On interpolation of reflexive variable Lebesgue spaces on which the Hardy-Littlewood maximal operator is bounded

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To Professor Stefan Samko on the occasion of his 80th birthday

Abstract. We show that if the Hardy-Littlewood maximal operator \( M \) is bounded on a reflexive variable exponent space \( L^{p(\cdot)}(\mathbb{R}^d) \), then for every \( q \in (1, \infty) \), the exponent \( p(\cdot) \) admits, for all sufficiently small \( \theta > 0 \), the representation \( 1/p(x) = \theta/q + (1 - \theta)/r(x) \), \( x \in \mathbb{R}^d \) such that the operator \( M \) is bounded on the variable Lebesgue space \( L^{r(\cdot)}(\mathbb{R}^d) \). This result can be applied for transferring properties like compactness of linear operators from standard Lebesgue spaces to variable Lebesgue spaces by using interpolation techniques.

1. Introduction

Let \( L^0(\mathbb{R}^d) \) denote the space of all (equivalence classes of) Lebesgue measurable complex-valued functions on \( \mathbb{R}^d \) with the topology of convergence in measure on sets of finite measure. Let \( p(\cdot) : \mathbb{R}^d \rightarrow [1, \infty] \) be a measurable a.e. finite function. By \( L^{p(\cdot)}(\mathbb{R}^d) \) we denote the set of all functions \( f \in L^0(\mathbb{R}^d) \) such that

\[
I_{p(\cdot)}(f/\lambda) := \int_{\mathbb{R}^d} |f(x)/\lambda|^{p(x)}dx < \infty
\]

for some \( \lambda > 0 \). This set becomes a Banach space when equipped with the Luxemburg-Nakano norm

\[
\|f\|_{p(\cdot)} := \inf \{ \lambda > 0 : I_{p(\cdot)}(f/\lambda) \leq 1 \}.
\]

It is easy to see that if \( p(\cdot) = p \) is constant, then \( L^{p(\cdot)}(\mathbb{R}^d) \) is nothing but the standard Lebesgue space \( L^p(\mathbb{R}^d) \). The space \( L^{p(\cdot)}(\mathbb{R}^d) \) is referred to as a variable Lebesgue space.

Let \( 1 \leq q < \infty \). Given \( f \in L^q_{\text{loc}}(\mathbb{R}^d) \), the \( q \)-th maximal operator is defined by

\[
(M_qf)(x) := \sup_{Q \ni x} \left( \frac{1}{|Q|} \int_Q |f(y)|^qdy \right)^{1/q},
\]

where the supremum is taken over all cubes \( Q \subset \mathbb{R}^d \) containing \( x \) (here, and throughout, cubes will be assumed to have their sides parallel to the coordinate axes). Note that \( M := M_1 \) is the usual Hardy-Littlewood maximal operator. By
\( \mathcal{B}_M(\mathbb{R}^d) \) denote the set of all measurable a.e. finite functions \( p(\cdot) : \mathbb{R}^d \to [1, \infty] \) such that the Hardy-Littlewood maximal operator is bounded on \( L^{p(\cdot)}(\mathbb{R}^d) \).

We will use the following standard notation:

\[
p_- := \text{ess inf}_{x \in \mathbb{R}^d} p(x), \quad p_+ := \text{ess sup}_{x \in \mathbb{R}^d} p(x).
\]

It is well known that the space \( L^{p(\cdot)}(\mathbb{R}^d) \) is reflexive if and only if \( 1 < p_- \leq p_+ < \infty \). In this case, its dual space is isomorphic to \( L^{p'(\cdot)}(\mathbb{R}^d) \), where

\[
\frac{1}{p(x)} + \frac{1}{p'(x)} = 1, \quad x \in \mathbb{R}^d
\]

(see, e.g., [5 Chap. 3]).

Suppose that \( 1 < p_- \leq p_+ < \infty \) and there exist constants \( c_0, c_\infty \in (0, \infty) \) and \( p_\infty \in (1, \infty) \) such that

\[
|p(x) - p(y)| \leq \frac{c_0}{\log(e + 1/|x - y|)}, \quad x, y \in \mathbb{R}^d, \tag{1.1}
\]

\[
|p(x) - p_\infty| \leq \frac{c_\infty}{\log(e + |x|)}, \quad x \in \mathbb{R}^d. \tag{1.2}
\]

Then \( p(\cdot) \in \mathcal{B}_M(\mathbb{R}^d) \) (see [3 Theorem 3.16] or [5 Theorem 4.3.8]). Following [5 Section 4.1] or [3 Section 2.1], we will say that \( p(\cdot) \) is globally log-Hölder continuous if conditions (1.1)–(1.2) are satisfied. The class of all globally log-Hölder continuous exponents will be denoted by \( \mathcal{P}^{\text{log}}(\mathbb{R}^d) \).

Conditions (1.1) and (1.2) are optimal for the boundedness of \( M \) in the sense of modulus of continuity; the corresponding examples are contained in [16] and [2]. However, neither (1.1) nor (1.2) is necessary for \( p(\cdot) \in \mathcal{B}_M(\mathbb{R}^d) \). Thus

\[\mathcal{P}^{\text{log}}(\mathbb{R}^d) \subsetneq \mathcal{B}_M(\mathbb{R}^d)\]

Here we mention results by Nekvinda [14, 15] and Lerner [13] and further discussion in the monographs [3 Chap. 4] and [5 Chaps. 4–5].

The following result was obtained in a somewhat more complete form by the first author (see [4 Theorem 8.1] or [5 Theorem 5.7.2]).

**Theorem 1.1.** Let \( p(\cdot) : \mathbb{R}^d \to [1, \infty] \) be a measurable function satisfying \( 1 < p_- \leq p_+ < \infty \). The following statements are equivalent:

(a) \( M \) is bounded on \( L^{p(\cdot)}(\mathbb{R}^d) \);
(b) \( M \) is bounded on \( L^{p'(\cdot)}(\mathbb{R}^d) \);
(c) there exists an \( s \in (1/p_-, 1) \) such that \( M \) is bounded on \( L^{sp(\cdot)}(\mathbb{R}^d) \);
(d) there exists a \( q \in (1, \infty) \) such that \( M_q \) is bounded on \( L^{p'(\cdot)}(\mathbb{R}^d) \).

Rabinovich and Samko [17] (see also [12 Section 9.1.2]) observed that if a variable exponent \( p(\cdot) \in \mathcal{P}^{\text{log}}(\mathbb{R}^d) \) satisfies \( 1 < p_- \leq p_+ < \infty \), then it can be decomposed as

\[
\frac{1}{p(x)} = \frac{\theta}{2} + \frac{1 - \theta}{r(x)}, \quad x \in \mathbb{R}^d, \tag{1.3}
\]

where \( 0 < \theta < 1 \) and \( r(\cdot) \) satisfies \( 1 < r_- \leq r_+ < \infty \) and belongs to \( \mathcal{P}^{\text{log}}(\mathbb{R}^d) \). This observation was important in the “transfer of the compactness techniques” from \( L^2(\mathbb{R}^d) \) to \( L^{p(\cdot)}(\mathbb{R}^d) \) by means of the one-sided interpolation of the compactness property between the spaces \( L^{r(\cdot)}(\mathbb{R}^d) \) (where an operator is merely bounded) and \( L^2(\mathbb{R}^d) \) (where an operator is compact).
The second author and Spitkovsky [11] exploited this idea for more general variable exponents \( p(\cdot) \in \mathcal{B}_M(\mathbb{R}^d) \) based on the following result obtained by the first author and published in [11] Theorem 4.1.

**Theorem 1.2.** Let \( p(\cdot) : \mathbb{R}^d \to [1, \infty] \) be a measurable function satisfying \( 1 < p_- \leq p_+ < \infty \). If \( p(\cdot) \in \mathcal{B}_M(\mathbb{R}^d) \), then there exist numbers \( q \in (1, \infty) \) and \( \theta \in (0, 1) \) such that the variable exponent \( r(\cdot) \) defined by

\[
\frac{1}{p(x)} = \frac{\theta}{q} + \frac{1 - \theta}{r(x)}, \quad x \in \mathbb{R}^d,
\]

(1.4)

belongs to \( \mathcal{B}_M(\mathbb{R}^d) \).

The above theorem has the disadvantage that the constant \( q \) depends on the variable exponent \( p(\cdot) \). It is desirable to avoid such a dependence and to find, for a given \( p(\cdot) \in \mathcal{B}_M(\mathbb{R}^d) \), a \( \theta \in (0, 1) \) such that [1,3] holds and \( r(\cdot) \) belongs to \( \mathcal{B}_M(\mathbb{R}^d) \). This would allow one to simplify formulations of several results in the literature, where it was supposed that \( p(\cdot) \) is of the form [1,3] with \( r(\cdot) \in \mathcal{B}_M(\mathbb{R}^d) \) and some sufficiently small \( \theta \in (0, 1) \) (see, e.g., [7 Corollary 2.1, Theorem 3.2], [8 Theorem 1.2], [9 Theorem 2.1]). The second author asked in [10 Section 4.4] whether for a given \( p(\cdot) \in \mathcal{B}_M(\mathbb{R}^d) \), satisfying \( 1 < p_- \leq p_+ < \infty \), one can find a number \( \tau_{p(\cdot)} \in (0, 1] \) such that the variable exponent \( r(\cdot) \) defined by [1,3] belongs to \( \mathcal{B}_M(\mathbb{R}^d) \) for every \( \theta \in (0, \tau_{p(\cdot)}) \).

Our main result is the following refinement of Theorem 1.2 which gives positive answers to the above questions.

**Theorem 1.3 (Main result).** Let \( p(\cdot) : \mathbb{R}^d \to [1, \infty] \) be a measurable function satisfying \( 1 < p_- \leq p_+ < \infty \). Then \( p(\cdot) \in \mathcal{B}_M(\mathbb{R}^d) \) if and only if for every \( q \in (1, \infty) \), there exists a number \( \Theta_{p(\cdot), q} \in (0, 1) \) such that for every \( \theta \in (0, \Theta_{p(\cdot), q}] \) the variable exponent \( r(\cdot) \) defined by [1,4] belongs to \( \mathcal{B}_M(\mathbb{R}^d) \).

Note that representation [1,4] implies that \( 0 < \theta/q \leq 1/p(x) \leq \theta/q + 1 - \theta < 1 \) for \( \theta > 0 \), whence \( 1 < p_- \leq p_+ < \infty \).

The paper is organized as follows. In Section 2, we formulate an interpolation lemma due to Cruz-Uribe [11], which immediately implies the proof of the sufficiency portion of Theorem 1.3. In Section 3, we show that if \( p(\cdot) \in \mathcal{B}_M(\mathbb{R}^d) \) satisfies \( 1 < p_- \leq p_+ < \infty \), then the variable exponents \( \left( \frac{1}{r(x)} \right)^{\frac{1}{s}} \) belong to \( \mathcal{B}_M(\mathbb{R}^d) \) for all \( s, t \geq 1 \) sufficiently close to 1. Based on this result, we complete the proof of the necessity portion of Theorem 1.3 in Section 4.

**2. Proof of the sufficiency portion of Theorem 1.3.**

The sufficiency portion is an immediate corollary of the following result obtained by Cruz-Uribe [1 Corollary 3] (see also [6 Corollary 2.5] for the case \( 1 < (p_j)_- \leq (p_j)_+ < \infty, j = 0, 1 \)) and the boundedness of the Hardy-Littlewood maximal operator \( M \) on the standard Lebesgue space \( L^q(\mathbb{R}^d) \) with \( q \in (1, \infty) \).

**Lemma 2.1.** If \( p_i(\cdot) \in \mathcal{B}_M(\mathbb{R}^d) \) for \( i = 0, 1 \), then for every \( \theta \in (0, 1) \), the variable exponent \( p_\theta(\cdot) \) defined by

\[
\frac{1}{p_\theta(x)} = \frac{\theta}{p_0(x)} + \frac{1 - \theta}{p_1(x)}, \quad x \in \mathbb{R}^d,
\]

(2.1)
belongs to $\mathcal{B}_M(\mathbb{R}^d)$ and
\[
\|M\|_{L^{p_1}(\cdot) \rightarrow L^{p_3}(\cdot)} \leq 96\|M\|_{L^{p_1}(\cdot) \rightarrow L^{p_2}(\cdot)}^\theta \|M\|_{L^{p_2}(\cdot) \rightarrow L^{p_1}(\cdot)}^{1-\theta}, \tag{2.2}
\]

Note that inequality (2.2) is stated in [11] with the constant 48, which seems to be a typo. This result was obtained as a consequence of the pointwise inequality $|T_f| \leq Mf \leq 2T_f|f|$, where each $T_f$ is a linear integral operator with a positive kernel. On the other hand, it was shown in [11] Theorem 1] that if $T$ is a linear integral operator with a positive kernel that satisfies $\|Tf\|_{p_i(\cdot)} \leq B_i\|f\|_{p_i(\cdot)}$ for $i = 0, 1$ and all $f \in L^{p_i(\cdot)}(\mathbb{R}^d)$ with $B_i$ independent of $f$, then
\[
\|Tf\|_{p_0(\cdot)} \leq 48B_0^\theta B_1^{1-\theta}\|f\|_{p_0(\cdot)}.
\]

3. Doubly iterated “left-openness and then duality” trick

Our construction is similar to that of the proof of [11] Theorem 4.1. It is based on the consecutive application of the “left-openness” of the class $\mathcal{B}_M(\mathbb{R}^d)$ (see Theorem 1.1(c)) and then the “duality” of the class $\mathcal{B}_M(\mathbb{R}^d)$ (see Theorem 1.1(b)).

In order to succeed, we repeat this procedure two times. The main novelty is that we can guarantee that the constructed exponents belong to $\mathcal{B}_M(\mathbb{R}^d)$ in certain ranges of parameters.

**Lemma 3.1.** If $p(\cdot) \in \mathcal{B}_M(\mathbb{R}^d)$ satisfies $1 < p_+ \leq p_0 < \infty$, then there exist $s_0, t_0 \in (1, \infty)$ such that
\[
\left(\frac{1}{t} \left(\frac{p(\cdot)}{s}\right)\right)' \in \mathcal{B}_M(\mathbb{R}^d) \quad \text{for all} \quad s \in [1, s_0], \quad t \in [1, t_0]. \tag{3.1}
\]

**Proof.** By Theorem 1.1(c), there exists a number $s_0 \in (1, \infty)$ such that $p(\cdot)/s_0 \in \mathcal{B}_M(\mathbb{R}^d)$. Then it follows from Theorem 1.1(b) that $p'(\cdot)$ and $(p(\cdot)/s_0)'$ belong to $\mathcal{B}_M(\mathbb{R}^d)$. Applying Theorem 1.1(c) once again, we see that there exist $t_1, t_2 \in (1, \infty)$ such that
\[
\frac{p'(\cdot)}{t_1} \in \mathcal{B}_M(\mathbb{R}^d), \quad \frac{1}{t_2} \left(\frac{p(\cdot)}{s_0}\right)' \in \mathcal{B}_M(\mathbb{R}^d).
\]
It follows from Lemma 2.1 (one can employ also a more elementary argument using Jensen’s inequality as in [6 p. 43]) that
\[
\frac{p'(\cdot)}{t} \in \mathcal{B}_M(\mathbb{R}^d) \quad \text{for all} \quad t \in [1, t_1], \quad \frac{1}{t} \left(\frac{p(\cdot)}{s_0}\right)' \in \mathcal{B}_M(\mathbb{R}^d) \quad \text{for all} \quad t \in [1, t_2].
\]
Put
\[t_0 := \min\{t_1, t_2\}.
\]
Then it is clear that $t_0 \in (1, \infty)$ and
\[
\frac{p'(\cdot)}{t} \in \mathcal{B}_M(\mathbb{R}^d), \quad \frac{1}{t} \left(\frac{p(\cdot)}{s_0}\right)' \in \mathcal{B}_M(\mathbb{R}^d) \quad \text{for all} \quad t \in [1, t_0]. \tag{3.2}
\]
Take any $s \in [1, s_0]$ and set $\theta := \frac{s_0 - s}{s_0 - 1} \in [0, 1]$. Then $s = \theta + (1 - \theta)s_0$. Further, for $x \in \mathbb{R}^d$, we have
\[
\left(\frac{1}{t} \left(\frac{p(x)}{s}\right)\right)^{-1} = t \left(\frac{p(x)/s}{p(x)/s - 1}\right)^{-1} = t \frac{p(x) - s}{p(x)}.
\]
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\[ tp(x)(\theta + (1 - \theta)) - (\theta + (1 - \theta) s_0) \]

\[ = t \theta p(x) - 1 + t(1 - \theta) p(x) - s_0 \]

\[ = \theta \left( \frac{p'(x)}{t} \right)^{-1} + (1 - \theta) \left( \frac{1}{t} \left( \frac{p(x)}{s_0} \right) \right)^{-1}. \] (3.3)

It follows from (3.2)–(3.3) and Lemma 2.1 that

\[ \frac{1}{t} \left( \frac{p(\cdot)}{s} \right)' \in B_M(\mathbb{R}^d) \quad \text{for all} \quad s \in [1, s_0], \quad t \in [1, t_0]. \]

Applying Theorem 1.1(b) one more time, one arrives at (3.1).

4. Proof of the necessity portion of Theorem 1.3

Suppose that \( q \in (1, \infty) \) and \( p(\cdot) \) satisfies \( 1 < p_- \leq p_+ < \infty \) and belongs to \( B_M(\mathbb{R}^d) \). We need to prove that \( r(\cdot) \) defined by (1.4) belongs to \( B_M(\mathbb{R}^d) \) for all sufficiently small positive values of \( \theta \). We will show that one can choose \( s \) and \( t \) in such a way that \( r(\cdot) = (\frac{1}{t} \left( \frac{p(\cdot)}{s} \right)')' \), and use Lemma 3.1 to conclude that \( r(\cdot) \in B_M(\mathbb{R}^d) \).

Now, (1.4) is equivalent to

\[ \frac{1}{r(x)} = \frac{1}{1 - \theta p(x)} - \frac{\theta}{1 - \theta q}, \quad x \in \mathbb{R}^d, \]

while

\[ \frac{1}{(\frac{1}{t} \left( \frac{p(\cdot)}{s} \right)')'} = 1 - t \left( 1 - \frac{s}{p(x)} \right) = st \frac{1}{p(x)} - (t - 1), \quad x \in \mathbb{R}^d. \]

So, we need to take \( s \) and \( t \) such that

\[ st = \frac{1}{1 - \theta} \quad \text{and} \quad t - 1 = \frac{\theta}{1 - \theta q}. \]

An easy calculation shows that these equations are equivalent to

\[ \theta = 1 - \frac{1}{st} \quad \text{and} \quad t = \frac{q - 1}{q - s}. \]

Let \( s_0 \in (1, \infty) \) and \( t_0 \in (1, \infty) \) be such that (3.1) holds. Put

\[ t(s) := \frac{q - 1}{q - s}, \quad 1 < s < q. \] (4.1)

Since \( 1 < t(s) \to 1 \) as \( s \to 1 \), there exists \( s_1 \in (1, s_0] \) such that

\[ 1 < t(s) \leq t_0 \quad \text{for all} \quad s \in (1, s_1], \]

Let

\[ \theta(s) := 1 - \frac{1}{st(s)} = 1 - \frac{q - s}{s(q - 1)} = \frac{q(s - 1)}{s(q - 1)} = \frac{q}{q - 1} \left( 1 - \frac{1}{s} \right). \] (4.2)

Then \( 0 < \theta(s) \to 0 \) as \( s \to 1 \). So, there exists \( s_2 \in (1, s_1] \) such that

\[ \Theta_{p(\cdot), q} := \theta(s_2) \in (0, 1). \]

It is clear from (4.2) that \( \theta(\cdot) \) is an increasing continuous function. Then

\[ \theta((1, s_2)] = (0, \Theta_{p(\cdot), q}]. \]
Take any $\theta \in (0, \Theta_{p(\cdot),q}]$. It follows from the above that there exists a unique $s \in (1, s_2]$ such that $\theta(s) = \theta$. For this $s$ and $t := t(s)$,

$$r_\theta(\cdot) := \left( \frac{1}{t} \left( \frac{p(\cdot)}{s} \right) \right)' \in \mathcal{B}_M(\mathbb{R}^d)$$

(4.3)

according to (3.1).

It follows from (4.1) that

$$q = \frac{st - 1}{t - 1}.$$  

(4.4)

Combining (4.2), (4.3), (4.4), we get for $x \in \mathbb{R}^d$,

$$\frac{\theta}{q} + \frac{1 - \theta}{r_\theta(x)} = \frac{\theta}{q} + (1 - \theta) \left( 1 - \frac{1}{t} \left( \frac{p(x)}{s} \right) \right)$$

$$= \left( 1 - \frac{1}{st} \right) \frac{t - 1}{st - 1} + \frac{1}{st} \left( 1 - t \left( 1 - \frac{s}{p(x)} \right) \right)$$

$$= \frac{t - 1}{st} + \frac{1}{st} (1 - t) + \frac{1}{st} \cdot \frac{ts}{p(x)}$$

$$= \frac{1}{p(x)}.$$  

Thus $r_\theta(\cdot)$ satisfies (1.4) for every $\theta \in (0, \Theta_{p(\cdot),q}]$. \qed

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