) is an arbitrary sequence converging to (t,x), stationary increments imply for each $c > 0$ that

$$\begin{aligned}&P(|X(t_n,x_n) - X(t,x)| > c) \\ &\quad\leqslant P(|X(t_n,x_n) - X(t,x_n)| > \tfrac{c}{2}) + P(|X(t,x_n) - X(t,x)| > \tfrac{c}{2}) \\ &\quad= P(|X(t_n - t,x_n)| > \tfrac{c}{2}) + P(|X(t,x_n) - X(t,x)| > \tfrac{c}{2}).\end{aligned}$$

Consequently, the above established continuity in probability shows the general case.

The fact that $X(z)$ is natural can be seen from the form of the characteristic function, where we have $p_z = t\gamma((0,x])$ for $z=(t,x)$. □

We are now able to state the link between Lévy-valued random measures and Lévy-valued additive sheets by formulating a result from Pedersen in [33] in our setting.

Theorem 3.5.

(a) *Let $(X(t,\cdot) : t \geqslant 0)$ be a Lévy-valued additive sheet. Then there exists a unique Lévy-valued random measure M on $\mathscr{B}_b(\mathbb{R}^d)$ with atomless control measure λ satisfying $M(t,(0,x]) = X(t,x)$ P-a.s. for each $t \geqslant 0$ and $x \in \mathbb{R}^d$.*

(b) *Let M be a Lévy-valued random measure on $\mathscr{B}_b(\mathbb{R}^d)$ with atomless control measure λ. Then any lamp and right-continuous modification of the stochastic process $X = (X(t,x) : t \geqslant 0,\, x \in \mathbb{R}^d)$ defined by*

$$X(t,x) := \begin{cases} 0, & \text{if } x_j = 0 \text{ for some } j = 1,\dots,d \\ M\big(t,(0,x]\big), & \text{else} \end{cases}$$

is a Lévy-valued additive sheet.

Proof. See [33]. □

Remark 3.6. Theorem 3.5 and its proof enables us to conclude a converse implication of Proposition 2.5. If M is a Lévy-valued random measure with atomless control measure λ, then it satisfies a Lévy-Itô decomposition of the form

$$\begin{aligned} M(t,A) &= t\gamma(A) + G\big((0,t]\times A\big) \\ &\quad + \int_{(0,t]\times A\times B_{\mathbb{R}}} y\,\widetilde{N}(\mathrm{d}s,\mathrm{d}x,\mathrm{d}y) + \int_{(0,t]\times A\times B_{\mathbb{R}}^c} y\,N(\mathrm{d}s,\mathrm{d}x,\mathrm{d}y), \end{aligned} \tag{3.1}$$

where γ is a signed measure on $\mathcal{O}$, G is a Gaussian random measure on $\mathscr{B}_b(\mathbb{R}_+ \times \mathcal{O})$ and N is an independent Poisson random measure on $\mathscr{B}_b(\mathbb{R}_+ \times \mathcal{O} \times \mathbb{R})$ with compensated part $\widetilde{N}$.

Furthermore, we see that one does not achieve larger generality by allowing an arbitrary measure space $(U,\mathcal{U},\nu)$ in Proposition 2.5, as the Poissonian components can be represented as integrals over $\mathbb{R}$.

4 Lévy-valued measures in the space of distributions

In this section, we embed the Lévy-valued random measure into the spaces of distributions and of tempered distributions. These embeddings are based on the integration theory for independently scattered infinitely divisible measures developed by Rajput and Rosinski in [36]. The multiplicative relation between the characteristics of the infinitely divisible random measures $M(1)$ and $M(t)$, remarked after Definition 2.2, enables us to apply directly the integration theory for infinitely divisible random measures to Lévy-valued random measures $(M(t) : t \geqslant 0)$ on $\mathscr{B}_b(\mathcal{O})$: for a simple function

$$f\colon \mathcal{O} \to \mathbb{R}, \qquad f(x) = \sum_{k=1}^{n} \alpha_k \mathbb{1}_{A_k}(x), \tag{4.1}$$

for $\alpha_k \in \mathbb{R}$ and pairwise disjoint sets $A_1, \dots, A_n \in \mathscr{B}_b(\mathcal{O})$, the integral is defined as

$$\int_A f(x)\, M(t, \mathrm{d}x) := \sum_{k=1}^{n} \alpha_k M(t, A \cap A_k) \qquad \text{for all } A \in \mathscr{B}(\mathcal{O}),\ t \geqslant 0. \tag{4.2}$$

An arbitrary measurable function $f\colon \mathcal{O} \to \mathbb{R}$ is said to be *M-integrable* if the following hold:

(1) there exists a sequence of simple functions $(f_n)_{n\in\mathbb{N}}$ of the form (4.1) such that f_n converges pointwise to f λ-a.e., where λ is the control measure of M;

(2) for each $A \in \mathscr{B}(\mathcal{O})$ and $t \geqslant 0$, the sequence $\left(\int_A f_n(x)\, M(t, \mathrm{d}x)\right)_{n\in\mathbb{N}}$ converges in probability.

In this case, the integral of f is defined as

$$\int_A f(x)\, M(t, \mathrm{d}x) := P\text{--}\lim_{n\to\infty} \int_A f_n(x)\, M(t, \mathrm{d}x). \tag{4.3}$$

It is clear, by the stationarity of the increments of Lévy processes, that Condition (2) above holds for all $t \geqslant 0$ if it holds for any $t > 0$. Furthermore, Theorem 3.3 in [36] identifies the set of M-integrable functions as the Musielak-Orlicz space

$$L_M(\mathcal{O}, \lambda) := \left\{ f \in L^0(\mathcal{O}, \lambda) : \int_{\mathcal{O}} \Phi_M(|f(x)|, x)\, \lambda(\mathrm{d}x) < \infty \right\},$$

where $\Phi_M : \mathbb{R} \times \mathcal{O} \to \mathbb{R}$ is defined as:

$$\Phi_M(u,x) := \sup_{|c|\leqslant 1} |U(cu,x)| + u^2 g(x) + \int_{\mathbb{R}} \Big(1 \wedge |uy|^2\Big)\, \rho(x, \mathrm{d}y), \tag{4.4}$$

$$\text{with } U(u,x) := ua(x) + u\int_{\mathbb{R}} y\big(\mathbb{1}_{B_{\mathbb{R}}}(uy) - \mathbb{1}_{B_{\mathbb{R}}}(y)\big)\,\rho(x,\mathrm{d}y),$$

$$a(x) := \frac{\mathrm{d}\gamma}{\mathrm{d}\lambda}(x), \qquad g(x) = \frac{\mathrm{d}\Sigma}{\mathrm{d}\lambda}(x).$$

Here, (γ, Σ, ν) denotes the characteristics of M. The measure $\rho(x,\cdot)$ is a disintegration of ν over λ, i.e. $\int_{\mathcal{O}\times\mathbb{R}} h(x,y)\,\nu(\mathrm{d}x,\mathrm{d}y) = \int_{\mathcal{O}} \Big(\int_{\mathbb{R}} h(x,y)\,\rho(x,\mathrm{d}y)\Big)\,\lambda(\mathrm{d}x)$ for each measurable function $h\colon \mathcal{O}\times\mathbb{R} \to \mathbb{R}_+$. The space $L_M(\mathcal{O},\lambda)$ is a complete, translation-invariant, linear metric space. Furthermore for all $t \geqslant 0$, the mapping

$$J(t) : L_M(\mathcal{O},\lambda) \to L^0(\Omega, P), \qquad J(t)f = \int_{\mathcal{O}} f(x)\, M(t, \mathrm{d}x), \tag{4.5}$$

is continuous. Finally, Proposition 2.6 in [36] allows us to immediately state the Lévy symbol of $J(\cdot)f$ as, for $u \in \mathbb{R}$,

$$\begin{aligned}\Psi_{J(\cdot)f}(u) =& iu\int_{\mathcal{O}} f(x)\,\gamma(\mathrm{d}x) - \tfrac{1}{2}u^2\int_{\mathcal{O}} f^2(x)\,\Sigma(\mathrm{d}x)\\ &+ \int_{\mathcal{O}\times\mathbb{R}} \Big(e^{iuf(x)y} - 1 - iuf(x)y\,\mathbb{1}_{B_{\mathbb{R}}}(y)\Big)\,\nu(\mathrm{d}x,\mathrm{d}y)\,.\end{aligned} \tag{4.6}$$

For an open set $\mathcal{O} \subseteq \mathbb{R}^d$ let $\mathcal{D}(\mathcal{O})$ denote the space of infinitely differentiable functions with compact support. We equip $\mathcal{D}(\mathcal{O})$ with the inductive topology, that is, the strict inductive limit of the Fréchet spaces $\mathcal{D}(K_i) := \{f \in C^\infty(\mathbb{R}^d) : \operatorname{supp}(f) \subseteq K_i\}$ where $\{K_i\}_{i\in\mathbb{N}}$ is a strictly increasing sequence of compact subsets of $\mathcal{O}$ such that $\mathcal{O} = \bigcup_{i\in\mathbb{N}} K_i$. The topological dual space $\mathcal{D}^*(\mathcal{O})$ is called the space of distributions, which we equip with the strong topology, that is the topology generated by the family of seminorms $\{\eta_B\}$, where for each bounded $B \subseteq \mathcal{D}(\mathcal{O})$ we define $\eta_B(f) := \sup_{\varphi\in B} |\langle\varphi, f\rangle|$ for $f \in \mathcal{D}^*(\mathcal{O})$. In these topologies $\mathcal{D}(\mathcal{O})$ is reflexive [41, p. 376].

Analogously as locally integrable functions and measures are identified with distributions, we proceed to relate a Lévy-valued random measure M on $\mathscr{B}_b(\mathcal{O})$ to a distribution-valued process. For this purpose, we define for each $t \geqslant 0$ the integral mapping

$$J_{\mathcal{D}}(t) : \mathcal{D}(\mathcal{O}) \to L^0(\Omega, P)\,, \qquad J_{\mathcal{D}}(t)f = \int_{\mathcal{O}} f(x)\, M(t, \mathrm{d}x). \tag{4.7}$$

In the proof of Theorem 4.1 below we show that $\mathcal{D}(\mathcal{O})$ is continuously embedded in $L_M(\mathcal{O},\lambda)$, and thus the mapping $J_{\mathcal{D}}(t)$ is well-defined.

In the following theorem as in the reminder of the article we use the phrase *genuine Lévy process in a space* F to emphasise that this is a Lévy process in the space F according to the usual definition, e.g. Definition 4.1 and Definition 14.2 in [34].

Theorem 4.1. *For a Lévy-valued random measure* M *on* $\mathscr{B}_b(\mathcal{O})$ *let* $J_{\mathcal{D}}$ *be defined by* (4.7)*. Then there exists a genuine Lévy process* $(Y(t) : t \geqslant 0)$ *in* $\mathcal{D}^*(\mathcal{O})$ *satisfying*

$$\langle f, Y(t)\rangle = J_{\mathcal{D}}(t)f \qquad \text{for all } f \in \mathcal{D}(\mathcal{O}),\, t \geqslant 0.$$

Our proof of this Theorem relies on the following two Lemmas.

Lemma 4.2. *For a Lévy-valued random measure* M *on* $\mathscr{B}_b(\mathcal{O})$ *let* J *be defined by* (4.5)*. Then, for any* $f_1, \dots, f_n \in L_M(\mathcal{O}, \lambda)$ *and* $n \in \mathbb{N}$*, we have that*

$$\big((J(t)f_1, \dots, J(t)f_n) : t \geqslant 0\big)$$

is a Lévy process in $\mathbb{R}^n$*.*

Proof. Let f_k for $k = 1, \dots, n$ be simple functions of the form

$$f_k \colon \mathcal{O} \to \mathbb{R}, \qquad f_k(x) = \sum_{j=1}^{m_k} \alpha_{k,j} \mathbb{1}_{A_{k,j}}(x),$$

for $\alpha_{k,j} \in \mathbb{R}$ and $A_{k,j} \in \mathscr{B}_b(\mathcal{O})$ with $A_{k,1}, \dots, A_{k,m_k}$ disjoint for each $k \in \{1, \dots, n\}$. By taking the intersections of all possible permutations of the sets $A_{k,j}$, we can assume that

$$f_k(x) = \sum_{j=1}^{m} \tilde{\alpha}_{k,j}\, \mathbb{1}_{\tilde{A}_j}(x) \quad \text{for all } x \in \mathcal{O},$$

for all $k = 1, \dots, n$, where $\tilde{\alpha}_{k,j} \in \mathbb{R}$ and disjoint sets $\tilde{A}_1, \dots, \tilde{A}_m \in \mathscr{B}_b(\mathcal{O})$ for some $m \in \mathbb{N}$. For each $0 \leqslant t_1 < \dots < t_n$ we obtain by the definition in (4.2) that

$$\begin{aligned}
J(t_1)f_1 &= \sum_{j=1}^{m} \tilde{\alpha}_{1,j} M\big(t_1, \tilde{A}_j\big), \\
(J(t_2) - J(t_1))f_2 &= \sum_{j=1}^{m} \tilde{\alpha}_{2,j} \big(M(t_2, \tilde{A}_j) - M(t_1, \tilde{A}_j)\big), \\
&\vdots \\
(J(t_n) - J(t_{n-1}))f_n &= \sum_{j=1}^{m} \tilde{\alpha}_{n,j} \big(M(t_n, \tilde{A}_j) - M(t_{n-1}, \tilde{A}_j)\big).
\end{aligned}$$

Independent increments of the Lévy process $\big(M(\cdot,\tilde{A}_1),\dots,M(\cdot,\tilde{A}_m)\big)$ together with independence of $M(t,\tilde{A}_i)$ and $M(t,\tilde{A}_j)$ for all $i,j=1,\dots,m$ with $i\neq j$ imply that the random variables

$$J(t_1)f_1,\big(J(t_2)-J(t_1)\big)f_2,\dots,\big(J(t_n)-J(t_{n-1})\big)f_n,$$

are independent. This property extends to arbitrary functions $f_1,\dots,f_n\in L_M(\mathcal{O},\lambda)$ by the definition of the integrals in (4.3) as a limit of the integral for simple functions. It follows that the n-dimensional stochastic process $\big((J(t)f_1,\dots,J(t)f_n):\, t\geqslant 0\big)$ has independent increments.

Furthermore, if f is a simple function of the form (4.1) then

$$J(t)f=\sum_{k=1}^{n}\alpha_k M(t,A_k) \tag{4.8}$$

is a Lévy process as it is the sum of independent Lévy processes $M(\cdot,A_k)$. Approximating an arbitrary function $f\in L_M(\mathcal{O},\lambda)$ by a sequence of simple functions and passing to the limit in (4.8) shows that $J(\cdot)f$ is a Lévy process.

Let $f_1,\dots,f_n$ be arbitrary functions in $L_M(\mathcal{O},\lambda)$. As $J(\cdot)f$ has stationary increments it follows that $\big((J(t)f_1,\dots,J(t)f_n):\, t\geqslant 0\big)$ has stationary increments by linearity. Furthermore, for each $c>0$ we have

$$\begin{aligned}P\big(\,|((J(t)f_1,\dots,J(t)f_n))|>c\big)&=P\big(\,|J(t)f_1|^2+\dots+|J(t)f_n|^2>c^2\big)\\&\leqslant\sum_{k=1}^{n}P\left(|J(t)f_k|^2\geqslant \tfrac{c^2}{n}\right),\end{aligned}$$

and thus the stochastic continuity of $J(\cdot)f$ implies that of $\big((J(t)f_1,\dots,J(t)f_n):\, t\geqslant 0\big)$. Consequently, the latter is verified as an n-dimensional Lévy process. □

Lemma 4.3. *Let M be a Lévy-valued random measure on $\mathscr{B}_b(\mathcal{O})$ with finite control measure λ. Then $L^2(\mathcal{O},\lambda)$ is continuously embedded into $L_M(\mathcal{O},\lambda)$.*

Proof. Denote the characteristics of M by (γ,Σ,ν). Note, that for arbitrary $g\in L^p(\mathcal{O},\lambda)$ and $p\in[1,2]$, we have

$$\begin{aligned}\int_{\mathcal{O}}g(x)\,\lambda(\mathrm{d}x)&=\int_{\mathcal{O}}g(x)\,\|\gamma\|_{TV}(\mathrm{d}x)+\int_{\mathcal{O}}g(x)\,\Sigma(\mathrm{d}x)\\&\quad+\int_{\mathcal{O}\times B_{\mathbb{R}}}g(x)\,|y|^2\,\nu(\mathrm{d}x,\mathrm{d}y)+\int_{\mathcal{O}\times B_{\mathbb{R}}^c}g(x)\,\nu(\mathrm{d}x,\mathrm{d}y).\end{aligned} \tag{4.9}$$

Let $f \in L^2(\mathcal{O}, \lambda)$ be given. It follows from (4.9) that

$$\int_{\mathcal{O}} |f(x)|^2 \, \Sigma(\mathrm{d}x) \leqslant \|f\|^2_{L^2(\mathcal{O},\lambda)} < \infty$$

and

$$\begin{aligned}\int_{\mathcal{O}\times\mathbb{R}} \big(1 \wedge |f(x)y|^2\big)\, \nu(\mathrm{d}x, \mathrm{d}y) &\leqslant \|f\|^2_{L^2(\mathcal{O},\lambda)} + \int_{\mathcal{O}\times B^c_{\mathbb{R}}} \big(1 \wedge |f(x)y|^2\big)\, \nu(\mathrm{d}x, \mathrm{d}y) \\ &\leqslant \|f\|^2_{L^2(\mathcal{O},\lambda)} + \lambda(\mathcal{O}) < \infty.\end{aligned}$$

Furthermore we obtain from the definition of U and (4.9), recalling that $L^1(\mathcal{O}, \lambda) \hookrightarrow L^2(\mathcal{O}, \lambda)$ as λ is finite, that

$$\begin{aligned}&\int_{\mathcal{O}} |U(|f(x)|, x)| \, \lambda(\mathrm{d}x) \\ &\quad= \int_{\mathcal{O}} \left| |f(x)| \Big(a(x) + \int_{\mathbb{R}} y\big(\mathbb{1}_{B_{\mathbb{R}}}(|f(x)|\, y) - \mathbb{1}_{B_{\mathbb{R}}}(y)\big)\, \rho(x, \mathrm{d}y)\Big)\right| \lambda(\mathrm{d}x) \\ &\quad\leqslant \int_{\mathcal{O}} |f(x)| \, \|\gamma\|_{TV}\, (\mathrm{d}x) + \int_{\mathcal{O}\times\mathbb{R}} |f(x)y| \, |\mathbb{1}_{B_{\mathbb{R}}}(|f(x)|\, y) - \mathbb{1}_{B_{\mathbb{R}}}(y)| \, \nu(\mathrm{d}x, \mathrm{d}y) \\ &\quad\leqslant \|f\|_{L^1(\mathcal{O},\lambda)} + \|f\|^2_{L^2(\mathcal{O},\lambda)} + \int_{\mathcal{O}\times B^c_{\mathbb{R}}} |f(x)y| \, \mathbb{1}_{B_{\mathbb{R}}}(|f(x)|\, y)\, \nu(\mathrm{d}x, \mathrm{d}y) \\ &\quad\leqslant \|f\|_{L^1(\mathcal{O},\lambda)} + \|f\|^2_{L^2(\mathcal{O},\lambda)} + \lambda(\mathcal{O}) < \infty. \end{aligned} \tag{4.10}$$

From Theorem 2.7 in [36] we obtain $f \in L_M(\mathcal{O}, \lambda)$ thus showing the stated embedding.

To show that the embedding is continuous, let (f_n) converge to 0 in $L^2(\mathcal{O}, \lambda)$. We firstly show that the functions $(x, y) \mapsto f_n(x)y$ converge to 0 in ν_1-measure where $\nu_1 := \nu|_{\mathcal{O}\times B^c_{\mathbb{R}}}$. For given $\varepsilon > 0$ define $M_n := \{(x, y) \in \mathcal{O} \times B^c_{\mathbb{R}} : |f_n(x)y| \geqslant \varepsilon\}$. As ν_1 is a finite measure, there exists a compact set $K \subseteq \mathcal{O}\times B^c_{\mathbb{R}}$ such that $\nu_1(\mathcal{O}\times B^c_{\mathbb{R}} \backslash K) < \frac{\varepsilon}{2}$. Let $C := \sup\{|y| : (x, y) \in K\}$. Define for $n \in \mathbb{N}, x \in \mathcal{O}$ and $y \in \mathbb{R}$ functions $g_n(x, y) := f_n(x)$. Since (f_n) converges in $L^1(\mathcal{O}, \lambda)$ it follows from (4.9) that (g_n) converges to 0 in $L^1(\mathcal{O} \times B^c_{\mathbb{R}}, \nu)$, and thus in ν_1-measure. Consequently, there exists $N \in \mathbb{N}$ such that, for $n \geqslant N$,

$$\nu_1\big(\{(x, y) \in \mathcal{O} \times B^c_{\mathbb{R}} : |f_n(x)| \geqslant \tfrac{\varepsilon}{C}\}\big) \leqslant \tfrac{\varepsilon}{2}\,.$$

Since $M_n \cap K \subseteq \big\{(x, y) \in \mathcal{O} \times B^c_{\mathbb{R}} : |f_n(x)| \geqslant \frac{\varepsilon}{C}\big\}$, we obtain

$$\nu_1(M_n) = \nu_1(M_n \cap K) + \nu_1(M_n\backslash K) \leqslant \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} \qquad \text{for all } n \geqslant N,$$

which shows the claim.

Since ν_1 is a finite measure, Lebesgue's theorem for dominated convergence in ν_1-measure implies

$$\lim_{n\to\infty} \int_{\mathcal{O}\times B_{\mathbb{R}}^c} |f_n(x)y| \, \mathbb{1}_{B_{\mathbb{R}}}(|f_n(x)|\, y)\, \nu(\mathrm{d}x, \mathrm{d}y) = 0.$$

Similar arguments show that

$$\lim_{n\to\infty} \int_{\mathcal{O}\times B_{\mathbb{R}}^c} \left(1 \wedge |f_n(x)y|^2\right) \nu(\mathrm{d}x, \mathrm{d}y) = 0.$$

For each $f \in L^2(\mathcal{O}, \lambda)$, by the definition in (4.4) of Φ_M and Lemma 2.8 in [36] we obtain

$$\begin{aligned}
&\int_{\mathcal{O}} \Phi_M(|f(x)|, x)\, \lambda(\mathrm{d}x) \\
&= \int_{\mathcal{O}} \sup_{|c|\leqslant 1} |U(c\,|f(x)|, x)|\; \lambda(\mathrm{d}x) + \int_{\mathcal{O}} |f(x)|^2\; \Sigma(\mathrm{d}x) + \int_{\mathcal{O}\times\mathbb{R}} \left(1 \wedge |f(x)y|^2\right) \nu(\mathrm{d}x, \mathrm{d}y) \\
&\leqslant \int_{\mathcal{O}} \big|U(|f(x)|, x)\big|\, \lambda(\mathrm{d}x) + 10\, \|f\|^2_{L^2(\mathcal{O},\lambda)} + 9 \int_{\mathcal{O}\times B_{\mathbb{R}}^c} \left(1 \wedge |f(x)y|^2\right) \nu(\mathrm{d}x, \mathrm{d}y).
\end{aligned}$$

Consequently, it follows from (4.10) that (f_n) converges in $L_M(\Phi, \lambda)$, which completes the proof. □

Proof of Theorem 4.1. We first show that the space $\mathcal{D}(K)$ is continuously embedded in $L_M(\mathcal{O}, \lambda)$ for each compact $K \subseteq \mathcal{O}$. Trivially, the space $\mathcal{D}(K)$ is continuously embedded in $L^\infty(K, \lambda)$. As $K \in \mathscr{B}_b(\mathcal{O})$, the control measure λ is finite on K, and it follows that $L^\infty(K, \lambda)$ is continuously embedded in $L^2(K, \lambda)$. The latter is continuously embedded in $L_M(K, \lambda)$ by Lemma 4.3. Because whenever $\mathrm{supp}(f) \subseteq K$ we have

$$\int_{\mathcal{O}} \Phi_M(|f(x)|, x)\, \lambda(\mathrm{d}x) = \int_K \Phi_M(|f(x)|, x)\, \lambda(\mathrm{d}x),$$

it follows that $\mathcal{D}(K)$ is continuously embedded in $L_M(\mathcal{O}, \lambda)$. As $\mathcal{D}(\mathcal{O})$ is the inductive limit of $\{\mathcal{D}(K_i)\}_{i\in\mathbb{N}}$, we thus conclude that $\mathcal{D}(\mathcal{O})$ is continuously embedded in $L_M(\mathcal{O}, \lambda)$.

Let $\iota\colon \mathcal{D}(\mathcal{O}) \to L_M(\mathcal{O}, \lambda)$ be the continuous embedding. Then the mapping $J_{\mathcal{D}}(t)\colon \mathcal{D}(\mathcal{O}) \to L^0(\Omega, P)$ can be represented as $J_{\mathcal{D}}(t) = J(t) \circ \iota$ for each $t \geqslant 0$, showing that $J_{\mathcal{D}}(t)$ is continuous. Lemma 4.2 shows that $J_{\mathcal{D}}$ is a cylindrical Lévy process in $\mathcal{D}^*(\mathcal{O})$ as defined in [20, Definition 3.6]. Furthermore, since $J_{\mathcal{D}}(t)$ is continuous, and $\mathcal{D}(\mathcal{O})$ is nuclear [41, Theorem 51.5] and ultrabornological [32, Page 447], Theorem 3.8 in [20] implies the existence of the $\mathcal{D}^*(\mathcal{O})$-valued Lévy process Y. □

In the second part of this section, we embed the Lévy-valued random measure into the space of tempered distribution $\mathcal{S}^*(\mathbb{R}^d)$. We introduce the Schwartz space

$$\mathcal{S}(\mathbb{R}^d) := \big\{ f \in C^\infty(\mathbb{R}^d) \colon \|f\|_{\mathcal{S}_k} < \infty \text{ for all } k \in \mathbb{N} \big\},$$

where the seminorms $\|\cdot\|_{\mathcal{S}_k}$, $k \in \mathbb{N}$ are defined by $\|f\|_{\mathcal{S}_k} := \max_{|s| \leqslant k} \sup_{x\in\mathbb{R}^d} (1 + |x|^2)^k \, |\partial^s f(x)|$. In particular, $\mathcal{S}(\mathbb{R}^d)$ is metrisable, and $f_n \to f$ in $\mathcal{S}(\mathbb{R}^d)$ means $\|f_n - f\|_{\mathcal{S}_k} \to 0$ for each $k \in \mathbb{N}$. The dual space of $\mathcal{S}(\mathbb{R}^d)$ is the space $\mathcal{S}^*(\mathbb{R}^d)$ of tempered distributions.

Define for each $t \geqslant 0$ the integral mapping

$$J_{\mathcal{S}}(t) : \mathcal{S}(\mathbb{R}^d) \to L^0(\Omega, P)\,, \qquad J_{\mathcal{S}}(t) f = \int_{\mathbb{R}^d} f(x)\, M(t, \mathrm{d}x). \tag{4.11}$$

Clearly, the mapping $J_{\mathcal{S}}(t)$ is only well defined if $\mathcal{S}(\mathbb{R}^d)$ is embedded in $L_M(\mathbb{R}^d, \lambda)$. The following theorem gives an equivalent condition for this.

Theorem 4.4. *Let M be a Lévy-valued random measure on $\mathscr{B}_b(\mathbb{R}^d)$ with control measure λ. Then the following are equivalent:*

(a) *$\mathcal{S}(\mathbb{R}^d)$ is continuously embedded in $L_M(\mathbb{R}^d, \lambda)$;*

(b) *there exists an $r > 0$ such that the function $x \mapsto (1 + |x|^2)^{-r}$ is in $L_M(\mathbb{R}^d, \lambda)$.*

In this case, the mapping $J_{\mathcal{S}}(t)$ as defined in (4.11) is well-defined and continuous for each $t \geqslant 0$. Furthermore, there exists a genuine Lévy process $(Y(t) : t \geqslant 0)$ in $\mathcal{S}^*(\mathbb{R}^d)$ satisfying

$$\langle f, Y(t) \rangle = J_{\mathcal{S}}(t) f \qquad \textit{for all } f \in \mathcal{S}(\mathbb{R}^d),\ t \geqslant 0.$$

Proof. We begin by showing the implication (b) $\Rightarrow$ (a), for which we suppose there exists $r > 0$ such that $x \mapsto \big(1 + |x|^2\big)^{-r}$ is in $L_M(\mathbb{R}^d, \lambda)$. For each $f \in \mathcal{S}(\mathbb{R}^d)$ there exists $K > 0$ such that $(1 + |x|^2)^r \, |f(x)| \leqslant K$ for all $x \in \mathbb{R}^d$. Since $\Phi_M(\cdot, x)$ is monotone for each $x \in \mathbb{R}^d$ according to [36, Lemma 3.1], we have

$$\Phi_M(|f(x)|\,, x) \leqslant \Phi_M(K(1 + |x|^2)^{-r}, x) \qquad \text{for each } x \in \mathbb{R}^d,$$

which implies $f \in L_M(\mathbb{R}^d, \lambda)$.

Let $(f_n)_{n\in\mathbb{N}} \subseteq \mathcal{S}(\mathbb{R}^d)$ be a sequence converging to 0 in $\mathcal{S}(\mathbb{R}^d)$. As the convergence is uniform in x, we have the existence of another $K > 0$ such that $(1 + |x|^2)^r \, |f_n(x)| \leqslant K$ for all $x \in \mathbb{R}^d$ and for all $n \in \mathbb{N}$. For fixed $x \in \mathbb{R}^d$ we have $\Phi_M(|f_n(x)|\,, x) \to \Phi_M(0, x) = 0$ by continuity [36, Lemma 3.1], and as

$\int_{\mathbb{R}^d} \Phi_M(K(1+|x|^2)^{-r}, x)\,\lambda(\mathrm{d}x) < \infty$, Lebesgue's theorem for dominated convergence implies

$$\int_{\mathbb{R}^d} \Phi_M(|f_n(x)|, x)\,\lambda(\mathrm{d}x) \to 0 \qquad \text{as } n \to \infty,$$

which completes the proof of the implication (b) $\Rightarrow$ (a).

Conversely, suppose $\mathcal{S}(\mathbb{R}^d)$ is continuously embedded in $L_M(\mathbb{R}^d, \lambda)$. Thus, the identity mapping $\iota\colon \mathcal{S}(\mathbb{R}^d) \to L_M(\mathbb{R}^d, \lambda)$ is continuous. Then, there exists a neighbourhood

$$U(0; k, \delta) := \{ f \in \mathcal{S}(\mathbb{R}^d) : \|f\|_{\mathcal{S}_k} < \delta \},$$

for some $k \in \mathbb{N}$ and $\delta > 0$ such that ι maps $U(0; k, \delta)$ into the open unit ball of $L_M(\mathbb{R}^d, \lambda)$. Let $(f_n)_{n\in\mathbb{N}} \subseteq \mathcal{S}(\mathbb{R}^d)$ be any sequence such that $\|f_n\|_{\mathcal{S}_k} \to 0$. Then, (f_n) is eventually in $U(0; k, \delta)$ and thus (ιf_n) is eventually in the unit ball and so is bounded in $L_M(\mathbb{R}^d, \lambda)$. By Proposition 4 of [25, p. 41] we have the continuity of ι in the semi-norm $\|\cdot\|_{\mathcal{S}_k}$, and thus we may extend ι by continuity to the completion of $\mathcal{S}(\mathbb{R}^d)$ in this semi-norm. We thus obtain the integrability condition by observing that the $C^\infty(\mathbb{R}^d)$ mapping $x \mapsto (1+|x|^2)^r$ has finite semi-norm $\|\cdot\|_{\mathcal{S}_k}$ for $r \leqslant -k$.

As in the proof of Theorem 4.1, an application of Lemma 4.2 and Theorem 3.8 in [20] establishes the existence of the Lévy process Y in $\mathcal{S}(\mathbb{R}^d)$. □

Remark 4.5. In Kabanava [26], it is shown that a Radon measure ζ can be identified with a tempered distribution in $\mathcal{S}^*(\mathbb{R}^d)$ if and only if there is a real number r such that $x \mapsto (1+|x|^2)^r$ is integrable over $\mathbb{R}^d$ with respect to ζ. Our condition for the mapping $J_{\mathcal{S}}$ in Theorem 4.4 is analogous.

Remark 4.6. By Proposition 2.3, we may also view the Lévy-valued random measure M as an infinitely divisible random measure M' on $\mathscr{B}_b(\mathbb{R}_+ \times \mathcal{O})$, and define the integral mapping

$$J'_{\mathcal{D}}\colon \mathcal{D}((0,\infty)\times\mathcal{O}) \to L^0(\Omega, P), \qquad J'_{\mathcal{D}} f = \int_{(0,\infty)\times\mathcal{O}} f(t,x)\, M'(\mathrm{d}t, \mathrm{d}x).$$

Analogously to Theorem 4.1 we obtain an infinitely divisible random variable Y' in $\mathcal{D}^*((0,\infty)\times\mathcal{O})$ satisfying $\langle f, Y'\rangle = J'_{\mathcal{D}} f$ for all $f \in \mathcal{D}((0,\infty)\times\mathcal{O})$. Similarly, under the conditions of Theorem 4.4, we may consider the operator J' on the space $\mathcal{S}(\mathbb{R}_+ \times \mathbb{R}^d)$.

Remark 4.7. In a series of papers, e.g. [4, 14, 18, 19], Dalang, Humeau, Unser and co-authors have studied the Lévy white noise Z defined as a distribution. Here, Z

is defined as a cylindrical random variable in $\mathcal{D}^*(\mathbb{R}^d)$, i.e. a linear and continuous mapping $Z\colon \mathcal{D}(\mathbb{R}^d) \to L^0(\Omega, P)$, with characteristic function

$$\varphi_Z\colon \mathcal{D}(\mathbb{R}^d) \to \mathbb{C}, \qquad \varphi_Z(f) = \exp\left(\int_{\mathbb{R}^d} \psi\big(f(x)\big)\,\mathrm{d}x\right),$$

where $\psi\colon \mathbb{R} \to \mathbb{C}$ is defined by

$$\psi(u) := ipu - \tfrac{1}{2}\sigma^2 u^2 + \int_{\mathbb{R}} \big(e^{iuy} - 1 - iuy\,\mathbb{1}_{B_{\mathbb{R}}}(y)\big)\,\nu_0(\mathrm{d}y), \tag{4.12}$$

for some constants $p \in \mathbb{R}$ and $\sigma^2 \in \mathbb{R}_+$ and a Lévy measure ν_0 on $\mathbb{R}$.

Let M be a Lévy-valued random measure on $\mathscr{B}_b(\mathbb{R}^d)$ with characteristics (γ, Σ, ν) and $J_{\mathcal{D}}(t)$ the corresponding operator defined in (4.7) for $t \geqslant 0$. By comparing the Lévy symbol in (4.6) with (4.12) it follows that, for fixed $t \geqslant 0$, the mapping $J_D(t)$ is a Lévy white noise in the above sense, if and only if

$$\gamma = p \cdot \mathrm{leb}, \qquad \Sigma = \sigma^2 \cdot \mathrm{leb}, \qquad \nu = \mathrm{leb} \otimes \nu_0,$$

for some $p \in \mathbb{R}$, $\sigma^2 \in \mathbb{R}_+$ and a Lévy measure ν_0 on $\mathbb{R}$. It follows that $M(t, A) \overset{\mathscr{D}}{=} M(t, B)$ for any sets $A, B \in \mathscr{B}_b(\mathbb{R}^d)$ with $\mathrm{leb}(A) = \mathrm{leb}(B)$. In this case, we call M *stationary in space*.

Dalang and Humeau have shown in [14] that a Lévy white noise in $\mathcal{D}^*(\mathbb{R}^d)$ with Lévy symbol (4.12) takes values in $\mathcal{S}^*(\mathbb{R}^d)$ P-a.s. if and only if

$$\int_{\mathbb{R}} \big(\,|y|^{\varepsilon} \wedge |y|^2\,\big)\,\nu_0(\mathrm{d}y) < \infty \quad \text{for some } \varepsilon > 0\,.$$

This result is analogous to our Theorem 4.4. However, as Lévy-valued random measures are not necessarily stationary in space, our condition is more complex. For example, even in the pure Gaussian case with characteristics $(0, \Sigma, 0)$, the measure Σ must be tempered; cf. Remark 4.5.

Regularity of the Lévy white noise Z in terms of Besov spaces is studied in [4]. Their results can be applied to a Lévy-valued random measure if it is additionally assumed to be stationary in space, i.e. which can be considered as a Lévy white noise in the above sense. We illustrate such an application in the following example.

Example 4.8. Let M be the α-stable random measure, $\alpha \in (0, 2)$, described in Example 2.6. For simplicity we consider the symmetric case, i.e. $p = q = \frac{1}{2}$. As the characteristics of M is given by $(0, 0, \mathrm{leb} \otimes \nu_\alpha)$, it follows that M is stationary in space. Thus, for a fixed time $t \geqslant 0$, the mapping $J_{\mathcal{D}}(t)$ or, equivalently the random variable $Y(t)$, where Y denotes the Lévy process derived in Theorem 4.1, can be considered as a Lévy white noise in $\mathcal{D}^*(\mathbb{R}^d)$; see Remark 4.7. Furthermore, since $\int_{\mathbb{R}} \left(|y|^\varepsilon \wedge |y|^2 \right) \nu_\alpha(\mathrm{d}y) < \infty$ for $\varepsilon < \alpha$, we have that $Y(t)$ is in $\mathcal{S}^*(\mathbb{R}^d)$ P-a.s. By applying the results from [4] we obtain the following: for $p \in (0,2) \cup 2\mathbb{N} \cup \{\infty\}$ and for all $t \geqslant 0$, we have, almost surely:

$$\text{if } \tau < d\Big(\frac{1}{\alpha} - 1\Big) \text{ and } \rho < -\frac{d}{p \wedge \alpha} \text{ then } Y(t) \in B_p^\tau(\mathbb{R}^d, \rho), \tag{4.13}$$

$$\text{if } \tau > d\Big(\frac{1}{\alpha} - 1\Big) \text{ or } \rho > -\frac{d}{p \wedge \alpha} \text{ then } Y(t) \notin B_p^\tau(\mathbb{R}^d, \rho)\,, \tag{4.14}$$

where $B_p^\tau(\mathbb{R}^d, \rho)$ is the weighted Besov space of integrability p, smoothness τ and asymptotic growth rate ρ. Furthermore, a modification of Y is a Lévy process in any Besov space satisfying (4.13), since its characteristic function is continuous in 0, guaranteeing stochastic continuity.

5 Cylindrical Lévy processes

The concept of cylindrical Lévy processes in Banach spaces is introduced in [2]. It naturally generalises the notation of cylindrical Brownian motion, based on the theory of cylindrical measures and cylindrical random variables. Here, a cylindrical random variable Z on a Banach space F is a linear and continuous mapping $Z\colon F^* \to L^0(\Omega, P)$, where F^* denotes the dual space of F. The characteristic function of Z is defined by $\varphi_Z(f) = E[\exp(iZf)]$ for all $f \in F^*$. In many cases, we will choose $F = L^p(\mathcal{O}, \zeta)$ for some $p \geqslant 1$ and an arbitrary locally finite Borel measure ζ. In this case $F^* = L^{p'}(\mathcal{O}, \zeta)$ for $p' := \frac{p}{p-1}$.

Definition 5.1. *A family $(L(t) : t \geqslant 0)$ of cylindrical random variables $L(t)\colon F^* \to L^0(\Omega, P)$ is called a* cylindrical Lévy process *if for all $f_1, ..., f_n \in F^*$ and $n \in \mathbb{N}$, the stochastic process $((L(t)f_1, ..., L(t)f_n) : t \geqslant 0)$ is a Lévy process in $\mathbb{R}^n$.*

The characteristic function of a cylindrical Lévy process $(L(t) : t \geqslant 0)$ is given by

$$\varphi_{L(t)}\colon F^* \to \mathbb{C}, \qquad \varphi_{L(t)}(f) = \exp\big(t\Psi_L(f)\big),$$

for all $t \geqslant 0$. Here, $\Psi_L\colon F^* \to \mathbb{C}$ is called the (cylindrical) symbol of L, and is of the form

$$\Psi_L(f) = ia(f) - \tfrac{1}{2}\langle f, Qf\rangle + \int_F \Big(e^{i\langle g,f\rangle} - 1 - i\langle g, f\rangle\, \mathbb{1}_{B_{\mathbb{R}}}(\langle g, f\rangle)\Big)\, \mu(\mathrm{d}g),$$

where $a\colon F^* \to \mathbb{R}$ is a continuous mapping with $a(0) = 0$, the mapping $Q\colon F^* \to F^{**}$ is a positive, symmetric operator and μ is a finitely additive measure on $\mathcal{Z}(F)$

satisfying

$$\int_F \big(\langle g,f\rangle^2 \wedge 1\big)\, \mu(\mathrm{d}g) < \infty \qquad \text{for all } f \in F^*.$$

Here, $\mathcal{Z}(F)$ is the algebra of all sets of the form $\{g \in F : (\langle g, f_1\rangle, \dots, \langle g, f_n\rangle) \in B\}$ for some $f_1, \dots, f_n \in F^*$, $B \in \mathscr{B}(\mathbb{R}^n\backslash\{0\})$ and $n \in \mathbb{N}$. We call (a, Q, μ) the *(cylindrical) characteristics of L*.

Theorem 5.2. *Let M be a Lévy-valued random measure on $\mathscr{B}_b(\mathcal{O})$ with characteristics (γ, Σ, ν) and control measure λ. If F is a Banach space for which F^* is continuously embedded into $L_M(\mathcal{O}, \lambda)$, and the simple functions are dense in F^*, then*

$$L(t)f := \int_{\mathcal{O}} f(x)\, M(t, \mathrm{d}x) \qquad \textit{for all } f \in F^*, \tag{5.1}$$

defines a cylindrical Lévy processes L in F. In this case, the characteristics (a, Q, μ) of L is given by

$$\begin{aligned} a(f) &= \int_{\mathcal{O}} f(x)\, \gamma(\mathrm{d}x) + \int_{\mathcal{O}\times\mathbb{R}} f(x)y\big(\mathbb{1}_{B_\mathbb{R}}(f(x)y) - \mathbb{1}_{B_\mathbb{R}}(y)\big)\, \nu(\mathrm{d}x, \mathrm{d}y), \\ \langle Qf, f\rangle &= \int_{\mathcal{O}} (f(x))^2\, \Sigma(\mathrm{d}x), \qquad\qquad \mu \circ \langle f, \cdot\rangle^{-1} = \nu \circ \chi_f^{-1}, \end{aligned}$$

for each $f \in F^*$, where $\chi_f \colon \mathcal{O} \times \mathbb{R} \to \mathbb{R}$ is defined by $\chi_f(x, y) := f(x)y$.

Proof. Lemma 4.2 shows that L is a cylindrical Lévy process in F. The claimed characteristics follows from (4.6) after rearranging the terms accordingly. □

The integration theory developed in [36] and briefly recalled in Section 4 guarantees that (5.1) is well defined for every $f \in L_M$. However, in order to be in the framework of cylindrical Lévy processes we need that the domain of $L(t)$ is the dual of a Banach space (or alternatively of a nuclear space). Since the Musielak-Orlicz space L_M is not in general the dual of a Banach space, for the hypothesis of Theorem 5.2 we require the existence of the Banach space F with F^* continuously embedded in L_M. If the control measure λ of M is finite on $\mathcal{O}$, then Lemma 4.3 gives us that $L^2(\mathcal{O}, \lambda)$ is continuously embedded in $L_M(\mathcal{O}, \lambda)$. It is possible as illustrated in the following example to relax the condition on finiteness of λ, but also the same example shows that there are cases where the finiteness of λ is necessary for any L^p space to be continuously embedded.

Example 5.3. We return again to Example 2.6; let M be the α-stable random measure for some $\alpha \in (0, 2)$, where now we consider the domain of definition to be

$\mathscr{B}_b(\mathcal{O})$ for a general $\mathcal{O} \in \mathscr{B}(\mathbb{R}^d)$. We consider the symmetric case $p = q = \frac{1}{2}$, where the characteristics of M is given by $(0, 0, \text{leb}\otimes\nu_\alpha)$ and the control measure by $\lambda(A) = \frac{2}{2-\alpha}\text{leb}(A)$, $A \in \mathscr{B}(\mathcal{O})$. One calculates from (4.4) that $L_M(\mathcal{O}, \lambda) = L^\alpha(\mathcal{O}, \text{leb})$; see [5, Lemma 4].

Thus, if $\alpha \in (1, 2)$ then we can always choose $F = L^{\alpha'}(\mathcal{O}, \text{leb})$. If $\alpha \in (0, 1]$ and $\mathcal{O}$ is bounded we can choose $F = L^p(\mathcal{O}, \text{leb})$ for any $p > 1$ since $|f(x)|^\alpha \leqslant 1 + |f(x)|^{p'}$. However, if $\text{leb}(\mathcal{O}) = \infty$ and $\alpha \leqslant 1$ then no L^p space is embedded in L_M for $p > 1$.

Assume $\alpha \in (1, 2)$. Then Theorem 5.2 implies that (5.1) defines a cylindrical Lévy process L in $F = L^{\alpha'}(\mathcal{O}, \text{leb})$, and its symbol is given by

$$\begin{aligned}\Psi_L(f) = i \int_{\mathcal{O}\times\mathbb{R}} & f(x)y\big(\mathbb{1}_{B_\mathbb{R}}(f(x)y) - \mathbb{1}_{B_\mathbb{R}}(y)\big)\,\mathrm{d}x\nu_\alpha(\mathrm{d}y) \\ &+ \int_\mathbb{R} \big(e^{iu} - 1 - iu\,\mathbb{1}_{B_\mathbb{R}}(u)\big)\ (\mu \circ \langle f, \cdot\rangle^{-1})(\mathrm{d}u) \\ = \int_{\mathcal{O}\times\mathbb{R}} & \Big(e^{if(x)y} - 1 - if(x)y\mathbb{1}_{B_\mathbb{R}}(y)\Big)\,\mathrm{d}x\nu_\alpha(\mathrm{d}y) = -C_\alpha \|f\|^\alpha_{L^\alpha(\mathcal{O},\text{leb})},\end{aligned}$$

where $C_\alpha = \frac{\Gamma(2-\alpha)}{1-\alpha}\cos\frac{\pi\alpha}{2}$ if $\alpha \neq 1$ and $C_\alpha = \frac{\pi}{2}$ if $\alpha = 1$.

We now turn to the question of which cylindrical Lévy processes induce Lévy-valued random measures. For this purpose we introduce the following:

Definition 5.4. *A cylindrical Lévy process* $(L(t) : t \geqslant 0)$ *in* $L^p(\mathcal{O}, \zeta)$ *for some* $p \geqslant 1$ *is called* independently scattered *if for any disjoint sets* $A_1, \dots, A_n \in \mathscr{B}_b(\mathcal{O})$ *and* $n \in \mathbb{N}$, *the random variables* $L(t)\mathbb{1}_{A_1}, \dots, L(t)\mathbb{1}_{A_n}$ *are independent for each* $t \geqslant 0$.

Theorem 5.5. *An independently scattered cylindrical Lévy process* $(L(t) : t \geqslant 0)$ *in* $L^p(\mathcal{O}, \zeta)$ *for some* $p \geqslant 1$ *defines by*

$$M(t, A) := L(t)\mathbb{1}_A \qquad \text{for all } t \geqslant 0, A \in \mathscr{B}_b(\mathcal{O}), \tag{5.2}$$

a Lévy-valued random measure M *on* $\mathscr{B}_b(\mathcal{O})$.

Proof. For each $t \geqslant 0$, the map $M(t, \cdot)\colon \mathscr{B}_b(\mathcal{O}) \to L^0(\Omega, P)$ is well-defined and $M(t, A)$ is an infinitely divisible random variable for each $A \in \mathscr{B}_b(\mathcal{O})$. Let $(A_k)_{k\in\mathbb{N}}$ be a sequence of disjoint sets in $\mathscr{B}_b(\mathcal{O})$ such that $A := \bigcup_{k\in\mathbb{N}} A_k \in \mathscr{B}_b(\mathcal{O})$. Then, for each $t \geqslant 0$, by the linearity and continuity of $L(t)$ we have

$$M(t, A) = \lim_{n\to\infty} L(t)\mathbb{1}_{\bigcup_{k=1}^n A_k} = \lim_{n\to\infty}\sum_{k=1}^n L(t)\mathbb{1}_{A_k} = \lim_{n\to\infty}\sum_{k=1}^n M(t, A_k),$$

with the limit in probability and thus almost surely by independence. Clearly, $M(t, \cdot)$ is independently scattered for each $t \geqslant 0$, and $\big(M(\cdot, A_1), \dots, M(\cdot, A_n)\big)$ is a Lévy process for each $A_1, \dots A_n \in \mathscr{B}_b(\mathcal{O})$. ∎

Theorem 5.6. *Let $(L(t) : t \geqslant 0)$ be a cylindrical Lévy process in $L^p(\mathcal{O}, \zeta)$ for some $p \geqslant 1$. Then L is independently scattered if and only if its symbol is of the form*

$$\begin{aligned}\Psi_L(f) = i \int_{\mathcal{O}} f(x)\,\gamma(\mathrm{d}x) - \tfrac{1}{2} \int_{\mathcal{O}} f^2(x)\,\Sigma(\mathrm{d}x) \\ + \int_{\mathcal{O}\times\mathbb{R}} \left(e^{if(x)y} - 1 - if(x)y\mathbb{1}_{B_{\mathbb{R}}}(y)\right) \nu(\mathrm{d}x, \mathrm{d}y)\,, \quad f \in L^{p'}(\mathcal{O}, \zeta),\end{aligned} \tag{5.3}$$

for a signed measure γ on $\mathscr{B}_b(\mathcal{O})$, a measure Σ on $\mathscr{B}_b(\mathcal{O})$ and a σ-finite measure ν on $\mathscr{B}(\mathcal{O} \times \mathbb{R})$ such that for each $B \in \mathscr{B}_b(\mathcal{O})$, $\nu(B \times \cdot)$ is a Lévy measure on $\mathbb{R}$.

Proof. If L is independently scattered then Theorem 5.5 implies that L defines a Lévy-valued random measure M by (5.2). Denote the characteristics of M by (γ, Σ, ν) and its control measure by λ. For a simple function f of the form (4.1) we obtain

$$L(t)(\mathbb{1}_A f) = \sum_{i=1}^n \alpha_i L(t)(\mathbb{1}_{A_i}\,\mathbb{1}_A) = \sum_{i=1}^n \alpha_i M(t, A_i \cap A) = \int_A f(x)\,M(t, \mathrm{d}x)\,. \tag{5.4}$$

For an arbitrary function $f \in L^{p'}(\mathcal{O}, \zeta)$ let $(f_n)_{n\in\mathbb{N}}$ be a sequence of simple functions converging to f both pointwise ζ-almost everywhere and in $L^{p'}(\mathcal{O}, \zeta)$. We note that, as $L(t)\,\mathbb{1}_A = 0$ whenever $\zeta(A) = 0$, ζ-null sets have null λ-measure, and thus we have $f_n \to f$ pointwise λ-almost everywhere. Since $L(t)\,\mathbb{1}_A\,f_n \to L(t)\,\mathbb{1}_A\,f$ in probability for each $A \in \mathscr{B}(\mathcal{O})$, it follows from (5.4) that $f \in L_M(\mathcal{O}, \lambda)$ and $L(t)f = \int_{\mathcal{O}} f(x)\,M(t, \mathrm{d}x)$. We obtain the stated form of the characteristic function of L by (4.6).

Conversely, if the Lévy symbol is given by (5.3), then this form implies for any disjoint sets $A_1, \dots, A_n \in \mathscr{B}_b(\mathcal{O})$ that

$$\Psi_L\left(\sum_{k=1}^n u_k\,\mathbb{1}_{A_k}\right) = \sum_{k=1}^n \Psi_L(u_k\,\mathbb{1}_{A_k}) \qquad \text{for all } u_1, \dots, u_n \in \mathbb{R}.$$

Consequently, we obtain for the characteristic function of the random vector $X := (L(1)\,\mathbb{1}_{A_1}, \dots, L(1)\,\mathbb{1}_{A_n})$ for all $u = (u_1, \dots u_n) \in \mathbb{R}^n$, that

$$\begin{aligned}\varphi_X(u) = \varphi_{L(1)}(u_1\mathbb{1}_{A_1} + \cdots + u_n\mathbb{1}_{A_n}) &= e^{i\Psi_L(u_1\mathbb{1}_{A_1}+\cdots+u_n\mathbb{1}_{A_n})} \\ &= \varphi_{L(1)\mathbb{1}_{A_1}}(u_1)\cdots\varphi_{L(1)\mathbb{1}_{A_n}}(u_n),\end{aligned}$$

which shows that L is independently scattered. □

Applying Theorem 5.5 to a given cylindrical Lévy process L on $L^p(\mathcal{O}, \zeta)$ gives the corresponding Lévy-valued random measure M, say with control measure λ.

The first part of the proof of Theorem 5.6 shows that $L^{p'}(\mathcal{O},\zeta)$ is a subspace of $L_M(\mathcal{O},\lambda)$. The following result guarantees that the embedding is continuous in non-degenerated cases.

Proposition 5.7. *Let L be an independently scattered cylindrical Lévy process in $L^p(\mathcal{O},\zeta)$ for some $p \geqslant 1$ with symbol of the form (5.3) and M the corresponding Lévy-valued random measure with control measure λ. If the measures γ, Σ and ν are such that for each $A \in \mathscr{B}_b(\mathcal{O})$ with $\Sigma(A) = 0$ and $\nu(A \times B) = 0$ for each $B \in \mathscr{B}(\mathbb{R})$ bounded away from 0, we have $\|\gamma\|_{\mathrm{TV}}(A) = 0$, then $L^{p'}(\mathcal{O},\zeta)$ is continuously embedded into $L_M(\mathcal{O},\lambda)$.*

Proof. By the first part of the proof of Theorem 5.6 we have $L^{p'}(\mathcal{O},\zeta) \subseteq L_M(\mathcal{O},\lambda)$, and, furthermore, the canonical injection $\iota\colon L^{p'}(\mathcal{O},\zeta) \to L_M(\mathcal{O},\lambda)$ is well defined, as the ζ-equivalence class of f is a subset of the λ-equivalence class of f. For each $t \geqslant 0$ we consider the operator $J(t)\colon L_M(\mathcal{O},\lambda) \to L^0(\Omega, P)$ defined in (4.5) and we see that $L(t)$ satisfies the factorisation $L(t) = J(t) \circ \iota$.

For establishing $\ker(J(t)) = \{0\}$, let $f \in L_M(\mathcal{O},\lambda)$ satisfy $J(t)f = 0$. Then, by considering only the real part of the characteristic function of $J(t)f$, we have for every $u \in \mathbb{R}$

$$-\tfrac{1}{2}u^2 \int_{\mathcal{O}} f^2(x)\,\Sigma(\mathrm{d}x) + \int_{\mathcal{O}\times\mathbb{R}} \big(\cos(uf(x)y) - 1\big)\,\nu(\mathrm{d}x,\mathrm{d}y) = 0.$$

As both terms are non-positive, we obtain that $f = 0$ Σ-a.e. and the function $z(x,y) := f(x)y$ satisfies $z = 0$ ν-a.e. In particular, for the set $A := \{x \in \mathcal{O} : f(x) \neq 0\}$ we have $\Sigma(A) = 0$ and $\nu(A \times B) = 0$ for any $B \in \mathscr{B}(\mathbb{R})$ bounded away from 0. The hypothesis on γ thus leads to $\lambda(A) = 0$, which shows $\ker(J(t)) = \{0\}$.

Let (f_n) be a sequence in $L^{p'}(\mathcal{O},\zeta)$ converging to $f \in L^{p'}(\mathcal{O},\zeta)$ and assume that ιf_n converges to some $g \in L_M(\mathcal{O},\lambda)$. As $\lim_{n\to\infty} J(t)(\iota f_n) = J(t)g$ and $\lim_{n\to\infty} L(t)f_n = L(t)f = J(t)(\iota f)$, we derive $J(t)(g - \iota f) = 0$. Since $J(t)$ is injective, we conclude $g = \iota f$ λ-a.e., and the closed graph theorem implies the continuity of ι. □

Example 5.8. Peszat and Zabczyk in [34, Section 7.2] define the impulsive cylindrical process in $L^2(\mathcal{O},\mathscr{B}(\mathcal{O}),\zeta)$ by

$$L(t)f := \int_0^t \int_{\mathcal{O}} \int_{\mathbb{R}} f(x)y\,\widetilde{N}(\mathrm{d}s,\mathrm{d}x,\mathrm{d}y),$$

where N is a Poisson random measure on $\mathbb{R}_+ \times \mathcal{O} \times \mathbb{R}$ with intensity $\mathrm{leb}\otimes\zeta\otimes\mu$ for a Lévy measure μ on $\mathscr{B}(\mathbb{R})$; see also [2, Ex. 3.6]. Since its symbol is given by

$$\Psi_L(f) = \int_{\mathcal{O}} \int_{\mathbb{R}} [e^{if(x)y} - 1 - if(x)y]\,\mu(\mathrm{d}y)\zeta(\mathrm{d}x),$$

Theorem 5.6 guarantees that L is independently scattered.

Finally, we note that the class of independently scattered cylindrical Lévy processes is a strict subclass, as the following counter-example shows:

Example 5.9. Let $(\ell_k)_{k\in\mathbb{N}}$ be a sequence of independent, identically distributed, real-valued Lévy processes. Assume for simplicity that ℓ_k is symmetric with characteristics $(0,0,\bar{\nu})$, with $\bar{\nu}\neq 0$, and satisfies $E[|\ell_k(1)|^2]<\infty$. Let $(e_k)_{k\in\mathbb{N}}$ be an orthonormal basis of $L^2((0,1),\mathrm{leb})$ such that $e_1\equiv 1$ (such bases include the standard polynomial and trigonometric bases). It follows from Lemma 4.2 in [37] that

$$L(t)f := \sum_{k=1}^{\infty} \langle f, e_k\rangle \ell_k(t), \qquad \text{for all } f\in L^2((0,1),\mathrm{leb}),$$

defines a cylindrical Lévy process L, say with characteristics $(0,0,\mu)$.

Assume for a contradiction that L is independently scattered and fix two disjoint sets $A,B\in\mathscr{B}((0,1))$ with $\mathrm{leb}(A)>0$ and $\mathrm{leb}(B)>0$. Thus, $\langle \mathbb{1}_A, e_1\rangle = \mathrm{leb}(A)>0$ and $\langle \mathbb{1}_B, e_1\rangle = \mathrm{leb}(B)>0$.

The Lévy measure of the Lévy process $\big((L(t)\,\mathbb{1}_A, L(t)\,\mathbb{1}_B) : t\geqslant 0\big)$ in $\mathbb{R}^2$ is given by $\mu\circ\pi^{-1}_{\mathbb{1}_A,\mathbb{1}_B}$. As $L(1)\,\mathbb{1}_A$ and $L(1)\,\mathbb{1}_B$ are independent, it follows from the uniqueness of the characteristic functions that

$$\mu\circ\pi^{-1}_{\mathbb{1}_A,\mathbb{1}_B} = ((\mu\circ\pi^{-1}_{\mathbb{1}_A})\otimes\delta_0) + (\delta_0\otimes\mu\circ\pi^{-1}_{\mathbb{1}_B})),$$

where $\mu\circ\pi^{-1}_{\mathbb{1}_A}$ is the Lévy measure of $(L(t)\,\mathbb{1}_A : t\geqslant 0)$ and $\mu\circ\pi^{-1}_{\mathbb{1}_B}$ is the Lévy measure of $(L(t)\,\mathbb{1}_B : t\geqslant 0)$. It follows in particular that

$$\mu\circ\pi^{-1}_{\mathbb{1}_A,\mathbb{1}_B}\big(\mathbb{R}\backslash\{0\}\times\mathbb{R}\backslash\{0\}\big) = 0. \tag{5.5}$$

On the other hand, Lemma 4.2 in [37] implies that

$$\mu\circ\pi^{-1}_{\mathbb{1}_A,\mathbb{1}_B} = \sum_{k=1}^{\infty}(\bar{\nu}\circ r_k^{-1}),$$

where $r_k\colon\mathbb{R}\to\mathbb{R}^2$ is defined by $r_k(x) = \big(\langle\mathbb{1}_A, e_k\rangle x, \langle\mathbb{1}_B, e_k\rangle x\big)$. It follows from (5.5) that

$$0 = \big(\bar{\nu}\circ r_1^{-1}\big)\big(\mathbb{R}\backslash\{0\}\times\mathbb{R}\backslash\{0\}\big) = \bar{\nu}(\mathbb{R}\backslash\{0\}) > 0\,,$$

which results in a contradiction.

6 Weak derivative of a Lévy-valued random measure

In this last section we establish the relation between a Lévy-valued random measure and a Lévy-valued additive sheet. For this purpose, we introduce a stochastic integral of deterministic functions $f\colon \mathbb{R}^d \to \mathbb{R}$ with respect to a Lévy-valued additive sheet. Instead of following the standard approach starting with simple functions and extending the integral operator by continuity, we utilise the correspondence between Lévy-valued additive sheets and Lévy valued random measures, established in Theorem 3.5, and refer to the integration for the latter developed in Rajput and Rosinksi [36] as presented in Section 4. For a Lévy-valued additive sheet $(X(t,x) : t \geqslant 0,\, x \in \mathbb{R}^d)$ let M denote the corresponding Lévy-valued random measure on $\mathscr{B}_b(\mathbb{R}^d)$ with control measure λ. Then we define for all $f \in L_M(\mathbb{R}^d, \lambda)$, $A \in \mathscr{B}(\mathbb{R}^d)$ and $t \geqslant 0$:

$$\int_A f(x)\,\mathrm{d}X(t,x) := \int_A f(x)\,M(t,\mathrm{d}x). \tag{6.1}$$

Let $(X(t,x) : x \in \mathbb{R}^d,\, t \geqslant 0)$ be a Lévy-valued additive sheet and $\mathcal{O} \subseteq \mathbb{R}^d$ be open. Then the definition in (6.1) allows us to define the same operator $J_{\mathcal{D}}(t)$ as introduced in (4.7) for a Lévy-valued additive sheet X:

$$J_{\mathcal{D}}(t)\colon \mathcal{D}(\mathcal{O}) \to L^0(\Omega, P), \quad J_{\mathcal{D}}(t)f = \int_{\mathcal{O}} f(x)\,\mathrm{d}X(t,x). \tag{6.2}$$

Theorem 4.1 guarantees that $J_{\mathcal{D}}$ is well-defined and even more, induces a genuine Lévy process Y in $\mathcal{D}^*(\mathcal{O})$. We define the operator

$$I_{\mathcal{D}}(t)\colon \mathcal{D}(\mathcal{O}) \to L^0(\Omega, P), \qquad I_{\mathcal{D}}(t)(f) = \int_{\mathcal{O}} f(x)X(t,x)\,\mathrm{d}x\,. \tag{6.3}$$

The mapping $I_{\mathcal{D}}$ is well defined because of the lamp property of $X(t,\cdot)$ for each $t \geqslant 0$ and as each $f \in \mathcal{D}(\mathcal{O})$ has compact support in $\mathcal{O}$. Lebesgue's dominated convergence theorem shows that $I_{\mathcal{D}}$ is continuous, as every convergent sequence in $\mathcal{D}(\mathcal{O})$ is uniformly bounded and compactly supported.

The following establishes the relation

$$(-1)^d I_{\mathcal{D}}(t)(\dot{f}) = J_{\mathcal{D}}(t)(f) \qquad \text{for all } f \in \mathcal{D}(\mathcal{O}).$$

In other words, if we neglect the embedding by the operators $I_{\mathcal{D}}$ and $J_{\mathcal{D}}$, we could interpret this result that M is the weak derivative of X. This is not very surprising, since, if we adapt notions from classical measure theory, the relation $M(t,(0,x]) = X(t,x)$ derived in Theorem 3.5, can be seen that X is the cumulative distribution function of the random measure M.

Theorem 6.1. *For a Lévy-valued additive sheet* $(X(t,\cdot) : t \geqslant 0)$ *and an open set* $\mathcal{O} \subseteq \mathbb{R}^d$ *let* $I_{\mathcal{D}}$ *be defined by* (6.3)*. Then there exists a stochastic process* $(V(t) : t \geqslant 0)$ *in* $\mathcal{D}^*(\mathcal{O})$ *satisfying*

$$\langle f, V(t)\rangle = I_{\mathcal{D}}(t)f \qquad \text{for all } f \in \mathcal{D}(\mathcal{O}),\ t \geqslant 0.$$

Furthermore, we have the equality

$$(-1)^d I_{\mathcal{D}}(t)(\dot{f}) = J_{\mathcal{D}}(t)(f) \qquad \text{for all } f \in \mathcal{D}(\mathcal{O})\,, \tag{6.4}$$

where $\dot{f} = \frac{\partial^d}{\partial x_1 \cdots \partial x_d} f$ *and* $J_{\mathcal{D}}(t)$ *denotes the operator in* (6.2)*.*

Proof. We show that, for each $f \in \mathcal{D}(\mathcal{O})$, the process $(I_{\mathcal{D}}(t)f : t \geqslant 0)$ has a càdlàg modification. First we consider a sequence (t_n) decreasing monotonically to some $t \geqslant 0$. Let K be the support of f. Then, as (t_n) is bounded, there exists a $C > 0$ such that $t_n \in [t, t + C]$ for each n. The lamp property of X implies that X is bounded on the compact set $[t, t + C] \times K$. Thus, since $X(t_n, x)$ converges to $X(t, x)$ in probability for each $x \in \mathcal{O}$, Lebesgue's dominated convergence theorem (for a stochastically convergent sequence) implies

$$\lim_{n\to\infty} \int_{\mathcal{O}} f(x)X(t_n, x)\,\mathrm{d}x = \int_{\mathcal{O}} f(x)X(t, x)\,\mathrm{d}x.$$

A similar argument establishes that the left limits exists.

The existence of the stochastic process V follows from Theorem 3.2 in [21] (as $\mathcal{D}(\mathcal{O})$ is nuclear [41, Theorem 51.5] and ultrabornological [32, Page 447]).

To show (6.4) we use ideas from [14]. By the fundamental theorem of calculus, as f has compact support,

$$f(x) = (-1)^d \int_{\mathcal{O}} \dot{f}(y)\mathbb{1}_{\{y \geqslant x\}}\,\mathrm{d}y \qquad \text{for all } x \in \mathcal{O}.$$

By utilising an analogue of Fubini's theorem for Lévy-valued random measures, as detailed below, we obtain

$$\begin{aligned} J_{\mathcal{D}}(t)f = \int_{\mathcal{O}} f(x)\,X(t,\mathrm{d}x) &= \int_{\mathcal{O}} \left((-1)^d \int_{\mathcal{O}} \dot{f}(y)\mathbb{1}_{\{y \geqslant x\}}\,\mathrm{d}y \right) X(t,\mathrm{d}x) \\ &= (-1)^d \int_{\mathcal{O}} \left(\int_{\mathcal{O}} \mathbb{1}_{\{y \geqslant x\}}\,X(t,\mathrm{d}x) \right) \dot{f}(y)\,\mathrm{d}y \\ &= (-1)^d \int_{\mathcal{O}} X(t,y)\dot{f}(y)\,\mathrm{d}y\,. \end{aligned}$$

We now show the analogue of Fubini's theorem to complete the proof. Let M denote the Lévy-valued random measure corresponding to X according to Theorem 3.5. The Lévy-Itô decomposition (3.1) yields that M admits the decomposition

$$M(t,A) = t\gamma(A) + G(t,A) + M_c(t,A) + M_p(t,A) \quad \text{for all } A \in \mathscr{B}_b(\mathbb{R}^d) \text{ and } t \geqslant 0.$$

Here, γ is a signed measure, G is a pure Gaussian Lévy-valued random measure with characteristics $(0,\Sigma,0)$, and

$$M_c(t,A) := \int_0^t \int_{A\times B_{\mathbb{R}}} y\, \widetilde{N}(\mathrm{d}s,\mathrm{d}x,\mathrm{d}y), \qquad M_p(t,A) := \int_0^t \int_{A\times B_{\mathbb{R}}^c} y\, N(\mathrm{d}s,\mathrm{d}x,\mathrm{d}y).$$

The classic Fubini theorem may be applied to γ. The Lévy-valued random measure M_p is a finite random sum and the Fubini result holds trivially.

For G and the compensated Poisson Lévy-valued random measure M_c we apply Theorem 2.6 in [42]. We note that $(G(t,\cdot) + M_c(t,\cdot) : t \geqslant 0)$ forms a martingale-valued measure. Furthermore, $G + M_c$ is orthogonal by the independence of the processes $(G(t,A) : t \geqslant 0)$, $(M_c(t,A) : t \geqslant 0)$, $(M_g(t,B) : t \geqslant 0)$ and $(M_c(t,B) : t \geqslant 0)$ whenever $A, B \in \mathscr{B}_b(\mathbb{R}^d)$ are disjoint. The covariance process is given by $Q_t(A,B) = t\big(\Sigma(A\cap B) + \int_{(A\cap B)\times B_{\mathbb{R}}} |y|^2\, \nu(\mathrm{d}x,\mathrm{d}y)\big)$. As the required dominating measure K in [42, Theorem 2.6] one can choose $K(A\times B\times(0,t]) = t\lambda(A\cap B)$. The required integrability condition follows as f is compactly supported and bounded. □

Remark 6.2. According to Proposition 3.4, a Lévy-valued additive sheet X defines a natural additive sheet $(X(t,x) : t \geqslant 0,\, x \in \mathbb{R}^d)$. Due to its lamp trajectories, we can define the mapping

$$I'_{\mathcal{D}} \colon \mathcal{D}((0,\infty)\times\mathbb{R}^d) \to L^0(\Omega,P), \qquad I'_{\mathcal{D}}(f) = \int_{(0,\infty)\times\mathbb{R}^d} f(t,x)X(t,x)\,\mathrm{d}t\,\mathrm{d}x.$$

On the other side, one can conclude as in Theorem 3.5 or by [33, Theorem 4.1], that there exists an infinitely divisible random measure M' on $\mathscr{B}_b(\mathbb{R}_+\times\mathbb{R}^d)$ satisfying $M((0,z]) = X(z)$ for all $z \in \mathbb{R}_+\times\mathbb{R}^d$. Thus, as in Remark 4.6, we can define

$$J'_{\mathcal{D}} \colon \mathcal{D}((0,\infty)\times\mathbb{R}^d) \to L^0(\Omega,P), \quad J'_{\mathcal{D}}(g) = \int_{(0,\infty)} \int_{\mathbb{R}^d} g(t,x)\, M'(\mathrm{d}t,\mathrm{d}x).$$

One can conclude as in the proof of Theorem 6.1 that there exists a genuine random variable W in $\mathcal{D}^*((0,\infty)\times\mathbb{R}^d)$ satisfying

$$\langle g, W\rangle = I'_{\mathcal{D}}(g) \qquad \text{for all } g \in \mathcal{D}((0,\infty)\times\mathbb{R}^d).$$

Furthermore, we have the equality

$$(-1)^d I'_{\mathcal{D}}(\dot{g}) = J'_{\mathcal{D}}(g) \qquad \text{for all } g \in \mathcal{D}((0,\infty)\times\mathbb{R}^d).$$

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