The structure of local Galois deformation rings

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A thesis presented for the degree of
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To Susheela Iyengar and Ranganayaki Padmanabhan, in loving memory.

Abstract

This thesis consists of two closely related projects concerning the structure of the universal lifting ring of a mod p representation of the Galois group of a p-adic local field. The first project, originally written in the published paper [Iye20], is the sole work of the author and focuses only on the trivial mod p representation. Under some mild conditions on p and the dimension, we prove that the universal lifting ring is a complete intersection of expected dimension with normal generic fibre and classify its irreducible components. The second project, originally written in the arXiv preprint [BIP21], was conceived and completed jointly with Gebhard Böckle and Vytautas Paškūnas. It strengthens and generalizes the previous result to all mod p Galois representations, unconditionally. The first project was completed before the second project was started, and though there is some overlap in ideas between the two projects, the methods used are substantially different.

Some terminology and notation has been slightly changed from [Iye20] and [BIP21] for internal consistency and readability of this thesis. The mathematical content is the same, except for some minor corrections to the published paper [Iye20].

Contents

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Chapter 1

Introduction

This thesis is about the deformation theory of local mod p Galois representations. One of the basic motivations for this theory is that it addresses the question:

in how many ways can you lift a mod p Galois representation to characteristic 0?

To answer this question the theory (originally due to Mazur in [Maz89b]) provides an object called the "universal lifting ring" which parametrizes all possible lifts simultaneously in a way that highlights the *variation* in the space of lifts.

1.1 Universal lifting ring

Let's recall the definition of this object. There are many conflicting conventions and notations in the literature, but we choose the following setup for this introduction. Fix p a prime number and $L/\mathbb{Q}_p$ a finite extension with ring of integers $\mathcal{O} \subset L$, uniformizer $\varpi \in \mathcal{O}$ and residue field $k := \mathcal{O}/\varpi$. Then for an arbitrary finite extension $K/\mathbb{Q}_p$ with absolute Galois group G_K (equipped with its usual profinite topology) we consider a continuous representation $\overline{\rho} : G_K \to \mathrm{GL}_d(k)$. The "lifting" in the question above refers to the problem of describing continuous representations $\rho : G_K \to \mathrm{GL}_d(\mathcal{O})$ which reduce to $\overline{\rho}$ under the natural reduction map $\mathcal{O} \to k$.

As previously mentioned, we describe such ρ by trying to find the universal family parametrizing all possible ρ. The usual algebro-geometric way to vary things in families is to create parameter spaces by turning constants into variables and then imposing relations between them as the situation prescribes. This allows one to pick out a member of the family by specializing the variables to a list of allowable numbers, specifically ones which satisfy the relations. Since for now we care only about formal deformation theory, we add these variables in the context of formal schemes and this leads us to consider continuous representations of the form

$$\rho : G_K \to \mathrm{GL}_d(\mathcal{O}[\![x_1, \ldots, x_r]\!]/(f_1, \ldots, f_s))$$

which lift $\overline{\rho}$ via the map $\mathcal{O}[\![x_1, \ldots, x_r]\!]/(f_1, \ldots, f_s) \to k$ sending $\varpi, x_1, \ldots, x_r \mapsto 0$.

The ring of coefficients above is exactly the form of an arbitrary complete local Noetherian $\mathcal{O}$-algebra with residue field k. So we can equivalently ask:

is there a universal lift of $\overline{\rho}$ to complete local Noetherian $\mathcal{O}$-algebras with residue field k?

The answer is yes. The following theorem is essentially due to Mazur in [Maz89b], although in this precise form it was first written down by Kisin in [Kis09a].

Theorem 1.1.0.1. *There is a complete local Noetherian ring $R^{\square}_{\overline{\rho}}$ (called the* ***universal lifting ring****) with residue field k and a continuous homomorphism $\rho^{\square} : G_K \to \mathrm{GL}_d(R^{\square}_{\overline{\rho}})$ such that for every complete local Noetherian ring A with residue field k[1] and every continuous homomorphism $\rho : G_K \to \mathrm{GL}_d(A)$ lifting $\overline{\rho}$, there exists a unique local $\mathcal{O}$-algebra map $f_\rho : R^{\square}_{\overline{\rho}} \to A$ such that ρ factors as*

[1]Strictly speaking, when we say a "complete local Noetherian ring A *with residue field* k" we are implicitly fixing an isomorphism $A/\mathfrak{m}_A \xrightarrow{\sim} k$, and all maps should be compatible with these isomorphisms.

$$\begin{array}{ccc} G_K & \xrightarrow{\quad\quad\rho\quad\quad} & \mathrm{GL}_d(A) \\ & {\scriptstyle \rho^{\square}}\searrow \qquad \nearrow{\scriptstyle \mathrm{GL}_d(f_\rho)} & \\ & \mathrm{GL}_d(R^{\square}_{\bar{\rho}}) & \end{array}$$

Thus $R^{\square}_{\bar{\rho}}$ parametrizes the universal family we were looking for.

Remark 1.1.0.2. The above theorem holds more generally for any profinite group G satisfying the technical condition that there are only finitely many continuous group homomorphisms $G \to \mathbb{Z}/p\mathbb{Z}$. We focus only on the groups G_K (for varying K), which satisfy this technical condition in view of local class field theory.

The main results of this thesis concerns the ring-theoretic properties of $R^{\square}_{\bar{\rho}}$. The reader should feel free to skip to Section 1.4 to see the statements.

1.2 Pseudorepresentations

In the second chapter of the thesis, we investigate $R^{\square}_{\bar{\rho}}$ in certain cases by working with representations themselves, while in the third chapter we perform a long-winded reduction to the theory of *pseudorepresentations* and apply work of Böckle–Juschka [BJ19] on the structure of deformations rings for pseudorepresentations. For the reader unfamiliar with such objects, we provide a history of their development here.

A motivating question:

to what extent is giving a representation the same as giving its traces?

In a first course on representation theory we learn that an irreducible representation ρ of a finite group with coefficients in an algebraically closed field E of characteristic 0 is uniquely determined by its associated *character* $\chi_\rho : g \mapsto \operatorname{tr} \rho(g)$, and that the characters form an orthonormal basis of the space of class functions on the group. This was generalized by

Helling in 1974 in [Hel74], where he proves that if G is *any* group[2] and $\chi : G \to E$ is a class function satisfying $\chi(1_G) = d$ and a certain permutation identity then there exists a finite dimensional representation $\rho : G \to \mathrm{GL}_d(E)$ satisfying $\chi(g) = \operatorname{tr}\rho(g)$ for all $g \in G$. At the risk of using one word to mean two things, we say that ρ "lifts" χ.

Wiles [Wil88] (for $d = 2$) and Taylor [Tay91] (for arbitrary d, generalizing Wiles's work) rediscovered Helling's result. They called class functions with values in an arbitrary ring satisfying $\chi(1_G) = d$ and Helling's permutation relation *d-dimensional pseudorepresentations*, and showed that every d-dimensional pseudorepresentation over an algebraically closed field of characteristic 0 lifts to a d-dimensional representation. They also showed that if G is a topological group, the coefficient field is a topological field and the pseudorepresentation is continuous then the representation lifting it is also continuous. While Wiles's proof is a short and explicit construction, Taylor's more general result relies on work of Procesi [Pro76] on the invariant theory of $n \times n$ matrices. Wiles and Taylor then used pseudorepresentations to construct Galois representations associated to Hilbert and Siegel modular forms, respectively, via congruences.

Rouquier [Rou96] proved the same results for d-dimensional representations when E is an algebraically closed field and $d! \in E^\times$, which weakens the requirement that E has characteristic 0. Using Roby's [Rob63] *polynomial laws* Chenevier [Che14] made a new definition of a pseudorepresentation valued in an arbitrary ring which he called a "determinant"[3], and extended the lifting result to an arbitrary algebraically closed field of any characteristic. Chenevier's determinants, which we will henceforth refer to as pseudorepresentations[4] by abuse of terminology, are equivalent to Taylor's over $\mathbb{Q}$-algebras, but differ in general.

[2] Helling actually proves a more general result for traces of representations of arbitrary associative algebras over E.

[3] This name is chosen due to the fact that a determinant keeps track of the whole characteristic polynomial of a representation in all cases, rather than just the trace.

[4] In the arXiv preprint [BIP21] we refer to pseudorepresentations as pseudo-characters, but we have decided to change it in this thesis.

In *loc. cit.* Chenevier also shows that his pseudorepresentations admit a good deformation theory, which will be relevant for us. If G is a profinite group and $\overline{\rho} : G \to \mathrm{GL}_d(k)$ is a continuous representation with associated pseudorepresentation $\overline{D}$, then he proves the existence of a complete local Noetherian $\mathcal{O}$-algebra $R^{\mathrm{ps}}_{\overline{D}}$ called the *pseudodeformation ring*, parametrizing lifts of $\overline{D}$ to complete local Noetherian $\mathcal{O}$-algebras in a manner completely analogous to Theorem 1.1.0.1. Any deformation of $\overline{\rho}$ maps to a deformation of $\overline{D}$, so there is always a map

$$R^{\mathrm{ps}}_{\overline{D}} \to R^{\square}_{\overline{\rho}},$$

and part of this thesis involves understanding this map in more detail; this is the "long-winded reduction" mentioned at the beginning of this section.

Remark 1.2.0.1. Although we do not use it in this work, Lafforgue [Laf18] has given a new definition of pseudorepresentations for arbitrary reductive groups as part of his work on the global Langlands correspondence for function fields.

1.3 Connection with the Emerton–Gee stack

The universal lifting ring of $\overline{\rho}$ parametrizes lifts of $\overline{\rho}$ from k to $\mathcal{O}$ in families. One can ask whether the $\overline{\rho}$ themselves also vary in families.

A first attempt might be to step back and consider the moduli space of representations[5] of a finitely presented group G with no topology. Given a presentation $G = \langle g_1, \ldots, g_a : r_1(g_1, \ldots, g_a), \ldots, r_b(g_1, \ldots, g_a)\rangle$, the moduli space of d-dimensional representations of G is simply the moduli space of tuples $(M_1, \ldots, M_a) \in \mathrm{GL}_d$ satisfying $r_i(M_1, \ldots, M_a) = \mathrm{id}_d$ for $i = 1, \ldots, b$. This moduli space is represented by a finite type affine scheme over $\operatorname{Spec} \mathbb{Z}$; one can fairly easily write out $\mathbb{Z}$-algebra generators for each matrix entry and mod out by the matrix equations coming from the r_i. So there is no essential obstacle to varying

[5] Strictly speaking we should say representations with a fixed basis, since we consider homomorphisms $G \to \mathrm{GL}_d(R)$, rather than homomorphisms up to conjugacy.

these kinds of representations in families; the only restrictions present are imposed by the relations defining the group, and they obstruct variation in families only to the extent that they obstruct the existence of the representations themselves.

But things become more subtle when one adds in a topology; see [EG20, Example 1.1.1] for a concrete example of the kind of issue that arises when considering even just unramified characters. In this case, the representations themselves exist, but they don't fit together in an algebraic family. A more general version of this issue is described in [Wan18, Section 3.1], which we recount now.

A result of Chenevier [Che14, Corollary 3.14] (see also [Wan13, Theorem 3.1.6.13]) shows that the functor (on admissible $\mathcal{O}$-algebras c.f. [Gro60, Chap. 0, 7.1.2])

$$\begin{aligned}\mathsf{PsR}^d_{G_K} &: \mathsf{Adm}_{\mathcal{O}} \to \mathsf{Set}\\ &\operatorname{Spf} A \mapsto \{\text{continuous pseudorepresentations } \mathcal{O}[\![G_K]\!] \otimes_{\mathcal{O}} A \to A \text{ of dimension } d\}\end{aligned}$$

is represented by an affine formal scheme and breaks up as a disjoint union

$$\mathsf{PsR}^d_{G_K} = \coprod_{\overline{D}} \operatorname{Spf} R^{\mathrm{ps}}_{\overline{D}}$$

where $\overline{D}$ runs over all distinct d-dimensional residual k-valued pseudorepresentations of G_K. The functor

$$\begin{aligned}\mathcal{R}\mathsf{ep}^{\square,d}_{G_K} &: \mathsf{Adm}_{\mathcal{O}} \to \mathsf{Set}\\ &\operatorname{Spf} A \mapsto \{\text{continuous representations } G_K \to \mathrm{GL}_d(A)\}\end{aligned}$$

admits a natural map $\mathcal{R}\mathsf{ep}^{\square,d}_{G_K} \to \mathsf{PsR}^d_{G_K}$, and thus we get a decomposition

$$\mathcal{R}\mathsf{ep}^{\square,d}_{G_K} = \coprod_{\overline{D}} \mathcal{R}\mathsf{ep}^{\square}_{G_K,\overline{D}}$$

where $\mathcal{R}\mathsf{ep}^{\square}_{G_K,\overline{D}} := \mathcal{R}\mathsf{ep}^{\square,d}_{G_K} \times_{\mathsf{PsR}^d_{G_K}} \operatorname{Spf} R^{\mathrm{ps}}_{\overline{D}}$. Furthermore, $\mathcal{R}\mathsf{ep}^{\square}_{G_K,\overline{D}}$ is represented by an affine formal scheme [Wan13, Theorem 3.2.4.1]. This is proved using an *algebraization* of $\mathcal{R}\mathsf{ep}^{\square}_{G_K,\overline{D}}$ defined using Cayley–Hamilton algebras; this algebraization will be important for us in Chapter 4, see Section 2.2.4 for a further discussion. Recall from Section 1.2 that any mod p representation $\overline{\rho}$ gives rise to a pseudorepresentation $\overline{D}$, and if $\overline{\rho}$ is absolutely irreducible then $\mathcal{R}\mathsf{ep}^{\square}_{G_K,\overline{D}} \cong \operatorname{Spf} R^{\square}_{\overline{D}}$.

The takeaway is that mod p pseudorepresentations don't vary in an algebraic family, and therefore the only mod p representations that one should expect to vary are ones that have the same residual pseudorepresentation, i.e. the same semisimplification. In particular absolutely irreducible mod p representations don't vary in an algebraic family.

On the other hand, Emerton and Gee in [EG19] fix this problem by embedding the category of mod p and p-adic representations (and more generally, representations with coefficients in a p-adically complete $\mathcal{O}$-algebra) into the category of (φ, Γ_K)-modules (see [EG19, Sections 2 and 3] for precise definitions), and varying those in families instead. They prove that the moduli stack $\mathcal{X}_{K,d}$ of rank d étale (φ, Γ_K)-modules is a formal algebraic stack (in the sense of [Eme]), and moreover that $R^{\square}_{\overline{\rho}}$ is a versal ring to the point $\overline{\rho} \in \mathcal{X}_{K,d}(k)$. Thus an understanding of the ring-theoretic properties of $R^{\square}_{\overline{\rho}}$ helps to describe the geometry of $\mathcal{X}_{K,d}$.

Remark 1.3.0.1. Emerton–Gee use their stack to prove the existence of crystalline lifts of mod p Galois representations. Forthcoming work of Dotto–Emerton–Gee [DEG] will categorify the known p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$ over the Emerton–Gee stack.

1.4 Main results

From now on assume that L is "large enough", i.e. contains all the embeddings of K into $\overline{\mathbb{Q}}_p$. Let $\mu := \mu_{p^\infty}(K)$ denote the (finite) group of p-power roots of unity in K, and let $q = |\mu|$.

Theorem 1.4.0.1 (Proposition 3.1.2.2, Corollary 4.2.5.7)**.** *The ring* $R^{\square}_{\overline{\rho}}$ *is a complete intersection and flat over* $\mathcal{O}$ *of dimension* $1 + d^2 + d^2[K : \mathbb{Q}_p]$.

Remark 1.4.0.2. In Corollary 4.2.5.7, the representation $\overline{\rho}$ gives rise to a closed point $x_{\overline{\rho}} \in X^{\mathrm{gen}}$, so we take $\Lambda = \mathcal{O}$ and use $x = x_{\overline{\rho}}$ in the statement of that Corollary. See Section 1.6.1 for an explanation of what X^{gen} is.

Remark 1.4.0.3. The second chapter of this thesis proves Theorem 1.4.0.1 when $\overline{\rho}$ is the trivial representation under the assumption[6] that $q \neq 2$. This is a relatively quick argument and makes essential use of an explicit presentation of the maximal pro-p quotient of G_K in terms of Demuškin equations. The general case is the technical heart of the third chapter of this thesis and takes a different approach that avoids any explicit presentations of the Galois group.

As an immediate corollary any $\overline{\rho}$ admits a lift to characteristic 0. This result was first proven by Emerton–Gee [EG19] using their stack, but our methods avoid it completely.

The second result describes the irreducible components of $R^{\square}_{\overline{\rho}}$ as follows. Since the determinant of any lift of $\overline{\rho}$ is a lift of $\det \overline{\rho}$, we obtain a natural local $\mathcal{O}$-algebra map $R^{\square}_{\det \overline{\rho}} \to R^{\square}_{\overline{\rho}}$.

Theorem 1.4.0.4 (Theorem 3.3.0.1, Corollary 4.3.3.8)**.** *The map* $\operatorname{Spec} R^{\square}_{\overline{\rho}} \to \operatorname{Spec} R^{\square}_{\det \overline{\rho}}$ *induces a bijection on irreducible components.*

[6] There should be no fundamental obstacle to addressing the case $q = 2$, but we chose not to in [Iye20]. The only difference is that the equations defining the maximal pro-2-quotient are slightly different, but we don't expect the arguments to change significantly.

Theorem 1.4.0.4 completely answers [BJ19, Question 1.10].

Using local class field theory it's fairly easy to show that $R^{\square}_{\det \overline{\rho}} \cong \mathcal{O}[\mu][\![\mathbb{Z}_p^{[K:\mathbb{Q}_p]+1}]\!]$. The latter is isomorphic to $\mathcal{O}[\mu][\![x_1, \ldots, x_{[K:\mathbb{Q}_p]+1}]\!]$, and therefore the irreducible components of $\operatorname{Spec} R^{\square}_{\det \overline{\rho}}$ are in bijection with $\{\chi : \mu \to \mathcal{O}^\times\}$, so by Theorem 1.4.0.4 the same holds for $\operatorname{Spec} R^{\square}_{\overline{\rho}}$.

Remark 1.4.0.5. In the second chapter we prove Theorem 1.4.0.4 only for the trivial representation, again under the hypothesis that $q \neq 2$ and also under the assumption that $p > d$. The proof reduces to the generic fibre and then exploits the explicit presentation of the Galois group to connect points in the preimages of irreducible components under the map $\operatorname{Spec} R^{\square}_{\overline{\rho}} \to \operatorname{Spec} R^{\square}_{\det \overline{\rho}}$. Similarly to Theorem 1.4.0.1, the proof in the general case given in the third chapter avoids the use of an explicit presentation and instead hinges on the fact that a normal local ring is an integral domain, as we will describe below.

1.5 Previous results

Mazur's seminal paper [Maz89b] introduced the formal deformation theory of continuous representations of profinite groups. Mazur, in *loc. cit.* (see also [Gou01, Lecture 4]) originally considered the question of whether a deformation ring for representations of a *global field* is a complete intersection of expected dimension rather than local fields, noting that for $d = 1$ the question is equivalent to Leopoldt's conjecture. Much of the original (and current) interest in deformation rings was focused on global deformations with local conditions, particularly with a view towards modularity lifting theorems. For instance, this plays a key role in the proof of Fermat's last theorem by Wiles [Wil95] and Taylor–Wiles [TW95] and its subsequent generalizations by many mathematicians.

Local deformation rings have always played an important role in the theory, as it was understood early on that the relations defining global deformation rings with local conditions come from local obstruction classes. This led to the extensive study of p-adic Hodge-theoretic conditions on local deformation rings by Wiles, Ramakrishna, Kisin and others. However, this thesis concerns the structure of local deformation rings without any local conditions, so we give a brief summary of previous work on such rings. Böckle in [Böc00] proved Theorem 1.4.0.1 for $d = 2$ via rather explicit methods, and later extended this to a proof of Theorem 1.4.0.4 for $d = 2$ and $p > 2$ in joint work with Ann-Kristin Juschka [BJ15]. Meanwhile, [CDP15] proved Theorem 1.4.0.1 and Theorem 1.4.0.4 for $p = 2$, $K = \mathbb{Q}_2$ and $\overline{\rho}$ the trivial representation. These results were extended by Babnik [Bab19] to 2-dimensional $\overline{\rho}$ with scalar semisimplification for $K = \mathbb{Q}_2$. The second chapter of this thesis proves the two theorems when $q \neq 2$ and $p > d$, while the third chapter proves the two theorems unconditionally, subsuming all other previously known results.

1.6 Methods

Here we give a brief summary of the methods used in both chapters to prove Theorem 1.4.0.1 and Theorem 1.4.0.4. We address each theorem independently.

1.6.1 Complete intersection

Any $\rho : G_K \to \mathrm{GL}_d(A)$ lifting the trivial representation $\overline{\rho}$ must factor through G_K^p, the maximal pro-p quotient of G_K. Lemma 3.1.1.1 gives an explicit presentation for this group in terms of generators and relations and then the structure of the deformation ring is easily read off, see Corollary 3.1.1.5. Then in Proposition 3.1.2.2 we kill some generators in order to compare $R_{\overline{\rho}}^{\square}$ with a local ring of the moduli space of pairs of matrices M_1, M_2 satisfying $M_2 M_1 M_2^{-1} = M_1^{q+1}$. Helm [Hel16], building on work of Choi [Cho09], (which in turn builds on work of Taylor [Tay08]) shows that this moduli space of almost-commuting pairs is a complete intersection, so we're done.

The case where $\overline{\rho}$ is non-trivial is addressed in the third chapter. We take a more conceptual approach that avoids explicit presentations. Start by observing that tangent-obstruction theory yields a non-canonical presentation

$$R_{\overline{\rho}}^{\square} \cong \mathcal{O}[\![x_1, \ldots, x_r]\!]/(f_1, \ldots, f_s)$$

where $r = \dim_k Z^1(G_K, \operatorname{ad}\overline{\rho})$ and $s = \dim_k H^2(G_K, \operatorname{ad}\overline{\rho})$; here Z^1 denotes the space of continuous 1-cocycles, H^2 is the second continuous group cohomology, and $\operatorname{ad}\overline{\rho}$ is the adjoint representation. The local Euler characteristic formula and the above presentation together imply $\dim R_{\overline{\rho}}^{\square} \geq 1 + r - s = 1 + d^2 + d^2[K : \mathbb{Q}_p]$. In the context of Galois deformation theory quantity $1 + r - s$ defined using tangent-obstruction theory is referred to as the *expected dimension*. So to show that $R_{\overline{\rho}}^{\square}$ is a complete intersection of the expected dimension it suffices to show the opposite inequality $\dim R_{\overline{\rho}}^{\square} \leq 1 + d^2 + d^2[K : \mathbb{Q}_p]$.

For this, we perform an elaborate reduction to the theory of deformations of pseudorepresentations. If $\overline{D}$ is the corresponding pseudorepresentation let R^{ps} denote its pseudodeformation ring (here and in the third chapter we omit the subscript $\overline{D}$ from the pseudodeformation ring to avoid notational overload). Work of Böckle–Juschka in [BJ19] provides us with a fairly complete description of the ring-theoretic properties of R^{ps} so we attempt to transport their results along this map by studying the fibres of the map $R^{\mathrm{ps}} \to R_{\overline{\rho}}^{\square}$ defined in Section 1.2. But this map is only *formally* of finite type and not literally of finite type, so it doesn't quite lend itself to the kind of algebro-geometric techniques we want to apply.

On the other hand, there is a way to factorize this map so that we can apply lemmas from algebraic geometry as well as some GIT theory. A construction originally due to Procesi in [Pro87] yields an affine scheme $X^{\mathrm{gen}} := \operatorname{Spec} A^{\mathrm{gen}}$ which represents an *algebraization* of $\mathcal{R}\mathsf{ep}_{G_K,\overline{D}}^{\square}$ as defined in [Wan18, Theorem 3.8], and can be roughly thought of as the moduli of representations with residual pseudorepresentation $\overline{D}$ (the "gen" is short for "generic matrices"; see Section 4.2 for more details). The relevant facts are that (1) A^{gen} factors the above map as $R^{\mathrm{ps}} \to A^{\mathrm{gen}} \to R^{\square}_{\overline{\rho}}$, (2) A^{gen} is of finite type over R^{ps}, and (3) Wang-Erickson proves in [Wan18, Theorem 2.20] that $X^{\mathrm{gen}} /\!\!/ \mathrm{GL}_d \to \operatorname{Spec} R^{\mathrm{ps}}$ is an *adequate homeomorphism* in the sense of [Alp14], where the double slash denotes the GIT quotient. Recall that the GIT quotient is simply represented the ring of GL_d-invariants in A^{gen}, and parametrizes stable orbits for the action of GL_d on X^{gen}. As we show, stable orbits consist of semisimple representations; see Lemma 4.2.2.2 for a more precise statement. Since $R^{\square}_{\overline{\rho}}$ is the completion of A^{gen} at the closed point corresponding to $\overline{\rho}$, it's enough to bound the dimension of A^{gen}.

Combining the above facts with Böckle–Juschka's work on R^{ps} and more specifically its special fibre R^{ps}/ϖ, we are able to show:

Theorem 1.6.1.1 (Theorem 4.2.4.8, Lemma 4.2.3.7)**.** *We have*

$$\dim A^{\mathrm{gen}}[1/\varpi] \leq \dim A^{\mathrm{gen}}/\varpi \leq d^2 + d^2[F : \mathbb{Q}_p].$$

The first inequality is straightforward commutative algebra; to prove the second inequality we use Böckle–Juschka's dimension estimates on strata of R^{ps}/ϖ indexed by "reducibility type" of the pseudorepresentation along with some GIT theory developed in Section 4.1. The GIT theory is used to study the fibers X^{gen}_y over geometric points $y \in \operatorname{Spec} R^{\mathrm{ps}}$, which are finite type affine schemes over an algebraically closed field, and thus lend themselves to the techniques of [Ses77]. Finally, along with more routine commutative algebra, one can use Theorem 1.6.1.1 to prove Theorem 1.4.0.1.

1.6.2 Irreducible components

The natural map $\mathcal{O}[\mu] \to R_{\det \overline{\rho}} \to R^{\square}_{\overline{\rho}}$ gives rise to quotients $R^{\square,\chi}_{\overline{\rho}} = R^{\square}_{\overline{\rho}} \otimes_{\mathcal{O}[\mu],\chi} \mathcal{O}$, and we wish to show that they are integral domains.

When $\overline{\rho}$ is the trivial representation, we first show that $R^{\square}_{\overline{\rho}}[1/\varpi]$ is normal via Serre's criterion. We know that $R^{\square}_{\overline{\rho}}[1/\varpi]$ is Cohen–Macaulay because it is a complete intersection, so we just need to show that the dimension of the singular locus is small. Proposition 3.2.0.2 shows that the singular locus is contained in the reducible locus, and in Section 3.2.1 we bound the dimension of the reducible locus by generalizing (from $d = 2$ to $d = n$) and slightly re-purposing a method of Geraghty in his thesis [Ger10]. Note that there is a resemblance between Section 3.2.1 in the second chapter and Section 4.2.2 in the third; both follow Geraghty's method. More specifically, notice the similarities between Proposition 3.2.2.3 with Proposition 4.2.2.6.

Then $\mathcal{O}$-flatness of $R^{\square}_{\overline{\rho}}$ allows us to reduce to showing that the generic fibre $\operatorname{Spec} R^{\square,\chi}_{\overline{\rho}}[1/\varpi]$ is irreducible. The irreducible and connected components of $R^{\square}_{\overline{\rho}}[1/\varpi]$ coincide since it is normal, so we just need to show that $\operatorname{Spec} R^{\square,\chi}_{\overline{\rho}}[1/\varpi]$ (which is at least a *union* of connected components of $R^{\square}_{\overline{\rho}}[1/\varpi]$) is connected. For this we pick two closed points ρ_1, ρ_2, i.e. two p-adic representations lifting $\overline{\rho}$ and satisfying $(\det \rho_1 \circ \mathrm{rec}_K)|_{\mu} = (\det \rho_2 \circ \mathrm{rec}_K)|_{\mu}$, and then explicitly find a "path" between two points. The basic idea is to exhibit a map into $\operatorname{Spec} R^{\square}_{\overline{\rho}}[1/\varpi]$ from an irreducible scheme whose image contains ρ_1 and ρ_2, but there are some reduction steps involved. Namely, in Section 3.3 we introduce some auxiliary points between ρ_1 and ρ_2 and connect them via a chain of irreducible subschemes. As noted above, we need to further restrict to the case where $p > d$ to avoid some technical complications that arise in the course of the proof.

For general $\overline{\rho}$, we exploit the fact that a normal local ring is an integral domain, so it suffices to show that $R^{\square,\chi}_{\overline{\rho}}$ is normal; this is done in Section 4.3. We first split this problem into normality of $R^{\square,\chi}_{\overline{\rho}}[1/\varpi]$ and $R^{\square,\chi}_{\overline{\rho}}/\varpi$, and then apply Serre's criterion to the generic and special fibres. By adapting a method due to Kisin of presenting global deformation rings over local deformation rings, we give a presentation of $R^{\square}_{\overline{\rho}}$ over $R_{\det \overline{\rho}}$. After specializing at a character χ we deduce from this presentation that $R^{\square,\chi}_{\overline{\rho}}[1/\varpi]$ and $R^{\square,\chi}_{\overline{\rho}}/\varpi$ are complete intersections of expected dimension as well, and thus it remains to show regularity in codimension 1 in both. For the generic fibre we use the fact that the singular locus is contained in the reducible locus, and the same bounds on dimensions of reducibility strata used to bound $\dim R^{\square}_{\overline{\rho}}$ apply here. In the special fibre the singular locus may leave the reducible locus. However, the singular locus of the special fibre is contained in the *Kummer-reducible* locus; the point here is that even if a singular point is irreducible, it is induced from a degree p extension, and we can use this fact to bound the dimension of the Kummer-reducible locus. See Section 4.A for more details.

1.7 Applications

In Section 3.4 and in the preprint [BIP22], we use the classification of irreducible components of $R^{\square}_{\overline{\rho}}$ given in Theorem 1.4.0.4 to deduce the following result. Let $\mathcal{X}^{\square}_{\overline{\rho}} := (\operatorname{Spf} R^{\square}_{\overline{\rho}})^{\mathrm{rig}}$ denote the rigid analytic generic fibre.

Theorem 1.7.0.1 (Theorem 3.4.2.4, [BIP22]). *The set of crystalline points in $\mathcal{X}^{\square}_{\overline{\rho}}$ is Zariski dense*[7]*.*

The proof goes in two steps: first one shows that the Zariski closure of crystalline points is a union of some subset of the irreducible components. This is done in Section 3.4 using results of Breuil–Hellmann–Schraen [BHS17b] on the trianguline variety, but the argument is essentially the same as Nakamura's [Nak14], which is in turn a direct generalization of work of Chenevier [Che13] [Che14] which takes inspiration from Gouvêa–Mazur's infinite fern [Maz97a].

Then it remains to prove that each connected component $(\operatorname{Spf} R^{\square,\chi}_{\overline{\rho}})^{\mathrm{rig}}$ contains a crystalline point. In other words, we need to show that $\overline{\rho}$ admits a crystalline lift ρ satisfying $\det \rho|_{\mu} = \chi$

[7]We mean *Zariski dense* in the usual rigid-analytic sense, i.e. that every rigid analytic function on $\mathcal{X}^{\square}_{\overline{\rho}}$ which vanishes at each crystalline point is identically zero.

for each $\chi : \mu \to \mathcal{O}^\times$. When $\overline{\rho}$ is trivial, it is fairly easy to combinatorially find explicit crystalline lifts of a given determinant. If $\overline{\rho}$ is any irreducible representation then it is the induction of a character of $G_{K'}$ for some unramified extension K'/K, and crystalline characters are well understood so it's possible to construct lifts in a direct way. When $\overline{\rho}$ is reducible, we use the Emerton–Gee stack to inductively find lifts. At present, we cannot find a way around using their stack to treat the reducible case.

Remark 1.7.0.2. By twisting, Nakamura also shows that there is a crystalline point in each component, but only under the fairly restrictive conditions that $R_{\overline{\rho}}^{\square,\chi}$ is formally smooth for each $\chi : \mu \to \mathcal{O}^\times$ and that $\overline{\rho}$ is irreducible.

Zariski density makes it possible to first prove something holds for crystalline representations lifting $\overline{\rho}$ and then extend it to all representations. Colmez [Col10] and Kisin [Kis10] used this to show that every 2-dimensional irreducible p-adic representation of $G_{\mathbb{Q}_p}$ lies in the image of the Colmez's Montreal functor, which has implications for the p-adic Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$. We hope that our results can be used in the p-adic Langlands program in more general cases.

Theorem 1.4.0.1 implies (by [GV18, Lemma 7.5]) that the derived deformation ring of $\overline{\rho}$ as introduced by Galatius and Venkatesh in [GV18], see also [Cai21], is homotopy discrete, which means the derived deformation theory of $\overline{\rho}$ does not contain more information than the usual deformation theory of $\overline{\rho}$. Theorem 1.4.0.1 will be used in forthcoming work of Matthew Emerton, Toby Gee and Xinwen Zhu on derived moduli stacks of global Galois representations.

Chapter 2

Preliminaries

2.1 Notation

Here we list a few notational conventions and assumptions made throughout the paper. We will restate this setup in both chapters, but we collect this here in the hope that the reader will find it convenient.

- p is a fixed prime number.
- $K/\mathbb{Q}_p$ is a finite extension of $\mathbb{Q}_p$ of degree[1] $د = [K : \mathbb{Q}_p]$.
- $G_K := \mathrm{Gal}(\overline{K}/K)$ is the absolute Galois group of K equipped with its usual profinite topology.
- $L/\mathbb{Q}_p$ is a finite extension that plays the role of a base field of coefficients for representations. We assume that L contains every embedding of K into $\overline{\mathbb{Q}}_p$.
- $\mathcal{O} \subset L$ is the ring of integers.

[1] د is the Arabic letter *dāl*, which is the equivalent of the Latin 'd'.

- $k = \mathcal{O}/\mathfrak{m}$ is the residue field.

- $\overline{\rho} : G_K \to \mathrm{GL}_d(k)$ is a continuous representation. We assume throughout that L is large enough so that every irreducible subquotient of $\overline{\rho}$ is absolutely irreducible.

- $\mathsf{Art}_{\mathcal{O}}$ is the category whose objects are pairs $(A, \varphi_A : A \twoheadrightarrow k)$ where A is a local Artinian $\mathcal{O}$-algebra and φ_A is an identification of the residue field with k, and morphisms are maps $f : A \to B$ satisfying $\varphi_A = \varphi_B \circ f$. We will often omit φ_A from the notation and think of A as an Artinian $\mathcal{O}$-algebra "with residue field k", although strictly speaking the precise identification φ_A is necessary for representability of moduli functors.

- Art_F is the category whose objects are pairs $(A, \varphi_A : A \twoheadrightarrow F)$ where A is a local Artinian F-algebra and φ_A is an identification of the residue field with F, and morphisms are maps $f : A \to B$ satisfying $\varphi_A = \varphi_B \circ f$. We similarly think of A as an Artinian F-algebra "with residue field F".

- If R is a commutative ring then Alg_R is the category of commutative R-algebras.

2.2 Pseudorepresentations

In Chapter 4 we use pseudorepresentations in a crucial way, so we provide some basic definitions here following [Che14], [Wan13], and [Wan18] closely. We note that much of this theory is heavily based on mid-to-late-century work of Roby in [Rob63], but more complete attributions can be found in [Che14].

2.2.1 Polynomial laws

If A is a unital commutative ring and M is an A-module, define the functor

$$\begin{aligned} \underline{M} : \mathsf{Alg}_A &\to \mathsf{Set} \\ B &\mapsto M \otimes_A B. \end{aligned}$$

Definition 2.2.1.1 ([Rob63], [Che14, Section 1.1])**.** If M and N are two A-modules then an A-*polynomial law* (or just *polynomial law* if A is clear from context) is a natural transformation $P : \underline{M} \to \underline{N}$. We will often write $P : M \to N$ without the underline, with the implicit understanding that this does *not* refer to an A-module homomorphism.

Remark 2.2.1.2. For any $B \in \mathsf{Alg}_A$, it is easy to see that an A-polynomial law $P : M \to N$ naturally induces a B-polynomial law $P \otimes_A B : M \otimes_A B \to N \otimes_A B$.

By functoriality, a polynomial law $P : M \to N$ is determined by $P_{A[T_1,\ldots,T_n]}$ for all n. To see this, note that a tensor $\sum_{i=1}^n m_i \otimes b_i \in M \otimes_A B$ is equal to the image of $\sum_{i=1}^n m_i \otimes T_i$ under the map $A[T_1,\ldots,T_n] \xrightarrow{T_i \mapsto b_i} B$, so its image in $N \otimes_A B$ is determined from the diagram

$$\begin{array}{ccc} M \otimes_A A[T_1,\ldots,T_n] & \xrightarrow{P_{A[T_1,\ldots,T_n]}} & N \otimes_A A[T_1,\ldots,T_n] \\ \downarrow & & \downarrow \\ M \otimes_A B & \xrightarrow{\quad P_B \quad} & N \otimes_A B \end{array}$$

Put differently, given a list $\vec{x} = (x_1, \ldots, x_n) \in M$ there exist some $y(\vec{x}; \vec{k}) \in N$ satisfying

$$P_{A[T_1,\ldots,T_n]}(\sum_i x_i \otimes T_i) = \sum_{\vec{k} \in (\mathbb{Z}_{\geq 0})^n} (y(\vec{x}; \vec{k}) \otimes T_1^{k_1} \cdots T_n^{k_n}).$$

Moreover, by [Rob63, Théorème 1.1] P is determined by the collection $\left\{ y(\vec{x}, \vec{k}) \right\}$; this justifies the name "polynomial law".

For a given $\vec{x}$ we have $y(\vec{x}; \vec{k}) \neq 0$ only for finitely many $\vec{k}$. If we group the $\vec{k}$ by their total degree, this lets us decompose

$$P = \sum_{d\geq 0} P_n$$

where P_d is the polynomial law defined by[2]

$$P_{d,A[T_1,\ldots,T_n]}(\sum_i x_i \otimes T_i) = \sum_{\sum \vec{k}=d} (y(\vec{x};\vec{k}) \otimes T^{k_1} \cdots T^{k_n}).$$

Definition 2.2.1.3 ([Che14, Section 1.1])**.** An A-polynomial law $P : M \to N$ is called *homogeneous of degree* $d \geq 0$ if[3] $P_B(x \otimes b) = b^d P_B(x \otimes 1)$ for all $B \in \mathsf{Alg}_A$, $b \in B$, and $x \in M$.

It follows that P_d is homogeneous of degree d, which means that we can reduce the study of polynomial laws to the study of homogeneous polynomial laws.

Example 2.2.1.4.

- If $P : M \to N$ is homogeneous of degree 0, then $P_B(x \otimes b) = b^0 P_B(x)$, so if $b = 0$ we see $P_B(x) = P_B(0) = P_A(0) \otimes 1_B$, so P is uniquely determined by the constant $P_A(0) \in N$. So we naturally identify the space of homogeneous degree 0 polynomial laws with N.

- If $P : M \to N$ is homogeneous of degree 1, then one can show that $P_B = P_A \otimes B$ and that P_A is a linear map, so we identify the space of homogeneous polynomials of degree 1 is $\mathrm{Hom}_A(M, N)$.

- If P is homogeneous of degree $d > 1$ then P_B will almost never be additive and the formulas become more complicated, but by considering the way that a polynomial law decomposes into homogeneous polynomial laws, one sees that P is determined by

[2] One has to show that this is a well-defined polynomial law; Roby does this in [Rob63, Section I.8].
[3] By convention, we take $0^0 = 1$.

$P_{A[T_1,\ldots,T_d]}$.

2.2.2 Pseudorepresentations

Now we specialize to the case we are most interested.

Definition 2.2.2.1 ([Che14, Section 1]).

- If $R \to S$ is a morphism of unital associative A-algebras and $P : R \to S$ is a degree d homogeneous A-polynomial law, then we say that P is *multiplicative* if $P_A(1) = 1$ and $P_B(xy) = P_B(x)P_B(y)$ for all B and $x, y \in R \otimes_A B$.

- If R is a unital associative A-algebra, a *degree d pseudorepresentation of R* is a homogeneous multiplicative polynomial law $D : R \to A$ of degree d.

Definition 2.2.2.2 ([Che14, Section 1.5]). The determinant maps $\det_B : M_d(A) \otimes_A B \to B$ define a degree d pseudorepresentation

$$\det : M_d(A) \to A.$$

If R is an associative A-algebra and $\rho : R \to M_d(A)$ is a representation then $\det \rho : R \to A$ is its *associated pseudorepresentation*, defined by

$$(\det \rho)_B : R \otimes_A B \to M_d(A) \otimes_A B \xrightarrow{\det_B} B.$$

This is a degree d pseudorepresentation.

Note that the pseudorepresentation $\det : M_d(A) \to A$ encodes the data of the entire characteristic polynomial of a matrix $M \in M_d(B)$ for $B \in \mathsf{Alg}_A$; just take $\chi(M,T) = \det_{B[T]}(\mathrm{id}_d \otimes T - M)$ for $m \in M - d(B)$.

Definition 2.2.2.3. More generally if R is an associative A-algebra and $D : R \to A$ is a degree d pseudorepresentation then for $s \in R \otimes_A B$ we can take

$$(\chi_D)_B(s, T) = D_{B[T]}(1_R \otimes T - s)$$

and $\chi_D : R \to A[T]$ is a multiplicative homogeneous polynomial law of degree d. [Wan13, Corollary 1.1.9.15] implies that D can be recovered from χ_D. Moreover, by "plugging elements into their own characteristic polynomials", we can define a homogeneous polynomial law $\chi_{D,\mathrm{id}} : R \to R$ of degree d by taking

$$\chi_{D,\mathrm{id}}(s) = (\chi_D)_B(s, s).$$

2.2.3 Cayley–Hamilton pseudorepresentations

As stated before, if $\varphi : R \to M_d(A)$ is a representation then associated pseudorepresentation $\det \varphi : R \to A$ encodes the characteristic polynomials of φ. In this case, by the Cayley–Hamilton theorem we see that $(\chi_D)_{\mathrm{id}}$ is identically zero. This property does not hold in general, but we can single out those for which it does.

Definition 2.2.3.1 ([Che14, Section 1.17])**.** A pseudorepresentation $D : R \to A$ of degree d is called *Cayley–Hamilton* if $\chi_{D,\mathrm{id}}$ is identically zero. In this case we call (R, D) a *Cayley–Hamilton algebra of degree d.*

Since D is multiplicative and determined by its value on $R[t_1, \dots, t_d]$, one can show that D is Cayley–Hamilton if and only if $\mathrm{CH}(D) = 0$, where $\mathrm{CH}(D)$ is the two-sided ideal in R generated by the coefficients of the polynomials

$$\chi_{D,\mathrm{id}}(t_1 r_1 + \cdots + t_d r_d)$$

for varying $r_i \in R$.

Let's now try to understand the significance of the notion of a Cayley–Hamilton pseudorepresentation. An arbitrary degree d pseudorepresentation $D : R \to A$ may not be associated with any representation $R \to M_d(A)$. On the other hand, [Che14, Lemma 1.21] implies that D factors as

$$R \xrightarrow{\rho_D^{\mathrm{CH}}} R/\operatorname{CH}(D) \xrightarrow{\widetilde{D}} A$$

Note that the induced pseudorepresentation $\widetilde{D}$ is Cayley–Hamilton. We also call ρ_D^{CH} the *Cayley–Hamilton representation* associated with D.

The Cayley–Hamilton representation can be thought of as some kind of replacement for a representation with associated pseudorepresentation D. To see this, we need an auxiliary result.

Theorem 2.2.3.2 ([Che14, Theorem 2.12])**.** *If k is an algebraically closed field and R is an associative k-algebra, then for every degree d pseudorepresentation $D' : R \to k$ there exists a unique (up to isomorphism) semisimple representation $\rho_{D'} : R \to M_d(k)$ with $D' = \det \circ \rho_{D'}$.*

So given a geometric point $x : A \twoheadrightarrow k$ we can specialize to the pseudorepresentation $D_x : R \otimes_A k \to k$ as in Remark 2.2.1.2, and we say that D is *irreducible at x* if ρ_{D_x} is irreducible.

Proposition 2.2.3.3 ([Che14, Corollary 2.23])**.** *If G is a group and $D : A[G] \to A$ is a degree d pseudorepresentation such that D_x is irreducible for all geometric points x, then $A[G]/\operatorname{CH}(D)$ is an Azumaya algebra of dimension d over A. In particular, D is étale-locally the determinant of a true representation of G.*

If D contains geometric points at which D_x is reducible, one can still recover the same result by restricting to the (absolutely) irreducible locus, which is Zariski-open.

2.2.4 Algebraization

Everything we've done so far involves no topology. Of course, G_K is a profinite group and we care about its continuous representations with coefficients in certain admissible topological algebras.

In Chapter 4 one of the main technical tools we use is a moduli space of Cayley–Hamilton representations satisfying certain properties, which we call X^{gen}. This is an *algebraization* of a formal moduli space of continuous Cayley–Hamilton representations. The precise role played by Cayley–Hamilton algebras and representations is a bit mysterious on first glance, so we now try to explain why they are unavoidable. For this we closely follow [Wan13] and [Wan18].

First we show that the moduli of representations of G_K "with residual pseudorepresentation $\overline{D}$" can be reinterpreted in terms of Cayley–Hamilton algebras. If $\overline{D} : k[[G_K]] \to k$ is a continuous Cayley–Hamilton pseudorepresentation then it has a universal pseudodeformation ring $R^{\text{ps}}_{\overline{D}}$, which admits a universal continuous pseudodeformation

$$D^u : R^{\text{ps}}_{\overline{D}}[[G_K]] \to R^{\text{ps}}_{\overline{D}}.$$

If we let $E := R^{\text{ps}}_{\overline{D}}[[G_K]]/\operatorname{CH}(D^u)$ then one can show that D^u factors through E and so the resulting pair (E, D^u) is a Cayley–Hamilton algebra.

Definition 2.2.4.1. As in Section 1.3 let

$$\begin{aligned} \mathcal{R}\mathsf{ep}^{\square,d}_E : \mathsf{Adm}_{\mathcal{O}} &\to \mathsf{Set} \\ A &\mapsto \{\text{continuous } A\text{-algebra homomorphisms } E \otimes_{\mathcal{O}} A \to M_d(A)\}. \end{aligned}$$

and

$$\mathsf{PsR}^d_E : \mathsf{Adm}_{\mathcal{O}} \to \mathsf{Set}$$
$$A \mapsto \{\text{continuous pseudorepresentations } E \otimes_{\mathcal{O}} A \to A \text{ of dimension } d\}\,.$$

As before there exists a map $\mathcal{R}\mathsf{ep}^{\square,d}_E \to \mathsf{PsR}^d_E$ defined by composing with the determinant pseudorepresentation. There is a an affine formal subscheme $\mathrm{Spf}\, R^{\mathrm{ps}}_{\overline{D}} \subset \mathsf{PsR}^d_E$ cutting out the pseudorepresentations with residual pseudorepresentation $\overline{D}$, and we can take the fiber $\mathcal{R}\mathsf{ep}^{\square}_{\overline{D}|_E} = \mathcal{R}\mathsf{ep}^{\square,d}_E \times_{\mathsf{PsR}^d_E} \mathrm{Spf}\, R^{\mathrm{ps}}_{\overline{D}}$.

Theorem 2.2.4.2 ([Wan13, Theorem 3.2.3.3])**.** *Any representation in $\mathcal{R}\mathsf{ep}^{\square}_{\overline{D}}(A)$ factors uniquely continuously through the Cayley–Hamilton representation $E \otimes_{\mathcal{O}} A \to A$, and this induces an isomorphism*

$$\mathcal{R}\mathsf{ep}^{\square}_{\overline{D}} \xrightarrow{\sim} \mathcal{R}\mathsf{ep}_{\overline{D}|_E}.$$

Now consider the functpr $\mathsf{Rep}_{\overline{D}|_E}$ on *non-topological* R^{ps}-algebras parametrizing representations $\rho : E \otimes_{R^{\mathrm{ps}}} B \to M_d(B)$ which are *compatible with* $\overline{D}$, i.e. such that $\det \rho = D^u \otimes_{R^{\mathrm{ps}}} B$.

Here is the key finiteness result:

Theorem 2.2.4.3 ([Wan13, Theorem 3.2.3.2])**.** *E is finitely generated as an $R^{\mathrm{ps}}_{\overline{D}}$-module.*

This implies ([Wan13, Corollary 3.2.4.3]) that $\mathsf{Rep}_{\overline{D}|_E}$ is both that $\mathsf{Rep}_{\overline{D}|_E}$ is an algebraization of $\mathcal{R}\mathsf{ep}_{\overline{D}|_E}$, i.e. its $\mathfrak{m}_{\overline{D}} \subset R^{\mathrm{ps}}_{\overline{D}}$-adic completion is isomorphic to $\mathcal{R}\mathsf{ep}_{\overline{D}|_E}$ over $\mathrm{Spf}\, R^{\mathrm{ps}}_{\overline{D}}$, but also that $\mathsf{Rep}_{\overline{D}|_E}$ is finite type over $R^{\mathrm{ps}}_{\overline{D}}$ by [Wan13, Theorem 1.4.1.3]. For the former, the key fact is that over admissible R^{ps}-algebras A, all representations of $E \otimes_{R^{\mathrm{ps}}} A$ are automatically $\mathfrak{m}_{\overline{D}}$-adically continuous; this is elaborated upon in [Wan13, Theorem 3.2.4.1].

In Chapter 4 we work with the finite type affine scheme $\mathsf{Rep}_{\overline{D}|_E}$, although we refer to it as

X^{gen}, which is a reference to a more explicit construction of this affine scheme originally due to Procesi in [Pro87].

In summary, Cayley–Hamilton algebras arise because they provide a natural way to algebraize moduli spaces of continuous representations, which allows us to apply certain technical lemmas from GIT theory. More precisely, the fiber X^{gen}_y over a point $y \in R^{\mathrm{ps}}_{\overline{D}}$ is an affine finite type scheme over the algebraically closed field $\kappa(y)$.

Chapter 3

The trivial representation

This chapter is based entirely on [Iye20], although some notation has been changed for the sake of consistency with the whole thesis. For instance, the dimension n in [Iye20] has been changed to d, and the degree $d = [K : \mathbb{Q}_p]$ has been changed to $\daleth$. Some minor corrections have been made, including a correction to the definition of the Robba ring in Section 3.4.

3.1 Galois deformation rings

Fix a prime p and a finite extension $K/\mathbb{Q}_p$ of degree $\daleth := [K : \mathbb{Q}_p]$. Fix another finite extension $L/\mathbb{Q}_p$ with ring of integers $\mathcal{O}$ and uniformizer ϖ, and let $k = \mathcal{O}/\varpi$ denote the residue field. We always assume L is "large enough", i.e. contains all embeddings of K into $\overline{\mathbb{Q}}_p$. Fix an integer $d > 1$. Let $\epsilon : G_K \to L^\times$ denote the p-adic cyclotomic character.

We are interested in deforming a continuous representation $\overline{\rho} : G_K \to \mathrm{GL}_d(k)$. Let $\mathsf{Art}_{\mathcal{O}}$ denote the category whose objects are local Artinian $\mathcal{O}$-algebras A together with a surjective reduction map $A \twoheadrightarrow k$, and whose morphisms are local $\mathcal{O}$-algebra homomorphisms $A \to B$

respecting the reduction maps to k. Then the lifting problem for $\overline{\rho}$ is the functor

$$\begin{aligned} D^{\square}_{\overline{\rho}} : \mathsf{Art}_{\mathcal{O}} &\to \mathsf{Set} \\ A &\mapsto \{\text{lifts of } \overline{\rho} \text{ to continuous representations } G_K \to \mathrm{GL}_d(A)\} \end{aligned}$$

and the deformation problem is the functor

$$\begin{aligned} D_{\overline{\rho}} : \mathsf{Art}_{\mathcal{O}} &\to \mathsf{Set} \\ A &\mapsto D^{\square}_{\overline{\rho}}(A)/(1 + \mathrm{Mat}_d(\mathfrak{m}_A))\text{-conjugacy}. \end{aligned}$$

The functor $D^{\square}_{\overline{\rho}}$ is always represented by a complete local Noetherian $\mathcal{O}$-algebra which we denote $R^{\square}_{\overline{\rho}}$. Note $D_{\overline{\rho}}$ may not be pro-representable, but it always has a versal hull, i.e. a complete local Noetherian ring $R^{\mathrm{ver}}_{\overline{\rho}}$ and a smooth[1] map $h_{R^{\mathrm{ver}}_{\overline{\rho}}} \to D_{\overline{\rho}}$ of deformation functors that induces an isomorphism on tangent spaces.

For any functor $D : \mathsf{Art}_{\mathcal{O}} \to \mathsf{Set}$, let $T(D) = D(k[x]/x^2)$ denote its tangent space.

Lemma 3.1.0.1. *The forgetful map $D^{\square}_{\overline{\rho}} \to D_{\overline{\rho}}$ factors through a (non-unique) smooth morphism $D^{\square}_{\overline{\rho}} \to h_{R^{\mathrm{ver}}_{\overline{\rho}}}$, which is an isomorphism if $\overline{\rho}$ is the trivial representation.*

Proof. Note $D^{\square}_{\overline{\rho}} \to D_{\overline{\rho}}$ is smooth, and is therefore a versal deformation. Then [Stacks, Tag 06T5(3)] gives us the smooth map $D^{\square}_{\overline{\rho}} \to h_{R^{\mathrm{ver}}_{\overline{\rho}}}$. Let $T(D) = D(k[x]/x^2)$ denote the tangent space to a deformation functor $D : \mathsf{Art}_{\mathcal{O}} \to \mathsf{Set}$. Then a standard computation shows that the induced map of tangent spaces $T(D^{\square}_{\overline{\rho}}) \to T(D_{\overline{\rho}})$ is just the quotient map $Z^1(G_K, \mathrm{ad}\,\overline{\rho}) \to H^1(G_K, \mathrm{ad}\,\overline{\rho})$ (where Z^1 denotes continuous cocycles and H^1 is continuous group cohomology). If $\overline{\rho}$ is trivial, then the coboundaries $B^1(G_K, \mathrm{ad}\,\overline{\rho}) = 0$, so $D^{\square}_{\overline{\rho}} \to D_{\overline{\rho}}$ induces an isomorphism on tangent spaces, and is thus a versal hull. □

[1] See [Sch68, Definition 2.2] for the definition of a smooth map of deformation functors.

3.1.1 Presentation of $R_{\overline{\rho}}^{\square}$

For the rest of this chapter, take $\overline{\rho}$ to be the trivial representation.

Since lifts $\rho \in D_{\overline{\rho}}^{\square}(A)$ reduce to $\overline{\rho}$, they must factor through $1+\mathrm{Mat}_d(\mathfrak{m}_A) \hookrightarrow \mathrm{GL}_d(A)$.

Lemma 3.1.1.1. *For any A in $\mathsf{Art}_{\mathcal{O}}$, the group $1 + \mathrm{Mat}_d(\mathfrak{m}_A)$ is a p-group.*

Proof. It suffices to count the number of elements in $\mathrm{Mat}_d(\mathfrak{m}_A)$, for which it suffices to count the number of elements in $\mathfrak{m}_A$. In the maximal ideal filtration $\mathfrak{m}_A \supset \mathfrak{m}_A^2 \supset \cdots \supset \mathfrak{m}_A^k = 0$, the successive quotients are finite k-vector spaces. The lemma follows. $\square$

So since any deformation $\rho : G_K \to 1 + \mathrm{Mat}_d(\mathfrak{m}_A)$ is a map from G_K into a p-group, it must factor through $G_K \twoheadrightarrow G_K^p$, where G_K^p denotes the maximal pro-p-quotient of G_K. Let $q = |\mu_{p^\infty}(K)|$. We note the following results of Shafarevich and Demuškin:

Theorem 3.1.1.2. *Recall that $\daleth := [K : \mathbb{Q}_p]$.*

1. *(Shafarevich [Sha47]) If $q = 1$, then G_K^p is a free pro-p-group of rank $\daleth + 1$.*
2. *(Demuškin [Dem59]) If $q \geq 3$ then G_K^p is isomorphic to the quotient of the free pro-p-group on $\daleth + 2$ generators $g_1, \ldots, g_{\daleth+2}$ by the relation*[2]

$$g_1^q[g_1, g_2][g_3, g_4] \cdots [g_{\daleth+1}, g_{\daleth+2}] = 1,$$

where $[g, h] = ghg^{-1}h^{-1}$.

When $q = 2$, the group $G_K^{(2)}$ is again cut out by one relation, which looks a lot like the one in part 2 of Theorem 3.1.1.2. This is due to Serre in [Ser95] when $\daleth$ is odd and Labute in [Lab67] when $\daleth$ is even. We suspect that one could prove that $X_{\overline{\rho}}^{\square}$ is normal in these

[2] This relation is well-defined because if $q \geq 3$ then $\daleth$ is even. For odd p this is because $[\mathbb{Q}_p(\mu_p) : \mathbb{Q}_p] = p-1$ and $\mathbb{Q}_p(\mu_p) \subseteq K$. For $p = 2$ this is because $[\mathbb{Q}_2(\mu_4) : \mathbb{Q}_2] = 2$ and $\mathbb{Q}_2(\mu_4) \subseteq K$.

exceptional cases using the same method, but since our crystalline density result assumes $p > d$, we have decided not to pursue this.

Remark 3.1.1.3. In [Dem59], Demuškin uses the convention that $[g,h] = g^{-1}h^{-1}gh$ rather than $[g,h] = ghg^{-1}h^{-1}$. However, following Serre's proof in [Ser95, Section 6], we may use either convention: the basis for G_K^p that we get depends on which convention we use to define the lower exponent-p central series of G_K^p in the proof of part 2 of Theorem 3.1.1.2. See also the discussion preceding [NSW08, Theorem 7.5.14].

We can use these presentations to determine the representing ring for $D_{\bar{\rho}}^{\square}$.

Proposition 3.1.1.4. *Suppose $G = \langle g_1, \dots, g_s : r(g_1, \dots, g_s) = 1 \rangle$ is the quotient of the free pro-p-group on s generators by the relation $r(g_1, \dots, g_s) = 1$. Then the framed deformation functor $D_{\bar{\rho}}^{\square}$ of the trivial representation ρ is pro-represented by the complete Noetherian local ring*

$$R_{\bar{\rho}}^{\square} = \mathcal{O}[\![X_1, \dots, X_s]\!]/(r(\widetilde{X}_1, \dots, \widetilde{X}_s) - I),$$

where each X_i is a $d \times d$ matrix of indeterminates, and $\widetilde{X}_i := X_i + I$.

Proof. For $A \in \mathsf{Art}_{\mathcal{O}}$, a continuous lift $\rho : G \to 1 + \mathrm{Mat}_d(\mathfrak{m}_A)$ is determined by where it sends $g_1, \dots, g_s$. Thus we can define a map

$$f_\rho : R_{\bar{\rho}}^{\square} \to A, X_i \mapsto \rho(g_i) - I$$

that is continuous because $\rho(g_i) - I \in \mathrm{Mat}_d(\mathfrak{m}_A)$, and well-defined because

$$r(\widetilde{X}_1, \dots, \widetilde{X}_s) \mapsto r(\rho(g_1), \dots, \rho(g_s)) = \rho(r(g_1, \dots, g_s)) = I.$$

Conversely, given a continuous map $f : R_{\bar{\rho}}^{\square} \to A$, we can define a continuous map $G \to 1 + \mathrm{Mat}_d(\mathfrak{m}_A)$ taking g_i to $f(\widetilde{X}_i)$. These are inverse constructions, and give an isomorphism of functors. $\square$

Let $\rho^\square : G_K \to \mathrm{GL}_d(R^\square_{\overline{\rho}})$ denote the universal representation.

Corollary 3.1.1.5.

$$R^\square_{\overline{\rho}} = \begin{cases} \mathcal{O}[\![X_1, \dots, X_{\text{د}+1}]\!] & q = 1 \\ \mathcal{O}[\![X_1, \dots, X_{\text{د}+2}]\!]/(\widetilde{X}_1^q[\widetilde{X}_1, \widetilde{X}_2] \cdots [\widetilde{X}_{\text{د}+1}, \widetilde{X}_{\text{د}+2}] - I) & q \geq 3 \end{cases}$$

We are interested in the irreducible components of $\operatorname{Spec} R^\square_{\overline{\rho}}[1/\varpi]$. If $q = 1$, then there is clearly only one such component. Since we have decided not to treat $q = 2$, we assume $q > 2$ in the remainder of this chapter.

The problem with studying the geometry of $\operatorname{Spec} R^\square_{\overline{\rho}}$ is that there is a particularly nasty singularity at the unique closed point. A nicer approach is to study the generic fibre, which has lots of closed points admitting a nice moduli description, and whose singularities are far easier to control. There are two (basically equivalent) ways of doing this: one can either study the rigid generic fibre $\mathcal{X}^\square_{\overline{\rho}}$ in the sense of Berthelot (see e.g. [Jon95, Section 7]), or one can just study the scheme-theoretic generic fibre

$$X^\square_{\overline{\rho}} := \operatorname{Spec} R^\square_{\overline{\rho}}[1/\varpi].$$

We will use both perspectives: to study irreducible components, it will suffice to use $X^\square_{\overline{\rho}}$ (although some of the path-connectedness arguments later on in the chapter are best thought of rigid analytically). Later, when showing density of crystalline points, we will use $\mathcal{X}^\square_{\overline{\rho}}$.

Remark 3.1.1.6. Restricting to the generic fibre still allows us to study irreducible components of $\operatorname{Spec} R^\square_{\overline{\rho}}$ itself, and thus of $\operatorname{Spf} R^\square_{\overline{\rho}}$ (with the maximal ideal topology): we will show that $R^\square_{\overline{\rho}}$ is $\mathcal{O}$-flat, which implies that the map $R^\square_{\overline{\rho}} \to R^\square_{\overline{\rho}}[1/\varpi]$ induces a bijection on irreducible components. To see this, note that there is a bijection between irreducible components of $X_{\bar\rho}^{\square}$ and irreducible components of $\operatorname{Spec} R_{\bar\rho}^{\square}$ that have nonempty intersection with $X_{\bar\rho}^{\square}$. Then note that each irreducible component intersects the generic fibre, since $R_{\bar\rho}^{\square}$ is $\mathcal{O}$-flat. Therefore, it suffices to study the irreducible components of $X_{\bar\rho}^{\square}$, but in fact we will first show that $X_{\bar\rho}^{\square}$ is normal, and then just study the connected components.

3.1.2 Dimension of $R_{\bar\rho}^{\square}$

First we note the following fact.

Lemma 3.1.2.1. *Let $M(d,q+1)_{\mathcal{O}} \subset \mathrm{GL}_{d,\mathcal{O}} \times_{\mathcal{O}} \mathrm{GL}_{d,\mathcal{O}}$ be the closed subspace of pairs of invertible matrices X, Y satisfying $XYX^{-1} = Y^{q+1}$. Then $M(d,q+1)_{\mathcal{O}}$ is a local complete intersection, and is Cohen-Macaulay and flat of relative dimension d^2 over $\operatorname{Spec}\mathcal{O}$.*

Proof. The proof is given in the first two paragraphs of [Sho18, Theorem 2.5], although Shotton attributes the proof to Helm in [Hel16, Proposition 4.2], who in turn attributes the argument to Choi in [Cho09]. □

Proposition 3.1.2.2. *The ring $R_{\bar\rho}^{\square}$ is a complete intersection ring of dimension $1+d^2(\text{د}+1)$, and $\mathcal{O}$-flat. In particular $R_{\bar\rho}^{\square}$ is Cohen-Macaulay.*

Proof. Note $\mathcal{O}[\![X_1,\dots,X_{\text{د}+2}]\!]$ has dimension $1+d^2(\text{د}+2)$ and we quotient out by d^2 equations so we would expect $R_{\bar\rho}^{\square}$ to have dimension $1+d^2(\text{د}+1)$. In fact, if we further quotient out by the $\text{د}d^2$ indeterminates defining $X_3,\dots,X_{\text{د}+2}$ we are left with

$$R' := \mathcal{O}[\![X_1, X_2]\!]/(\widetilde{X}_1^q[\widetilde{X}_1, \widetilde{X}_2] - I).$$

This can be rewritten as

$$R' = \mathcal{O}[\![X_1, X_2]\!]/(\widetilde{X_2}\widetilde{X_1}\widetilde{X_2}^{-1} - \widetilde{X_1}^{q+1}),$$

and $\operatorname{Spf} R$ is the formal completion of $M(d, q+1)_{\mathcal{O}}$ at the closed k-point x_0 defined by $X = Y = I$; one can show this by noting that $M(d,q+1)^{\wedge x_0}_{\mathcal{O}}$ represents a functor $\mathsf{Art}_{\mathcal{O}} \to \mathsf{Set}$ which is also representable by R'. Thus by Lemma 3.1.2.1, $\operatorname{Spec} R'$ is flat of relative dimension d^2 over $\operatorname{Spec} \mathcal{O}$, and thus has dimension d^2+1. Therefore, $\operatorname{Spec} R'/\varpi$ has dimension d^2. In summary, taking the quotient of $\mathcal{O}[\![X_1, \ldots, X_{\text{د}+2}]\!]$ by the $1 + d^2(\text{د}+1)$ equations

$$\widetilde{X_2}\widetilde{X_1}\widetilde{X_2}^{-1} - \widetilde{X_1}^{q+1}, X_3, \ldots, X_{\text{د}+2}, \varpi$$

gives a ring of dimension d^2, so these $1 + d^2(\text{د}+1)$ equations must form a regular sequence (in any order since $\mathcal{O}[\![X_1, \ldots, X_{\text{د}+2}]\!]$ is local Noetherian) and thus $\dim R^{\square}_{\overline{\rho}} = 1 + d^2(\text{د}+1)$. We conclude that $R^{\square}_{\overline{\rho}}$ is complete intersection and therefore also Cohen-Macaulay. Note ϖ is a nonzerodivisor so $R^{\square}_{\overline{\rho}}$ is $\mathcal{O}$-torsion free, and in particular $\mathcal{O}$-flat since $\mathcal{O}$ is a DVR. $\square$

3.2 Normality

In this section, we show that $X^{\square}_{\overline{\rho}}$ is a normal scheme, so that in fact its irreducible components are just the connected components.

For any closed point $x \in X^{\square}_{\overline{\rho}}$ we may define the residual representation

$$\rho_x : G_K \xrightarrow{\rho^{\square}} \mathrm{GL}_d(R^{\square}_{\overline{\rho}}) \to \mathrm{GL}_d(R^{\square}_{\overline{\rho}}[1/\varpi]) \to \mathrm{GL}_d(\kappa(x))$$

where $\kappa(x)$ is the residue field of the stalk at $x \in X^{\square}_{\overline{\rho}}$.

Lemma 3.2.0.1. *If $x \in X^{\square}_{\overline{\rho}}$ is a closed point, then the residue field $\kappa(x)$ is a finite extension* *of L, and the image of ρ_x lands in $\mathrm{GL}_d(\mathscr{O}_{\kappa(x)})$, where $\mathscr{O}_{\kappa(x)} \subset \kappa(x)$ is the ring of integers.*

Proof. This proof is based on [BM02, Lemma 5.1.1]. Let $x = \mathfrak{m}$ be some maximal ideal in $R^{\square}_{\bar{\rho}}[1/\varpi]$, and let $\mathfrak{p}$ denote the corresponding prime ideal in $R^{\square}_{\bar{\rho}}$. Note $(R^{\square}_{\bar{\rho}}/\mathfrak{p})[1/\varpi] \cong R^{\square}_{\bar{\rho}}[1/\varpi]/\mathfrak{m}$, which is a field, so in particular $\dim((R^{\square}_{\bar{\rho}}/\mathfrak{p})[1/\varpi]) = 0$. By $\mathcal{O}$-flatness, $\varpi \in R^{\square}_{\bar{\rho}}$ is not nilpotent and $R^{\square}_{\bar{\rho}}/\mathfrak{p}$ is a local Noetherian domain, hence equidimensional, so [CDP15, Lemma 2.3] tells us that $\dim(R^{\square}_{\bar{\rho}}/\mathfrak{p}) = 1$. Since $\varpi \notin \mathfrak{p}$, it follows that $\dim R^{\square}_{\bar{\rho}}/(\varpi, \mathfrak{p}) = 0$, i.e. $R^{\square}_{\bar{\rho}}/(\varpi, \mathfrak{p})$ is a local Artinian $\mathcal{O}$-algebra, hence its underlying set is finite.

Now fix a (finite) set $\widetilde{S}$ consisting of a lift in $R^{\square}_{\bar{\rho}}/\mathfrak{p}$ for each residue class in $R^{\square}_{\bar{\rho}}/(\varpi, \mathfrak{p})$. Then given an element $a_0 \in R^{\square}_{\bar{\rho}}/\mathfrak{p}$, by reducing mod ϖ we can find b_0 generated by elements of $\widetilde{S}$ over $\mathcal{O}$ such that

$$a_0 - b_0 \in \varpi R^{\square}_{\bar{\rho}}/\mathfrak{p}.$$

So there is some $a_1 \in R^{\square}_{\bar{\rho}}/\mathfrak{p}$ such that $a_0 - b_0 = \varpi a_1$. Repeating this for a_1, then a_2, etc, we find after rearranging that

$$a_0 = b_0 + \varpi b_1 + \varpi^2 b_2 + \cdots,$$

which converges, and by rearranging the terms we can express a_0 in terms of elements of $\widetilde{S}$ over $\mathcal{O}$.

Therefore, $R^{\square}_{\bar{\rho}}/\mathfrak{p}$ is a finitely generated $\mathcal{O}$-module and thus

$$(R^{\square}_{\bar{\rho}}/\mathfrak{p})[1/\varpi] = \kappa(x),$$

is a finite extension of L. Furthermore, the image of $R^{\square}_{\bar{\rho}}$ in $\kappa(x)$ lands in $\mathscr{O}_{\kappa(x)}$: this follows from the remarks in [Jon95, Section 7.1.8]. $\square$

Thus we may write $\rho_x : G_K \to \mathrm{GL}_d(\mathscr{O}_{\kappa(x)})$.

Proposition 3.2.0.2. *If $x = \mathfrak{m} \in X^{\square}_{\bar{\rho}}$ is a singular (i.e. not regular) closed point, then ρ_x*

is reducible.

Proof. [Kis09b, Section 2.3] (in particular Lemma 2.3.3 and Proposition 2.3.5) shows that the $\mathfrak{m}$-adic completion $(R^{\square}_{\overline{\rho}})^{\wedge}_{\mathfrak{m}}$ represents the framed deformation functor $D^{\square}_{\rho_x} : \mathsf{Art}_{\kappa(x)} \to \mathsf{Set}$. The point x is singular so $(R^{\square}_{\overline{\rho}})^{\wedge}_{\mathfrak{m}}$ cannot be formally smooth over $\mathcal{O}$, and thus the deformation problem for ρ_x is obstructed. Therefore

$$H^2(G_K, \operatorname{ad}\rho_x) \xlongequal{\text{local Tate duality}} H^0(G_K, \operatorname{ad}\rho_x \otimes \epsilon) = \operatorname{Hom}_{G_K}(\rho_x, \rho_x \otimes \epsilon)$$

is nonzero, so there exists some nonzero map $\psi : \rho_x \to \rho_x \otimes \epsilon$ of G_K-representations. But ψ cannot be an isomorphism because $\det(\rho_x)$ and $\det(\rho_x \otimes \epsilon) = \epsilon^d \det(\rho_x)$ are non-isomorphic characters, so ρ_x is reducible. ☐

3.2.1 The reducible locus

To control the singular locus, we imitate Geraghty's approach in [Ger10, Section 3] and try to bound the locus of points of $X^{\square}_{\overline{\rho}}$ whose corresponding representation is reducible. The only real differences between what we do here and what Geraghty does in his thesis is that we will need to consider flag varieties for every parabolic subgroup (not just the Borel), and we need to parametrize pairs $(\rho, \operatorname{Fil}^\bullet)$ where ρ fixes $\operatorname{Fil}^\bullet$ but does not fix a finer flag (see Remark 3.2.1.4).

Pick a k-tuple of positive integers $\underline{d} = (d_1, \ldots, d_k)$ such that $\sum_{i=1}^k d_i = d$, and let $\mathcal{F}_{\underline{d}} \in \mathsf{Sch}_{\mathcal{O}}$ be the flag variety associated with $\underline{d}$, i.e. the scheme representing the functor $\mathcal{F}_{\underline{d}} : \mathsf{Alg}_{\mathcal{O}} \to \mathsf{Set}$ defined by

$$\mathcal{F}_{\underline{d}} : A \mapsto \left\{ \begin{matrix} \text{filtrations } 0 \subset \operatorname{Fil}^1 \subset \cdots \subset \operatorname{Fil}^k = A^d \text{ by projective} \\ A\text{-submodules that are locally direct summands,} \\ \text{and that satisfy } \operatorname{rank}_A(\operatorname{Fil}^i) = d_1 + \cdots + d_i \end{matrix} \right\}.$$

Then $\operatorname{Spec} R_{\overline{\rho}}^{\square} \times_{\mathcal{O}} \mathcal{F}_{\underline{d}}$ is an $\mathcal{O}$-scheme whose A-points (for $A \in \mathsf{Alg}_{\mathcal{O}}$) are pairs $(f, \mathrm{Fil}^{\bullet})$, where $\mathrm{Fil}^{\bullet}$ is as above, and $f : R_{\overline{\rho}}^{\square} \to A$ is an $\mathcal{O}$-algebra morphism. Note f induces a representation

$$\rho_f : G_K \xrightarrow{\rho^{\square}} \mathrm{GL}_d(R_{\overline{\rho}}^{\square}) \xrightarrow{f} \mathrm{GL}_d(A).$$

Define a subfunctor $\mathcal{G}_{\underline{d}} \hookrightarrow \operatorname{Spec} R_{\overline{\rho}}^{\square} \times_{\mathcal{O}} \mathcal{F}_{\underline{d}}$ by

$$\mathcal{G}_{\underline{d}}(A) = \{(f, \mathrm{Fil}^{\bullet}) : \text{the action of } G_K \text{ on } A^d \text{ via } \rho_f \text{ preserves } \mathrm{Fil}^{\bullet}\}.$$

Proposition 3.2.1.1. *$\mathcal{G}_{\underline{d}}$ is represented by a closed subscheme of* $\operatorname{Spec} R_{\overline{\rho}}^{\square} \times_{\mathcal{O}} \mathcal{F}_{\underline{d}}$.

Proof. It suffices to show that for any $A \in \mathsf{Alg}_{\mathcal{O}}$ and any A-point of $\operatorname{Spec} R_{\overline{\rho}}^{\square} \times_{\mathcal{O}} \mathcal{F}_{\underline{d}}$, there exists an ideal $I \subseteq A$ and a map $\mathsf{Alg}_{\mathcal{O}}(A/I, -) \to \mathcal{G}_{\underline{d}}$ such that

$$\begin{array}{ccc} \mathsf{Alg}_{\mathcal{O}}(A/I, -) & \longrightarrow & \mathcal{G}_{\underline{d}} \\ \big\downarrow & & \big\downarrow \\ \mathsf{Alg}_{\mathcal{O}}(A, -) & \longrightarrow & \operatorname{Spec} R_{\overline{\rho}}^{\square} \times_{\mathcal{O}} \mathcal{F}_{\underline{d}} \end{array}$$

is Cartesian.

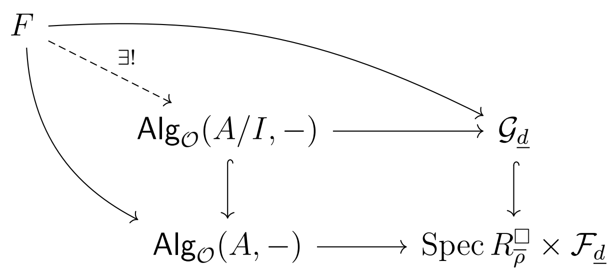
Fix an A-point of $\operatorname{Spec} R_{\overline{\rho}}^{\square} \times_{\mathcal{O}} \mathcal{F}_{\underline{d}}$, which gives a pair $(f, \mathrm{Fil}^{\bullet})$ as before. Then since $\mathrm{Fil}^{\bullet}$ is a filtration of direct summands, we can fix complementary A-submodules $N_i \subset A^d$ such that $\mathrm{Fil}^i \oplus N_i = A^d$. These come with surjective projection maps $A^d \xrightarrow{\pi_i} N_i$.

We can now define the ideal $I \subseteq A$ generated by the coefficients of $\pi_i \rho_f(g) v$ with respect to the standard basis of A^d for all $g \in G_K$ and all $v \in \mathrm{Fil}^i$, for each $i = 1, \ldots, r$. If we write

$$\rho_{f,I} : G_K \xrightarrow{\rho_f} \mathrm{GL}_d(A) \to \mathrm{GL}_d(A/I)$$

then $(R_{\overline{\rho}}^{\square} \xrightarrow{f} A \to A/I, \mathrm{Fil}^{\bullet} \otimes_A A/I)$ is an A/I-point of $\mathcal{G}_{\underline{d}}$ (because $\rho_{f,I}$ is now forced to fix

$\mathrm{Fil}^\bullet \otimes_A A/I$), which thus gives us the desired map $\mathsf{Alg}_{\mathcal{O}}(A/I, -) \to G_{\underline{d}}$. Now given a diagram



one checks easily that we get a unique map $F \to \mathsf{Alg}_{\mathcal{O}}(A/I, -)$. □

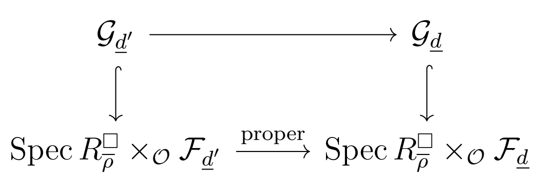
Definition 3.2.1.2. For a positive integer m, denote by $P(m)$ the finite set of ordered partitions (viewed as ordered tuples) of the integer m. If $\underline{m} = (m_1, \dots, m_k)$ is a k-tuple of integers for $k \geq 1$, then let $P(\underline{m})$ be the image of the natural concatenation map

$$P(m_1) \times \cdots \times P(m_k) \to P(m_1 + \cdots + m_k)$$

Finally, let $P(\underline{m})^\circ = P(\underline{m}) \setminus \{\underline{m}\}$ and $P(m)^\circ = P(m) \setminus \{(m)\}$.

Lemma 3.2.1.3. *If $\underline{d}' \in P(\underline{d})^\circ$, then the natural map $\mathcal{G}_{\underline{d}'} \to \mathcal{G}_{\underline{d}}$ (taking a filtration of shape $\underline{d}'$ and only remembering that it gives a filtration of shape $\underline{d}$) is proper.*

Proof. The partial flag varieties $\mathcal{F}_{\underline{d}'}$ and $\mathcal{F}_{\underline{d}}$ are proper over $\operatorname{Spec} \mathcal{O}$, so the map $\mathcal{F}_{\underline{d}'} \to \mathcal{F}_{\underline{d}}$ is proper. We want the top arrow in the following diagram (which is not Cartesian!) to be proper:



But all the other arrows are proper, so the top is as well. □

Putting together these maps, we obtain a map

$$\bigsqcup_{\underline{d}'\in P(\underline{d})^\circ} \mathcal{G}_{\underline{d}'} \to \mathcal{G}_{\underline{d}}.$$

Since each $\mathcal{G}_{\underline{d}'} \to \mathcal{G}_{\underline{d}}$ is closed (by properness), and the disjoint union is taken over a finite set, the (set-theoretic) image of this map is closed: denote by $\mathcal{G}_{\underline{d}}^{\mathrm{irr}}$ its open complement with the natural subscheme structure.

Remark 3.2.1.4. To motivate this definition, note that $\mathcal{G}_{\underline{d}}^{\mathrm{irr}}$ should parametrize pairs $(f, \mathrm{Fil}^\bullet)$ where the induced representation ρ_f fixes $\mathrm{Fil}^\bullet$ but does not fix any finer filtration in $\mathcal{F}_{\underline{d}'}$ for $\underline{d}' \in P(\underline{d})^\circ$ after base changing to an algebraic closure: in fact, $\mathcal{G}_{\underline{d}}^{\mathrm{irr}}$ represents the functor that takes an $\mathcal{O}$-algebra A to the set of pairs $(f, \mathrm{Fil}^\bullet)$ such that ρ_f fixes $\mathrm{Fil}^\bullet$ and such that for all geometric points $\overline{s}$ of $\mathrm{Spec}\, A$, the representation $\rho_{f,\overline{s}}$ does not fix any filtration strictly refining $\mathrm{Fil}^\bullet_{\overline{s}}$.

Finally, consider the map

$$\bigsqcup_{\underline{d}\in P(d)^\circ} \mathcal{G}_{\underline{d}}^{\mathrm{irr}} \to \bigsqcup_{\underline{d}\in P(d)^\circ} \mathcal{G}_{\underline{d}} \to \bigsqcup_{\underline{d}\in P(d)^\circ} (\mathrm{Spec}\, R_{\overline{\rho}}^{\square} \times_{\mathcal{O}} \mathcal{F}_{\underline{d}}) \to \mathrm{Spec}\, R_{\overline{\rho}}^{\square}.$$

After passing to the generic fibre, we obtain a map

$$\bigsqcup_{\underline{d}\in P(d)^\circ} \mathcal{G}_{\underline{d}}^{\mathrm{irr}}[1/\varpi] \to X_{\overline{\rho}}^{\square}.$$

The image is closed: to see this note the image is the same as the image of

$$\bigsqcup_{\underline{d}\in P(d)^\circ} \mathcal{G}_{\underline{d}}[1/\varpi] \to X_{\overline{\rho}}^{\square},$$

(this follows from the moduli description of $\mathcal{G}_{\underline{d}}^{\mathrm{irr}}$ given in Remark 3.2.1.4) and then note that $\mathcal{G}_{\underline{d}} \to \mathrm{Spec}\, R_{\overline{\rho}}^{\square} \times_{\mathcal{O}} \mathcal{F}_{\underline{d}} \to \mathrm{Spec}\, R_{\overline{\rho}}^{\square}$ is proper. So the scheme-theoretic image is a closed subscheme $X_{\overline{\rho}}^{\square,\mathrm{red}} \subset X_{\overline{\rho}}^{\square}$. Note furthermore, that since $R_{\overline{\rho}}^{\square}$ is excellent (as it is complete local Noetherian) the singular locus $X_{\overline{\rho}}^{\square,\mathrm{sing}} \subset X_{\overline{\rho}}^{\square}$ is closed. In fact,

Corollary 3.2.1.5. $X_{\overline{\rho}}^{\square,\mathrm{sing}} \subseteq X_{\overline{\rho}}^{\square,\mathrm{red}}$.

Proof. [Gro66, Corollaire 10.5.9 and Proposition 10.3.2] imply that $X_{\overline{\rho}}^{\square}$ is a Jacobson scheme so it again follows from Proposition 10.3.2 that the closed subset $X_{\overline{\rho}}^{\square,\mathrm{sing}}$ is Jacobson, which implies that it suffices to show that singular closed points are contained in $X_{\overline{\rho}}^{\square,\mathrm{red}}$. So pick $x \in X_{\overline{\rho}}^{\square}$ that is a singular closed point. By Proposition 3.2.0.2, ρ_x stabilizes a flag $\mathrm{Fil}_{\overline{x}}^{\bullet} \in \mathcal{F}_{\underline{d}}(\overline{\kappa(x)})$ of some shape determined by $\underline{d} \in P(d)^{\circ}$ and we can assume that $\underline{d}$ is minimal for this property (extending to a larger algebraically closed field will not affect minimality), so we get a point $(f_{\overline{x}}, \mathrm{Fil}_{\overline{x}}^{\bullet}) \in \mathcal{G}_{\underline{d}}^{\mathrm{irr}}(\overline{\kappa(x)})$, where f_x is the map to the algebraic closure of the residue field. Thus, ρ_x is in the image of the map $\mathcal{G}_{\underline{d}}^{\mathrm{irr}}[1/\pi_L] \to X_{\overline{\rho}}^{\square}$. $\square$

Therefore,

$$\dim X_{\overline{\rho}}^{\square,\mathrm{sing}} \leq \dim X_{\overline{\rho}}^{\square,\mathrm{red}} \leq \max_{\underline{d} \in P(d)^{\circ}} \dim \mathcal{G}_{\underline{d}}^{\mathrm{irr}}[1/\varpi],$$

so in the remainder of this section, we will bound the dimension of each $\mathcal{G}_{\underline{d}}^{\mathrm{irr}}$: later, in Proposition 3.2.2.3, we will see why we need to restrict to $\mathcal{G}_{\underline{d}}^{\mathrm{irr}}$ inside $\mathcal{G}_{\underline{d}}$.

3.2.2 Dimension counting

To compute the dimension of $\mathcal{G}_{\underline{d}}^{\mathrm{irr}}[1/\varpi]$, we can compute

$$\max_{x \in \mathcal{G}_{\underline{d}}^{\mathrm{irr}}[1/\varpi] \text{ closed}} \dim \mathscr{O}_{\mathcal{G}_{\underline{d}}^{\mathrm{irr}}[1/\varpi],x} = \max_{x \in \mathcal{G}_{\underline{d}}^{\mathrm{irr}}[1/\varpi] \text{ closed}} \dim \mathscr{O}_{\mathcal{G}_{\underline{d}},x} = \max_{x \in \mathcal{G}_{\underline{d}}^{\mathrm{irr}}[1/\varpi] \text{ closed}} \dim \widehat{\mathscr{O}}_{\mathcal{G}_{\underline{d}},x}.$$

We showed in Lemma 3.2.0.1 that the residue field of a closed point $x \in X_{\overline{\rho}}^{\square}$ is a finite extension of L. Given a closed point $x \in \mathcal{G}_{\underline{d}}^{\mathrm{irr}}[1/\varpi]$, what can we say about its residue field? The map

$$\mathcal{G}_{\underline{d}}[1/\varpi] \hookrightarrow (\operatorname{Spec} R_{\overline{\rho}}^{\square} \times_{\mathcal{O}} \mathcal{F}_{\underline{d}})[1/\pi_L] \to X_{\overline{\rho}}^{\square}$$

is locally of finite type, so by [Gro66, Corollaire 10.4.7], the image of x in $X^{\square}_{\overline{\rho}}$ is a closed point, and thus we can take the field of definition F of x to be a finite extension of the residue field of its image in $X^{\square}_{\overline{\rho}}$, which is in turn a finite extension of L. Note

$$\mathcal{G}_{\underline{d}} \hookrightarrow \operatorname{Spec} R^{\square}_{\overline{\rho}} \times_{\mathcal{O}} \mathcal{F}_{\underline{d}} \to \operatorname{Spec} R^{\square}_{\overline{\rho}}$$

is proper, so we apply the valuative criterion of properness to the diagram

$$\begin{array}{ccc} \operatorname{Spec} F & \xrightarrow{x} & \mathcal{G}_{\underline{d}} \\ \downarrow & \nearrow & \downarrow \\ \operatorname{Spec} \mathcal{O}_F & \longrightarrow & \operatorname{Spec} R^{\square}_{\overline{\rho}} \end{array}$$

to get a lift $x : \operatorname{Spec} \mathcal{O}_F \to \mathcal{G}_{\underline{d}}$ (by abuse of notation we call both points x). But this corresponds to some map $f_x : R^{\square}_{\overline{\rho}} \to \mathcal{O}_F$ and some $\mathrm{Fil}^{\bullet}_x \in \mathcal{F}_{\underline{d}}(\mathcal{O}_F)$. Note f_x induces a representation

$$\rho_x : G_K \to \mathrm{GL}_d(\mathcal{O}_F).$$

We also get a map

$$\operatorname{Spec} \widehat{\mathscr{O}}_{\mathcal{G}_{\underline{d}},x} \to \operatorname{Spec} \mathscr{O}_{\mathcal{G}_{\underline{d}},x} \to \mathcal{G}_{\underline{d}},$$

which determines a representation $\widehat{\rho}_x : G_K \to \mathrm{GL}_d(\widehat{\mathscr{O}}_{\mathcal{G}_{\underline{d}},x})$ and a filtration $\widehat{\mathrm{Fil}}^{\bullet}_x \in \mathcal{F}_{\underline{d}}(\widehat{\mathscr{O}}_{\mathcal{G}_{\underline{d}},x})$.

Let Art_F denote the category whose objects are local Artinian F-algebras A together with a surjective reduction map $A \twoheadrightarrow F$, and whose morphisms are local homomorphisms $A \to B$ respecting the reduction maps to F. Now let $D^{\square}_{\rho_x,\underline{d}} : \mathsf{Art}_F \to \mathsf{Set}$ be the functor taking B to the set of pairs $(\rho, \mathrm{Fil}^{\bullet})$ of continuous $\rho : G_K \to \mathrm{GL}_d(B)$ lifting ρ_x and $\mathrm{Fil}^{\bullet} \in \mathcal{F}_{\underline{d}}(B)$ lifting $\mathrm{Fil}^{\bullet}_x$ such that ρ preserves $\mathrm{Fil}^{\bullet}$.

Proposition 3.2.2.1. *$D^{\square}_{\rho_x,\underline{d}}$ is pro-representable by $\widehat{\mathscr{O}}_{\mathcal{G}_{\underline{d}},x}$ with the universal representation $\widehat{\rho_x}$ and universal filtration $\widehat{\mathrm{Fil}}^{\bullet}_x$.*

Before proving the proposition, we note a lemma.

Lemma 3.2.2.2. *If E/L is a finite extension, then the universal framed deformation problem $D^{\square}_{\overline{\rho}_E} : \mathsf{Art}_{\mathcal{O}_E} \to \mathsf{Set}$ for the trivial representation $\overline{\rho}_E : G_K \to \mathrm{GL}_d(k_E)$ is pro-represented by $R^{\square}_{\overline{\rho}} \otimes_{\mathcal{O}} \mathcal{O}_E$. In particular, a lift of $\overline{\rho}_E$ to $A \in \mathsf{Art}_{\mathcal{O}_E}$ is given by a unique $\mathcal{O}_E$-algebra map $R^{\square}_{\overline{\rho}} \otimes_{\mathcal{O}} \mathcal{O}_E \to A$ that descends uniquely to an $\mathcal{O}$-algebra map $R^{\square}_{\overline{\rho}} \to A$.*

Proof. In this case, one can see the first part by, for example, looking at Proposition 3.1.1.4 and comparing the deformation rings. The second part is clear from the fact that $R^{\square}_{\overline{\rho}} \otimes_{\mathcal{O}} \mathcal{O}_E \to A$ is an $\mathcal{O}_E$-algebra map. $\square$

Proof of Proposition 3.2.2.1. Given an F-algebra map $\widehat{\mathscr{O}}_{\mathcal{G}_{\underline{d}},x} \to B$, we can push forward $\widehat{\rho_x}$ and $\widehat{\mathrm{Fil}}^{\bullet}_x$ to get a pair $(\rho, \mathrm{Fil}^{\bullet}) \in D^{\square}_{\rho_x,\underline{d}}(B)$.

Conversely, suppose we are given $(\rho, \mathrm{Fil}^{\bullet}) \in D^{\square}_{\rho_x,\underline{d}}(B)$. The idea is to try to use universality of $R^{\square}_{\overline{\rho}}$, but B is an F-algebra and not a $\mathcal{O}_F$-algebra, so we cannot immediately reduce mod $\mathfrak{m}_F$. But in fact, we can first show that ρ factors through a finitely generated local $\mathcal{O}_F$-subalgebra $A \subset B$ with a surjective map onto $\mathcal{O}_F$: this is exactly Kisin's argument in [Kis03a, Proposition 9.5].

If we take the composition $G_K \xrightarrow{\rho} \mathrm{GL}_d(A) \to \mathrm{GL}_d(A/\mathfrak{m}_A)$ we get ρ_x. But note $R^{\square}_{\overline{\rho}} \xrightarrow{f_x} \mathcal{O}_F$ is a local homomorphism, so commutativity of

$$\begin{array}{ccc} & & \mathrm{GL}_d(A) \\ & \nearrow & \downarrow \\ \mathrm{GL}_d(R^{\square}_{\overline{\rho}}) & \xrightarrow[\rho_x]{} & \mathrm{GL}_d(\mathcal{O}_F) \\ \downarrow & & \downarrow \\ \mathrm{GL}_d(k) & \hookrightarrow & \mathrm{GL}_d(k_F) \end{array}$$

shows that ρ reduces to the trivial representation valued in k_F. By Lemma 3.2.2.2, ρ is induced by a local $\mathcal{O}$-algebra map $a : R^{\square}_{\overline{\rho}} \to A$.

Suppose we have a different $a' : R_{\overline{\rho}}^{\square} \to B$ inducing ρ. Then $R_{\overline{\rho}}^{\square} \xrightarrow{a'} B \to B/\mathfrak{m}_B = F$ is f_x and thus factors through $\mathcal{O}_F$. The aforementioned argument of Kisin in [Kis03a] also implies in this case that a' factors through a finitely generated $\mathcal{O}_F$-subalgebra $A' \subset B$, which we can take large enough so that it contains A. Then by universality of $R_{\overline{\rho}}^{\square} \otimes_{\mathcal{O}} \mathcal{O}_F$, we have $a = a'$.

The map $a : R_{\overline{\rho}}^{\square} \to B$ specializes to f_x under the reduction map $B \twoheadrightarrow F$. Similarly $\mathrm{Fil}^{\bullet}$ specializes to $\mathrm{Fil}_x^{\bullet}$, so in other words, we have constructed a B-point of $\mathcal{G}_{\underline{d}}$ that specializes to x. Thus, we get a map

$$\mathscr{O}_{\mathcal{G}_{\underline{d}},x} \to B$$

that factors through the completion, since B is complete. □

To compute the dimension of $\widehat{\mathscr{O}}_{\mathcal{G}_{\underline{d}},x}$, we can find another object representing $D_{\rho_x,\underline{d}}^{\square}$ whose dimension can be computed explicitly. Note $D_{\rho_x,\underline{d}}^{\square}$ contains data about lifting representations, but also data about lifting filtrations, and we can consider these separately, essentially by picking a basis (which we do by fixing a parabolic determined by a fixed element of $\mathcal{F}_{\underline{d}}$).

Let P denote the parabolic subgroup of $\mathrm{GL}_{d,F}$ corresponding to $\mathrm{Fil}_x^{\bullet}$, so that ρ_x naturally lands in $P(F)$. Let $\mathfrak{p}$ denote the Lie algebra of P, which naturally comes equipped with a G_K-action via the adjoint action of $P(F)$ on $\mathfrak{p}$: in other words $\sigma \cdot M = \rho_x(\sigma) M \rho_x(\sigma)^{-1}$.

We define a functor

$$D_{\rho_x,P}^{\square} : \mathsf{Art}_F \to \mathsf{Set}, B \mapsto \{\rho : G_K \xrightarrow{\text{cts}} P(B) : \rho \text{ lifts } \rho_x\}$$

parametrizing P-deformations of ρ_x to local Artinian F-algebras with residue field F.

Proposition 3.2.2.3. *The deformation problem $D_{\rho_x,P}^{\square}$ is pro-represented by a complete local*

Noetherian F-algebra $R^{\square}_{\rho_x,P}$ such that

$$\dim R^{\square}_{\rho_x,P} \leq (\text{د} + 1)(\dim \mathfrak{p}) + (d^2 - \dim \mathfrak{p})$$

Proof. The existence of $R^{\square}_{\rho_x,P}$ is standard, and follows from (for example) a slightly modified version of [Böc13, Proposition 1.3.1], replacing GL_d with P. The tangent space of $D^{\square}_{\rho_x,P}$ consists of the $F[\epsilon]/(\epsilon^2)$-points of $D^{\square}_{\rho_x,P}$. A standard argument shows that any such lift $\rho \in D^{\square}_{\rho_x,P}(F[\epsilon]/(\epsilon^2))$ can be written uniquely as $\sigma \mapsto (1 + c(\sigma)x)\rho_x(\sigma)$ for some continuous 1-cocycle $c : G_K \to \mathfrak{p}$. Therefore, the tangent space at the closed point has dimension $\dim Z^1(G_K, \mathfrak{p})$, and another standard argument says that $R^{\square}_{\rho_x,P}$ can be written as a quotient of a power series ring over F in $\dim Z^1(G_K, \mathfrak{p})$ variables. But we have

$$\begin{aligned}
\dim Z^1(G_K, \mathfrak{p}) &= \dim H^1(G_K, \mathfrak{p}) + \dim B^1(G_K, \mathfrak{p}) \\
&= \dim H^1(G_K, \mathfrak{p}) + (\dim \mathfrak{p} - \dim \mathfrak{p}^{G_K}) \\
&= \dim H^1(G_K, \mathfrak{p}) + \dim \mathfrak{p} - \dim H^0(G_K, \mathfrak{p}) \\
&= (\text{د} + 1) \dim \mathfrak{p} + \dim H^2(G_K, \mathfrak{p})
\end{aligned}$$

by the local Euler characteristic formula: here Z^1 denotes 1-cocycles and B^1 denotes 1-coboundaries.

It now suffices to bound $\dim H^2(G_K, \mathfrak{p})$, which by local Tate duality is $\dim H^0(G_K, \mathfrak{p}^\vee \otimes \epsilon)$. If $\mathfrak{n}_P$ is the Lie algebra of the unipotent radical of P, then we have short exact sequence of $F[G_K]$-modules

$$0 \to \mathfrak{n}_P \to \mathfrak{p} \to \mathfrak{l}_P \to 0,$$

where $\mathfrak{l}_P$ is the Levi quotient. Dualizing, twisting by the cyclotomic character ϵ, and taking the associated G_K-cohomology long exact sequence, we get

$$0 \to H^0(G_K, \mathfrak{l}_P^\vee \otimes \epsilon) \to H^0(G_K, \mathfrak{p}^\vee \otimes \epsilon) \to H^0(G_K, \mathfrak{n}_P^\vee \otimes \epsilon) \to \cdots .$$

Note $\dim H^0(G_K, \mathfrak{n}_P^\vee \otimes \epsilon) \leq \dim \mathfrak{n}_P = d^2 - \dim \mathfrak{p}$, so we are done if we can show that

$$H^0(G_K, \mathfrak{l}_P^\vee \otimes \epsilon) = 0.$$

Since $x \in \mathcal{G}_{\underline{d}}^{\mathrm{irr}}(F)$, we can find a basis respecting $\mathrm{Fil}_x^\bullet$ in which

$$\rho_x \cong \begin{pmatrix} \alpha_1 & * & * \\ & \ddots & * \\ & & \alpha_m \end{pmatrix}$$

for some (absolutely) irreducible representations α_i of dimension d_i. One can compute that $\mathfrak{l}_P$ is isomorphic, as a G_K-representation, to $\bigoplus_{i=1}^m \mathfrak{gl}_{d_i}$, where each $\mathfrak{gl}_{d_i}$ is the Lie algebra of $\mathrm{GL}_{d_i}(F)$ equipped with the G_K-action induced by $G_K \xrightarrow{\alpha_i} \mathrm{GL}_{d_i}(F) \xrightarrow{\mathrm{ad}} \mathrm{GL}(\mathfrak{gl}_{d_i})$. Thus we have an isomorphism of G_K-representations

$$\mathfrak{l}_P^\vee \otimes \epsilon \cong \bigoplus_{i=1}^m (\mathfrak{gl}_{d_i}^\vee \otimes \epsilon),$$

and we conclude that

$$H^0(G_K, \mathfrak{l}_P^\vee \otimes \epsilon) = \bigoplus_{i=1}^m H^0(G_K, \mathfrak{gl}_{d_i}^\vee \otimes \epsilon) = \bigoplus_{i=1}^m \mathrm{Hom}_{F[G_K]}(\alpha_i, \alpha_i \otimes \epsilon) = 0,$$

where the last equality follows from the fact that the α_i are irreducible and $\alpha_i \not\cong \alpha_i \otimes \epsilon$ (e.g. they have nonisomorphic determinant). $\square$

Now define a functor

$$D_{\mathrm{Fil}_x^\bullet} : \mathsf{Art}_F \to \mathsf{Set}, B \mapsto \{\text{lifts of } \mathrm{Fil}_x^\bullet \text{ in } B^n\}$$

This is represented by the completed local ring $\widehat{\mathscr{O}}_{\mathcal{F}_{\underline{d}}, \mathrm{Fil}_x^\bullet}$, which is isomorphic to a power series ring over F in $d^2 - \dim \mathfrak{p}$ variables (the flag variety is smooth and isomorphic to $\mathrm{GL}_{d,F} / P$).

Let $\widehat{\mathrm{Fil}}_x^\bullet \in D_{\mathrm{Fil}_x^\bullet}(\widehat{\mathscr{O}}_{\mathcal{F}_{\underline{d}},\mathrm{Fil}_x^\bullet})$ denote the universal filtration for this deformation problem. Each $\widehat{\mathrm{Fil}}_x^i$ is free since $\widehat{\mathscr{O}}_{\mathcal{F}_{\underline{d}},\mathrm{Fil}_x^\bullet}$ is local, and moreover there exists some $\varphi \in \mathrm{GL}_d(\widehat{\mathscr{O}}_{\mathcal{F}_{\underline{d}},\mathrm{Fil}_x^\bullet})$ such that

$$\widehat{\mathrm{Fil}}_x^\bullet = \varphi(\mathrm{Fil}_x^\bullet \otimes_F \widehat{\mathscr{O}}_{\mathcal{F}_{\underline{d}},\mathrm{Fil}_x^\bullet}),$$

and such that φ reduces to the identity map mod the maximal ideal of $\widehat{\mathscr{O}}_{\mathcal{F}_{\underline{d}},\mathrm{Fil}_x^\bullet}$.

Proposition 3.2.2.4. *There is an isomorphism of functors*

$$D^\square_{\rho_x,\underline{d}} = D^\square_{\rho_x,P} \times D_{\mathrm{Fil}_x^\bullet}.$$

Proof. Fix a point $(\rho, \mathrm{Fil}^\bullet) \in D^\square_{\rho_x,\underline{d}}(A)$ lifting $(\rho_x, \mathrm{Fil}_x^\bullet)$ so that ρ fixes $\mathrm{Fil}^\bullet$. The filtration $\mathrm{Fil}^\bullet$ is induced by a map $\widehat{\mathscr{O}}_{\mathcal{F}_{\underline{d}},\mathrm{Fil}_x^\bullet} \to A$, and if we push forward φ along this map then we get some $\varphi_A \in \mathrm{GL}_d(A)$ such that $\mathrm{Fil}^\bullet = \varphi_A(\mathrm{Fil}_x^\bullet \otimes_F A)$, and such that φ_A reduces to 1 mod $\mathfrak{m}_A$. Therefore, $\varphi_A^{-1}\rho\varphi_A$ fixes $\mathrm{Fil}_x^\bullet$ and lands in $P(A)$. Then the map $D^\square_{\rho_x,\underline{d}} \to D^\square_{\rho_x,P} \times D_{\mathrm{Fil}_x^\bullet}$ given by $(\rho, \mathrm{Fil}^\bullet) \mapsto (\varphi_A^{-1}\rho\varphi_A, \mathrm{Fil}^\bullet)$ is a functorial bijection, with inverse $(\rho, \mathrm{Fil}^\bullet) \mapsto (\varphi_A\rho\varphi_A^{-1}, \mathrm{Fil}^\bullet)$. □

Corollary 3.2.2.5. *The ring $\widehat{\mathscr{O}}_{\mathcal{G}_{\underline{d}},x}$ is isomorphic to a power series ring over $R^\square_{\rho_x,P}$ in $d^2 - \dim\mathfrak{p}$ variables.*

Now we can simply compute. Recall that we assumed $x \in \mathcal{G}_{\underline{d}}^{\mathrm{irr}}$.

Proposition 3.2.2.6. *The ring $\widehat{\mathscr{O}}_{\mathcal{G}_{\underline{d}},x}$ satisfies*

$$\dim \widehat{\mathscr{O}}_{\mathcal{G}_{\underline{d}},x} \leq (\daleth + 1)(\dim\mathfrak{p}) + 2(d^2 - \dim\mathfrak{p}).$$

Proof. This follows from Proposition 3.2.2.3 and Corollary 3.2.2.5. □

The upshot is the following theorem.

Theorem 3.2.2.7. *Assume $q > 2$. Then $X_{\overline{\rho}}^{\square}$ is regular in codimension 1. In particular, since it is Cohen-Macaulay, $X_{\overline{\rho}}^{\square}$ is normal (by Serre's criterion for normality).*

Proof. We have shown that

$$\dim X_{\overline{\rho}}^{\square,\mathrm{sing}} \leq \max_{\underline{d}\in P(d)^{\circ}} \dim \mathcal{G}_{\underline{d}}^{\mathrm{irr}} \leq \max_{\underline{d}\in P(d)^{\circ}} (\daleth+1)(\dim \mathfrak{p}_{\underline{d}}) + 2(d^2 - \dim \mathfrak{p}_{\underline{d}}).$$

where $\mathfrak{p}_{\underline{d}}$ is the Lie algebra of a parabolic $P_{\underline{d}}$ of shape determined by $\underline{d}$. But note that $\dim X_{\overline{\rho}}^{\square} = d^2(\daleth+1)$ by Proposition 3.1.2.2, so the singular locus has codimension

$$\dim X_{\overline{\rho}}^{\square} - \dim X_{\overline{\rho}}^{\square,\mathrm{sing}} \geq \min_{\underline{d}\in P(d)^{\circ}} (\daleth-1)(d^2 - \dim \mathfrak{p}_{\underline{d}})$$

which is strictly bigger than 1 whenever $\daleth > 2$. If our bound is exactly 1, then we must have $\daleth = 2$. This means that either $p = 2$ and $K = \mathbb{Q}_2(\mu_4)$, or $p = 3$ and $K = \mathbb{Q}_3(\mu_3)$. In either case we must have $d = 2$.

One can still prove the theorem in these cases, but it requires a slight modification, so we say a few words about how to do this. By [CDP15, Proposition 4.2], if $x \in X_{\overline{\rho}}^{\square}(F)$ is a singular closed point, there is an exact sequence $0 \to \delta \to \rho_x \to \delta \otimes \epsilon \to 0$. In other words, ρ_x fixes some full flag $0 \subset \mathrm{Fil}^1 \subset \mathrm{Fil}^2 = F^2$, and $\rho_x|_{\mathrm{Fil}^1} \otimes \epsilon = \rho_x|_{\mathrm{Fil}^2/\mathrm{Fil}^1}$. So we can leverage this to get a sharper bound on $\dim H^2(G_K, \mathfrak{p})$ in Proposition 3.2.2.3.

First of all, proper parabolics are Borels, so

$$\bigsqcup_{\underline{d}\in P(2)^{\circ}} \mathcal{G}_{\underline{d}}^{\mathrm{irr}} = \mathcal{G}_{(1,1)}.$$

We can define a closed subspace $\mathcal{G}_{(1,1)}^{\epsilon} \subset \mathcal{G}_{(1,1)}$.

$$\mathcal{G}_{(1,1)}^{\epsilon}(A) = \{(f, \mathrm{Fil}^{\bullet}) \in \mathcal{G}_{(1,1)} : \rho_f|_{\mathrm{Fil}^1} = \rho_f|_{\mathrm{Fil}^2/\mathrm{Fil}^1} \otimes \epsilon\}.$$

By the discussion above, the scheme theoretic image of $\mathcal{G}^{\epsilon}_{(1,1)}[1/\varpi] \to X^{\square}_{\overline{\rho}}$ still contains $X^{\square,\mathrm{sing}}_{\overline{\rho}}$. Then if B is the Borel corresponding to $\mathrm{Fil}^{\bullet}$, one can do a direct matrix computation and show that $H^0(G_K, \mathfrak{n}_B^{\vee} \otimes \epsilon) = 0$, which implies that $H^2(G_K, \mathfrak{b}) = 0$ where $\mathfrak{b}$ is the Lie algebra of B. Thus $\dim X^{\square}_{\overline{\rho}} = 12$, but the singular locus is at most 10-dimensional. $\square$

3.3 Irreducible components

The goal of this section is to prove that the irreducible components of $X^{\square}_{\overline{\rho}}$ are exactly parametrized by the qth roots of unity.

To see why this might be true, notice that the equation

$$\widetilde{X}_1^q[\widetilde{X}_1, \widetilde{X}_2] \cdots [\widetilde{X}_{\text{د}+1}, \widetilde{X}_{\text{د}+2}] = I \tag{3.1}$$

implies that $\det(\widetilde{X}_1)^q = 1$ in $R^{\square}_{\overline{\rho}}[1/\varpi]$ and thus induces a map

$$\pi : X^{\square}_{\overline{\rho}} \to \mu_{q,L} := \operatorname{Spec} L[x]/(x^q - 1).$$

Since L is large enough $\mu_{q,L}$ is just the disjoint union of q copies of $\operatorname{Spec} L$, one for each qth root of unity in L. Since the image of any connected component of $X^{\square}_{\overline{\rho}}$ is connected, it must be sent to one of these points. Thus, for each point $\zeta \in \mu_{q,L}$, $X_\zeta := X^{\square}_{\overline{\rho}} \times_{\mu_{q,L},\zeta} \operatorname{Spec} L$ is a union of connected components of $X^{\square}_{\overline{\rho}}$. In fact, we will show that Equation 3.1 provides enough leverage to connect every pair of points in X_ζ, and thus to conclude that X_ζ is connected. Thus, the goal is to prove:

Theorem 3.3.0.1. *Assume $q \neq 2$, and further assume $p > d$ if $q > 2$. Then $X^{\square}_{\overline{\rho}}$ breaks into the union of q connected (and therefore irreducible, since $X^{\square}_{\overline{\rho}}$ is normal) components*

$$X^{\square}_{\overline{\rho}} = \bigsqcup_{\zeta \in \mu_q(L)} X_\zeta.$$

Remark 3.3.0.2. As mentioned in the introduction, the assumption that $p > d$ is a hypothesis that we hope to be able to remove; for now, we do not see an easy way to deal with the extra technicalities that arise.

3.3.1 Connectedness

Now fix a qth root of unity $\zeta \in L$. In [CDP15], the notion of arc-connectedness between points is introduced to prove that a scheme is connected. The idea is that for any two closed points $x_0, x_1 \in X_\zeta(F)$ (where F/L is a finite extension) one should find a path from x_0 to x_1 by exhibiting $x_0 = x(0)$ and $x_1 = x(1)$ for some point $x \in X_\zeta(T_F)$, where T_F is the Tate algebra in one variable over F. From the rigid analytic viewpoint, this amounts to connecting x_0 and x_1 via a path parametrized by a closed unit disk.

More generally, suppose $x_0, x_1 \in X_\zeta(F)$ are two closed points. If we can find a connected L-scheme Y, and a map $Y \to X_\zeta$ whose image contains both x_0 and x_1 then they are contained in the same connected component of X_ζ.

There are two examples of Y (as in the definition) that we will use.

- In [CDP15] the following example is used: let T_F denote the Tate algebra in one variable, which is the subring of $F[\![t]\!]$ consisting of power series $\sum_n c_n t^n$ for which $|c_n| \to 0$ as $n \to \infty$, equipped with the sup norm taken over the coefficients. We then take $Y = \operatorname{Spec} T_F$. As mentioned before, this is essentially the method of connecting the two points in the associated rigid analytic space attached to X_ζ via the rigid closed unit disk.

- Consider the following affinoid version of GL_d:

$$\mathscr{O}_{\mathrm{GL}_d,F} = (\mathcal{O}_F[(x_{ij})_{i,j=1,\dots,d}, b]^\wedge_{\mathfrak{m}_F}/(\det(x_{ij})b - 1))[1/\varpi_F].$$

We then take $Y = \operatorname{Spec} \mathscr{O}_{\mathrm{GL}_d,F}$. Note that $Y(F) = \mathrm{GL}_d(\mathcal{O}_F)$ and thus the $\mathrm{GL}_d(\mathcal{O}_F)$-orbit of an F-valued deformation (which is essentially given by some tuple of invertible matrices and here $\mathrm{GL}_d(\mathcal{O}_F)$ acts by conjugating such a tuple) is contained in a single connected component.

The rest of this section will be devoted to proving the following Proposition.

Proposition 3.3.1.1. *For any finite extension F/L, any two closed points in $X_\zeta(F)$ are contained in the same connected component of X_ζ.*

We note the following corollary:

Proof of Theorem 3.3.0.1. If X_ζ is not connected, then since X_ζ is Jacobson, we can find two *closed* points $x, y \in X_\zeta(F)$ living on different connected components, where F/L can be taken to be finite by Lemma 3.2.0.1. The result then follows immediately from Proposition 3.3.1.1. □

3.3.2 Restriction to a closed subspace

For F/L a finite extension, an F-point of X_ζ is the data of a tuple $(M_1, \ldots, M_{d+2}) \subset 1 + \mathrm{Mat}_d(\mathfrak{m}_F)$ satisfying the equations

$$M_1^q[M_1, M_2] \cdots [M_{\text{د}+1}, M_{\text{د}+2}] = I, \det(M_1) = \zeta.$$

It is difficult to get any useful intuition for this equation in its full form, so it is helpful to first restrict to a certain nicely chosen closed subspace of X_ζ whose F-points satisfy a more useful equation. To see that this suffices, we note the following fact.

Proposition 3.3.2.1 ([CDP15, Proposition 5.1]). *Suppose A is a Cohen-Macaulay Noetherian local ring and $x_1, \ldots, x_k, x$ is a regular sequence in A. Then every irreducible component* *of* $\operatorname{Spec} A[1/x]$ *meets the closed subset* $\operatorname{Spec} A/(x_1, \ldots, x_k)[1/x]$.

In the proof of Proposition 3.1.2.2 we showed that the coefficients of $(X_3, \ldots, X_{\text{ɔ}+2})$ (in any order) along with ϖ form a regular sequence in $R^{\square}_{\bar{\rho}}$, so in particular, we want to consider the closed subspace $V \subset X^{\square}_{\bar{\rho}}$ defined by

$$V = \operatorname{Spec} R^{\square}_{\bar{\rho}}/(X_3, \ldots, X_{\text{ɔ}+2})[1/\varpi].$$

Let $V_\zeta = X_\zeta \times_{X^{\square}_{\bar{\rho}}} V$.

Corollary 3.3.2.2. *Fix a finite extension* F/L*. If any two closed points in* $V_\zeta(F)$ *are contained in the same connected component of* X_ζ*, then any two closed points in* $X_\zeta(F)$ *are contained in the same connected component of* X_ζ*.*

Proof. This is implied by Proposition 3.3.2.1, as follows: Let x_0, x_1 denote two closed points in $X_\zeta(F)$ and let Z_0 and Z_1 denote the connected component of X_ζ containing them, respectively. Note these are also connected components of $X^{\square}_{\bar{\rho}}$. By normality of $X^{\square}_{\bar{\rho}}$ and Proposition 3.3.2.1, Z_0 and Z_1 both meet $V \cap X_\zeta = V_\zeta$. Note Z_0 and Z_1 are both connected, so we can replace x_0, x_1 with some closed (note V_ζ is Jacobson) points $v_0, v_1 \in V_\zeta(F)$, after possibly extending F by a finite extension. But these are contained in the same connected component by assumption. □

In the remainder of the section, we show that any two points in $V_\zeta(F)$ are contained in the same connected component of X_ζ. To do this, we connect every point in $V_\zeta(F)$ to the point corresponding to

$$M_1 = \operatorname{diag}(\zeta, 1, \ldots, 1), M_2 = \cdots = M_{\text{ɔ}+2} = I.$$

3.3.3 Constructing paths

Fix a point $x \in V_\varsigma(F)$ for some finite extension F/L. This is the same as giving a pair $M_1, M_2 \in 1 + \mathrm{Mat}_d(\mathfrak{m}_F)$ such that

$$M_2 M_1 M_2^{-1} = M_1^{q+1}.$$

(all other M_i are equal to I in this subspace). Enlarge F if needed so that F contains every eigenvalue of M_1. The equation above implies that the $(q+1)$-power map from the set of eigenvalues of M_1 to itself is a bijection, so if λ is an eigenvalue of M_1, then there exists some a in the range $[1, d]$ such that $\lambda^{(q+1)^a} = \lambda$. In other words, $\lambda^{q(qm+a)} = 1$ for some positive integer m. So λ^q is a $(qm+a)$th root of unity that reduces to 1 mod $\mathfrak{m}_F$, but $p > d$ and thus $p \nmid a$, so actually $\lambda^q = 1$ by Hensel's Lemma and the fact that $\gcd(qm + a, p) = 1$.

Remark 3.3.3.1. It is the possibility of the eigenvalues being higher order p-power roots of unity that forces us, for the time being, to use the assumption that $p > d$. We hope to be able to find another argument that works for higher order eigenvalues of M_1.

Proposition 3.3.3.2. *The point $x \in V_\varsigma(F)$ is in the same connected component of X_ς as the point defined by $(\mathrm{diag}(\lambda_1, \dots, \lambda_d), I)$, where the λ_i is some ordering of the eigenvalues of M_1.*

Proof. We regard M_1, M_2 as elements in $\mathrm{GL}(F^d)$. Since the eigenvalues of M_1 are contained in $\mu_q(F)$, we have a decomposition into generalized eigenspaces

$$F^d = \bigoplus_{\lambda \in \mu_q(F)} W_\lambda.$$

For any eigenvalue $\lambda \in \mu_q(F)$, consider the filtration $\mathrm{Fil}_\lambda^\bullet$ defined by

$$0 \subset \mathrm{Fil}_\lambda^1 = \ker(X_1 - \lambda)^1 \subset \cdots \subset \mathrm{Fil}_\lambda^{m_\lambda} = \ker(X_1 - \lambda)^{m_\lambda} = W_\lambda,$$

where m_λ is the maximum size of a Jordan block of M_1 with eigenvalue λ. Let

$$f(x) = \frac{x^{q+1} - \lambda}{x - \lambda} = \prod_{i=1}^{q}(x - \zeta_{q+1}^i \lambda)$$

where ζ_{q+1} is a primitive $(q+1)$th root of unity in F (enlarge F if needed). Note $\zeta_{q+1}^i = 1$ if and only if $\zeta_{q+1}^i \equiv 1 \mod \mathfrak{m}_F$ by Hensel's Lemma. Then $M_1^{q+1} - \lambda = f(M_1)(M_1 - \lambda)$, and each of the $M_1 - \zeta_{q+1}^i \lambda$ are invertible because $\zeta_{q+1}^i \lambda \not\equiv 1 \mod \mathfrak{m}_F$, so $f(M_1)$ is invertible. If $v \in \ker(M_1 - \lambda)^a$ then

$$f(M_1)^a (M_1 - \lambda)^a M_2 v = (M_1^{q+1} - \lambda)^a M_2 v = M_2 (M_1 - \lambda)^a v = 0,$$

so multiplying by $f(M_1)^{-a}$ shows that M_2 preserves both W_λ and the filtration Fil_λ. Let $d_\lambda = \dim W_\lambda$ and let $e_1, \ldots, e_{d_\lambda}$ be a basis of W_λ constructed by picking a basis for each Fil_λ^i for which M_2 is upper triangular, and then concatenating them together in order. Note M_1 clearly respects Fil_λ and acts via the scalar λ on $\mathrm{Fil}_\lambda^i / \mathrm{Fil}_\lambda^{i-1}$ so in particular M_1 is upper triangular for $e_1, \ldots, e_{d_\lambda}$. Thus if $E \in \mathrm{GL}_d(F)$ is the change of basis matrix that takes the standard basis of F^d into the basis determined by the e_i, then EM_1E^{-1} and EM_2E^{-1} are both upper triangular. But by the Iwasawa decomposition we can write $E = NE_0$ where $N \in B(F)$ is an element of the standard Borel and $E_0 \in \mathrm{GL}_d(\mathcal{O}_F)$, and thus $M_1' := E_0 M_1 E_0^{-1}$ and $M_2' := E_0 M_2 E_0^{-1}$ are still upper triangular. But this allows us to define a map

$$\operatorname{Spec} \mathscr{O}_{\mathrm{GL}_d, F} \to V_\zeta$$

taking $g \mapsto (gM_1g^{-1}, gM_2g^{-1})$. Specializing at $g = I$ gives x, and specializing at $g = E_0 \in \mathrm{GL}_d(\mathcal{O}_F)$ gives a point $x' \in V_\zeta(F)$ determined by M_1' and M_2', so both (M_1, M_2) and (M_1', M_2') live in the same connected component of V_ζ. Now we define a path

$$\operatorname{Spec} T_F \to V_\zeta$$

by $t \mapsto (g(t)M_1'g(t)^{-1}, g(t)M_2'g(t)^{-1})$ where

$$g(t) = \mathrm{diag}(t^{d-1}, t^{d-2}, \dots, t, 1).$$

This is well defined since M_1', M_2' are upper-triangular (in particular, the $g(t)$-conjugated matrix has no negative powers of t and doesn't affect the diagonal). and has the effect of "killing the strictly upper-triangular part", in the sense that specializing at $t = 1$ gives the point x' and specializing at $t = 0$ gives the point $x^\star$ with corresponding matrices

$$M_1^\star = \mathrm{diag}(\lambda_1, \dots, \lambda_d) \mathrm{and} M_2^\star = \mathrm{diag}(1 + m_1, \dots, 1 + m_d)$$

for some $m_i \in \mathfrak{m}_F$. But then the path $M_1^\star(t) = M_1^\star$ and $M_2^\star(t) = \mathrm{diag}(1 + tm_1, \dots, 1 + tm_d)$ connects $M_2^\star$ to I. □

Proposition 3.3.3.3. *If F/L is a finite extension and $x \in X_\zeta(F)$ is the point*

$$M_1 = \mathrm{diag}(\lambda_1, \dots, \lambda_n), M_2 = \dots = M_{\daleth+2} = I$$

with $\lambda_1 \cdots \lambda_d = \zeta$ then x is in the same connected component of X_ζ as the point $x_\zeta \in X_\zeta(F)$ corresponding to

$$M_1 = \mathrm{diag}(\zeta, 1, \dots, 1), M_2 = \dots = M_{\daleth+2} = I.$$

Proof. The idea is to treat the $d = 2$ case and then focus on the 2×2 diagonal blocks and replace $M_1 = \mathrm{diag}(\lambda_1, \dots, \lambda_d)$ by $\mathrm{diag}(\lambda_1, \dots, \lambda_{d-1}\lambda_d, 1)$ and then $\mathrm{diag}(\lambda_1, \dots, \lambda_{d-2}\lambda_{d-1}\lambda_d, 1, 1)$ etc.

For each $\lambda \in \mu_q(F)$, define a character

$$c_\lambda : G_K \to (G_K^p)^{\mathrm{ab}} \cong \langle x_1, \ldots, x_{\text{د}+2} : x_1^q = 1 \rangle^{\mathrm{ab}} \to \mathcal{O}^\times$$

$$(x_1, x_2, \ldots, x_{\text{د}+2}) \mapsto (\lambda, 1, \ldots, 1)$$

and let $R_{\overline{\rho},2}^{\square,\lambda}$ denote the 2-dimensional universal framed deformation ring for $\overline{\rho}$ with fixed determinant c_λ, which comes with a universal lift $\rho^\lambda : G_K \to \mathrm{GL}_2(R_{\overline{\rho},2}^{\square,\lambda})$ and generic fibre $X_{\overline{\rho},2}^{\square,\lambda} = \operatorname{Spec} R_{\overline{\rho},2}^{\square,\lambda}[1/p]$.

By universality, the representation

$$c_{\lambda_1} \oplus \cdots \oplus c_{\lambda_{d-2}} \oplus \rho^{\lambda_{d-1}\lambda_d} : G_K \to \mathrm{GL}_d(R_{\overline{\rho},2}^{\square,\lambda_{d-1}\lambda_d})$$

induces a map $R_{\overline{\rho}}^{\square} \to R_{\overline{\rho},2}^{\square,\lambda_{d-1}\lambda_d}$ and thus a map $X_{\overline{\rho},2}^{\square,\lambda_{d-1}\lambda_d} \to X_{\overline{\rho}}^{\square}$ which, by definition, descends to a map $X_{\overline{\rho},2}^{\square,\lambda_{d-1}\lambda_d} \to X_\zeta$. By [BJ15, Theorem 1.5 and Remark 1.7] (combined with the fact that "versal" and "framed" are interchangeable for the trivial representation, cf. Lemma 3.1.0.1), $R_{\overline{\rho},2}^{\square,\lambda_{d-1}\lambda_d}$ is an integral domain, so $X_{\overline{\rho},2}^{\square,\lambda_{d-1}\lambda_d}$ is irreducible. But this $(X_{\overline{\rho},2}^{\square,\lambda_{d-1}\lambda_d})$-point of X_ζ specializes to both x and the point given by

$$M_1 = \mathrm{diag}(\lambda_1, \ldots, \lambda_{d-1}\lambda_d, 1), M_2 = \cdots = M_{\text{د}+2} = I$$

so these points are contained in the same irreducible component of X_ζ. Now we repeat the process with $c_{\lambda_1} \oplus \cdots \oplus c_{\lambda_{d-3}} \oplus \rho^{\lambda_{d-2}\lambda_{d-1}\lambda_d} \oplus 1$, then $c_{\lambda_1} \oplus \cdots \oplus c_{\lambda_{d-4}} \oplus \rho^{\lambda_{d-3}\lambda_{d-2}\lambda_{d-1}\lambda_d} \oplus 1 \oplus 1$, etc. After $d-1$ iterations of this process, we specialize to x_ζ. □

Remark 3.3.3.4. In the previous version [Iye19] of this chapter, we constructed an explicit path between the two points, which worked under the hypothesis $[K : \mathbb{Q}_p] \geq 6$. At the suggestion of the anonymous journal referee, we used the results from [BJ15] to remove this extra assumption. We thank the referee for this suggestion.

Proof of Proposition 3.3.1.1. By Proposition 3.3.3.2 and Proposition 3.3.3.3, any two points in $V_\zeta(F)$ are contained in the same connected component of X_ζ. Then apply Corollary 3.3.2.2. $\square$

3.3.4 Deforming 1

To prove Theorem 1.4.0.4 for the trivial representation, we need to describe $X^{\square}_{\det \overline{\rho}} = X^{\square}_{\mathbf{1}}$.

Let $\mathbf{1} : G_K^{\mathrm{ab}} \to k^\times$ denote the trivial character, and let $D_{\mathbf{1}} : \mathsf{Art}_{\mathcal{O}} \to \mathsf{Set}$ denote the universal deformation problem for $\mathbf{1}$. Deformations of the trivial character to a ring $A \in \mathsf{Art}_{\mathcal{O}}$ are valued in the p-group $1 + \mathfrak{m}_A$, so they are really representations of $(G_K^p)^{\mathrm{ab}}$, which by part 2 of Theorem 3.1.1.2 is the free abelian pro-p group on generators $g_1, \ldots, g_{\text{د}+2}$ subject to the single relation $g_1^q = 1$. Therefore, Proposition 3.1.1.4 (with $d = 1$) implies the following lemma.

Lemma 3.3.4.1. *$D_{\mathbf{1}}$ is pro-represented by the complete local Noetherian ring*

$$R_{\mathbf{1}} = \mathcal{O}[\![x_1, \ldots, x_{\text{د}+2}]\!]/((1 + x_1)^q - 1).$$

Corollary 3.3.4.2. *$R_{\mathbf{1}}$ is $\mathcal{O}$-torsion free and has q irreducible components, given by $x_1 = \zeta - 1$ for each $\zeta \in \mu_q(L)$. These are also the connected components.*

Proof. Note ϖ does not divide $(1 + x_1)^q - 1$, so $R_{\mathbf{1}}$ is $\mathcal{O}$-torsion free. We have

$$\mathcal{O}[\![x_1, \ldots, x_{\text{د}+2}]\!]/((1 + x_1)^q - 1) \cong \mathcal{O}[x_1]/((1 + x_1)^q - 1) \otimes_{\mathcal{O}} \mathcal{O}[\![x_2, \ldots, x_{\text{د}+2}]\!],$$

but $\mathcal{O}[\![x_2, \ldots, x_{\text{د}+2}]\!]$ is an integral domain, so it suffices to describe the irreducible components of $\mathcal{O}[x_1]/((1 + x_1)^q - 1)$. Furthermore, since $R_{\mathbf{1}}$ is $\mathcal{O}$-torsion free it is $\mathcal{O}$-flat, so it suffices to check the description of the irreducible components on the generic fibre

$L[x_1]/((1+x_1)^q - 1)$. But Spec of this ring is just the finite set of L-points $x_1 = \zeta - 1$ for each $\zeta \in \mu_q(L)$. $\square$

Remark 3.3.4.3. The local reciprocity map $\widehat{K^\times} \xrightarrow{\sim} G_K^{\mathrm{ab}} \twoheadrightarrow (G_K^p)^{\mathrm{ab}}$ sends some primitive qth root of unity ζ to g_1. The irreducible component containing a closed point $x \in X_{\mathbf{1}}$ with residue field F/L and corresponding character $\chi_x : G_K^{\mathrm{ab}} \to R_{\mathbf{1}}^\times \to F^\times$ is determined by the element $\chi_x(g_1) = (\chi_x \circ \mathrm{rec}_K)(\zeta) \in \mu_q(L)$.

3.3.5 Proof of Theorem 3.3.0.1

Proof. If $q = 1$, then $R_{\overline{\rho}}^\square$ and $R_{\det \overline{\rho}}^\square$ are formally smooth over $\mathcal{O}$, so there is nothing to prove. Assume $q > 2$. Then Theorem 3.2.2.7 says that $\mathcal{X}_{\overline{\rho}}^\square$ is normal. If $p > d$, the set $\mu_q(K)$ classifies the connected components of both $X_{\det \overline{\rho}}^\square$ and $X_{\overline{\rho}}^\square$, and this classification is visibly compatible with respect to the determinant map $d : X_{\det \overline{\rho}}^\square \to X_{\overline{\rho}}^\square$. Since the preimage of a connected component is the union of connected components, d must induce a bijection between the connected components, which are irreducible. $\square$

3.4 Crystalline density

In this section we prove that the crystalline closed points of $X_{\overline{\rho}}^\square$ (i.e. closed points whose induced p-adic representations are crystalline) are Zariski dense. In particular, we show that each irreducible component contains a crystalline point, and the work of Nakamura in [Nak14] (following Chenevier and Gouvêa-Mazur) shows that the Zariski closure of the crystalline points is the union of some collection of the irreducible components. However, Nakamura assumes that $\mathrm{End}_{k[G_K]}(\overline{\rho}) = k$ and works with the unframed deformation space, so we need to slightly modify the arguments to work in the framed case. To do so, we use work of Breuil-Hellmann-Schraen in [BHS17b] (and related papers) on the *trianguline deformation space* that is related to the finite slope subspace used by Nakamura, and is already framed.

Note throughout this section we will actually work with the *rigid* generic fibre $\mathcal{X}_{\overline{\rho}}^{\square}$.

3.4.1 Review of trianguline deformation theory

Here we recall the definition of the trianguline deformation space, following [BHS17b]. From now on, assume that L contains the Galois closure of K, i.e. L contains every embedding of K into $\overline{\mathbb{Q}_p}$. Let $f = [K_0 : \mathbb{Q}_p]$, where K_0 is the maximal unramified extension of $\mathbb{Q}_p$ contained in K. Let Rig_L denote the category of rigid analytic spaces over L. We define $\mathcal{T} : \mathsf{Rig}_L \to \mathsf{Ab}$ taking

$$\mathcal{T}(X) = \{\text{continuous characters } K^\times \to \mathscr{O}_X(X)^\times\}$$

whose group structure is given by multiplication of characters. This is represented by a rigid analytic group variety, also denoted $\mathcal{T} \in \mathsf{Rig}_L$. If $\mathbf{h} \in \mathbb{Z}^{\mathrm{Hom}(K,L)}$, then we define the *algebraic* character

$$(\cdot)^{\mathbf{h}} : K^\times \mapsto L^\times, z \mapsto \prod_{\tau : K \hookrightarrow L} \tau(z)^{h_\tau}.$$

Fix a uniformizer ϖ_K of K. Let $|\cdot|_K$ denote the normalized ϖ_K-adic absolute value on K, and let val_K denote the corresponding valuation. Then the set of L-points

$$\{(\cdot)^{-\mathbf{h}}, |\cdot|_K (\cdot)^{\mathbf{h+1}} : \mathbf{h} \in (\mathbb{Z}_{\geq 0})^{\mathrm{Hom}(K,L)}\}$$

is Zariski closed in $\mathcal{T}$, and we define $\mathcal{T}_{\mathrm{reg}}$ to be its open complement inside $\mathcal{T}$. Further, let $\mathcal{T}^d_{\mathrm{reg}}$ denote the Zariski open subset of $\mathcal{T}^d$ defined by

$$\mathcal{T}^d_{\mathrm{reg}} = \{(\delta_1, \ldots, \delta_d) : \delta_i/\delta_j \in \mathcal{T}_{\mathrm{reg}} \text{ whenever } i \neq j\}.$$

The theory of trianguline representations is part of the theory of (φ, Γ_K)-modules over the Robba ring, where $\Gamma_K = \mathrm{Gal}(K(\mu_{p^\infty})/K)$. Let $K_0' := W(k_{K(\mu_{p^\infty})})[1/p]$ where $k_{K(\mu_{p^\infty})}$ denotes the residue field of $K(\mu_{p^\infty})$ and W denotes p-typical Witt vectors, and set[3]

$$R_K = \{f(z) \in K_0'[\![z, z^{-1}]\!] : f \text{ converges on } \{z \in \mathbf{C}_p : r \leq |z| < 1\} \text{ for some } 0 < r < 1\}$$

In the notation of [KPX14, Definition 2.2.2], our R_K is their $\mathcal{R}_{\mathbb{Q}_p}(\pi_K)$ for some choice of indeterminate π_K, and the action[4] of φ and Γ_K on R_K is defined as in *loc. cit.* Then if A is an L-module-finite L-algebra, then define the relative Robba ring $R_{A,K} := R_K \otimes_{\mathbb{Q}_p} A$. If L'/L is a finite extension and $\rho : G_K \to \mathrm{GL}_d(L')$ is a Galois representation, we denote by $\mathbf{D}_{\mathrm{rig}}(\rho)$ the associated (φ, Γ_K)-module over $R_{L',K}$.

Proposition 3.4.1.1 ([KPX14, Construction 6.2.4 and Theorem 6.2.14])**.** *If L'/L is a finite extension, then there is a canonical bijection $\delta \mapsto R_{L',K}(\delta)$ between $\mathcal{T}(L')$ and the set of isomorphism classes of rank 1 (φ, Γ_K)-modules over $R_{L',K}$.*

Definition 3.4.1.2. Let $\rho : G_K \to \mathrm{GL}_d(L')$ be a continuous representation, and fix a d-tuple $\delta = (\delta_1, \ldots, \delta_d) \in \mathcal{T}^d(L')$. Then we say that ρ is *trianguline of parameter δ* if the (φ, Γ_K)-module $\mathbf{D}_{\mathrm{rig}}(\rho)$ admits a full flag $\mathrm{Fil}^\bullet$ of sub-(φ, Γ_K)-modules (which are free and direct summands as $R_{L',K}$-modules) such that each graded piece $\mathrm{Fil}^i / \mathrm{Fil}^{i-1}$ is isomorphic to $R_{L',K}(\delta_i)$ (c.f. Proposition 3.4.1.1). We call $\mathrm{Fil}^\bullet$ a *triangulation* of ρ.

Suppose $\rho : G_K \to \mathrm{GL}_d(L')$ is crystalline. Then the associated filtered φ-module $\mathbf{D}_{\mathrm{cris}}(\rho)$ is finite free rank of rank d over $(K_0 \otimes_{\mathbb{Q}_p} L') \cong \bigoplus_{\tau : K_0 \hookrightarrow L'} L'$, and breaks up under this identification as

$$\mathbf{D}_{\mathrm{cris}}(\rho) = \bigoplus_{\tau : K_0 \hookrightarrow L'} \mathbf{D}_{\mathrm{cris},\tau}(\rho)$$

where $\mathbf{D}_{\mathrm{cris},\tau}(\rho) = (B_{\mathrm{cris}} \otimes_{K_0,\tau} \rho)^{G_K}$, such that φ^f acts L'-linearly on each $\mathbf{D}_{\mathrm{cris},\tau}$. But these

[3] In [Iye20] we gave an incorrect definition of the Robba ring; this is fixed here.

[4] The action depends on the choice of π_K, so we make an arbitrary choice.

τ-labeled φ^f operators are all mutually conjugate, so the characteristic polynomial of φ^f is independent of τ, and thus we get a multi-set of φ^f-eigenvalues $\{\alpha_1, \ldots, \alpha_d\}$ living in $\overline{\mathbb{Q}_p}$, independent of $\tau : K_0 \hookrightarrow L'$.

There is an alternative characterization of triangulations of a trianguline crystalline representation.

Definition 3.4.1.3. Let $\rho : G_K \to \mathrm{GL}_d(L')$ be a crystalline representation. Then a *refinement* of ρ is a full filtration $F^\bullet$ of $\mathbf{D}_{\mathrm{cris}}(\rho)$ by free and φ-stable $(K_0 \otimes_{\mathbb{Q}_p} L')$-modules.

In fact, refinements are the same as triangulations: given a triangulation $\mathrm{Fil}^\bullet$ of ρ, one can construct a refinement

$$F^i := (\mathrm{Fil}^i[\frac{1}{t}])^{\Gamma_K},$$

(where $t = \log(1+z) \in R_K$ is the usual crystalline period: recall $\mathbf{D}_{\mathrm{cris}}(\rho) = (\mathbf{D}_{\mathrm{rig}}(\rho)[\frac{1}{t}])^{\Gamma_K})$ and [BC09a, Proposition 2.4.1][5] shows that this map is a bijection.

In particular, suppose the φ^f-eigenvalues $\{\alpha_i\}$ are distinct. By enlarging L', we may assume that $\{\alpha_i\} \subset L'$. By picking a corresponding eigenvector in $\mathbf{D}_{\mathrm{cris},\tau}(\rho)$ for each $\tau : K_0 \hookrightarrow L'$ and putting them all together, we get a $(K_0 \otimes_{\mathbb{Q}_p} L')$-linear decomposition

$$\mathbf{D}_{\mathrm{cris}}(\rho) = (K_0 \otimes_{\mathbb{Q}_p} L')e_1 \oplus \cdots \oplus (K_0 \otimes_{\mathbb{Q}_p} L')e_d,$$

such that $\varphi^f(e_i) = \alpha_i e_i$. Since refinements are required to be φ-stable, every refinement must be of the form

$$F^i_\sigma = \bigoplus_{j=1}^{i} (K_0 \otimes_{\mathbb{Q}_p} L')e_{\sigma(j)}$$

for some $\sigma \in \Sigma_d$, where Σ_d is the symmetric group on d letters. We denote the corresponding triangulation $\mathrm{Fil}^\bullet_\sigma$ for each $\sigma \in \Sigma_d$: these are all the triangulations.

[5] In [BC09a] this is proven when $K = \mathbb{Q}_p$, but the proof generalizes straightforwardly to our case: the point is that an ordering on the Hodge-Tate weights and on the φ^f-eigenvalues determine the parameter δ of the triangulation.

Lemma 3.4.1.4 ([BHS17a, Lemma 2.1])**.** *If $\rho : G_K \to \mathrm{GL}_d(L')$ is crystalline and trianguline of parameter δ, then there is some ordering $(h_{\tau,1}, \ldots, h_{\tau,d})_{\tau:K\hookrightarrow L'}$ of the labeled Hodge-Tate weights of ρ and some ordering $(\alpha_1, \ldots, \alpha_n)$ of the φ^f-eigenvalues such that*

$$\delta_i = (\cdot)^{\mathbf{h}_i}\,\mathrm{unr}(\alpha_i)$$

where $\mathrm{unr}(\alpha_i)$ is the unramified character of $K^\times$ taking ϖ_K to α_i.

In particular, if the φ^f-eigenvalues are all distinct, then there exists a unique triangulation of ρ with parameter δ.

We note a few useful types of crystalline representation. Our definition of Hodge-Tate weights is normalized so that the cyclotomic character has weight $+1$.

Definition 3.4.1.5. Let $\rho : G_K \to \mathrm{GL}_d(L')$ be a crystalline representation with τ-labeled Hodge-Tate weights $\{h_{\tau,1} \geq \cdots \geq h_{\tau,d}\}_{\tau:K\hookrightarrow L'}$ and φ^f-eigenvalues $\{\alpha_1, \ldots, \alpha_d\}$.

- If $h_{\tau,i} \neq h_{\tau,j}$ for all $i \neq j$ and all τ, then we say ρ is *regular* or *regular crystalline*.
- If $\alpha_i \neq \alpha_j$ for all $i \neq j$, we say ρ is φ^f*-generic*.
- If ρ is regular and $\mathrm{Fil}^\bullet$ is a triangulation of ρ such that the Hodge-Tate weights of Fil^i are exactly $\{h_{\tau,1} > \cdots > h_{\tau,i}\}$ for each τ, we say that $\mathrm{Fil}^\bullet$ is *noncritical*.
- If ρ is regular and every triangulation of ρ is noncritical, then we say that ρ is *noncritical*.
- If ρ is regular, φ^f-generic, and noncritical, and if additionally $\alpha_i \neq p^{\pm f}\alpha_j$ for all $i \neq j$, then we say that ρ is *benign*.

Now recall that a point of $\mathcal{X}_{\bar{\rho}}^{\square}$ is the same as a surjection $f : R_{\bar{\rho}}^{\square}[1/p] \twoheadrightarrow L'$ for some finite extension L'/L, which gives rise to a p-adic representation $\rho_f : G_K \to \mathrm{GL}_d(L')$. We define the subset

$$\mathcal{U}^{\square}_{\mathrm{tri},\overline{\rho}} = \{(f,\delta) \in \mathcal{X}^{\square}_{\overline{\rho}} \times_L \mathcal{T}^d_{\mathrm{reg}} : \rho_f \text{ is trianguline of parameter } \delta\}.$$

Then the *trianguline deformation space* $\mathcal{X}^{\square}_{\mathrm{tri},\overline{\rho}}$ is the Zariski closure of $\mathcal{U}^{\square}_{\mathrm{tri},\overline{\rho}}$ in $\mathcal{X}^{\square}_{\overline{\rho}} \times_L \mathcal{T}^d_{\mathrm{reg}}$.

Recall that at any point $f \in \mathcal{X}^{\square}_{\overline{\rho}}$, the completion $\mathscr{O}^{\wedge}_{\mathcal{X}^{\square}_{\overline{\rho}},f}$ is the universal framed deformation ring for the induced representation $\rho_f : G_K \to \mathrm{GL}_d(L')$. In fact, a similar result holds for a point $x = (f_x, \delta_x) \in \mathcal{X}^{\square}_{\mathrm{tri},\overline{\rho}}$ such that $\rho_x := \rho_{f_x}$ is benign, as we recall now.

Since ρ_x is in particular φ^f-generic, Lemma 3.4.1.4 says that there is a *unique* triangulation $\mathrm{Fil}^{\bullet}_x$ of ρ_x with parameter δ_x. So we define the deformation problem

$$D^{\square}_{\rho_x,\mathrm{Fil}^{\bullet}_x[\frac{1}{t}]} : \mathsf{Art}_{L'} \to \mathsf{Set}$$

taking an Artinian local L'-algebra A to the set of pairs $(\rho, \mathrm{Fil}^{\bullet})$ where $\rho : G_K \to \mathrm{GL}_d(A)$ of ρ lifts ρ_x, and $\mathrm{Fil}^{\bullet}$ is a full filtration (by direct summands) of $\mathbf{D}_{\mathrm{rig}}(\rho)[\frac{1}{t}]$ by finite free $R_{A,K}$-modules whose successive quotients are of the form $R_{A,K}(\delta)[\frac{1}{t}]$, and which lifts the filtration $\mathrm{Fil}^{\bullet}_x[\frac{1}{t}]$.

Proposition 3.4.1.6. *The natural forgetful map $D^{\square}_{\rho_x,\mathrm{Fil}^{\bullet}_x[\frac{1}{t}]} \to D^{\square}_{\rho_x}$ is injective and relatively representable. We denote the representing ring $R^{\square}_{\rho_x,\mathrm{Fil}^{\bullet}_x[\frac{1}{t}]}$.*

Proof. Since ρ_x is φ^f-generic, δ_x is regular in the sense of [HMS18, (3.1)]. The result then follows from [HMS18, Proposition 3.5] and the identification (see [BHS19, Section 3.6] for more details)

$$D^{\square}_{\rho_x,\mathrm{Fil}^{\bullet}_x[\frac{1}{t}]} \cong D^{\square}_{\rho_x} \times_{D_{\mathbf{D}_{\mathrm{rig}}(\rho_x)}} D_{\mathbf{D}_{\mathrm{rig}}(\rho_x),\mathrm{Fil}^{\bullet}_x[\frac{1}{t}]},$$

□

Proposition 3.4.1.7. *There is an isomorphism*

$$\mathscr{O}^{\wedge}_{\mathcal{X}^{\square}_{\mathrm{tri},\overline{\rho}},x} \cong R^{\square}_{\rho_x,\mathrm{Fil}^{\bullet}_x[\frac{1}{t}]}$$

of integral domains.

Proof. Let $G = \operatorname{Spec} L' \times_{\operatorname{Spec} \mathbb{Q}_p} \operatorname{Res}_{K/\mathbb{Q}_p} \mathrm{GL}_{d,K} \cong \prod_{\tau: K \hookrightarrow L'} \mathrm{GL}_{d,L'}$ with Weyl group $W = \prod_{\tau:K\hookrightarrow L'} \Sigma_d$. In [BHS19], the authors show that one can associate to $(\rho_x, \mathrm{Fil}^{\bullet}_x[\frac{1}{t}])$ an element $w_x \in W$ such that the irreducible components of $\operatorname{Spec} R^{\square}_{\rho_x,\mathrm{Fil}^{\bullet}_x[\frac{1}{t}]}$ are in bijection with the set $\{w \in W : w \succeq w_x\}$, where $\succeq$ denotes the Bruhat ordering on W (see Theorem 3.6.2 and the proof of Corollary 4.3.2). Roughly speaking, w_x measures the relative position of $\mathrm{Fil}^{\bullet}_x[\frac{1}{t}]$ with respect to the Hodge filtration on $\mathbf{D}_{\mathrm{dR}}(\rho_x)$.

However, the assumption that ρ_x is noncritical ensures that w_x is the maximal element for the Bruhat ordering, i.e. each $w_{x,\tau}$ is the order-reversing permutation. Therefore, $\operatorname{Spec} R^{\square}_{\rho_x,\mathrm{Fil}^{\bullet}_x[\frac{1}{t}]}$ is irreducible, so [BHS19, Corollary 3.7.8] exactly says that

$$\mathscr{O}^{\wedge}_{\mathcal{X}^{\square}_{\mathrm{tri},\overline{\rho}},x} \cong R^{\square}_{\rho_x,\mathrm{Fil}^{\bullet}_x[\frac{1}{t}]},$$

and [BHS19, Theorem 3.6.2(a)] says that this ring is reduced. □

3.4.2 Proof of density

In this section we prove Theorem 3.4.2.4.

Definition 3.4.2.1. Let

$$\mathcal{X}^{\square}_{\mathrm{reg-cris}} = \{x \in \mathcal{X}^{\square}_{\overline{\rho}} : \rho_x \text{ is regular crystalline}\}$$

and let $\overline{\mathcal{X}}^{\square}_{\mathrm{reg-cris}}$ be its Zariski closure in $\mathcal{X}^{\square}_{\overline{\rho}}$. Let

$$\mathcal{X}_{\mathrm{cris}} = \{(f', \delta') \in \mathcal{X}^{\square}_{\mathrm{tri}, \overline{\rho}} : \rho_{f'} \text{ is regular crystalline, } \varphi^f\text{-generic, and noncritical}\}$$

We will need the following result about density of crystalline points in the trianguline deformation space:

Proposition 3.4.2.2 ([BHS19, Proposition 4.1.4])**.** *Suppose $x = (f_x, \delta_x) \in \mathcal{X}^{\square}_{\mathrm{tri}, \overline{\rho}}$ is benign. Then there exists an affinoid open neighborhood $\mathcal{U} \subseteq \mathcal{X}^{\square}_{\mathrm{tri}, \overline{\rho}}$ of x such that $\mathcal{U} \cap \mathcal{X}_{\mathrm{cris}}$ is Zariski dense in $\mathcal{U}$.*

Proposition 3.4.2.3. *$\overline{\mathcal{X}}^{\square}_{\mathrm{reg-cris}}$ is a union of irreducible components of $\mathcal{X}^{\square}_{\overline{\rho}}$.*

Proof. Let $\mathcal{Z}$ be an irreducible component of $\overline{\mathcal{X}}^{\square}_{\mathrm{reg-cris}}$. Note the singular locus $\mathcal{Z}_{\mathrm{sing}} \subset \mathcal{Z}$ is a proper Zariski closed subset, so its complement $\mathcal{U}$ is an admissible open in $\mathcal{Z}$. Thus, we may pick a smooth point $x \in \mathcal{U}$ with corresponding representation $\rho_x : G_K \to \mathrm{GL}_d(L')$, and by density we assume $x \in \mathcal{X}^{\square}_{\mathrm{reg-cris}}$. By Corollary 2.7.7[6] in [Kis08], there is a Zariski closed subspace $\mathcal{X}^{\square, \mathbf{k}_x}_{\overline{\rho}, \mathrm{cris}} \subset \mathcal{X}^{\square}_{\overline{\rho}}$ consisting of the crystalline representations with Hodge-Tate weights $\mathbf{k}_x$, where $\mathbf{k}_x$ denotes the Hodge-Tate weights of ρ_x, as above. By [Nak14, Lemma 4.2] (with $U = \mathcal{U} \cap \mathcal{X}^{\square, \mathbf{k}_x}_{\overline{\rho}, \mathrm{cris}}$) we may assume ρ_x is benign.

In fact, x is smooth in $\mathcal{X}^{\square}_{\overline{\rho}}$. To see this, note that

$$\mathscr{O}^{\wedge}_{\mathcal{X}^{\square}_{\overline{\rho}}, x} \cong R^{\square}_{\rho_x},$$

where $R^{\square}_{\rho_x}$ is the universal deformation ring of the framed deformation problem for ρ_x. Thus, it suffices to show that $H^2(G_K, \mathrm{ad}\, \rho_x) = 0$, which is the same as showing that

[6] In fact, we use a slight modification: we first use the space of semi-stable deformations with fixed Hodge-Tate weights constructed in [Kis08, Corollary 2.6.2], and then consider the zero locus of the monodromy operator N.

$\mathrm{Hom}_{\kappa(\rho_x)[G_K]}(\rho_x, \rho_x \otimes \epsilon) = 0$ by local Tate duality. But a morphism $g : \rho_x \to \rho_x \otimes \epsilon$ induces a φ^f-equivariant map $\mathbf{D}_{\mathrm{cris}}(\rho_x) \to \mathbf{D}_{\mathrm{cris}}(\rho_x \otimes \epsilon)$. If $\{\alpha_i\}_i$ denotes the set of (distinct) eigenvalues of φ^f on $\mathbf{D}_{\mathrm{cris}}(\rho_x)$, then the eigenvalues of φ^f on $\mathbf{D}_{\mathrm{cris}}(\rho_x \otimes \epsilon)$ are exactly $\{p^f \alpha_i\}_i$. But ρ_x is benign, so in particular, $\alpha_i \neq p^{\pm f} \alpha_j$ for $i \neq j$, and thus $g = 0$.

The irreducible set $\mathcal{Z}$ admits a closed immersion $\mathcal{Z} \hookrightarrow \mathcal{V}$ where $\mathcal{V}$ is an irreducible component of $\mathcal{X}_{\overline{\rho}}^{\square}$. We wish to show that this map is an equality, for which it suffices to show that $\dim \mathcal{Z} = \dim \mathcal{V}$. Since dimension can be computed as the dimension of the tangent space at smooth points, it suffices to show that the natural injection

$$T_x \mathcal{Z} \hookrightarrow T_x \mathcal{X}_{\overline{\rho}}^{\square} = T_x \mathcal{V}$$

is an isomorphism.

As noted above, the triangulations of ρ_x are exactly parametrized by $\sigma \in \Sigma_d$, and we write them as $\mathrm{Fil}_\sigma^\bullet$. Each pair $(x, \mathrm{Fil}_\sigma^\bullet)$ defines a point $y_\sigma \in \mathcal{X}_{\mathrm{tri},\overline{\rho}}^{\square}$. Since ρ_x is benign, Proposition 3.4.1.7 says that $\mathscr{O}^{\wedge}_{\mathcal{X}_{\mathrm{tri},\overline{\rho}}^{\square}, y_\sigma} \cong R^{\square}_{\rho_x, \mathrm{Fil}_\sigma^\bullet[\frac{1}{t}]}$ is an integral domain. This implies that there is a unique irreducible component of $\mathcal{X}_{\mathrm{tri},\overline{\rho}}^{\square}$ containing y_σ, which we call $\mathcal{Y}_\sigma$. By Proposition 3.4.2.2, there exists an affinoid open neighborhood $\mathcal{U}_\sigma$ of y_σ such that $\mathcal{U}_\sigma \cap \mathcal{X}_{\mathrm{cris}}$ is dense in $\mathcal{U}_\sigma$. But then $\mathcal{U}_\sigma \cap \mathcal{X}_{\mathrm{cris}} \cap \mathcal{Y}_\sigma$ is dense in $\mathcal{U}_\sigma \cap \mathcal{Y}_\sigma$, which is a nonempty open and thus dense in $\mathcal{Y}_\sigma$. Thus $\mathcal{X}_{\mathrm{cris}} \cap \mathcal{Y}_\sigma$ is dense in $\mathcal{Y}_\sigma$.

Under the natural projection

$$\mathcal{X}_{\mathrm{tri},\overline{\rho}}^{\square} \to \mathcal{X}_{\overline{\rho}}^{\square} \times_L \mathcal{T}_{\mathrm{reg}}^d \to \mathcal{X}_{\overline{\rho}}^{\square},$$

the subset $\mathcal{X}_{\mathrm{cris}}$ lands in $\mathcal{X}_{\mathrm{reg-cris}}$ and $\mathcal{Y}_\sigma$ lands in $\mathcal{V}$. By density, this descends to a map

$\mathcal{Y}_\sigma \to \mathcal{Z}$, taking $y_\sigma \mapsto x$. Therefore, by considering all $\sigma \in \Sigma_d$, we get maps

$$\bigoplus_{\sigma\in\Sigma_d} T(R^{\square}_{\rho_x,\mathrm{Fil}^\bullet_\sigma[\frac{1}{t}]}) = \bigoplus_{\sigma\in\Sigma_d} T_{y_\sigma}\mathcal{Y}_\sigma \to T_x\mathcal{Z} \hookrightarrow T_x\mathcal{V} \cong T(R^{\square}_{\rho}).$$

which are actually induced by the forgetful maps $D^{\square}_{\rho_x,\mathrm{Fil}^\bullet_\sigma[\frac{1}{t}]} \to D^{\square}_{\rho_x}$.

Let D_{ρ_x} and $D_{\rho_x,\mathrm{Fil}^\bullet_\sigma[\frac{1}{t}]}$ be the unframed deformation functors of ρ_x and $(\rho_x, \mathrm{Fil}^\bullet_\sigma[\frac{1}{t}])$ respectively. Let $\pi_\sigma : D_{\rho_x,\mathrm{Fil}^\bullet_\sigma[\frac{1}{t}]} \to D_{\rho_x}$ denote the map forgetting the filtration. Then $D^{\square}_{\rho_x} \to D_{\rho_x}$ is smooth of relative dimension $e = d^2 - \dim H^0(G_K, \mathrm{ad}\,\rho_x)$. So we may write $T(R^{\square}_{\rho_x}) = T(D_{\rho_x}) \oplus V^{\square}$ where $V^{\square}$ is some e-dimensional L'-vector space, and such that

$$T(R^{\square}_{\rho_x}) = T(D_{\rho_x}) \oplus V^{\square} \twoheadrightarrow T(D_{\rho_x})$$

is just the natural projection map. There is a canonical isomorphism of functors

$$D^{\square}_{\rho_x,\mathrm{Fil}^\bullet_\sigma[\frac{1}{t}]} \cong D^{\square}_{\rho_x} \times_{D_{\rho_x}} D_{\rho_x,\mathrm{Fil}^\bullet_\sigma[\frac{1}{t}]},$$

so the map $\bigoplus_{\sigma\in\Sigma_d} T(R^{\square}_{\rho_x,\mathrm{Fil}^\bullet_\sigma[\frac{1}{t}]}) \to T(R^{\square}_{\rho_x})$ factors as

$$\bigoplus_{\sigma\in\Sigma_d} T(R^{\square}_{\rho_x,\mathrm{Fil}^\bullet_\sigma[\frac{1}{t}]}) \cong \bigoplus_{\sigma\in\Sigma_d} (T(D_{\rho_x,\mathrm{Fil}^\bullet_\sigma[\frac{1}{t}]}) \oplus V^{\square}) \xrightarrow{\sum(T(\pi_\sigma)\oplus \mathrm{id}_{V^\square})} T(D_{\rho_x}) \oplus V^{\square} \cong T(R^{\square}_{\rho_x})$$

By [HMS18, Corollary 3.13], $\sum_{\sigma\in\Sigma_d} T(\pi_\sigma)$ is surjective, and the above map $\bigoplus_{\sigma\in\Sigma_d} V^{\square} \xrightarrow{\sum \mathrm{id}_{V^\square}} V^{\square}$ is clearly surjective, so we conclude that the map is surjective, which is exactly what we wanted. Thus, $T_x\mathcal{Z} \hookrightarrow T_x\mathcal{V}$ is an isomorphism. □

Theorem 3.4.2.4. *Under the assumptions made in Theorem 3.3.0.1,*

$$\overline{\mathcal{X}}^{\square}_{\mathrm{reg-cris}} = \mathcal{X}^{\square}_{\overline{\rho}}$$

Proof. By Proposition 3.4.2.3, it suffices to show the existence of a regular crystalline point in each irreducible component of $\mathcal{X}_{\overline{\rho}}^{\square}$: by restricting algebraic characters to $\mathscr{O}_K^\times$, we just construct them explicitly.

Let $\mathrm{rec}_K : \widehat{K^\times} \xrightarrow{\sim} G_K^{\mathrm{ab}}$ be the local reciprocity map, normalized so that ϖ_K is sent to a lift of the geometric Frobenius. Let $I = \{i \in \mathbb{Z} : 0 < i < q \text{ and } \gcd(i,q) = 1\}$ and $J = \{1, \dots, |\operatorname{Hom}_{\mathbb{Q}_p(\mu_q)}(K,L)|\}$, and pick some bijection $\iota : \operatorname{Hom}(K,L) \xrightarrow{\sim} I \times J$ such that if $\tau_1|_{\mathbb{Q}_p(\mu_q)} = \tau_2|_{\mathbb{Q}_p(\mu_q)}$, then $p_I(\iota(\tau_1)) = p_I(\iota(\tau_2))$, where $p_I : I \times J \to I$ is the projection. Then for $\mathbf{h} = (h_{i,j}) \in \mathbb{Z}_{>0}^{I \times J}$, we define the character $\chi^{\mathbf{h}} : G_K^{\mathrm{ab}} \to L^\times$ by setting $\chi^{\mathbf{h}}(\mathrm{rec}_K(\varpi_K)) := 1$, and setting $(\chi^{\mathbf{h}} \circ \mathrm{rec}_K)|_{\mathscr{O}_K^\times} := (\cdot)^{\iota^*(\mathbf{h})}|_{\mathscr{O}_K^\times}$ where $(\cdot)^{\iota^*(\mathbf{h})}$ is the algebraic character in $\mathcal{T}(L)$ defined at the beginning of this subsection. This is crystalline with labeled Hodge-Tate weights $\iota^*(\mathbf{h})$.

The point is that if $\zeta_0 \in K$ is some choice of primitive qth root of unity and $\tau_0 \in \operatorname{Hom}(K,L)$ is some embedding such that $p_I(\iota(\tau_0)) = 1$, then

$$\chi^{\mathbf{h}}(\mathrm{rec}_K(\zeta_0)) = \tau_0(\zeta_0)^{\sum_{(i,j)\in I\times J} i h_{i,j}}.$$

So pick some $\mathbf{h} \in \mathbb{Z}_{>0}^{I \times J}$ such that $\sum_{(i,j)\in I\times J} i h_{i,j}$ is coprime to q: this is possible since $(i,q) = 1$ for all $i \in I$. Then $\chi^{\mathbf{h}}(\mathrm{rec}_K(\zeta_0))$ is actually a primitive qth root of unity. Furthermore,

$$(\chi^{\mathbf{h}})^{\otimes(p^f-1)} \equiv 1 \mod \varpi,$$

which implies that $(\chi^{\mathbf{h}})^{\otimes(p^f-1)}$ is induced by a point in the generic fibre $\mathcal{X}_{\mathbf{1}}$ of the deformation space of the trivial character.

Let $\chi_0 = (\chi^{\mathbf{h}})^{\otimes(p^f-1)}$. For $m = 0, \dots, q-1$ define

$$\rho_m = \left(\bigoplus_{i=1}^{d-1} \chi_0^{\otimes i}\right) \oplus \chi_0^{\otimes(d+m)}.$$

Then ρ_m is induced by a point in $\mathcal{X}_{\overline{\rho}}^{\square}$. It is crystalline and its τ-labeled Hodge-Tate weights are exactly

$$(p^f - 1)h_\tau, 2(p^f - 1)h_\tau, \ldots, (d-1)(p^f - 1)h_\tau, (d+m)(p^f - 1)h_\tau,$$

which are distinct and nonzero since $h_\tau > 0$ by assumption (here $h_\tau := \iota^*(\mathbf{h})_\tau$) so ρ_m is actually regular. Finally note that

$$\{\det(\rho_m(\mathrm{rec}_K(\zeta_0)))\}_{m=0,\ldots,q-1} = \mu_q(L).$$

By Theorem 3.3.0.1 and Remark 3.3.4.3, we have thus found regular crystalline points in each of the irreducible components of $\mathcal{X}_{\overline{\rho}}^{\square}$. $\square$

Chapter 4

The general case

This chapter is based entirely on [BIP21], which is joint work with Gebhard Böckle and Vytautas Paškūnas. As mentioned in the abstract, some notation has been slightly altered.

4.1 Geometric invariant theory

We assume the set up of [Ses77]. Let R be a Noetherian ring and let $S = \operatorname{Spec} R$. Let G be a reductive group scheme over S, so that G is an affine group scheme over S, $G \to S$ is smooth and the geometric fibres are connected reductive groups. In the application $G = S \times_{\operatorname{Spec}\mathbb{Z}} \mathrm{GL}_d$ and $G = S \times_{\operatorname{Spec}\mathbb{Z}} \mathbb{G}_m^r$ so that these conditions hold.

Let V be a free R-module of finite rank r endowed with a G-module structure, let $\check{V} = \operatorname{Hom}_R(V, R)$ and let $\operatorname{Sym}(\check{V})$ be the symmetric algebra. The G-module structure on V induces an action of G on $\operatorname{Spec}(\operatorname{Sym}(\check{V})) = \mathbb{A}_S^r$. Let X be a closed G-invariant subscheme of $\operatorname{Spec}(\operatorname{Sym}(\check{V}))$. The G-action on X induces an action on B, the ring of functions on X. The GIT quotient $X /\!\!/ G$ is represented by the ring of invariants B^G.

Let $y = \operatorname{Spec} \kappa$ be a geometric point of $X /\!\!/ G$. We may identify the fibre X_y with a closed G-invariant subscheme of X.

Lemma 4.1.0.1. *Let $x \in X_y(\kappa)$ be such that the orbit $G \cdot x$ is closed in X_y then*

$$\dim X_y \leq \dim_\kappa T_x(X_y).$$

Proof. Since X_y is Noetherian it has finitely many irreducible components and the G_y-action permutes them, but since G_y is connected every irreducible component is G_y-invariant and thus the permutation is the identity. This can be seen as follows: let U be the open subscheme of X_y obtained by removing all the intersections of irreducible components. Then it is enough to show that the connected components of U are G_y-invariant. If U' is a connected component of U then the image of $G \times_S U'$ in U under the action map is connected and contains U', so is equal to U'.

Let Z be an irreducible component of X_y such that $\dim Z = \dim X_y$. As explained above Z is both closed in X_y and G-invariant. Then by [Ses77, Theorem 3] both Z and X_y have a unique closed G-orbit, hence those orbits must be equal. Therefore $x \in Z$ so since Z is irreducible,

$$\dim X_y = \dim Z \leq \dim_\kappa T_x(Z) \leq \dim_\kappa T_x(X_y).$$

□

4.2 $R_{\overline{\rho}}^{\square}$ is complete intersection

Let $\overline{\rho} : G_K \to \mathrm{GL}_d(k)$[1] be a continuous representation with universal lifting ring $R_{\overline{\rho}}^{\square}$. The proof of [Maz97b, Proposition 21.1] shows that the tangent space to $D_{\overline{\rho}}^{\square}$ is $Z^1(G_K, \operatorname{ad} \overline{\rho})$ and

[1]Recall that in Section 2.1 we assumed that the irreducible subquotients of $\overline{\rho}$ are absolutely irreducible (which can always be arranged by possibly enlarging k and $\mathcal{O}$).

it follows from the proof of [Maz89a, Sec. 1.6, Proposition 2] that there is a presentation

$$R_{\overline{\rho}}^{\square} \cong \mathcal{O}[\![x_1, \dots, x_r]\!]/(f_1, \dots, f_s), \tag{4.1}$$

where

$$r = \dim_k Z^1(G_K, \operatorname{ad} \overline{\rho}) = d^2 - \dim_k H^0(G_K, \operatorname{ad} \overline{\rho}) + \dim_k H^1(G_K, \operatorname{ad} \overline{\rho})$$

and

$$s = \dim_k H^2(G_K, \operatorname{ad} \overline{\rho}).$$

If we can show that the Krull dimension of $R_{\overline{\rho}}^{\square}$ is equal to $1 + r - s$ then [Stacks, Tag 02JN] implies that $f_1, \dots, f_s$ forms a regular sequence and thus $R_{\overline{\rho}}^{\square}$ is a complete intersection ring. We also remark that the local Euler characteristic formula implies that

$$1 + r - s = 1 + d^2 + d^2[K : \mathbb{Q}_p].$$

Let $\overline{D} : k[\![G_K]\!] \to k$ be the pseudorepresentation associated with $\overline{\rho}$ (see Definition 2.2.2.2). Let $D^{\mathrm{ps}} : \mathsf{Art}_{\mathcal{O}} \to \mathsf{Set}$ be the functor, such that for $(A, \mathfrak{m}_A) \in \mathsf{Art}_{\mathcal{O}}$, $D^{\mathrm{ps}}(A)$ is the set of continuous A-valued d-dimensional pseudorepresentations of $A[G_K]$, which reduce to $\overline{D}$ modulo $\mathfrak{m}_A$. The functor D^{ps} is pro-representable by a complete local Noetherian $\mathcal{O}$-algebra $(R^{\mathrm{ps}}, \mathfrak{m}_{R^{\mathrm{ps}}})$, see [Che14, Section 3.1].

Mapping a deformation of $\overline{\rho}$ to its determinant induces a natural transformation $D_{\overline{\rho}}^{\square} \to D^{\mathrm{ps}}$ and thus a map of local $\mathcal{O}$-algebras $R^{\mathrm{ps}} \to R_{\overline{\rho}}^{\square}$. The ring R^{ps} has been well understood in the recent work of Böckle–Juschka [BJ19]. Our basic idea is to study $R_{\overline{\rho}}^{\square}$ by studying the fibres of this map. In fact it is technically more convenient to introduce an intermediate ring $R^{\mathrm{ps}} \to A^{\mathrm{gen}} \to R_{\overline{\rho}}^{\square}$, depending on $\overline{D}$ and not on $\overline{\rho}$ itself, such that A^{gen} is of finite type over R^{ps} and $R_{\overline{\rho}}^{\square}$ is a completion of A^{gen} at a maximal ideal. Since $\dim R_{\overline{\rho}}^{\square} \leq \dim A^{\mathrm{gen}}$, it is enough to bound the dimension of A^{gen}.

4.2.1 Generic matrices

Let $D^u : R^{\mathrm{ps}}[\![G_K]\!] \to R^{\mathrm{ps}}$ be the universal pseudorepresentation lifting $\overline{D}$. Let $\mathrm{CH}(D^u)$ be the closed two-sided ideal of $R^{\mathrm{ps}}[\![G_K]\!]$ defined in [Che14, §1.17], so that $E := R^{\mathrm{ps}}[\![G_K]\!]/\,\mathrm{CH}(D^u)$ is the associated Cayley–Hamilton algebra (see Definition 2.2.3.1 for a definition). Implicit here is the fact that D^u descends to a pseudorepresentation $D^u|_E : E \to R^{\mathrm{ps}}$, but we abusively call this D^u as well. Then E is a finitely generated R^{ps}-module, [Wan18, Proposition 3.6].

Definition 4.2.1.1. If B is a commutative R^{ps}-algebra and $f : E \to M_d(B)$ is an R^{ps}-algebra homomorphism then we say f is a *homomorphism of Cayley–Hamilton algebras* if the associated pseudorepresentation $\det f : E \otimes_{R^{\mathrm{ps}}} B \to B$ (with scalars extended to B, as in the notation) is equal to $D^u \otimes B : E \otimes_{R^{\mathrm{ps}}} B \to B$.

The superscript *gen* in A^{gen} stands for *generic matrices*, and the following construction appears in the work of Procesi [Pro87]; Lemma 4.2.1.2, Lemma 4.2.1.4, Lemma 4.2.1.5 are contained in [Wan18, Theorem 3.8], but we write direct proofs here as well.

Lemma 4.2.1.2. *There is a finitely generated commutative R^{ps}-algebra A^{gen} together with a homomorphism of Cayley–Hamilton R^{ps}-algebras $j : E \to M_d(A^{\mathrm{gen}})$, satisfying the following universal property: if $f : E \to M_d(B)$ is a homomorphism of Cayley–Hamilton R^{ps}-algebras for a commutative R^{ps}-algebra B then there is a unique R^{ps}-algebra homomorphism $\tilde{f} : A^{\mathrm{gen}} \to B$ of such that $f = M_d(\tilde{f}) \circ j$.*

Proof. By writing down a generic $d \times d$-matrix for each R^{ps}-generator of E and quotienting out by the relations the generators satisfy in E, one obtains a commutative R^{ps}-algebra C and a homomorphism of R^{ps}-algebras $j : E \to M_d(C)$. More formally, C is a quotient of $R^{\mathrm{ps}} \otimes_{\mathbb{Z}} \mathrm{Sym}(W)$, where W is a direct sum of n copies of $\mathrm{End}(\mathrm{Std})^*$, where Std is the standard representation of GL_d over $\mathbb{Z}$, n is the size of a generating set of E as an R^{ps}-module and $\mathrm{Sym}(W)$ is the symmetric algebra over $\mathbb{Z}$. If we would only require the homomorphisms to be homomorphisms of R^{ps}-algebras (i.e., if we would not impose the Cayley–Hamilton condition) then the triple $j: E \to M_d(C)$ would have the required universal property. To ensure that the Cayley–Hamilton condition is satisfied we have to consider the quotient of C constructed as follows. Let $\Lambda_i : E \to R^{\mathrm{ps}}$, $0 \leq i \leq d$ be the coefficients of the characteristic polynomial of D^u - these are homogeneous polynomial laws satisfying $D^u(t-a) = \sum_{i=0}^{n} (-1)^i \Lambda_i(a) t^{d-i}$ in $R^{\mathrm{ps}}[t]$, see [Che14, Section 1.10]. For each $a \in E$ let $c_i(j(a))$ be the i-th coefficient of the characteristic polynomial of the matrix $j(a) \in M_d(C)$. Let I be the ideal of C generated by $\Lambda_i(a) - c_i(j(a))$ for all $a \in E$ and $0 \leq i \leq d$ and let $A^{\mathrm{gen}} := C/I$. Since the coefficients of the characteristic polynomial determine pseudorepresentations uniquely, [Che14, Corollary 1.14], [Wan13, p. 1.1.9.15], the composition $E \to M_d(C) \to M_d(A^{\mathrm{gen}})$ is a homomorphism of Cayley–Hamilton algebras, and the universal property of $j: E \to M_d(C)$ implies the universal property for $j: E \to M_d(A^{\mathrm{gen}})$. Since E is finitely generated as an R^{ps}-module, C and hence A^{gen} are of finite type over R^{ps}. □

Definition 4.2.1.3. Let

- $X^{\mathrm{gen}} := \operatorname{Spec} A^{\mathrm{gen}}$ and $\overline{X}^{\mathrm{gen}} = X^{\mathrm{gen}} \times_{\mathcal{O}} k$, and
- $X^{\mathrm{ps}} := \operatorname{Spec} R^{\mathrm{ps}}$ and $\overline{X}^{\mathrm{ps}} \times_{\mathcal{O}} k$

Let us make a connection to the GIT theory in Section 4.1. If E is generated by n generators as an R^{ps}-module, then, as explained in the proof of Lemma 4.2.1.2, A^{gen} is a quotient of $R^{\mathrm{ps}} \otimes_{\mathbb{Z}} \operatorname{Sym}(W)$. The group $G := \mathrm{GL}_d$ acts on W by conjugation, and this induces an action of GL_d on X^{gen}. For every R^{ps}-algebra B, a point in $X^{\mathrm{gen}}(B)$ corresponds to an n-tuple of $d \times d$-matrices with entries in B satisfying certain relations, and $\mathrm{GL}_d(B)$ acts on $X^{\mathrm{gen}}(B)$ by conjugating the matrices. The scheme X^{gen} is isomorphic to $\mathsf{Rep}^{\square}_{D} = \mathsf{Rep}^{\square}_{E,D^u}$ as defined in [Wan18, Theorem 3.8].

The GIT quotient $X^{\mathrm{gen}} /\!\!/ G$ is represented by the ring of invariants $(A^{\mathrm{gen}})^G$. The map

$R^{\mathrm{ps}} \to A^{\mathrm{gen}}$ is G-invariant and induces a homomorphism $R^{\mathrm{ps}} \to (A^{\mathrm{gen}})^G$. It follows from [Wan18, Theorem 2.20] that the induced map

$$X^{\mathrm{gen}} /\!\!/ G \to X^{\mathrm{ps}} \tag{4.2}$$

is an adequate homeomorphism, i.e. an integral, universal homeomorphism which is a local isomorphism around points with characteristic zero residue field, see [Alp14, Definition 3.3.1]. The same argument shows that

$$\overline{X}^{\mathrm{gen}} /\!\!/ G \to \overline{X}^{\mathrm{ps}}$$

is an adequate homeomorphism.

We equip R^{ps} with the $\mathfrak{m}_{R^{\mathrm{ps}}}$-adic topology. Since the residue field is finite R^{ps} is a compact ring.

Lemma 4.2.1.4. *Let B be a topological R^{ps}-algebra. If $f : E \to M_d(B)$ is any homomorphism of R^{ps}-algebras then the composition $G_K \to E^\times \xrightarrow{f} \mathrm{GL}_d(B)$ defines a continuous representation of G_K.*

Proof. Since R^{ps} is a compact ring for every finitely generated R^{ps}-module M there is a unique topology on M making M into a topological R^{ps}-module, see [AU95, Corollary 1.10].

We equip $R^{\mathrm{ps}}[\![G_K]\!]$ with its projective limit topology, E with the quotient topology, and its group of units $E^\times$ with the subspace topology via the embedding $E^\times \hookrightarrow E \times E$, $x \mapsto (x, x^{-1})$. Since the map $G_K \to R^{\mathrm{ps}}[\![G_K]\!]$ is continuous, the map $G_K \to E^\times$ is also continuous.

Since E is a finitely generated R^{ps}-module, its topology coincides with $\mathfrak{m}_{R^{\mathrm{ps}}}$-adic topology. Let $M = f(E) \subset M_d(B)$. Then M is again a finitely generated R^{ps}-module, thus the quotient topology on M coincides with the subspace topology induced by the topology on B. Thus $f : E \to M_d(B)$ is continuous and hence induces a continuous group homomorphism

$E^\times \to M_d(B)^\times = \mathrm{GL}_d(B)$. □

The representation $\overline{\rho} : G_K \to \mathrm{GL}_d(k)$ induces a map of R^{ps}-algebras $E \to M_d(k)$ and thus a homomorphism of R^{ps}-algebras $A^{\mathrm{gen}} \to k$. It follows from the universal property of A^{gen} that $R^{\square}_{\overline{\rho}}$ is isomorphic to the completion of A^{gen} with respect to the kernel of this map, see Proposition 4.2.5.3. Conversely, we have the following Lemma.

Lemma 4.2.1.5. *Let $x \in X^{\mathrm{gen}}$ be a closed point above the closed point of X^{ps} and let $\rho_x : G_K \to \mathrm{GL}_d(\kappa(x))$ be the representation obtained by composing*

$$G_K \to R^{\mathrm{ps}}[\![G_K]\!] \to E \xrightarrow{j} M_d(A^{\mathrm{gen}}) \to M_d(\kappa(x)).$$

Then the pseudorepresentation associated to ρ_x is equal to $\overline{D} \otimes_k \kappa(x)$. In particular, ρ_x and $\overline{\rho} \otimes_k \kappa(x)$ have the same semi-simplification.

Proof. Since $D^u \otimes_{R^{\mathrm{ps}}} k = \overline{D}$ the first part follows immediately from the definition of A^{gen}. The second part follows from [Che14, Theorem 2.12]. Note that since we have assumed that all irreducible subquotients of $\overline{\rho}$ are absolutely irreducible, it is enough to prove that ρ_x and $\overline{\rho}$ have the same semi-simplification after extending scalars to the algebraic closure of k. □

Remark 4.2.1.6. We note that one needs to impose the Cayley–Hamilton condition in the definition of A^{gen} for Lemma 4.2.1.5 to hold. For example, if $\overline{D} = \chi_1 + \chi_2$, where $\chi_1, \chi_2 : G_K \to k^\times$ are distinct characters, then $E \otimes_{R^{\mathrm{ps}}} k \cong k \times k$ by Equation (8) in the proof of [BC09b, Lemma 1.4.3], let $\pi_1 : E \to k$ be the map obtained by projecting to the first component. Then the map $E \to M_2(k)$, $a \mapsto \mathrm{diag}(\pi_1(a), \pi_1(a))$ is a map of R^{ps}-algebras, and hence induces a map of R^{ps}-algebras $x : C \to k$, where C is the algebra introduced in the proof of Lemma 4.2.1.2. The representation ρ_x obtained by specializing $j : E \to M_2(C)$ at x is isomorphic to $\chi_1 + \chi_1$; hence ρ_x is not equal to $\chi_1 + \chi_2$.

4.2.2 Bounding the dimension of the fibres

Our goal is to show that $\dim R^{\square}_{\bar\rho} \leq 1 + d^2 + d^2[K : \mathbb{Q}_p]$. To do this we will actually show that $\dim \overline{X}^{\mathrm{gen}} \leq d^2 + d^2[K : \mathbb{Q}_p]$. This is done in two steps. First, we study the dimension of the fiber of X^{gen} over geometric points $y \in \operatorname{Spec} R^{\mathrm{ps}}$; that is the content of this section. The next section will study R^{ps} itself.

Let $\mathfrak{p}$ be a prime ideal of R^{ps} such that $\dim R^{\mathrm{ps}}/\mathfrak{p} \leq 1$. Let κ be an algebraic closure of the residue field of $\mathfrak{p}$ and let $y : R^{\mathrm{ps}} \to \kappa$ denote the corresponding homomorphism of R^{ps}-algebras. The goal of this subsection (Proposition 4.2.2.10) is to bound the dimension of the fibre

$$X_y^{\mathrm{gen}} := X^{\mathrm{gen}} \times_{X^{\mathrm{ps}},y} \operatorname{Spec} \kappa.$$

We let $E_y := E \otimes_{R^{\mathrm{ps}},y} \kappa$ and let D_y be the specialisation of the universal pseudorepresentation along $y : R^{\mathrm{ps}} \to \kappa$. Since κ is algebraically closed we may write

$$D_y = \prod_{i=1}^{r} D_i,$$

where each D_i is an irreducible pseudorepresentation of dimension d_i. (We follow the convention of [Che14], so that a pseudorepresentation of a direct sum of representations is a product of their pseudorepresentations; the papers [BJ19] and [Wan18] refer to a direct sum instead.) We define an equivalence relation on the set $\{D_i : 1 \leq i \leq r\}$ by $D_i \sim D_j$ if $D_i = D_j(m)$ for some $m \in \mathbb{Z}$. Let k be the number of the equivalence classes, n_i the number of elements in the i-th equivalence class.

Moreover, for $1 \leq i \leq r$ we fix representations $\rho_i : G_K \to \mathrm{GL}_{d_i}(\kappa)$ such that D_i is the pseudorepresentation associated to ρ_i. These representations are uniquely determined up to an isomorphism, but by ρ_i we really mean a group homomorphism into $\mathrm{GL}_{d_i}(\kappa)$ and not the equivalence class.

Lemma 4.2.2.1. *If $i \neq j$ then*

$$\dim_\kappa \operatorname{Ext}^1_{E_y}(\rho_i, \rho_j) = \dim_\kappa \operatorname{Ext}^1_{G_K}(\rho_i, \rho_j).$$

Proof. Given an extension $0 \to \rho_j \to W \to \rho_i \to 0$ of G_K-representations, we let $V = W \oplus \bigoplus_{l \neq i,j} \rho_l$. Then the G_K-action on V will factor through the action of E_y. Hence, W is a representation of E_y, which implies that $\operatorname{Ext}^1_{E_y}(\rho_i, \rho_j) = \operatorname{Ext}^1_{G_K}(\rho_i, \rho_j)$. □

Since (4.2) is an adequate homeomorphism there is a unique point $y' \in X^{\mathrm{gen}} /\!\!/ G$ above y and $X^{\mathrm{gen}}_{y'} \to X^{\mathrm{gen}}_y$ is a homeomorphism. The group G acts on X^{gen}_y. Moreover, X^{gen}_y is of finite type over κ and $X^{\mathrm{gen}}_y(\kappa)$ is in bijection with the set of representations $\rho : G_K \to \mathrm{GL}_d(\kappa)$ such that the semi-simplification of ρ is isomorphic to $\rho_1 \oplus \ldots \oplus \rho_r$.

Lemma 4.2.2.2. *The fibre X^{gen}_y is connected and the unique closed G-orbit in X^{gen}_y consists of the points corresponding to semi-simple representations. If the ρ_i are pairwise non-isomorphic, then its dimension is equal to $d^2 - r$.*

Proof. It follows from [Ses77, Theorem 3] that $X^{\mathrm{gen}}_{y'}$ (and hence X^{gen}_y, by the remark in the paragraph above) contains a unique closed G-orbit. Thus it is enough to show that the closure of every G-orbit contains a semi-simple representation. If $x \in X^{\mathrm{gen}}_y(\kappa)$ then after conjugation we may assume that x corresponds to a representation $\rho : G_K \to \mathrm{GL}_d(\kappa)$ such that the image of ρ is block-upper-triangular, and the blocks on the diagonal are given by $\mathrm{diag}(\rho_{\sigma(1)}(g), \ldots, \rho_{\sigma(r)}(g))$ for some permutation $\sigma \in S_r$. By extending scalars to $\kappa[T]$, conjugating ρ by $\mathrm{diag}(T^{r-1}\,\mathrm{id}_{d_{\sigma(1)}}, T^{r-2}\,\mathrm{id}_{d_{\sigma(2)}}, \ldots, \mathrm{id}_{d_{\sigma(r)}})$ and specializing at $T = 0$ we see that the closure of the G-orbit contains a semi-simple representation. The action of G on X^{gen}_y leaves the connected components invariant, see the proof of Lemma 4.1.0.1. Hence, every connected component of X^{gen}_y will contain the closed point corresponding to the representation $g \mapsto \mathrm{diag}(\rho_1(g), \ldots, \rho_r(g))$. Thus X^{gen}_y is connected.

The stabilizer of a semi-simple representation with distinct irreducible factors in GL_d is isomorphic to $\mathbb{G}_m^r$: a copy of $\mathbb{G}_m$ is embedded as scalar matrices inside of each block. Hence, the dimension of the closed G-orbit is given by $\dim \mathrm{GL}_d - \dim \mathbb{G}_m^r = d^2 - r$. □

We fix a permutation $\sigma \in S_r$ and let P be the block-upper-triangular parabolic of GL_d with the i-th diagonal block of the size $d_{\sigma(i)} \times d_{\sigma(i)}$, let N be its unipotent radical and L be Levi subgroup consisting of block diagonal matrices and let $Z_L \cong \mathbb{G}_m^r$ denote the centre of L. We denote their Lie algebras by $\mathfrak{p}$, $\mathfrak{n}$, $\mathfrak{l}$ and $\mathfrak{z}_L$, respectively. Let $\mathfrak{gl}$ be the Lie algebra of GL_d. We have

$$\dim \mathfrak{gl} = d^2, \quad \dim \mathfrak{l} = \sum_{i=1}^{r} d_i^2, \quad \dim \mathfrak{z}_L = r, \tag{4.3}$$

$$\dim \mathfrak{n} = \frac{1}{2}(\dim \mathfrak{gl} - \dim \mathfrak{l}) = \sum_{1 \le i < j \le r} d_i d_j. \tag{4.4}$$

Remark 4.2.2.3. We note that although $\mathfrak{p}$, $\mathfrak{n}$, $\mathfrak{l}$ and $\mathfrak{z}_L$ depend on σ, their dimensions do not.

Let $\rho_\sigma : G_K \to \mathrm{GL}_d(\kappa)$ be the representation $g \mapsto \mathrm{diag}(\rho_{\sigma(1)}(g), \dots, \rho_{\sigma(r)}(g))$.

Lemma 4.2.2.4. *There exists a closed subscheme $X_{y,\sigma}^{\mathrm{gen}} \subset X_y^{\mathrm{gen}}$ representing the functor sending a κ-algebra B to the set of homomorphisms of Cayley–Hamilton κ-algebras $\varphi : E_y \to \mathfrak{p} \otimes_\kappa B$ such that projection onto the ith diagonal block is $\rho_{\sigma(i)}$ for $1 \le i \le r$.*

Proof. The universal map $j : E \to M_d(A^{\mathrm{gen}})$ induces a map

$$j_y : E_y \to M_d(A^{\mathrm{gen}} \otimes_{R^{\mathrm{ps}},y} \kappa).$$

Let $I_{\rho,\sigma}$ be the ideal of $A^{\mathrm{gen}} \otimes_{R^{\mathrm{ps}},y} \kappa$ generated by the matrix entries of $j_y(a)$ for all $a \in E_y$, which lie below the diagonal blocks of P, and by all the elements on the block diagonal of the matrices $(j_y(a) - \rho_\sigma(a))$ for all $a \in E_y$. Let

$$X_{y,\sigma}^{\mathrm{gen}} := \mathrm{Spec}((A^{\mathrm{gen}} \otimes_{R^{\mathrm{ps}},y} \kappa)/I_{\rho,\sigma}).$$

Then $X_{y,\sigma}^{\mathrm{gen}}$ is a closed subscheme of X_y^{gen}, and its defining ideal $I_{\rho,\sigma}$ was constructed precisely so that a B-point of X_y^{gen} factors through $X_{y,\sigma}^{\mathrm{gen}}$ if and only if it lands in $\mathfrak{p} \otimes_\kappa B$ and matches the ρ_i on the diagonals for $1 \le i \le r$. □

The adjoint action (i.e. via conjugation) of $Z_L N$ on $\mathfrak{p}$ induces an action of $Z_L N$ on $X_{y,\sigma}^{\mathrm{gen}}$.

Lemma 4.2.2.5. *The unique closed Z_L-orbit in $X_{y,\sigma}^{\mathrm{gen}}$ is the singleton $\{\rho_\sigma\}$.*

Proof. Just as in Lemma 4.2.2.2 we show that the closure of every Z_L-orbit contains ρ_σ. Again, if $x \in X_{y,\sigma}^{\mathrm{gen}}(\kappa)$ then after extending scalars to $\kappa[T]$, conjugating ρ by

$$\mathrm{diag}(T^{r-1}\,\mathrm{id}_{d_{\sigma(1)}}, T^{r-2}\,\mathrm{id}_{d_{\sigma(2)}}, \ldots, \mathrm{id}_{d_{\sigma(r)}})$$

and specializing at $T = 0$ we see that the closure of the G-orbit of x contains ρ_σ. Thus [Ses77, Theorem 3(ii)] implies that $X_{y,\sigma}^{\mathrm{gen}} /\!\!/ Z_L$ is a singleton, and [Ses77, Theorem 3(iii)] then implies that it has a unique closed orbit, which must be $\{\rho_\sigma\}$. □

Proposition 4.2.2.6. *Let $x \in X_{y,\sigma}^{\mathrm{gen}}$ be the point corresponding to the representation ρ_σ. Then*

$$\begin{aligned} \dim T_x(X_{y,\sigma}^{\mathrm{gen}}) &= \dim \mathfrak{n} + (\dim \mathfrak{n})[K:\mathbb{Q}_p] + \sum_{1 \le i < j \le r} \dim \mathrm{Hom}_{G_K}(\rho_{\sigma(i)}, \rho_{\sigma(j)}(1)) \\ &\le \dim \mathfrak{n} + (\dim \mathfrak{n})[K:\mathbb{Q}_p] + \sum_{i=1}^{k} \binom{n_i}{2}. \end{aligned} \tag{4.5}$$

Proof. Using Lemma 4.2.2.4 and the decomposition $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{n}$ we may identify $T_x(X_{y,\sigma}^{\mathrm{gen}})$ with the κ-vector space of κ-algebra homomorphisms $\varphi : E_y \to M_d(\kappa[\varepsilon])$ of the form $\varphi = \rho_\sigma + \varepsilon\beta$, where $\beta : E_y \to \mathfrak{n}$ is a κ-linear map. If $\beta : E_y \to \mathfrak{n}$ is any κ-linear map then $\varphi := \rho_\sigma + \varepsilon\beta$ is a homomorphism of κ-algebras if and only if

$$\beta(aa') = \rho_\sigma(a)\beta(a') + \beta(a)\rho_\sigma(a'), \quad \forall a, a' \in E_y \tag{4.6}$$

For $1 \leq i \leq r$ we let $\mathbf{1}_i \in M_d(\kappa)$ be the block diagonal matrix with the identity matrix on the i-th block and zeros everywhere else. Since $\rho_\sigma(g)$ commutes with $\mathbf{1}_i$ for all i, we have an isomorphism

$$T_x(X_{y,\sigma}^{\mathrm{gen}}) \cong \bigoplus_{1\leq i<j\leq r} V_{ij},$$

where V_{ij} is the space of functions $\beta : E_y \to \mathbf{1}_i\mathfrak{n}\mathbf{1}_j$ satisfying (4.6). We may identify $\mathbf{1}_i\mathfrak{n}\mathbf{1}_j$ with $\mathrm{Hom}_\kappa(\rho_{\sigma(j)}, \rho_{\sigma(i)})$. Then V_{ij} is precisely the space of 1-cocycles for the Hochschild cohomology of E_y with values in $\mathrm{Hom}_\kappa(\rho_{\sigma(j)}, \rho_{\sigma(i)})$. Thus

$$\begin{aligned} \dim_\kappa V_{ij} &= \dim_\kappa HH^1(E_y, \mathrm{Hom}_\kappa(\rho_{\sigma(j)}, \rho_{\sigma(i)})) + \dim_\kappa \mathrm{Hom}_\kappa(\rho_{\sigma(j)}, \rho_{\sigma(i)}) \\ &\quad - \dim_\kappa HH^0(E_y, \mathrm{Hom}_\kappa(\rho_{\sigma(j)}, \rho_{\sigma(i)})) \\ &= \dim_\kappa \mathrm{Ext}^1_{E_y}(\rho_{\sigma(j)}, \rho_{\sigma(i)}) + d_id_j - \dim_\kappa \mathrm{Hom}_{E_y}(\rho_{\sigma(j)}, \rho_{\sigma(i)}) \\ &= d_id_j + [K : \mathbb{Q}_p]d_id_j + \dim_\kappa \mathrm{Ext}^2_{G_K}(\rho_{\sigma(j)}, \rho_{\sigma(i)}), \end{aligned} \tag{4.7}$$

where the first equality follows from [CE99, Proposition IX.4.4.1], the second from [CE99, Corollary IX.4.4.4], the third from Lemma 4.2.2.1 together with the local Euler characteristic formula in this context, see [BJ19, Theorem 3.4.1 (c)]. Thus

$$\dim_\kappa T_x(X_{y,\sigma}^{\mathrm{gen}}) = \dim \mathfrak{n} + (\dim \mathfrak{n})[K : \mathbb{Q}_p] + \sum_{1\leq i<j\leq r} \dim_\kappa \mathrm{Ext}^2_{G_K}(\rho_{\sigma(j)}, \rho_{\sigma(i)}).$$

It follows from Tate local duality (with mixed characteristic *or* equicharacteristic coefficients as in Nekovář [Nek06], see also [BJ19, Theorem 3.4.1 (b)] for an explanation) that

$$\dim_\kappa \mathrm{Ext}^2_{G_K}(\rho_{\sigma(j)}, \rho_{\sigma(i)}) = \dim_\kappa \mathrm{Hom}_{G_K}(\rho_{\sigma(i)}, \rho_{\sigma(j)}(1)).$$

Thus if this term is non-zero then it is equal to 1 and $\rho_{\sigma(i)}$ and $\rho_{\sigma(j)}$ belong to the same equivalence class. ☐

Remark 4.2.2.7. If $\mathrm{char}(\kappa) = p$ and $\zeta_p \in K$ then $D_i \sim D_j$ if and only if $D_i = D_j$ and the bound is sharp in this case.

Corollary 4.2.2.8. $\dim X^{\mathrm{gen}}_{y,\sigma} \leq \dim_\kappa T_x(X^{\mathrm{gen}}_{y,\sigma}) \leq \dim \mathfrak{n} + (\dim \mathfrak{n})[K : \mathbb{Q}_p] + \sum_{i=1}^k \binom{n_i}{2}$.

Proof. This follows from Lemma 4.2.2.5 and Lemma 4.1.0.1 applied with $G = Z_L$ and $X = X^{\mathrm{gen}}_{y,\sigma}$, noting that $X^{\mathrm{gen}}_{y,\sigma} /\!\!/ Z_L$ is a singleton. ☐

Lemma 4.2.2.9. *If $f : X \to Y$ is a finite type and dominant morphism of Noetherian Jacobson universally catenary schemes, then $\dim Y \leq \dim X$.*

Proof. Passing to reduced subschemes does not affect Krull dimension, so we may assume that X and Y are both reduced.

First assume X and Y are irreducible. Pick dense open affines $U \subset Y$, $V \subset X$ such that $f(V) \subset U$. It follows from [Stacks, Tag 0CC1] that $A := \mathcal{O}_Y(U) \hookrightarrow B := \mathcal{O}_X(V)$ is injective. Since A is an integral domain, Noether normalization [Stacks, Tag 07NA] implies that the map factors as

$$A \hookrightarrow A[x_1, \ldots, x_m] \hookrightarrow B' \hookrightarrow B,$$

with B' finite over $A[x_1, \ldots, x_m]$ and $B'_g \cong B_g$ for some non-zero $g \in A$. Then [Stacks, Tag 0DRT] and [Mat80, 13.C, Theorem 20] imply that

$$\dim X = \dim B = \dim B_g = \dim B'_g = \dim B' = \dim A + m = \dim Y + m$$

so $\dim Y \leq \dim X$.

For the general case we argue as in the proof of [Stacks, Tag 01RM]. Write $X = \bigcup_j Z_j$ as the union of its irreducible components. Because f is dominant, we have $Y = \bigcup_j \overline{f(Z_j)}$. Clearly the $\overline{f(Z_j)}$ have to be irreducible, and so the irreducible components of Y have to be among them. The Z_j and $\overline{f(Z_j)}$ are again Noetherian, Jacobson and universally catenary, and hence by the case already treated we have

$$\dim Y = \max_j \dim \overline{f(Z_j)} \leq \max_j \dim Z_j = \dim X.$$

□

Proposition 4.2.2.10. $\dim X_y^{\text{gen}} \leq \dim \mathfrak{gl} - r + (\dim \mathfrak{n})[K : \mathbb{Q}_p] + \sum_{i=1}^{k} \binom{n_i}{2}$.

Proof. We want to apply Lemma 4.2.2.9 to

$$\coprod_{\sigma \in S_r} G \times^{Z_{L_\sigma} N_\sigma} X_{y,\sigma}^{\text{gen}} \to X_y^{\text{gen}}. \tag{4.8}$$

If $x \in X_y^{\text{gen}}(\kappa)$ and $\varphi : E_y \to M_d(\kappa)$ is the corresponding κ-algebra homomorphism then there will exist $\sigma \in S_r$ such that κ^d will admit a filtration by subspaces $0 = V_0 \subset V_1 \subset \ldots \subset V_r = V$, which is invariant under the action of E_y via φ, satisfying $V_i/V_{i-1} \cong \rho_{\sigma(i)}$ for $1 \leq i \leq r$. Thus there is $g \in G(\kappa)$ such that $g\varphi g^{-1}$ will lie in $X_{y,\sigma}^{\text{gen}}(\kappa)$, and hence (4.8) induces a surjection on κ-points. But (4.8) is also a map of finite type κ-schemes, and therefore is a dominant map of Noetherian Jacobson universally catenary schemes, so we can apply Lemma 4.2.2.9.

The fibre bundles $G \times^{Z_{L_\sigma} N_\sigma} X_{y,\sigma}^{\text{gen}}$ have dimension equal to

$$\dim G + \dim X_{y,\sigma}^{\text{gen}} - \dim(Z_{L_\sigma} N_\sigma) = \dim \mathfrak{gl} - r + \dim X_{y,\sigma}^{\text{gen}} - \dim \mathfrak{n}.$$

The bound in Corollary 4.2.2.8 gives the required assertion. □

Corollary 4.2.2.11. *If $r = 1$ then X_y^{gen} is smooth of dimension* $\dim \mathfrak{gl} - 1$.

Proof. If $r = 1$ then $E_y \cong M_d(\kappa)$ and thus has a unique irreducible representation ρ (up to isomorphism). Thus all the points in $X_y^{\mathrm{gen}}(\kappa)$ lie in the same G-orbit. Fix such a point x. Since the G-stabiliser of x is equal to Z_G we obtain $\dim X_y^{\mathrm{gen}} = \dim G - \dim Z_G = \dim \mathfrak{gl} - 1$.

Since E_y is semi-simple we have $\mathrm{Ext}^1_{E_y}(\rho, \rho) = 0$ and thus an argument as in the first paragraph of the proof of Proposition 4.2.2.6 gives us

$$\dim_\kappa T_x(X_y^{\mathrm{gen}}) = \dim_\kappa \mathrm{End}_\kappa(\rho) - \dim_\kappa \mathrm{End}_{E_y}(\rho) = \dim X_y^{\mathrm{gen}}.$$

Thus x is a smooth point of X_y^{gen}, and since G acts transitively on $X_y^{\mathrm{gen}}(\kappa)$ all the points in $X_y^{\mathrm{gen}}(\kappa)$ are smooth. Since X_y^{gen} is of finite type over κ, we deduce that X_y^{gen} is smooth. $\square$

4.2.3 Bounding the dimension of the space: some commutative algebra

In the following two sections we combine the previous results on the fibers X_y^{gen} with Böckle–Juschka's results on the dimension of strata in $\mathrm{Spec}\, R^{\mathrm{ps}}$ to bound the dimension of $\overline{X}^{\mathrm{gen}}$; see Theorem 4.2.4.8 for the conclusion.

This section will consist of some general commutative algebra lemmas. For a ring R we set $P_1 R = \{\mathfrak{p} \in \mathrm{Spec}\, R : \dim R/\mathfrak{p} = 1\}$.

Lemma 4.2.3.1. *Let $(R, \mathfrak{m}_R)$ be a complete local Noetherian $\mathcal{O}$-algebra with finite residue field k'. If $\mathfrak{p} \in P_1 R$ then $\kappa(\mathfrak{p})$ is either a finite extension of L or a local field of characteristic p. Moreover, $R/\mathfrak{p}$ is contained in the ring of integers $\mathcal{O}_{\kappa(\mathfrak{p})}$ of $\kappa(\mathfrak{p})$ and the quotient topology on $R/\mathfrak{p}$ induced by the $\mathfrak{m}_R$-adic topology on R coincides with the subspace topology induced by the topology on $\mathcal{O}_{\kappa(\mathfrak{p})}$.*

Proof. It follows from Cohen's structure theorem that if $\operatorname{char}(R/\mathfrak{p}) = 0$ then $\mathcal{O} \subset R/\mathfrak{p}$ and $R/\mathfrak{p}$ is a finitely generated $\mathcal{O}$-module. Thus $\kappa(\mathfrak{p})$ is a finite extension of L and $R/\mathfrak{p}$ is contained in the integral closure of $\mathcal{O}$ in $\kappa(\mathfrak{p})$, which is equal to $\mathcal{O}_{\kappa(\mathfrak{p})}$. If $\operatorname{char}(R/\mathfrak{p}) = p$ then $R/\mathfrak{p}$ is finite over a subring isomorphic to $k'[\![t]\!]$ and the same argument carries over. Moreover, $\mathcal{O}_{\kappa(\mathfrak{p})}$ is a finitely generated $R/\mathfrak{p}$-module, and this implies that the topologies coincide. □

The next lemma is particularly useful for moving back and forth between X^{gen} and R^{ps}, and subspaces of both.

Lemma 4.2.3.2. *Let $(R, \mathfrak{m}_R)$ be a complete local Noetherian ring and $\varphi : R \to S$ a ring map of finite type. Let U be a non-empty open subscheme of $U_{\max} := (\operatorname{Spec} R) \setminus \{\mathfrak{m}_R\}$, let V (resp. $V_{\max}$) be the preimage of U (resp. $U_{\max}$) in $\operatorname{Spec} S$, let Z (resp. $Z_{\max}$) be the closure of V (resp. $V_{\max}$) in $\operatorname{Spec} S$ and let Y be the preimage of $\{\mathfrak{m}_R\}$ in $\operatorname{Spec} S$. Then*

1. *V is Jacobson;*
2. *the set of closed points of V is $V \cap \{\text{closed points of } V_{\max}\}$;*
3. *if x is a closed point of V then its image y in $\operatorname{Spec} R$ is a closed point of U and the field extension $\kappa(x)/\kappa(y)$ is finite;*
4. *the set of closed points of U is $U \cap P_1 R$;*
5. *if every irreducible component of $\operatorname{Spec} S$ meets Y non-trivially then $\dim Z = \dim V + 1$;*
6. *$\dim V \leq \dim U + \max_{y \in U \cap P_1 R} \dim \varphi^{-1}(\{y\})$.*

Proof. We will first prove parts (1), (2) and (3). If $R = S$ and if $U = U_{\max}$ then (1) follows from [Stacks, Tag 02IM] and both (2) and (3) hold trivially. If $R = S$ and if U is arbitrary then $U = V$ and (1), (2) follow from the previous case together with [Stacks, Tag 005W] and (3) holds trivially. The case of general φ now follows from [Stacks, Tag 00GB] together with [Stacks, Tag 01P4], because the map $V \to U$ induced from φ is of finite type.

Part (4) follows from (2) applied with $S = R$, using that $\mathfrak{m}_R$ is the unique maximal ideal of R, so that the set of closed points of $U_{\max}$ is equal to $P_1 R$.

For (5) note first that since V is open in $\operatorname{Spec} S$ the set of generic points of V is a subset of the set of generic points of $\operatorname{Spec} S$. Thus Z is union of irreducible components of $\operatorname{Spec} S$. Let $Z' = \operatorname{Spec} S'$ be an irreducible component of Z with the induced reduced subscheme structure so that S' is a domain, let $V' = Z' \cap V$, let R' be the image of R in S'. The rings R' and S' are excellent and hence universally catenary by [Stacks, Tag 07QW]. If $\mathfrak{q} \in \operatorname{Spec} S'$ and $\mathfrak{p} = \mathfrak{q} \cap R'$ then

$$\begin{aligned} \dim S'_{\mathfrak{q}} &= \dim R'_{\mathfrak{p}} + \operatorname{trdeg}_{R'} S' - \operatorname{trdeg}_{\kappa(\mathfrak{p})} \kappa(\mathfrak{q}) \\ &= \dim R' + \operatorname{trdeg}_{R'} S' - \dim R'/\mathfrak{p} - \operatorname{trdeg}_{\kappa(\mathfrak{p})} \kappa(\mathfrak{q}), \end{aligned} \tag{4.9}$$

where trdeg stands for transcendence degree, the first equality is [Stacks, Tag 02IJ], and the second is [Mat89, Theorem 31.4]. It follows from (4.9) that

$$\dim S'_{\mathfrak{q}} \leq \dim R' + \operatorname{trdeg}_{R'} S' \tag{4.10}$$

and the equality in (4.10) holds if and only if $\mathfrak{q}$ maps to the maximal ideal of R' and $\mathfrak{q}$ is a maximal ideal of S'. Since $Z' \cap Y$ is non-empty by assumption, such $\mathfrak{q}$ exists and so

$$\dim S' = \dim R' + \operatorname{trdeg}_{R'} S'.$$

Let $\mathfrak{q}$ be a closed point of V' and let $\mathfrak{p} = \mathfrak{q} \cap R'$. Since V' is open in Z' we have $\mathcal{O}_{V',\mathfrak{q}} = S'_{\mathfrak{q}}$.

It follows from (3) that $\operatorname{trdeg}_{\kappa(\mathfrak{p})} \kappa(\mathfrak{q}) = 0$ and $\dim R/\mathfrak{p} = 1$. Thus (4.9) gives us

$$\dim \mathcal{O}_{V',\mathfrak{q}} = \dim R' + \operatorname{trdeg}_{R'} S' - 1.$$

Since this holds for all closed points of V' we deduce that

$$\dim V' = \dim R' + \operatorname{trdeg}_{R'} S' - 1.$$

This implies part (5).

Let x be a closed point of V and let y be its image in U. Then y is also a closed point of U. We have

$$\dim \mathcal{O}_{V,x} \leq \dim \mathcal{O}_{U,y} + \dim(\mathcal{O}_{V,x} \otimes_{\mathcal{O}_{U,y}} \kappa(y)) \leq \dim U + \dim \varphi^{-1}(\{y\}),$$

where the first inequality is given by [Mat89, Theorem 15.1 (i)]. Since

$$\dim V = \max_x \dim \mathcal{O}_{V,x},$$

where the maximum is taken over all closed points x of V we get (6). □

Remark 4.2.3.3. We caution the reader that the equality $\dim Z = \dim V + 1$ might fail if one drops the assumption that Y meets every irreducible component non-trivially. For example, if $R = \mathbb{Z}_p$ and $S = \mathbb{Z}_p[x]/(px-1) = \mathbb{Q}_p$ then Y is empty and $V_{\max} = Z_{\max} = \operatorname{Spec} S$.

Remark 4.2.3.4. Here is another cautionary example. If R and S are as in Lemma 4.2.3.2, $\mathfrak{q}$ is a prime of S and S is a domain then it need not be true that $\dim S_{\mathfrak{q}} + \dim S/\mathfrak{q} = \dim S$. For example, if $R = \mathbb{Z}_p$, $S = \mathbb{Z}_p[x]$ and $\mathfrak{q} = (px-1)$ then $S/\mathfrak{q} = \mathbb{Q}_p$ and $S_{\mathfrak{q}}$ is a DVR, so that $\dim S_{\mathfrak{q}} + \dim S/\mathfrak{q} = 1$ and $\dim S = 2$. We also note that $\mathfrak{q}$ is a closed point of $\operatorname{Spec} S$ but it does not map to a closed point of $\operatorname{Spec} R$. Further, if $\mathfrak{q}' = (p, x)$ then $S/\mathfrak{q}' = \mathbb{F}_p$ and p, x is a regular sequence of parameters in $S_{\mathfrak{q}'}$, and thus $\dim S_{\mathfrak{q}'} = 2$. Thus $\mathfrak{q}$ and $\mathfrak{q}'$ are closed points of an irreducible scheme, but their local rings have different dimensions.

Lemma 4.2.3.5. *Let Y be the preimage of $\{\mathfrak{m}_{R^{\text{ps}}}\}$ in X^{gen}, let W be a closed non-empty GL_d-invariant subscheme of X^{gen} and let Z be an irreducible component of W. Then $Y \cap Z$ is non-empty. Moreover, if x is a closed point of Z then the following hold:*

1. *if $x \in Y$ then $\dim \mathcal{O}_{Z,x} = \dim Z$;*

2. *if $x \notin Y$ then $\dim \mathcal{O}_{Z,x} = \dim Z - 1$.*

Proof. Let x be a closed point of Z and let y be its image in $\operatorname{Spec} R^{\text{ps}}$. If y is not the maximal ideal of R^{ps} then $x \in Z \setminus Y$.

If $Z \cap Y$ is empty then Lemma 4.2.3.2 (3) applied with $R = R^{\text{ps}}$, $U = U_{\max}$ and $Z = \operatorname{Spec} S$ implies that $V_{\max} = Z$ and hence Z is Jacobson. Let W' be the union of irreducible components of W, different from Z. Then $Z \setminus W'$ is a non-empty open subscheme of Z. Let x be a closed point of $Z \setminus W'$ then x is also a closed point of Z by [Stacks, Tag 005W]. By construction Z is the unique irreducible component of W containing x.

It follows from Lemma 4.2.3.2 (4) that $y \in P_1 R^{\text{ps}}$ and $\kappa(x)$ is a finite extension of $\kappa(y)$. Thus $\kappa(x)$ is either a finite extension of L or a local field of characteristic p. Let $\rho_x : G_K \to \mathrm{GL}_d(\kappa(x))$ be the corresponding Galois representation. Since G_K is compact, there is a matrix $M \in \mathrm{GL}_d(\kappa(x))$, such that $M\rho_x(g)M^{-1} \in \mathrm{GL}_d(\mathcal{O}_{\kappa(x)})$ for all $g \in G_K$. Since conjugation does not change the characteristic polynomial there is an R^{ps}-algebra homomorphism $x' : A^{\text{gen}} \to \kappa(x)$, such that $\rho_{x'}(g) = M\rho_x(g)M^{-1}$ for all $g \in G_K$.

We claim that x and x' lie on the same irreducible component of $W_y := W \times_{R^{\text{ps}}} \kappa(y)$. If B is a $\kappa(x)$-algebra then the map $\mathrm{GL}_d(B) \to X^{\text{gen}}(B)$, $N \mapsto [g \mapsto N\rho_x(g)N^{-1}]$ defines a map of schemes over $\operatorname{Spec} \kappa(x)$, $\mathrm{GL}_d \to W_y \times_{\kappa(y)} \kappa(x)$ and both x and x' are contained in its scheme theoretic image. Note that W (and hence the fibre W_y) are GL_d-invariant by assumption. Since GL_d over $\kappa(x)$ is irreducible we obtain the claim. We conclude that both x and x' lie on the same irreducible component of W, which is Z.

Let $\mathfrak{q}$ be the kernel of $x' : A^{\mathrm{gen}} \to \kappa(x)$. Since the image of $\rho_{x'}$ is contained in $\mathrm{GL}_d(\mathcal{O}_{\kappa(x)})$, $A^{\mathrm{gen}}/\mathfrak{q}$ is a subring of $\mathcal{O}_{\kappa(x)}$. Let k' be the residue field of $\mathcal{O}_{\kappa(x)}$ and let $z \in X^{\mathrm{gen}}(k')$ be the composition $z : A^{\mathrm{gen}} \to A^{\mathrm{gen}}/\mathfrak{q} \to \mathcal{O}_{\kappa(x)} \to k'$. Then z maps to the closed point in $\operatorname{Spec} R^{\mathrm{ps}}$ (i.e. $z \in Y$), and is contained in the closure of x' in X^{gen}. Since Z is closed in X^{gen} and contains x' we deduce that $z \in Z$, contradiction.

The claims about $\dim \mathcal{O}_{Z,x}$ follows from the proof of part (5) in Lemma 4.2.3.2. □

Example 4.2.3.6. Let us illustrate Lemma 4.2.3.5 with a concrete example. Let $\overline{D}$ be the pseudorepresentation of the 2-dimensional trivial representation of the group $\Gamma := \mathbb{Z}_p$. It follows from [Che14, Theorem 1.15] that $R^{\mathrm{ps}} \cong \mathcal{O}[\![t,d]\!]$ and

$$E \cong \frac{R^{\mathrm{ps}}[\![T]\!]}{((1+T)^2 - (2+t)(1+T) + 1 + d)},$$

where the map $\Gamma \to R^{\mathrm{ps}}[\![\Gamma]\!] \twoheadrightarrow E$ sends a fixed topological generator γ of Γ to $1+T$. Then E is a free R^{ps}-module with basis $1+T, 1$ and so

$$A^{\mathrm{gen}} = \frac{R^{\mathrm{ps}}[x_{11}, x_{12}, x_{21}, x_{22}]}{(x_{11} + x_{22} - (2+t), x_{11}x_{22} - x_{12}x_{21} - (1+d))},$$

and $j : E \to M_2(A^{\mathrm{gen}})$ sends $1+T$ to the matrix $\left(\begin{smallmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{smallmatrix}\right)$. Let $x : A^{\mathrm{gen}} \to L$ be the homomorphism corresponding to the representation $\rho : E \to M_2(L)$, such that $\rho(\gamma) = \left(\begin{smallmatrix} 1 & p^{-1} \\ 0 & 1 \end{smallmatrix}\right)$. Then x is a closed point of X^{gen} with residue field L, thus it does not map to the closed point in X^{ps}. Indeed, $A^{\mathrm{gen}}/(x_{11}-1, x_{21}, x_{22}-1) \cong \mathcal{O}[x_{12}]$, so we are in the situation considered in Remark 4.2.3.4.

Lemma 4.2.3.7. *Let W be a closed non-empty GL_d-invariant subscheme of X^{gen} and write*

$W[1/p]$ and $\overline{W}$ for the generic and special fibre. Then $\dim W[1/p] \leq \dim \overline{W}$. *In particular,* $\dim X^{\text{gen}}[1/p] \leq \dim \overline{X}^{\text{gen}}$.

Proof. We may assume that $W[1/p]$ is non-empty. Let $Z = \operatorname{Spec} A^{\text{gen}}/\mathfrak{p}$ be an irreducible component of W such that $Z[1/p]$ is non-empty and let $\overline{Z} = \operatorname{Spec} A^{\text{gen}}/(\mathfrak{p}, \varpi)$. Lemma 4.2.3.5 implies that there is a closed point $x \in Z$, which maps to the closed point in X^{ps}. Lemma 4.2.3.2 (5) implies that $\dim Z[1/p] = \dim Z - 1$.

Since Z is irreducible and $Z[1/p] \neq \emptyset$ the local ring $\mathcal{O}_{Z,x}$ is a domain and multiplication by ϖ is injective. Since $\operatorname{char}(\kappa(x)) = p$, ϖ cannot be a unit in $\mathcal{O}_{Z,x}$. Thus $\dim \mathcal{O}_{\overline{Z},x} = \dim \mathcal{O}_{Z,x} - 1$. It follows from Lemma 4.2.3.5 that $\dim \overline{Z} = \dim Z - 1$. Since $\overline{Z}$ is a closed subset of $\overline{W}$ we have $\dim \overline{W} \geq \dim \overline{Z} = \dim Z - 1 = \dim Z[1/p]$, and hence $\dim \overline{W} \geq \dim W[1/p]$. □

4.2.4 Bounding the dimension of the space

Now we apply our previous work to $\overline{X}^{\text{gen}}$. We note also that some of the computations done in this section will be useful for bounding the singular locus in X^{gen} later in Section 4.3.

Recall that $\overline{D} : G_K \to k$ is the specialization of the universal pseudorepresentation $D^u : G_K \to R^{\text{ps}}$ at the maximal ideal of R^{ps}. We may write $\overline{D} = \prod_{i=1}^m \overline{D}_i$, where $\overline{D}_i$ are absolutely irreducible pseudorepresentations. Let $\mathcal{P}$ be an (unordered) partition of the set $\{1, \ldots, m\}$ into r disjoint subsets Σ_j, and let $\underline{\Sigma} = (\Sigma_1, \ldots, \Sigma_r)$ be an ordering of the subsets in $\mathcal{P}$. For each $1 \leq j \leq r$ let $\overline{D}'_j = \prod_{i \in \Sigma_j} \overline{D}_i$, and let d_j be the dimension of $\overline{D}'_j$. We define an equivalence relation on the set of pseudorepresentations $\{\overline{D}'_j : 1 \leq j \leq r\}$ by $\overline{D}'_j \sim \overline{D}'_{j'}$ if $\overline{D}'_j = \overline{D}'_{j'}(t)$ for some $t \in \mathbb{Z}$. Let k' be the number of the equivalence classes, n'_i be the number of elements in the i-th equivalence class, c_i be the dimension of the pseudorepresentations in the i-th equivalence class. We have

$$\sum_{i=1}^{k'} n'_i = r, \quad \sum_{i=1}^{k'} c_i n'_i = d.$$

We define

$$l_{\mathcal{P}} := \sum_{j=1}^{r} d_j^2 = \sum_{i=1}^{k'} n_i' c_i^2, \quad p_{\mathcal{P}} := l_{\mathcal{P}} + n_{\mathcal{P}} = \sum_{j=1}^{r} d_j^2 + \sum_{1 \le j < j' \le r} d_j d_{j'}, \tag{4.11}$$

where

$$n_{\mathcal{P}} = \frac{1}{2}(d^2 - l_{\mathcal{P}}) = \sum_{1 \le j < j' \le r} d_j d_{j'} = \sum_{1 \le i < i' \le k'} c_i c_{i'} n_i' n_{i'}' + \sum_{i=1}^{k'} c_i^2 \binom{n_i'}{2}. \tag{4.12}$$

The notation is motivated by (4.3) and (4.4), see also Remark 4.2.2.3.

For each $1 \le j \le r$ let R_j^{ps} be the universal deformation ring of $\overline{D}_j'$ and let $X_j^{\mathrm{ps}} := \operatorname{Spec} R_j^{\mathrm{ps}}$. The functor $\mathcal{F}_{\underline{\Sigma}}$, which sends a local Artinian $\mathcal{O}$-algebra $(A, \mathfrak{m}_A)$ with residue field k to the set of ordered r-tuples $(D_1, \dots, D_r)$ of pseudorepresentations with each D_i a deformation of $\overline{D}_i'$ to A is represented by the completed tensor product

$$R_{\underline{\Sigma}}^{\mathrm{ps}} := R_1^{\mathrm{ps}} \widehat{\otimes}_{\mathcal{O}} \dots \widehat{\otimes}_{\mathcal{O}} R_r^{\mathrm{ps}}.$$

We let $X_{\underline{\Sigma}}^{\mathrm{ps}} := \operatorname{Spec} R_{\underline{\Sigma}}^{\mathrm{ps}}$ and denote by $\overline{X}_{\underline{\Sigma}}^{\mathrm{ps}} := \operatorname{Spec} R_{\underline{\Sigma}}^{\mathrm{ps}}/\varpi$ to its special fibre. By mapping an r-tuple of pseudorepresentations to their product we obtain a map

$$\iota_{\underline{\Sigma}} : \overline{X}_{\underline{\Sigma}}^{\mathrm{ps}} \to \overline{X}^{\mathrm{ps}}.$$

Lemma 4.2.4.1. *The map $R^{\mathrm{ps}} \to R_{\underline{\Sigma}}^{\mathrm{ps}}$ is finite.*

Proof. By topological Nakayama's lemma it is enough to show that the fibre ring $C := k \otimes_{R^{\mathrm{ps}}} R_{\underline{\Sigma}}^{\mathrm{ps}}$ is a finite dimensional k-vector space. Let $\mathcal{F}$ be the closed subfunctor of $\mathcal{F}_{\underline{\Sigma}}$ defined by C. If $(A, \mathfrak{m}_A)$ is a local Artinian k-algebra then $\mathcal{F}(A)$ is in bijection with the set of r-tuples $(D_1, \dots, D_r)$, each D_i lifting $\overline{D}_i'$ to A such that $\prod_{i=1}^r D_i = (\prod_{i=1}^r \overline{D}_i') \otimes_k A$.

Since C is a complete local Noetherian ring, it is enough to show that its Krull dimension is

0. If this is not the case then there is $\mathfrak{p} \in \operatorname{Spec} C$ such that $\dim C/\mathfrak{p} = 1$. Let $(D_{1,y}, \dots, D_{r,y})$ be the specialization of the universal object of $\mathcal{F}_{\underline{\Sigma}}$ along $y : R^{\mathrm{ps}}_{\underline{\Sigma}} \to \kappa(\mathfrak{p})$. It follows from [Che14, Corollary 1.14] that the coefficients of the polynomials $D_{i,y}(t-a)$, for all $a \in E$ and $1 \le i \le r$ will generate a dense subring of $R^{\mathrm{ps}}_{\underline{\Sigma}}/\mathfrak{p}$. Since $R^{\mathrm{ps}}_{\underline{\Sigma}}/\mathfrak{p}$ is a complete local k-algebra of dimension 1, there will exist $a \in E$ and index i such that the coefficients of $D_{i,y}(t-a)$ will generate a transcendental extension of k inside $\kappa(\mathfrak{p})$. Since $\mathfrak{p} \in \operatorname{Spec} C$ we have

$$\prod_{i=1}^{r} D_{i,y}(t-a) = \prod_{i=1}^{r} \overline{D}'_i(t-a).$$

Thus all the roots of $D_{i,y}(t-a)$ in the algebraic closure of $\kappa(\mathfrak{p})$ are algebraic over k. Since $D_{i,y}(t-a)$ is a monic polynomial, we conclude that all the coefficients are also algebraic over k, giving a contradiction. $\square$

Let $\overline{X}^{\mathrm{ps}}_{\mathcal{P}}$ be the scheme theoretic image of $\iota_{\underline{\Sigma}}$. We note that $\overline{X}^{\mathrm{ps}}_{\mathcal{P}}$ depends only on $\mathcal{P}$ and not on the chosen ordering $\underline{\Sigma}$. It follows from Lemma 4.2.4.1 that

$$\dim \overline{X}^{\mathrm{ps}}_{\mathcal{P}} = \dim \overline{X}^{\mathrm{ps}}_{\underline{\Sigma}} = \sum_{i=1}^{r} \dim \overline{X}^{\mathrm{ps}}_i = r + l_{\mathcal{P}}[K : \mathbb{Q}_p],$$

where the last equality is given by [BJ19, Theorem 5.4.1(i)].

We define a partial order on the set of partitions of $\{1, \dots, m\}$ by $\mathcal{P} \le \mathcal{P}'$ if $\mathcal{P}'$ is a refinement of $\mathcal{P}$. The partition $\mathcal{P}_{\min}$ consisting of 1 part is the minimal element and the partition $\mathcal{P}_{\max}$ consisting of m parts is the maximal element with respect to this partial ordering. If $\mathcal{P} \le \mathcal{P}'$ then $\overline{X}^{\mathrm{ps}}_{\mathcal{P}'}$ is a closed subscheme of $\overline{X}^{\mathrm{ps}}_{\mathcal{P}}$ and $\overline{X}^{\mathrm{ps}}_{\mathcal{P}_{\min}} = \overline{X}^{\mathrm{ps}}$. Let

$$U_{\mathcal{P}} := \overline{X}^{\mathrm{ps}}_{\mathcal{P}} \setminus (\{\mathfrak{m}_{R^{\mathrm{ps}}}\} \cup \bigcup_{\mathcal{P} < \mathcal{P}'} \overline{X}^{\mathrm{ps}}_{\mathcal{P}'})$$

and let $V_{\mathcal{P}}$ be the preimage of $U_{\mathcal{P}}$ in $\overline{X}^{\mathrm{gen}}$ and let $Z_{\mathcal{P}}$ be the closure of $V_{\mathcal{P}}$ in $\overline{X}^{\mathrm{gen}}$. Let $\overline{X}^{\mathrm{gen}}_{\mathcal{P}}$ be the preimage of $\overline{X}^{\mathrm{ps}}_{\mathcal{P}}$ in $\overline{X}^{\mathrm{gen}}$. Then $\overline{X}^{\mathrm{gen}}_{\mathcal{P}}$ is closed in $\overline{X}^{\mathrm{gen}}$ and contains $V_{\mathcal{P}}$, hence we are in the situation of Lemma 4.2.3.2 with $\operatorname{Spec} R = \overline{X}^{\mathrm{ps}}_{\mathcal{P}}$ and $\operatorname{Spec} S = Z_{\mathcal{P}}$. Note that Lemma 4.2.3.5 implies that every irreducible component of $\overline{X}^{\mathrm{gen}}_{\mathcal{P}}$ contains a closed point mapping to $\mathfrak{m}_{R^{\mathrm{ps}}}$. Thus the condition in part (5) of Lemma 4.2.3.2 is satisfied and hence $\dim Z_{\mathcal{P}} = \dim V_{\mathcal{P}} + 1$; the same conclusion applies to closures of various loci considered below. Moreover, we have

$$\overline{X}^{\mathrm{gen}}_{\mathcal{P}} = Y \cup \bigcup_{\mathcal{P} \leq \mathcal{P}'} Z_{\mathcal{P}'}, \tag{4.13}$$

where Y is the preimage of $\{\mathfrak{m}_{R^{\mathrm{ps}}}\}$ in $\overline{X}^{\mathrm{gen}}$.

We will also need a variant of the situation above. This is used later when we bound the dimension of the singular locus in $\overline{X}^{\mathrm{gen}}$; the point is that the non-vanishing of a certain H^2 will imply that certain Jordan–Holder factors of a representations are twists of one another. We will need to leverage this fact to get a good enough bound.

Assume that $r > 1$ and let i and j be distinct indices with $1 \leq i, j \leq r$. Let $\mathcal{F}^{ij}_{\underline{\Sigma}}$ be a subfunctor of $\mathcal{F}_{\underline{\Sigma}}$ parameterizing the deformations $(D_1, \ldots, D_r)$ of the ordered r-tuple $(\overline{D}'_1, \ldots, \overline{D}'_r)$ such that $D_i = D_j(1)$. Then $\mathcal{F}^{ij}_{\underline{\Sigma}}$ is a closed subfunctor of $\mathcal{F}_{\underline{\Sigma}}$ and we let $R^{\mathrm{ps},ij}_{\underline{\Sigma}}$ be the quotient of $R^{\mathrm{ps}}_{\underline{\Sigma}}$ representing it. If $\overline{D}'_i \neq \overline{D}'_j(1)$ then $R^{\mathrm{ps},ij}_{\underline{\Sigma}}$ is the zero ring, otherwise

$$\dim R^{\mathrm{ps},ij}_{\underline{\Sigma}}/\varpi = \dim R^{\mathrm{ps}}_{\underline{\Sigma}}/\varpi - \dim R^{\mathrm{ps}}_{i}/\varpi \leq r + l_{\mathcal{P}}[K : \mathbb{Q}_p] - (1 + [K : \mathbb{Q}_p]).$$

Let $\overline{X}^{\mathrm{ps},ij}_{\mathcal{P}}$ be the scheme theoretic image of $\operatorname{Spec} R^{\mathrm{ps},ij}_{\underline{\Sigma}}$ in $\overline{X}^{\mathrm{ps}}$ under $\iota_{\underline{\Sigma}}$. Then

$$\dim \overline{X}^{\mathrm{ps},ij}_{\mathcal{P}} \leq r + l_{\mathcal{P}}[K : \mathbb{Q}_p] - (1 + [K : \mathbb{Q}_p]). \tag{4.14}$$

Let $U^{ij}_{\mathcal{P}} := U_{\mathcal{P}} \cap \overline{X}^{\mathrm{ps},ij}_{\mathcal{P}}$, let $V^{ij}_{\mathcal{P}}$ be the preimage of $U^{ij}_{\mathcal{P}}$ in $\overline{X}^{\mathrm{gen}}$ and let $Z^{ij}_{\mathcal{P}}$ be the closure of $V^{ij}_{\mathcal{P}}$ in $\overline{X}^{\mathrm{gen}}$.

Lemma 4.2.4.2. *If y is a geometric point over a closed point in $U_{\mathcal{P}}$ then*

$$\dim X_y^{\mathrm{gen}} \leq d^2 - r + n_{\mathcal{P}}[K:\mathbb{Q}_p] + \sum_{i=1}^{k'} \binom{n_i'}{2}.$$

If we additionally assume that $y \notin U_{\mathcal{P}}^{ij}$ for any $i \neq j$ then

$$\dim X_y^{\mathrm{gen}} \leq d^2 - r + n_{\mathcal{P}}[K:\mathbb{Q}_p].$$

Proof. We may write $D_y = D_1 + \ldots + D_r$ with D_i lifting $\overline{D}_i'$. We note that all the D_i are absolutely irreducible, since otherwise $y \in X_{\mathcal{P}'}^{\mathrm{ps}}$ for some $\mathcal{P}' > \mathcal{P}$. Let k and n_i be the numbers defined in Section 4.2.2. Proposition 4.2.2.10 implies that

$$\dim X_y^{\mathrm{gen}} \leq d^2 - r + n_{\mathcal{P}}[K:\mathbb{Q}_p] + \sum_{i=1}^{k} \binom{n_i}{2}.$$

If $D_i = D_j(m)$ for some $m \in \mathbb{Z}$ then also $\overline{D}_i' = \overline{D}_j'(m)$. This implies that

$$\sum_{i=1}^{k} \binom{n_i}{2} \leq \sum_{i=1}^{k'} \binom{n_i'}{2},$$

which implies the first assertion. We note that if $a_i, \ldots, a_s$ are positive integers then $\sum_{i=1}^{s} \binom{a_i}{2} \leq \binom{\sum_{i=1}^{s} a_i}{2}$.

If $y \notin U_{\mathcal{P}}^{ij}$ for any $i \neq j$ then $D_i \neq D_j(1)$ for any $i \neq j$ and the Hom terms in (4.5) vanish. The assertion follows from Proposition 4.2.2.10 using this improved bound. $\square$

Proposition 4.2.4.3. $\dim Z_{\mathcal{P}}^{ij} \leq d^2 + p_{\mathcal{P}}[K:\mathbb{Q}_p] + \sum_{i=1}^{k'} \binom{n_i'}{2} - (1 + [K:\mathbb{Q}_p])$.

Proof. It follows from Lemma 4.2.3.2 (5) that the closure of $U_{\mathcal{P}}^{ij}$ has dimension $\dim U_{\mathcal{P}}^{ij} + 1$.

Thus

$$\dim U_{\mathcal{P}}^{ij} + 1 \leq \dim X_{\mathcal{P}}^{\mathrm{ps},ij} \leq r + l_{\mathcal{P}}[K:\mathbb{Q}_p] - (1 + [K:\mathbb{Q}_p]),$$

where the last inequality is (4.14). Parts (5) and (6) of Lemma 4.2.3.2 together with Lemma 4.2.4.2 imply that

$$\dim Z_{\mathcal{P}}^{ij} \leq (r + l_{\mathcal{P}}[K:\mathbb{Q}_p] - (1 + [K:\mathbb{Q}_p])) + (d^2 - r + n_{\mathcal{P}}[K:\mathbb{Q}_p] + \sum_{i=1}^{k'} \binom{n'_i}{2}),$$

which imply the assertion. □

Proposition 4.2.4.4. *Let* $\delta_{\mathcal{P}} = \max\{0, \sum_{i=1}^{k'} \binom{n'_i}{2} - (1 + [K:\mathbb{Q}_p])\}$. *Then*

$$\dim Z_{\mathcal{P}} \leq d^2 + p_{\mathcal{P}}[K:\mathbb{Q}_p] + \delta_{\mathcal{P}}.$$

Proof. Let $U'_{\mathcal{P}} := U_{\mathcal{P}} \setminus \bigcup_{i \neq j} U_{\mathcal{P}}^{ij}$, let $V'_{\mathcal{P}}$ be the preimage of $U'_{\mathcal{P}}$ in $\overline{X}^{\mathrm{gen}}$ and let $Z'_{\mathcal{P}}$ denote the closure of $V'_{\mathcal{P}}$ in $\overline{X}^{\mathrm{gen}}$. If y is a closed point of $U'_{\mathcal{P}}$ then $\dim \overline{X}_y^{\mathrm{gen}} \leq d^2 - r + n_{\mathcal{P}}[K:\mathbb{Q}_p]$ by Lemma 4.2.4.2. Thus Lemma 4.2.3.2 implies that

$$\dim Z'_{\mathcal{P}} \leq \dim \overline{X}_{\mathcal{P}}^{\mathrm{ps}} + (d^2 - r + n_{\mathcal{P}}[K:\mathbb{Q}_p]) = d^2 + p_{\mathcal{P}}[K:\mathbb{Q}_p]. \tag{4.15}$$

Since $Z_{\mathcal{P}} = Z'_{\mathcal{P}} \cup \bigcup_{i \neq j} Z_{\mathcal{P}}^{ij}$ we have $\dim Z_{\mathcal{P}} = \max_{i \neq j}\{\dim Z'_{\mathcal{P}}, \dim Z_{\mathcal{P}}^{ij}\}$ and the assertion follows from Proposition 4.2.4.3. □

Proposition 4.2.4.5. $\dim Z_{\mathcal{P}_{\min}} \leq d^2 + d^2[K:\mathbb{Q}_p]$.

Proof. In this case $r = 1$ so $Z_{\mathcal{P}} = Z'_{\mathcal{P}}$ and the assertion follows from (4.15). □

Lemma 4.2.4.6. *Assume that* $\mathcal{P} \neq \mathcal{P}_{\min}$. *If* $d = 2$ *then*

$$d^2 + d^2[K:\mathbb{Q}_p] - \dim Z_{\mathcal{P}} \geq [K:\mathbb{Q}_p],$$

and

$$d^2 + d^2[K:\mathbb{Q}_p] - \dim Z_{\mathcal{P}} \geq 1 + [K:\mathbb{Q}_p],$$

otherwise.

Proof. Proposition 4.2.4.4 implies that

$$d^2 + d^2[K:\mathbb{Q}_p] - \dim Z_{\mathcal{P}} \geq n_{\mathcal{P}}[K:\mathbb{Q}_p] - \delta_{\mathcal{P}}.$$

If $d > 2$ then $n_{\mathcal{P}} \geq 2$ and if $d = 2$ then $n_{\mathcal{P}} = 1$, which implies the assertion if $\delta_{\mathcal{P}} = 0$. Let us assume that $\delta_{\mathcal{P}} \neq 0$. Then using (4.12) we may write

$$n_{\mathcal{P}}[K:\mathbb{Q}_p] - \delta_{\mathcal{P}} = \sum_{1 \leq i < j \leq k'} c_i c_j n'_i n'_j [K:\mathbb{Q}_p] + \sum_{i=1}^{k'} (c_i^2 [K:\mathbb{Q}_p] - 1) \binom{n'_i}{2} + 1 + [K:\mathbb{Q}_p],$$

which implies the assertion. □

Lemma 4.2.4.7. *Let Y be the preimage of $\{\mathfrak{m}_{R^{\mathrm{ps}}}\}$ in $\overline{X}^{\mathrm{gen}}$. Then*

$$\dim Y \leq d^2 + n_{\mathcal{P}_{\max}}[K:\mathbb{Q}_p] + n_{\mathcal{P}_{\max}} - 1.$$

In particular, $d^2 + d^2[K:\mathbb{Q}_p] - \dim Y \geq 1 + l_{\mathcal{P}_{\max}}[K:\mathbb{Q}_p] \geq 1 + 2[K:\mathbb{Q}_p]$.

Proof. Proposition 4.2.2.10 implies that

$$\dim Y \leq d^2 - m + n_{\mathcal{P}_{\max}}[K:\mathbb{Q}_p] + \sum_{i=1}^{k'} \binom{n'_i}{2}.$$

As already explained in the proof of Lemma 4.2.4.6 we have $\sum_{i=1}^{k'} \binom{n'_i}{2} \leq n_{\mathcal{P}_{\max}}$. This implies the assertion. □

Finally we arrive at our desired bound.

Theorem 4.2.4.8. $\dim \overline{X}^{\text{gen}} \leq d^2 + d^2[K : \mathbb{Q}_p]$.

Proof. Since $\overline{X}^{\text{ps}} = \{\mathfrak{m}_{R^{\text{ps}}}\} \cup \bigcup_{\mathcal{P}} U_{\mathcal{P}}$ we have $\overline{X}^{\text{gen}} = Y \cup \bigcup_{\mathcal{P}} Z_{\mathcal{P}}$. Since Y and $Z_{\mathcal{P}}$ are closed in $\overline{X}^{\text{gen}}$ we have

$$\dim \overline{X}^{\text{gen}} = \max_{\mathcal{P}}\{\dim Y, \dim Z_{\mathcal{P}}\} \leq d^2 + d^2[K : \mathbb{Q}_p],$$

by Proposition 4.2.4.5 and Lemma 4.2.4.6 and Lemma 4.2.4.7. □

4.2.5 Completions at maximal ideals and deformation problems

Let x be a closed point of X^{gen} and let y be its image in $\operatorname{Spec} R^{\text{ps}}$. It follows from Lemma 4.2.3.1 and Lemma 4.2.3.2 that $\kappa(x)$ is a finite extension of $\kappa(y)$ and there are the following possibilities:

1. $\kappa(x)$ is a finite extension of k;
2. $\kappa(x)$ is a finite extension of L;
3. $\kappa(x)$ is a local field of characteristic p.

The universal property of A^{gen} gives us a continuous Galois representation

$$\rho_x : G_K \to \mathrm{GL}_d(\kappa(x)).$$

In this section we want to relate the completion of the local ring $\mathcal{O}_{X^{\text{gen}},x}$ to a deformation problem for ρ_x.

We will introduce some notation to formulate the deformation problem for ρ_x. More generally, let $\rho : G_K \to \mathrm{GL}_d(\kappa)$ be a continuous representation, where κ is as above. In case

(1) let L' be an unramified extension of L with residue field κ, let $\Lambda := \mathcal{O}_{L'}$ be the ring of integers in L'. In case (2) let $\Lambda = \kappa$, let Λ^0 be the ring of integers in Λ and let $t = \varpi$. In case (3), let $\mathcal{O}_\kappa$ be the ring of integers in κ and let k' be its residue field. Since $\mathrm{char}(\kappa) = p$ by choosing a uniformizer we obtain an isomorphism $\mathcal{O}_\kappa \cong k'[\![t]\!]$. Let L' be an unramified extension of L with residue field k', let $\Lambda^0 := \mathcal{O}_{L'}[\![t]\!]$ and let Λ be the p-adic completion of $\Lambda^0[1/t]$. Then Λ is a complete DVR with uniformiser ϖ and residue field κ. We equip Λ^0 with its (ϖ, t)-adic topology, this induces a topology on $\Lambda^0[1/t]$ and $\Lambda^0[1/t]/p^n\Lambda^0[1/t]$ for all $n \geq 1$. We equip $\Lambda = \varprojlim_n \Lambda^0[1/t]/p^n\Lambda^0[1/t]$ with the projective limit topology.

Remark 4.2.5.1. In case (3), if Λ' is an $\mathcal{O}$-algebra, which is a complete DVR with uniformiser ϖ and residue field κ then it follows from [Bou06, Ch. IX, §2.3, Prop. 4] that Λ' is non-canonically isomorphic to Λ. We will refer to Λ' (and Λ) as an $\mathcal{O}$-Cohen ring of κ.

Let Art_Λ be the category of local Artinian Λ-algebras with residue field κ. Let $(A, \mathfrak{m}_A) \in \mathsf{Art}_\Lambda$. In case (1) A is a finite $\mathcal{O}/\varpi^n$-module for some $n \gg 0$, and we just put discrete topology on A, in case (2) A is a finite dimensional L-vector space and we put the p-adic topology on A, in case (3) A is a $\Lambda^0[1/t]/\varpi^n\Lambda^0[1/t]$ -module of finite length, for some $n \gg 0$ and we put the induced topology on A.

Let $D^\square_\rho(A)$ be the set of continuous group homomorphisms $\rho_A : G_K \to \mathrm{GL}_d(A)$, such that $\rho_A \pmod{\mathfrak{m}_A} = \rho$.

Proposition 4.2.5.2. *The functor* $D^\square_\rho : \mathsf{Art}_\Lambda \to \mathsf{Set}$ *is represented by a complete local Noetherian* Λ*-algebra* $R^\square_\rho$*. Moreover, there is a presentation*

$$R^\square_\rho \cong \Lambda[\![x_1, \dots, x_r]\!]/(f_1, \dots, f_s) \tag{4.16}$$

with $r = \dim_\kappa Z^1(G_K, \mathrm{ad}\,\rho)$ *and* $s = \dim_\kappa H^2(G_K, \mathrm{ad}\,\rho)$*.*

Proof. Lecture 6 in [Con10] contains a very nice exposition of the result if κ is a finite extension of either k or L. The same argument works if κ is a local field of characteristic p. □

If we let $h^i := \dim_\kappa H^i(G_K, \operatorname{ad}\rho)$ then

$$r - s = \dim_\kappa(\operatorname{ad}\rho) - h^0 + h^1 - h^2 = d^2 + d^2[K : \mathbb{Q}_p], \tag{4.17}$$

where the last equality follows from Euler characteristic formula in this setting, see [BJ19, Theorem 3.4.1].

Proposition 4.2.5.3. *Let $\mathfrak{q}$ be the kernel of the map*

$$\Lambda \otimes_{\mathcal{O}} A^{\mathrm{gen}} \to \kappa(x), \quad \lambda \otimes a \mapsto \bar{\lambda}\bar{a},$$

where $\bar{\lambda}$ and $\bar{a}$ denote the images of λ and a in $\kappa(x)$. Then the completion of $(\Lambda \otimes_{\mathcal{O}} A^{\mathrm{gen}})_{\mathfrak{q}}$ with respect to the maximal ideal is naturally isomorphic to $R^{\square}_{\rho_x}$.

Proof. We will prove the proposition, when $\kappa(x)$ is a local field of characteristic p. The other cases are similar and are left to the reader.

Let $\widehat{B}$ be the completion of $(\Lambda \otimes_{\mathcal{O}} A^{\mathrm{gen}})_{\mathfrak{q}}$. It follows from the first line of Lemma 4.2.5.5 below that $\widehat{B}/\varpi\widehat{B}$ (and hence $\widehat{B}$) is Noetherian. Thus $\widehat{B}/\mathfrak{q}^n\widehat{B} \in \mathsf{Art}_\Lambda$ for all $n \geq 1$. The composition

$$\Lambda \otimes_{\mathcal{O}} E \xrightarrow{\mathrm{id}\otimes j} \Lambda \otimes_{\mathcal{O}} M_d(A^{\mathrm{gen}}) \to M_d(\widehat{B}/\mathfrak{q}^n\widehat{B})$$

induces a continuous representation $G_K \to \mathrm{GL}_d(\widehat{B}/\mathfrak{q}^n\widehat{B})$ by Lemma 4.2.1.4, which is a deformation of ρ_x to $\widehat{B}/\mathfrak{q}^n\widehat{B}$, and hence a map of local Λ-algebras $R^{\square}_{\rho_x} \to \widehat{B}/\mathfrak{q}^n\widehat{B}$. By passing to the projective limit over n we obtain a continuous representation $\hat{\rho} : G_K \to \mathrm{GL}_d(\widehat{B})$ and a map of local Λ-algebras $R^{\square}_{\rho_x} \to \widehat{B}$.

Let $(A, \mathfrak{m}_A) \in \mathsf{Art}_\Lambda$ and let $\rho : G_K \to \mathrm{GL}_d(A)$ be a continuous representation such that $\rho \pmod{\mathfrak{m}_A} = \rho_x$. We claim that there is a unique homomorphism of local Λ-algebras $\varphi : \widehat{B} \to A$, such that ρ is equal to the composition $\mathrm{GL}_d(\varphi) \circ \hat{\rho}$. The claim implies that the map $R^{\square}_{\rho_x} \to \widehat{B}$ constructed above is an isomorphism.

The proof of the claim is based on [Kis03b, Proposition 9.5]. Following its proof, we may construct an ascending chain of local open Λ^0-subalgebras A^0_n of A for $n \geq 1$, such that for all n the following hold: $A^0_n[1/t] = A$, the image of A^0_n under the projection $b : A \to \kappa(x)$ is equal to $\mathcal{O}_{\kappa(x)}$ and $\bigcup_{n \geq 1} A^0_n = b^{-1}(\mathcal{O}_{\kappa(x)})$. Let $M \in \mathrm{GL}_d(\kappa(x))$ be a matrix such that the image of G_K under $M\rho_x M^{-1}$ is contained in $\mathrm{GL}_d(\mathcal{O}_{\kappa(x)})$. Let $x' \in X^{\mathrm{gen}}$ correspond to the representation $M\rho_x M^{-1}$. Then $\kappa(x') = \kappa(x)$ and the image of $x' : A^{\mathrm{gen}} \to \kappa(x)$ is contained in $\mathcal{O}_{\kappa(x)}$. Let $z \in X^{\mathrm{gen}}$ be the composition $z : A^{\mathrm{gen}} \xrightarrow{x'} \mathcal{O}_{\kappa(x)} \to k'$, where k' is the residue field of $\mathcal{O}_{\kappa(x)}$, let $\widetilde{M} \in \mathrm{GL}_d(A)$ be a matrix lifting M and let $\rho' := \widetilde{M}\rho\widetilde{M}^{-1}$. Since G_K is compact $\rho'(G_K)$ will be contained in some $\mathrm{GL}_d(A^0_n)$ for $n \gg 0$. We may consider $\rho' : G_K \to \mathrm{GL}_d(A^0_n)$ as a deformation of ρ_z to A^0_n. Since the pseudorepresentation of ρ_z is equal to $\overline{D} \otimes_k k'$ by Lemma 4.2.1.5, the pseudorepresentation of $\rho' : G_K \to \mathrm{GL}_d(A^0_n)$ is a deformation of $\overline{D} \otimes_k k'$ to A^0_n and hence induces a map of local $\mathcal{O}$-algebras $R^{\mathrm{ps}} \to A^0_n$. Thus ρ' factors through the map $\Lambda^0 \otimes_{\mathcal{O}} R^{\mathrm{ps}}[\![G_K]\!] \to M_d(A^0_n)$, which will factor through the Cayley–Hamilton quotient $(\Lambda^0 \otimes_{\mathcal{O}} R^{\mathrm{ps}}[\![G_K]\!])/\operatorname{CH}(\Lambda^0 \otimes_{\mathcal{O}} D^u) \to M_d(A^0_n)$. It follows from [Che14, Section 1.22] or [Wan13, Lemma 1.1.8.6] that

$$\Lambda^0 \otimes_{\mathcal{O}} E \cong (\Lambda^0 \otimes_{\mathcal{O}} R^{\mathrm{ps}}[\![G_K]\!])/\operatorname{CH}(\Lambda^0 \otimes_{\mathcal{O}} D^u).$$

After inverting t and conjugating by $\widetilde{M}^{-1}$ we obtain a map of $\Lambda^0[1/t]$-algebras $\Lambda^0[1/t] \otimes_{\mathcal{O}} E \to M_d(A)$, such that if we compose this map with the map induced by $G_K \to R^{\mathrm{ps}}[\![G_K]\!] \to E$ then we get back ρ. Since A is an Artinian Λ-algebra, $\varpi^n \Lambda^0[1/t]$ will be mapped to zero for $n \gg 0$, and thus the map extends to a map of Λ-algebras $\alpha : \Lambda \otimes_{\mathcal{O}} E \to M_d(A)$. The universal property of $j : E \to M_d(A^{\mathrm{gen}})$ implies that there is a unique map of Λ-algebras

$\varphi : \widehat{B} \to A$, such that $M_d(\varphi) \circ (\mathrm{id} \otimes j) = \alpha$.

It remains to show the uniqueness of the map φ, which is equivalent to showing that there is at most one map of $\Lambda \otimes_{\mathcal{O}} R^{\mathrm{ps}}$-algebras $\alpha : \Lambda \otimes_{\mathcal{O}} E \to M_d(A)$ such that the composition with $G_K \to \Lambda \otimes_{\mathcal{O}} E$ gives ρ. It follows from the Cayley–Hamilton theorem in $M_d(A)$ and [Che14, Corollary 1.14], that the map $\Lambda \otimes_{\mathcal{O}} R^{\mathrm{ps}} \to \Lambda \otimes_{\mathcal{O}} E \xrightarrow{\alpha} A$ is uniquely determined by ρ. Thus α is uniquely determined on the image of $\Lambda \otimes_{\mathcal{O}} R^{\mathrm{ps}}[G_K]$ in $\Lambda \otimes_{\mathcal{O}} E$. The map $R^{\mathrm{ps}}[G_K] \to E$ is surjective, since the image is dense and closed as E is a finitely generated R^{ps}-module, hence α is uniquely determined by ρ. □

The following Lemma is a mild generalization of [BJ19, Lemma 3.3.3].

Lemma 4.2.5.4. *Let R be a complete local Noetherian k-algebra with residue field k, let A be a finitely generated R-algebra, let $\mathfrak{p} \in \operatorname{Spec} A$ such that its image in $\operatorname{Spec} R$ lies in $P_1 R$, and let $\mathfrak{q}$ be the kernel of the map*

$$B := \kappa(\mathfrak{p}) \otimes_k A \to \kappa(\mathfrak{p}), \quad x \otimes a \mapsto x(a + \mathfrak{p}).$$

Then $\hat{B}_{\mathfrak{q}} \cong \hat{A}_{\mathfrak{p}}[\![T]\!]$. In particular, $A_{\mathfrak{p}}$ is regular (resp. complete intersection) if and only if $\hat{B}_{\mathfrak{q}}$ is.

Proof. Let $\mathfrak{p}'$ be the image of $\mathfrak{p}$ in $\operatorname{Spec} R$. Since by assumption $\mathfrak{p}' \in P_1 R$, the residue field $\kappa(\mathfrak{p}')$ a local field of characteristic p. Since A is finitely generated over R, $\kappa(\mathfrak{p})$ is a finite extension of $\kappa(\mathfrak{p}')$ and thus is also a local field of characteristic p. The proof of [BJ19, Lemma 3.3.3] goes through verbatim by replacing R with A everywhere. □

Lemma 4.2.5.5. *Let R be a complete local Noetherian $\mathcal{O}$-algebra with residue field k, let A be a finitely generated R-algebra, let $\mathfrak{p} \in \operatorname{Spec} A$ such that $\kappa(\mathfrak{p})$ is a local field of characteristic*

p, and let $\mathfrak{q}$ be the kernel of the map

$$B := \Lambda \otimes_{\mathcal{O}} A \to \kappa(\mathfrak{p}), \quad \lambda \otimes a \mapsto \bar{\lambda}(a + \mathfrak{p}),$$

If $\hat{B}_{\mathfrak{q}}$ or $A_{\mathfrak{p}}$ is ϖ-torsion free then $\hat{B}_{\mathfrak{q}} \cong \hat{A}_{\mathfrak{p}}[\![T]\!]$. In particular, $A_{\mathfrak{p}}$ is regular (resp. complete intersection) if and only if $\hat{B}_{\mathfrak{q}}$ is.

Proof. It follows from Lemma 4.2.5.4 that the map $A \to B$, $a \mapsto 1 \otimes a$ induces a map of local rings $\hat{A}_{\mathfrak{p}} \to \hat{B}_{\mathfrak{q}}$, such that $\hat{B}_{\mathfrak{q}}/\varpi \cong (\hat{A}_{\mathfrak{p}}/\varpi)[\![T]\!]$. By choosing $b \in \hat{B}_{\mathfrak{q}}$, which maps to T under this isomorphism, we obtain a map $\hat{A}_{\mathfrak{p}}[\![T]\!] \to \hat{B}_{\mathfrak{q}}$, which induces an isomorphism modulo ϖ. Nakayama's lemma implies that the map is surjective. If $\hat{B}_{\mathfrak{q}}$ is ϖ-torsion free then another application of Nakayama's lemma shows that the kernel is zero. If $A_{\mathfrak{p}}$ is $\mathcal{O}$-torsion free then $(\Lambda \otimes_{\mathcal{O}} A)_{\mathfrak{q}} = (\Lambda \otimes_{\mathcal{O}} A_{\mathfrak{p}})_{\mathfrak{q}}$ is also $\mathcal{O}$-torsion free, and the same applies to the completion $\hat{B}_{\mathfrak{q}}$. □

Lemma 4.2.5.6. *Let R be a complete local Noetherian $\mathcal{O}$-algebra with residue field k, let A be a finitely generated R-algebra, let $\mathfrak{p} \in \operatorname{Spec} A$ such that $\kappa(\mathfrak{p})$ is either a finite extension of L, in which case we let $\Lambda = \kappa(\mathfrak{p})$, or a finite extension of k, in which case we let Λ be the ring of integers in the finite unramified extension of L with residue field $\kappa(\mathfrak{p})$. Let $\mathfrak{q}$ be the kernel of the map*

$$B := \Lambda \otimes_{\mathcal{O}} A \to \kappa(\mathfrak{p}), \quad \lambda \otimes a \mapsto \bar{\lambda}(a + \mathfrak{p}).$$

Then $\hat{B}_{\mathfrak{q}} \cong \hat{A}_{\mathfrak{p}}$.

Proof. The completion of $\Lambda \otimes_{\mathcal{O}} \Lambda$ with respect to the kernel of $\Lambda \otimes_{\mathcal{O}} \Lambda \to \Lambda$, $x \otimes y \mapsto xy$ is just Λ (and that is why we don't get an extra variable T like in Lemma 4.2.5.4, see [BJ19, Lemma 3.3.4].) The rest of the proof is the same as the proof of Lemma 4.2.5.4. □

Having established a relationship between A^{gen} and $R^{\square}_{\bar{\rho}_x}$ for varying $x \in X^{\mathrm{gen}}$, we now establish the main result of this chapter.

Corollary 4.2.5.7. *Let x be a closed point of X^{gen}. Then the following hold:*

1. *$R^{\square}_{\rho_x}$ is a flat Λ-algebra of relative dimension $d^2 + d^2[K : \mathbb{Q}_p]$ and is complete intersection;*

2. *if $\mathrm{char}(\kappa(x)) = p$ then $R^{\square}_{\rho_x}/\varpi$ is complete intersection of dimension $d^2 + d^2[K : \mathbb{Q}_p]$.*

Proof. Let us assume that $\kappa(x)$ is a finite extension of k. It follows from Lemma 4.2.5.6 that $R^{\square}_{\rho_x}/\varpi \cong \widehat{\mathcal{O}}_{\overline{X}^{\mathrm{gen}},x}$, the completion of the local ring of $\overline{X}^{\mathrm{gen}}$ at x with respect to the maximal ideal. We have $\dim \widehat{\mathcal{O}}_{\overline{X}^{\mathrm{gen}},x} = \dim \mathcal{O}_{\overline{X}^{\mathrm{gen}},x} \leq \dim \overline{X}^{\mathrm{gen}}$, and thus

$$\dim R^{\square}_{\rho_x}/\varpi \leq \dim \overline{X}^{\mathrm{gen}} \leq d^2 + d^2[K : \mathbb{Q}_p] = r - s,$$

where the last equality is (4.17). It follows from (4.16) that $\dim R^{\square}_{\rho_x}/\varpi \geq r-s$ and $\dim R^{\square}_{\rho_x} \geq 1+r-s$. Thus the lower bounds of the dimensions are equalities, and $\varpi, f_1, \ldots, f_s$ are a part of system of parameters in $\Lambda[\![x_1, \ldots, x_r]\!]$. Thus they form a regular sequence in $\Lambda[\![x_1, \ldots, x_r]\!]$ and so $R^{\square}_{\rho_x}$ and $R^{\square}_{\rho_x}/\varpi$ are complete intersections of the claimed dimensions. Moreover, since Λ is a DVR with uniformiser ϖ, flatness is equivalent to ϖ-torsion-freeness, and hence $R^{\square}_{\rho_x}$ is flat over Λ.

If $\kappa(x)$ is a finite extension of L then Lemma 4.2.5.6 implies that $R^{\square}_{\rho_x} \cong \widehat{\mathcal{O}}_{X^{\mathrm{gen}},x} = \widehat{\mathcal{O}}_{X^{\mathrm{gen}}[1/p],x}$. Lemma 4.2.3.7 implies that $\dim R^{\square}_{\rho_x} \leq \dim X^{\mathrm{gen}}[1/p] \leq \dim \overline{X}^{\mathrm{gen}}$. Then the same argument goes through.

Let us assume that $\kappa(x)$ is a local field of characteristic p. Lemma 4.2.5.4 implies that $R^{\square}_{\rho_x}/\varpi \cong \widehat{\mathcal{O}}_{\overline{X}^{\mathrm{gen}},x}[\![T]\!]$ and Lemma 4.2.3.5 implies that $\dim \widehat{\mathcal{O}}_{\overline{X}^{\mathrm{gen}},x} \leq \dim \overline{X}^{\mathrm{gen}} - 1$. Thus $\dim R^{\square}_{\rho_x} \leq \dim \overline{X}^{\mathrm{gen}}$ and the same argument as above goes through. $\square$

Corollary 4.2.5.8. *Let x be a closed point in X^{gen} and let $\widehat{\mathcal{O}}_{X^{\mathrm{gen}},x}$ be the completion with respect to the maximal ideal of the local ring at x. If $\kappa(x)$ is a finite extension of k or L then*

$\widehat{\mathcal{O}}_{X^{\mathrm{gen}},x} \cong R^{\square}_{\rho_x}$. *If* $\kappa(x)$ *is a local field of characteristic* p *then* $R^{\square}_{\rho_x} \cong \widehat{\mathcal{O}}_{X^{\mathrm{gen}},x}[\![T]\!]$.

Proof. If $\kappa(x)$ is a finite extension of k or L then the assertion follows from Proposition 4.2.5.3 and Lemma 4.2.5.6. If $\kappa(x)$ is a local field of characteristic p then $R^{\square}_{\rho_x}$ is $\mathcal{O}$-torsion free by Corollary 4.2.5.7, and the assertion follows from Proposition 4.2.5.3 and Lemma 4.2.5.5. □

Corollary 4.2.5.9. *The following hold:*

1. A^{gen} *is* $\mathcal{O}$*-torsion free, equi-dimensional of dimension* $1+d^2+d^2[K:\mathbb{Q}_p]$ *and is locally complete intersection;*

2. A^{gen}/ϖ *is equi-dimensional of dimension* $d^2+d^2[K:\mathbb{Q}_p]$ *and is locally complete intersection.*

Proof. Let us prove (1) as the proof of (2) is identical. Corollary 4.2.5.8 together with Corollary 4.2.5.7 implies that the local rings at closed points of X^{gen} are $\mathcal{O}$-torsion free and complete intersection. This implies that A^{gen} is $\mathcal{O}$-torsion free and A^{gen} is locally complete intersection by [Stacks, Tag 09Q5].

Let Z be an irreducible component of X^{gen}. Lemma 4.2.3.5 implies that there is a closed point $x \in Z$ such that x maps to the closed point of X^{ps}. Moreover, $\dim Z = \dim \mathcal{O}_{Z,x}$. Since $\mathcal{O}_{X^{\mathrm{gen}},x}$ is complete intersection, it is equi-dimensional and thus $\dim \mathcal{O}_{Z,x} = \dim \mathcal{O}_{X^{\mathrm{gen}},x} = d^2+d^2[K:\mathbb{Q}_p]+1$, where the last equality follows from Corollary 4.2.5.7 and Corollary 4.2.5.8. □

Proposition 4.2.5.10. *Let* $x \in P_1 R^{\square}_{\overline{\rho}}$*, where* $R^{\square}_{\overline{\rho}}$ *is the framed deformation ring of* $\overline{\rho} : G_K \to \mathrm{GL}_d(k')$*, where* k' *is finite extension of* k*. Let* $\rho_x : G_K \to \mathrm{GL}_d(\kappa(x))$ *be the representation obtained by specializing the universal framed deformation of* $\overline{\rho}$ *at* x*. Let* $\mathfrak{q}$ *be the kernel of the map*

$$\Lambda \otimes_{\mathcal{O}} R^{\square}_{\overline{\rho}} \to \kappa(x), \quad \lambda \otimes a \mapsto \bar{\lambda}\bar{a},$$

where Λ is the ring defined at the beginning of the subsection. Then the completion of $(\Lambda \otimes_{\mathcal{O}} R^{\square}_{\bar{\rho}})_{\mathfrak{q}}$ with respect to the maximal ideal is naturally isomorphic to $R^{\square}_{\rho_x}$.

Proof. The proof is similar to the proof of Proposition 4.2.5.3, but easier, since the setting is much closer to the setting of [Kis03b, Proposition 9.5], [BJ19, Theorem 3.3.1], where an analogous result is proved for versal deformation rings. We leave the details to the reader. □

Let x be a closed point of X^{gen} such that $\kappa(x)$ is a local field. Since G_K is compact there is a matrix $M \in \mathrm{GL}_d(\kappa(x))$, such that the image of $M\rho_x M^{-1}$ is contained in $\mathrm{GL}_d(\mathcal{O}_{\kappa(x)})$. Let $x' : A^{\mathrm{gen}} \to \mathcal{O}_{\kappa(x)}$ be the R^{ps}-algebra homomorphism corresponding to the representation $E \to M_d(\mathcal{O}_{\kappa(x)})$, $a \mapsto M\rho_x(a)M^{-1}$. We will denote the corresponding Galois representation by $\rho^0_{x'} : G_K \to \mathrm{GL}_d(\mathcal{O}_{\kappa(x)})$ and let $\rho_{x'}$ be the composition $\rho_{x'} : G_K \xrightarrow{\rho^0_{x'}} \mathrm{GL}_d(\mathcal{O}_{\kappa(x)}) \to \mathrm{GL}_d(\kappa(x))$. We note that $\kappa(x') = \kappa(x)$ and let Λ be the coefficient ring defined at the beginning of the subsection. Let k' be the residue field of $\mathcal{O}_{\kappa(x)}$ and let $\rho_z : G_K \to \mathrm{GL}_d(k')$ be the representation corresponding to $z : A^{\mathrm{gen}} \xrightarrow{x'} \mathcal{O}_{\kappa(x)} \to k'$. Then $\rho^0_{x'}$ is a deformation of ρ_z to $\mathcal{O}_{\kappa(x)}$, thus the map $x' : A^{\mathrm{gen}} \to \mathcal{O}_{\kappa(x)}$ factors through $x' : R^{\square}_{\rho_z} \to \mathcal{O}_{\kappa(x)}$.

Corollary 4.2.5.11. *There is an isomorphism of local Λ-algebras between $R^{\square}_{\rho_x}$, $R^{\square}_{\rho_{x'}}$ and the completion of $(\Lambda \otimes_{\mathcal{O}} R^{\square}_{\rho_z})_{\mathfrak{q}}$ with respect to the maximal ideal, where $\mathfrak{q}$ is as in Proposition 4.2.5.10 with respect to $x' : R^{\square}_{\rho_z} \to \mathcal{O}_{\kappa(x)}$.*

Proof. Let $\widetilde{M}$ be any lift of M to $M_d(\Lambda)$. Since Λ is a local ring $\det \widetilde{M}$ is a unit in Λ and hence $\widetilde{M} \in \mathrm{GL}_d(\Lambda)$. Conjugation by $\widetilde{M}$ induces an isomorphism between the deformation problems for ρ_x and $\rho_{x'}$ and hence between the deformation rings. Proposition 4.2.5.10 implies that these rings are also isomorphic to the completion of $(\Lambda \otimes_{\mathcal{O}} R^{\square}_{\rho_z})_{\mathfrak{q}}$. □

Remark 4.2.5.12. Corollary 4.2.5.11 enables us to study local properties of X^{gen}, by studying the completions of local rings at closed points above $\mathfrak{m}_{R^{\mathrm{ps}}}$. For example, if we could show that $R^{\square}_{\rho_z}$ is regular, we could conclude that the local ring at x', $(R^{\square}_{\rho_z})_{x'}$ is regular, and hence that the completion $\widehat{(R^{\square}_{\rho_z})}_{x'}$ is regular. If $\kappa(x)$ is a local field of characteristic p then Proposition 4.2.5.3, Corollary 4.2.5.11 and Lemma 4.2.5.5 imply that

$$\widehat{\mathcal{O}}_{X^{\mathrm{gen}},x}[\![T]\!] \cong R^{\square}_{\rho_x} \cong R^{\square}_{\rho_{x'}} \cong \widehat{(R^{\square}_{\rho_z})}_{x'}[\![T]\!].$$

If $\kappa(x)$ is a finite extension of L then Proposition 4.2.5.3, Corollary 4.2.5.11 and Lemma 4.2.5.6 imply that

$$\widehat{\mathcal{O}}_{X^{\mathrm{gen}},x} \cong R^{\square}_{\rho_x} \cong R^{\square}_{\rho_{x'}} \cong \widehat{(R^{\square}_{\rho_z})}_{x'}.$$

Thus in both cases we can deduce that $\widehat{\mathcal{O}}_{X^{\mathrm{gen}},x}$ and hence $\mathcal{O}_{X^{\mathrm{gen}},x}$ are regular. Thus if we can show that $R^{\square}_{\rho_z}$ is regular for all closed points $z \in X^{\mathrm{gen}}$ above $\mathfrak{m}_{R^{\mathrm{ps}}}$ then we can conclude that $\mathcal{O}_{X^{\mathrm{gen}},x}$ is regular for all closed points $x \in X^{\mathrm{gen}}$ and thus X^{gen} is regular.

Of course, one may also reverse the logic of this argument: if X^{gen} is regular then all its local rings and their completions are regular and hence $R^{\square}_{\rho_z}$ is regular for all closed points $z \in X^{\mathrm{gen}}$ above $\mathfrak{m}_{R^{\mathrm{ps}}}$.

Corollary 4.2.5.13. *Let $\rho : G_K \to \mathrm{GL}_d(\kappa)$ be a continuous representation with κ a local field. Then the conclusion of Corollary 4.2.5.7 holds for $R^{\square}_{\rho}$.*

Proof. After conjugation we may assume that $\rho(G_K) \subset \mathrm{GL}_d(\mathcal{O}_\kappa)$. Let $\overline{\rho}$ be the representation obtained by reducing the matrix entries modulo a uniformizer of $\mathcal{O}_\kappa$ and let $\overline{D}$ be the associated pseudorepresentation. Corollary 4.2.5.7 applies to $R^{\square}_{\overline{\rho}}$. Since ρ corresponds to an $x \in P_1 R^{\square}_{\overline{\rho}}$, Proposition 4.2.5.10 together with Lemma 4.2.5.6, Lemma 4.2.5.5 allows us to bound the dimension of $R^{\square}_{\rho}$ from above. Then the proof of Corollary 4.2.5.7 carries over. □

As a corollary of flatness, we get existence of lifts.

Corollary 4.2.5.14. *Every representation $\overline{\rho} : G_K \to \mathrm{GL}_d(k)$ can be lifted to characteristic zero.*

Proof. It follows from Corollary 4.2.5.7 that $R^{\square}_{\overline{\rho}}[1/p]$ is non-zero. We obtain a lift by specializing the universal framed deformation along any $\mathcal{O}$-algebra homomorphism $x : R^{\square}_{\overline{\rho}} \to \overline{\mathbb{Q}}_p$. $\square$

4.2.6 Bounding the maximally reducible semi-simple locus

Later, when we bound the singular locus of $\overline{X}^{\mathrm{gen}}$ in Proposition 4.3.2.6, we will need to study the reducible semi-simple locus in order to deal with one exceptional case for $d = 2$ and $K = \mathbb{Q}_2$. The reader is strongly encouraged to skip this section for now.

Writing $\overline{D} = \prod_{i=1}^{m} \overline{D}_i$ with $\overline{D}_i$ absolutely irreducible pseudorepresentations, we now take $\mathcal{P} = \mathcal{P}_{\max}$ and consider the finite (by Lemma 4.2.4.1) R^{ps}-algebra $R^{\mathrm{ps}}_{\underline{\Sigma}}$, where $\underline{\Sigma}$ amounts to some choice of ordering of $\{1, \ldots, m\}$. Note that if $\overline{\rho}_i : G_K \to \mathrm{GL}_d(k)$ is an (absolutely irreducible) representation with pseudorepresentation $\overline{D}_i$ then

$$R^{\mathrm{ps}}_{\underline{\Sigma}} \cong R_{\overline{\rho}_1} \widehat{\otimes}_{\mathcal{O}} \cdots \widehat{\otimes}_{\mathcal{O}} R_{\overline{\rho}_m}$$

where $R_{\overline{\rho}_i}$ denotes the universal deformation ring of $\overline{\rho}_i$. So let $\rho_i^{\mathrm{univ}} : G_K \to \mathrm{GL}_{d_i}(R_{\overline{\rho}_i})$ denote a representative of the strict equivalence class of the universal representation for each $i = 1, \ldots, m$. If we let M denote the universal invertible matrix in $\mathrm{GL}_d(\mathcal{O}_{\mathrm{GL}_d}(\mathrm{GL}_d))$, then the representation

$$M \times \mathrm{diag}(\rho_1^{\mathrm{univ}}, \ldots, \rho_m^{\mathrm{univ}}) \times M^{-1} : G_K \to \mathrm{GL}_d(R^{\mathrm{ps}}_{\underline{\Sigma}} \otimes_{\mathcal{O}} \mathcal{O}_{\mathrm{GL}_d}(\mathrm{GL}_d))$$

gives rise to a map of Cayley–Hamilton algebras $E \to M_d(R^{\mathrm{ps}}_{\underline{\Sigma}} \otimes_{\mathcal{O}} \mathcal{O}_{\mathrm{GL}_d}(\mathrm{GL}_d))$ which satisfies the universal property of A^{gen} and so defines a map of R^{ps}-schemes

$$\mathrm{GL}_d \times_{\mathcal{O}} X^{\mathrm{ps}}_{\underline{\Sigma}} \to X^{\mathrm{gen}}$$

which descends to a map of R^{ps}-schemes

$$\eta_{\underline{\Sigma}} : \mathrm{GL}_d / Z_L \times_{\mathcal{O}} X^{\mathrm{ps}}_{\underline{\Sigma}} \to X^{\mathrm{gen}}$$

where $L := L_{\underline{\Sigma}}$ denotes the standard Levi subgroup of GL_d with blocks corresponding to $\underline{\Sigma}$ and Z_L denotes its center.

Definition 4.2.6.1. The *maximally reducible semi-simple locus* $X^{\mathrm{mrs}} \subset X^{\mathrm{gen}}$ is the scheme-theoretic image of $\eta_{\underline{\Sigma}} : \mathrm{GL}_d / Z_L \times_{\mathcal{O}} X^{\mathrm{ps}}_{\underline{\Sigma}} \to X^{\mathrm{gen}}$.

Lemma 4.2.6.2. *Let $x \in X^{\mathrm{gen}}$ and let y be the image of x in X^{ps}. If y lies in $X^{\mathrm{ps}}_{\mathcal{P}_{\max}}$ and ρ_x is semi-simple then $x \in X^{\mathrm{mrs}}$. Moreover, such points are dense in X^{mrs}.*

Proof. We first note that if $x \in X^{\mathrm{gen}}$ maps to $X^{\mathrm{ps}}_{\mathcal{P}_{\max}}$ and ρ_x is semi-simple then $\rho_x \cong \rho_1 \oplus \ldots \oplus \rho_m$, with each ρ_i an irreducible representation of G_K lifting $\overline{\rho}_i$. By conjugating by an element of $h \in \mathrm{GL}_d(\kappa(x))$ we may ensure that $h^{-1}\rho_x(g)h = \mathrm{diag}(\rho_1(g), \ldots, \rho_m(g))$ for all $g \in G_K$ and this implies that $x \in X^{\mathrm{mrs}}$.

Since $\eta_{\underline{\Sigma}}$ is a map of affine schemes, it is affine and hence quasi-compact, see [Stacks, Tag 01S5]. It follows from [Stacks, Tag 01R8] that the set theoretic image of $\eta_{\underline{\Sigma}}$ is dense in X^{mrs}. □

Proposition 4.2.6.3. $\dim X^{\mathrm{mrs}} \leq 1 + d^2 + [K : \mathbb{Q}_p] \sum_{i=1}^{m} d_i^2$.

Proof. The open subscheme $U_{\max} = X^{\mathrm{ps}} \setminus \{\mathfrak{m}_{R^{\mathrm{ps}}}\} \subset \overline{X}^{\mathrm{ps}}$ is Jacobson by Lemma 4.2.3.2, as is $V_{\max} := X^{\mathrm{mrs}} \times_{X^{\mathrm{ps}}} U_{\max}$. Let $Z_{\max}$ denote the closure of $V_{\max}$ in X^{mrs}. The formation of scheme-theoretic images commutes with restriction to opens, so the map

$$(\mathrm{GL}_d / Z_L \times_{\mathcal{O}} X^{\mathrm{ps}}_{\underline{\Sigma}}) \times_{X^{\mathrm{ps}}} U_{\max} \to V_{\max}$$

is a dominant map of Jacobson Noetherian excellent schemes. Applying Lemma 4.2.2.9 we see that

$$\dim V_{\max} \leq \dim((\mathrm{GL}_d/Z_L \times_{\mathcal{O}} X^{\mathrm{ps}}_{\underline{\Sigma}}) \times_{X^{\mathrm{ps}}} U_{\max}).$$

Since X^{mrs} is by definition a non-empty closed GL_d-invariant subscheme of X^{gen}, Lemma 4.2.3.5 implies that every irreducible component of X^{mrs} has a point in common with the preimage of $\mathfrak{m}_{R^{\mathrm{ps}}}$ in X^{mrs}. Therefore, Lemma 4.2.3.2 (5) implies that

$$\dim Z_{\max} = \dim V_{\max} + 1$$

Furthermore, GL_d/Z_L is flat over $\operatorname{Spec} \mathcal{O}$ with geometrically irreducible fibres, so the projection $\mathrm{GL}_d/Z_L \times_{\mathcal{O}} X^{\mathrm{ps}}_{\underline{\Sigma}} \to X^{\mathrm{ps}}_{\underline{\Sigma}}$ is a flat (and hence open) map with irreducible fibres. It follows from [Stacks, Tag 037A] that this map induces a bijection between the sets of irreducible components. Since $R^{\mathrm{ps}}_{\underline{\Sigma}}$ is a local ring, we deduce that $\mathrm{GL}_d/Z_L \times_{\mathcal{O}} X^{\mathrm{ps}}_{\underline{\Sigma}}$ satisfies the assumptions of Lemma 4.2.3.2 (5) and thus Lemma 4.2.3.2 (5) implies that

$$\dim \mathrm{GL}_d/Z_L \times_{\mathcal{O}} X^{\mathrm{ps}}_{\underline{\Sigma}} = \dim((\mathrm{GL}_d/Z_L \times_{\mathcal{O}} X^{\mathrm{ps}}_{\underline{\Sigma}}) \times_{X^{\mathrm{ps}}} U_{\max}) + 1$$

Since $\dim X^{\mathrm{ps}}_{\underline{\Sigma}} = 1 + \sum_{i=1}^{m}(1 + d_i^2[K : \mathbb{Q}_p])$ and the relative dimension of GL_d/Z_L over $\mathcal{O}$ is $d^2 - m$ we get that

$$\dim Z_{\max} \leq \dim \mathrm{GL}_d/Z_L \times_{\mathcal{O}} X^{\mathrm{ps}}_{\underline{\Sigma}} = 1 + d^2 + [K : \mathbb{Q}_p] \sum_{i=1}^{m} d_i^2.$$

Let Y^{mrs} be the scheme theoretic image of $\mathrm{GL}_d/Z_L \times_{\mathcal{O}} \{\mathfrak{m}_{R^{\mathrm{ps}}}\} \to Y$. Since Y is of finite type over k the same argument as above shows that

$$\dim Y^{\mathrm{mrs}} \leq \dim(\mathrm{GL}_d/Z_L \times_{\mathcal{O}} \{\mathfrak{m}_{R^{\mathrm{ps}}}\}) = d^2 - m.$$

Now $Z_{\max} \cup Y^{\mathrm{mrs}}$ is a closed subscheme of X^{gen} containing the image of $\eta_{\underline{\Sigma}}$. It follows from Lemma 4.2.6.2 that $Z_{\max} \cup Y^{\mathrm{mrs}}$ will contain X^{mrs}. Hence,

$$\dim X^{\mathrm{mrs}} \leq \max\{\dim Z_{\max}, \dim Y^{\mathrm{mrs}}\} = \dim Z_{\max}.$$

□

Corollary 4.2.6.4. $\dim \overline{X}^{\mathrm{mrs}} = \dim X^{\mathrm{mrs}} - 1 \leq d^2 + [K : \mathbb{Q}_p] \sum_{i=1}^{m} d_i^2$.

Proof. It follows from Corollary 4.2.5.7 that $R^{\mathrm{ps}}_{\underline{\Sigma}}$ is $\mathcal{O}$-torsion free, which implies that $\mathrm{GL}_d / Z_L \times_{\mathcal{O}} X^{\mathrm{ps}}_{\underline{\Sigma}}$ is flat over $\operatorname{Spec} \mathcal{O}$ and the same applies for X^{mrs}. (Here we are simply saying that a subring of $\mathcal{O}$-torsion free ring is $\mathcal{O}$-torsion free.) Thus for all $x \in \overline{X}^{\mathrm{mrs}}$, ϖ is a regular element in $\mathcal{O}_{X^{\mathrm{mrs}},x}$ and so $\dim \mathcal{O}_{\overline{X}^{\mathrm{mrs}},x} = \dim \mathcal{O}_{X^{\mathrm{mrs}},x} - 1$. This implies $\dim \overline{X}^{\mathrm{mrs}} = \dim X^{\mathrm{mrs}} - 1$ and the inequality follows from the Proposition 4.2.6.3. □

Remark 4.2.6.5. One could study the closure of the reducible semi-simple locus corresponding to more general partitions using a similar argument. We don't pursue this here, since we need the bound only for $d = 2$ and $K = \mathbb{Q}_2$, see Case 3 of Proposition 4.3.2.6 below.

4.2.7 Density of the irreducible locus

In this section we study the *irreducible locus* and the *special locus* in the special fiber of X^{gen}. The special locus is a special fiber analog of the reducible locus used in Chapter 3 to bound the singular locus. We prove dimension bounds in this section that will be useful for proving normality of components of $\overline{X}^{\mathrm{gen}}$ and $X^{\mathrm{gen}}[1/p]$ in the next section.

Let us first unravel the definitions of $U_{\mathcal{P}_{\min}}$ and $V_{\mathcal{P}_{\min}}$ in Section 4.2.4. We have that $U_{\mathcal{P}_{\min}}$ is an open subscheme of $\overline{X}^{\mathrm{ps}}$ such that the closed points of $U_{\mathcal{P}_{\min}}$ are in bijection with $\mathfrak{p} \in P_1(R^{\mathrm{ps}}/\varpi)$, such that the specialization of the universal pseudorepresentation along

$R^{\mathrm{ps}} \to \kappa(\mathfrak{p})$ is absolutely irreducible. Now $V_{\mathcal{P}_{\min}}$ is the preimage of $U_{\mathcal{P}_{\min}}$ in $\overline{X}^{\mathrm{gen}}$, so that it is an open subscheme of $\overline{X}^{\mathrm{gen}}$ and its closed points are in bijection $\mathfrak{q} \in \overline{X}^{\mathrm{gen}}$, which map to $P_1(R^{\mathrm{ps}}/\varpi)$ in $\overline{X}^{\mathrm{ps}}$, such that the representation

$$E \xrightarrow{j} M_d(A^{\mathrm{gen}}) \to M_d(\kappa(\mathfrak{q}))$$

is absolutely irreducible.

Proposition 4.2.7.1. *$V_{\mathcal{P}_{\min}}$ is dense in $\overline{X}^{\mathrm{gen}}$.*

Proof. We have

$$\overline{X}^{\mathrm{gen}} \setminus V_{\mathcal{P}_{\min}} = Y \cup \bigcup_{\mathcal{P}_{\min} < \mathcal{P}} Z_{\mathcal{P}}$$

and it follows from Lemma 4.2.4.6, Lemma 4.2.4.7 that $\overline{X}^{\mathrm{gen}} \setminus V_{\mathcal{P}_{\min}}$ has positive codimension in $\overline{X}^{\mathrm{gen}}$. Since $\overline{X}^{\mathrm{gen}}$ is equi-dimensional by Corollary 4.2.5.9 we conclude that $V_{\mathcal{P}_{\min}}$ is dense in $\overline{X}^{\mathrm{gen}}$. In particular, the inequality in Proposition 4.2.4.5 is an equality. □

We will now prove a stronger version of the above result. Following [BJ19, Definition 5.1.2] we call $y \in U_{\mathcal{P}_{\min}}$ *special* if either $\zeta_p \notin K$ and $D_y = D_y(1)$ or $\zeta_p \in K$ and the restriction D_y to $G_{K'}$ is reducible for some degree p Galois extension K' of K. Otherwise, y is called *non-special*. According to [BJ19, Lemma 5.1.3.] there is a closed subscheme U^{spcl} of $U_{\mathcal{P}_{\min}}$ such that the closed points of U^{spcl} are precisely the closed special points of $U_{\mathcal{P}_{\min}}$. Let V^{spcl} denote the preimage of U^{spcl} in $\overline{X}^{\mathrm{gen}}$ and let Z^{spcl} denote the closure of V^{spcl}.

Similarly let $U^{\mathrm{Kirr}} \subset U_{\mathcal{P}_{\min}}$ be the Kummer-irreducible locus defined in Section 4.A. Let U^{Kred} denote its complement in $U_{\mathcal{P}_{\min}}$, let V^{Kred} be the preimage of U^{Kred} in $\overline{X}^{\mathrm{gen}}$ and let Z^{Kred} denote the closure of V^{Kred}. We have $Z^{\mathrm{Kred}} \subseteq Z^{\mathrm{spcl}}$ with equality if $\zeta_p \in K$.

Lemma 4.2.7.2. *We have*

$$\dim \overline{X}^{\mathrm{gen}} - \dim Z^{\mathrm{spcl}} \geq \frac{1}{2}[K:\mathbb{Q}_p]d^2, \quad \dim \overline{X}^{\mathrm{gen}} - \dim Z^{\mathrm{Kred}} \geq [K:\mathbb{Q}_p]d.$$

Proof. It follows from [BJ19, Theorem 5.3.1 (i)] that the dimension of the Zariski closure of U^{spcl} in $\overline{X}^{\mathrm{ps}}$ is at most $1+\frac{1}{2}[K:\mathbb{Q}_p]d^2$. If $y \in U^{\mathrm{spcl}}$ then its fibre X^{gen}_y has dimension d^2-1 by Corollary 4.2.2.11. Thus Lemma 4.2.3.2 implies that

$$\dim Z^{\mathrm{spcl}} \leq d^2 + \frac{1}{2}[K:\mathbb{Q}_p]d^2.$$

Since $\dim \overline{X}^{\mathrm{gen}} = d^2 + d^2[K:\mathbb{Q}_p]$ by Corollary 4.2.5.9 the assertion follows. Similarly Proposition 4.A.0.8 implies that the dimension of the closure of U^{Kred} in $\overline{X}^{\mathrm{ps}}$ is at most $1+(d^2-d)[K:\mathbb{Q}_p]$. The same argument gives the required bound for the codimension of Z^{Kred}. □

Let $U^{\mathrm{nspcl}} := U_{\mathcal{P}_{\min}} \setminus U^{\mathrm{spcl}}$ and let V^{nspcl} the preimage of U^{nspcl} in $\overline{X}^{\mathrm{gen}}$. Let V^{Kirr} be the preimage of U^{Kirr} in $\overline{X}^{\mathrm{gen}}$. We have an inclusion $V^{\mathrm{Kirr}} \subset V^{\mathrm{nspcl}}$ and the subschemes coincide if $\zeta_p \in K$.

Proposition 4.2.7.3. *V^{Kirr} is Zariski dense in $\overline{X}^{\mathrm{gen}}$. Moreover, the following hold:*

1. *if $d=2$ then $\dim \overline{X}^{\mathrm{gen}} - \dim(\overline{X}^{\mathrm{gen}} \setminus V^{\mathrm{Kirr}}) \geq [K:\mathbb{Q}_p]$;*
2. *if $d>2$ then $\dim \overline{X}^{\mathrm{gen}} - \dim(\overline{X}^{\mathrm{gen}} \setminus V^{\mathrm{Kirr}}) \geq 1+[K:\mathbb{Q}_p]$.*
3. *if $d>1$ is arbitrary but $\overline{D}$ is absolutely irreducible (i.e. $m=1$) then $\dim \overline{X}^{\mathrm{gen}} - \dim(\overline{X}^{\mathrm{gen}} \setminus V^{\mathrm{Kirr}}) \geq d[K:\mathbb{Q}_p]$.*

Proof. Since $V_{\mathcal{P}_{\min}}$ is dense in $\overline{X}^{\mathrm{gen}}$ by Proposition 4.2.7.1, we have $\overline{X}^{\mathrm{gen}} = Z_{\mathcal{P}_{\min}} = Z^{\mathrm{Kred}} \cup Z^{\mathrm{Kirr}}$, where Z^{Kirr} is the closure of V^{Kirr}. Since $\dim Z^{\mathrm{Kred}} < \dim \overline{X}^{\mathrm{gen}}$ by Lemma 4.2.7.2 and

$\overline{X}^{\text{gen}}$ is equi-dimensional we get that $\overline{X}^{\text{gen}} = Z^{\text{Kirr}}$. Moreover,

$$\overline{X}^{\text{gen}} \setminus V^{\text{Kirr}} = Y \cup Z^{\text{Kred}} \cup \bigcup_{\mathcal{P}_{\min} < \mathcal{P}} Z_{\mathcal{P}}$$

and the assertion follows from Lemma 4.2.4.7, Lemma 4.2.4.6, Lemma 4.2.7.2, noting that if $\overline{\rho}$ is absolutely irreducible then $\{\mathcal{P} : \mathcal{P}_{\min} < \mathcal{P}\} = \emptyset$. □

We now want to transfer the density results from $\overline{X}^{\text{gen}}$ to $R^{\square}_{\overline{\rho}}/\varpi$.

Lemma 4.2.7.4. *Let $A \to B$ be a flat ring homomorphism and let U be an open subscheme of* $\operatorname{Spec} A$ *and let V be the preimage of U in* $\operatorname{Spec} B$*. If U is dense in* $\operatorname{Spec} A$ *then V is dense in* $\operatorname{Spec} B$*.*

Proof. Let $\mathfrak{q}$ be a minimal prime of B and let $\mathfrak{p}$ be its image in $\operatorname{Spec} A$. Since the map is flat, it satisfies going down, and so $\mathfrak{p}$ is a minimal prime of A. Since U is dense, it will contain $\mathfrak{p}$ and hence V will contain $\mathfrak{q}$. Thus V contains all the minimal primes of B and so is dense in $\operatorname{Spec} B$. □

Proposition 4.2.7.5. *Let* $(\operatorname{Spec}(R^{\square}_{\overline{\rho}}/\varpi))^{\text{Kirr}}$ *be the preimage of* V^{Kirr} *in* $\operatorname{Spec}(R^{\square}_{\overline{\rho}}/\varpi)$*. Then* $(\operatorname{Spec}(R^{\square}_{\overline{\rho}}/\varpi))^{\text{Kirr}}$ *is dense in* $\operatorname{Spec}(R^{\square}_{\overline{\rho}}/\varpi)$*.*

Proof. The map $A^{\text{gen}}/\varpi \to R^{\square}_{\overline{\rho}}/\varpi$ is flat, since it is a localization followed by a completion. The assertion then follows from Lemma 4.2.7.4 and Proposition 4.2.7.3. □

Remark 4.2.7.6. Since $(\operatorname{Spec}(R^{\square}_{\overline{\rho}}/\varpi))^{\text{Kirr}}$ is also the preimage of U^{Kirr} in $\operatorname{Spec} R^{\square}_{\overline{\rho}}/\varpi$ we may characterise it as an open subscheme of $\operatorname{Spec} R^{\square}_{\overline{\rho}}/\varpi$, such that its closed points are in bijection with $x \in P_1(R^{\square}_{\overline{\rho}}/\varpi)$, which map to $P_1(R^{\text{ps}}/\varpi)$ in $\operatorname{Spec} R^{\text{ps}}$ and for which the representation

$$\rho_x : G_K \to \operatorname{GL}_d(R^{\square}_{\overline{\rho}}/\varpi) \to \operatorname{GL}_d(\kappa(x))$$

remains absolutely irreducible after restriction to $G_{K'}$ for all degree p Galois extensions K' of $K(\zeta_p)$. Lemma 4.A.0.2 implies that $H^2(G_K, \operatorname{ad}^0 \rho_x) = 0$ for such x.

We will now prove similar results for the generic fibres. For each partition $\mathcal{P}$ as in Section 4.2.4 let $X^{\mathrm{ps}}_{\mathcal{P}}$ be the scheme theoretic image of $X^{\mathrm{ps}}_{\underline{\Sigma}}$ in X^{ps} and let $X^{\mathrm{gen}}_{\mathcal{P}}$ be the preimage of $X^{\mathrm{ps}}_{\mathcal{P}}$ in X^{gen}. We warn the reader that contrary to our usual notational conventions it is not clear that $\overline{X}^{\mathrm{ps}}_{\mathcal{P}}$ considered in Section 4.2.4 is the special fibre of $X^{\mathrm{ps}}_{\mathcal{P}}$. However, the following still holds.

Lemma 4.2.7.7. $\dim X^{\mathrm{gen}}_{\mathcal{P}}[1/p] \leq \dim \overline{X}^{\mathrm{gen}}_{\mathcal{P}}$.

Proof. Let $\mathfrak{a}_{\mathcal{P}}$ be the R^{ps}-annihilator of $R^{\mathrm{ps}}_{\underline{\Sigma}}$, and let $\mathfrak{b}_{\mathcal{P}}$ be the R^{ps}-annihilator of $R^{\mathrm{ps}}_{\underline{\Sigma}}/\varpi$. We may write

$$X^{\mathrm{gen}}_{cP} = \operatorname{Spec} A^{\mathrm{gen}}/\mathfrak{a}_{\mathcal{P}} A^{\mathrm{gen}}, \quad \overline{X}^{\mathrm{gen}}_{\mathcal{P}} = \operatorname{Spec} A^{\mathrm{gen}}/\mathfrak{b}_{\mathcal{P}} A^{\mathrm{gen}}.$$

Since $R^{\mathrm{ps}}_{\underline{\Sigma}}$ is a finite R^{ps}-module by Lemma 4.2.4.1, we have $\sqrt{\mathfrak{b}_{\mathcal{P}}} = \sqrt{(\mathfrak{a}_{\mathcal{P}}, \varpi)}$. In particular, the special fibre of $X^{\mathrm{gen}}_{\mathcal{P}}$ has dimension equal to $\dim \overline{X}^{\mathrm{gen}}_{\mathcal{P}}$. The assertion follows from Lemma 4.2.3.7. □

Proposition 4.2.7.8. *Let*

$$V^{\mathrm{irr}} := X^{\mathrm{gen}}[1/p] \setminus \bigcup_{\mathcal{P}_{\min} < \mathcal{P}} X^{\mathrm{gen}}_{\mathcal{P}}[1/p].$$

Then V^{irr} is an open dense subset of $X^{\mathrm{gen}}[1/p]$. Moreover, the following hold:

1. *if $d = 2$ then $\dim X^{\mathrm{gen}}[1/p] - \dim(X^{\mathrm{gen}}[1/p] \setminus V^{\mathrm{irr}}) \geq [K : \mathbb{Q}_p]$;*
2. *if $d > 2$ then $\dim X^{\mathrm{gen}}[1/p] - \dim(X^{\mathrm{gen}}[1/p] \setminus V^{\mathrm{irr}}) \geq 1 + [K : \mathbb{Q}_p]$;*
3. *if $d > 1$ is arbitrary but $\overline{D}$ is absolutely irreducible (i.e. $m = 1$) then $X^{\mathrm{gen}}[1/p] = V^{\mathrm{irr}}$.*

Proof. It follows from Corollary 4.2.5.9 that $\dim X^{\text{gen}}[1/p] = d^2 + d^2[K : \mathbb{Q}_p] = \dim \overline{X}^{\text{gen}}$. Lemma 4.2.7.7 and Lemma 4.2.4.6 together with (4.13) imply that for $\mathcal{P} > \mathcal{P}_{\min}$ we have

$$\dim X^{\text{gen}}[1/p] - \dim X^{\text{gen}}_{\mathcal{P}}[1/p] \geq \dim \overline{X}^{\text{gen}} - \dim \overline{X}^{\text{gen}}_{\mathcal{P}}. \tag{4.18}$$

It follows from (4.13) that $\overline{X}^{\text{gen}} \setminus V^{\text{Kirr}} = Y \cup Z^{\text{Kred}} \cup \bigcup_{\mathcal{P}_{\min} < \mathcal{P}} \overline{X}^{\text{gen}}_{\mathcal{P}}$. Thus it follows from (4.18) and the definition of V^{irr} that

$$\dim X^{\text{gen}}[1/p] - \dim(X^{\text{gen}}[1/p] \setminus V^{\text{irr}}) \geq \dim \overline{X}^{\text{gen}} - \dim(\overline{X}^{\text{gen}} \setminus V^{\text{Kirr}}) \tag{4.19}$$

and the lower bounds for the codimension of $X^{\text{gen}}[1/p] \setminus V^{\text{irr}}$ follow from Proposition 4.2.7.3.

Thus the dimension of the closure of V^{irr} is equal to $\dim X^{\text{gen}}[1/p]$. Since A^{gen} is $\mathcal{O}$-torsion free and equi-dimensional, $X^{\text{gen}}[1/p]$ is equi-dimensional, and so V^{irr} is dense in $X^{\text{gen}}[1/p]$.

If $\overline{D}$ is absolutely irreducible then ρ_x is absolutely irreducible for all closed points $x \in X^{\text{gen}}[1/p]$ and so $X^{\text{gen}}[1/p] = V^{\text{irr}}$. $\square$

Corollary 4.2.7.9. *Let* $(\operatorname{Spec} R^{\square}_{\overline{\rho}}[1/p])^{\text{irr}}$ *be the preimage of* V^{irr} *in* $\operatorname{Spec} R^{\square}_{\overline{\rho}}[1/p]$*. Then* $(\operatorname{Spec} R^{\square}_{\overline{\rho}}[1/p])^{\text{irr}}$ *is dense in* $\operatorname{Spec} R^{\square}_{\overline{\rho}}[1/p]$*.*

Proof. As explained in the proof of Proposition 4.2.7.5 the map $A^{\text{gen}} \to R^{\square}_{\overline{\rho}}$ is flat. Hence, the localization $A^{\text{gen}}[1/p] \to R^{\square}_{\overline{\rho}}[1/p]$ is also flat. The assertion follows from Lemma 4.2.7.4 and Proposition 4.2.7.8. $\square$

Remark 4.2.7.10. Similarly to Remark 4.2.7.6 we may characterize $(\operatorname{Spec} R^{\square}_{\overline{\rho}}[1/p])^{\text{irr}}$ as an open subscheme of $\operatorname{Spec} R^{\square}_{\overline{\rho}}[1/p]$ such that its closed points correspond to maximal ideals $\mathfrak{p}$ of $R^{\square}_{\overline{\rho}}[1/p]$ for which the representation

$$\rho_{\mathfrak{p}} : G_K \to \mathrm{GL}_d(R^{\square}_{\overline{\rho}}[1/p]) \to \mathrm{GL}_d(\kappa(\mathfrak{p}))$$

is absolutely irreducible.

Corollary 4.2.7.11. *The characteristic zero lift of $\overline{\rho}$ in Corollary 4.2.5.14 maybe chosen to be absolutely irreducible.*

Proof. It follows from Corollary 4.2.5.14 that $\operatorname{Spec} R^{\square}_{\overline{\rho}}[1/p]$ is non-empty, and Corollary 4.2.7.9 implies that $(\operatorname{Spec} R^{\square}_{\overline{\rho}}[1/p])^{\mathrm{irr}}$ is non-empty. A closed point in $(\operatorname{Spec} R^{\square}_{\overline{\rho}}[1/p])^{\mathrm{irr}}$ gives the desired lift of $\overline{\rho}$ to characteristic zero. □

Corollary 4.2.7.12. *Let* $\Sigma \subset \operatorname{MaxSpec} R^{\square}_{\overline{\rho}}[1/p]$ *be dense in* $\operatorname{Spec} R^{\square}_{\overline{\rho}}[1/p]$. *Then*

$$\Sigma^{\mathrm{irr}} := \Sigma \cap (\operatorname{Spec} R^{\square}_{\overline{\rho}}[1/p])^{\mathrm{irr}}$$

is also dense in $\operatorname{Spec} R^{\square}_{\overline{\rho}}[1/p]$.

Proof. It follows from the proof of Proposition 4.2.7.8 that $\Sigma \setminus \Sigma^{\mathrm{irr}}$ is contained in a closed subset of $\operatorname{Spec} R^{\square}_{\overline{\rho}}[1/p]$ of positive codimension. Since $\operatorname{Spec} R^{\square}_{\overline{\rho}}[1/p]$ is equi-dimensional Σ^{irr} is dense. □

4.3 Irreducible components

Let $\mu := \mu_{p^\infty}(K)$ be the subgroup of p-power roots of unity in K. We note that μ is a finite group of p-power order. Let $\bar{\psi} : G_K \to k^\times$ be a character and let $\psi^{\mathrm{univ}} : G_K \to \mathrm{GL}_1(R_{\bar{\psi}})$ be its universal deformation. Local class field theory gives a group homomorphism

$$\mu \to K^\times \xrightarrow{\mathsf{Art}_K} G_K^{\mathrm{ab}} \xrightarrow{\psi^{\mathrm{univ}}} \mathrm{GL}_1(R_{\bar{\psi}})$$

and the map induces a homomorphism of $\mathcal{O}$-algebras $\mathcal{O}[\mu] \to R_{\bar{\psi}}$, where $\mathcal{O}[\mu]$ is the group algebra of μ over $\mathcal{O}$.

Lemma 4.3.0.1. $R_{\bar\psi} \cong \mathcal{O}[\mu][\![y_1, \dots, y_{[K:\mathbb{Q}_p]+1}]\!]$.

Proof. It follows from local class field theory that the pro-p completion of G_K^{ab} is isomorphic to $\mu_{p^\infty}(K) \times \mathbb{Z}_p^{[K:\mathbb{Q}_p]+1}$ and the assertion follows from [Gou01, Proposition 3.13]. $\square$

Let $R_{\overline{\rho}}^{\square}$ be the framed deformation ring of $\overline{\rho} : G_K \to \mathrm{GL}_d(k)$ and let $R_{\det \overline{\rho}}$ denote the universal deformation ring of the one dimensional representation $\det \overline{\rho}$. Mapping a deformation of $\overline{\rho}$ to its determinant induces a natural map $R_{\det \overline{\rho}} \to R_{\overline{\rho}}^{\square}$, which makes $R_{\overline{\rho}}^{\square}$ into an $\mathcal{O}[\mu]$-algebra by applying the above discussion to $\bar\psi = \det \overline{\rho}$. The algebra $\mathcal{O}[\mu][1/p]$ is semi-simple and its maximal ideals are in bijection with characters $\chi : \mu \to \mathcal{O}^\times$. We thus have

$$R_{\overline{\rho}}^{\square}[1/p] \cong \prod_{\chi:\mu\to\mathcal{O}^\times} R_{\overline{\rho}}^{\square,\chi}[1/p], \tag{4.20}$$

where $R_{\overline{\rho}}^{\square,\chi} := R_{\overline{\rho}}^{\square} \otimes_{\mathcal{O}[\mu],\chi} \mathcal{O}$.

Warning 4.3.0.2. We emphasize that $R_{\overline{\rho}}^{\square,\chi}$ is *not* the fixed determinant deformation ring, but is rather constructed by fixing the value of the determinant only on the subgroup $\mathsf{Art}_K(\mu) \subset G_K^{\mathrm{ab}}$.

The aim of this section is to show that the rings $R_{\overline{\rho}}^{\square,\chi}$ are $\mathcal{O}$-torsion free integral domains by showing that they are normal. Since we already know that $R_{\overline{\rho}}^{\square}$ is $\mathcal{O}$-torsion free by Corollary 4.2.5.7, this implies that the map $R_{\det \overline{\rho}} \to R_{\overline{\rho}}^{\square}$ induces a bijection between the sets of irreducible components, which answers affirmatively a question raised by Böckle–Juschka in [BJ15]. We also determine the irreducible components of A^{gen} and R^{ps}.

We begin by showing that $R_{\overline{\rho}}^{\square}$ is a "relative complete intersection" over $R_{\det \overline{\rho}}$.

4.3.1 Relative complete intersection

Proposition 4.3.1.1. *There is an isomorphism*

$$R_{\det\overline{\rho}}[\![x_1,\ldots,x_r]\!]/(f_1,\ldots,f_t)\xrightarrow{\simeq}R^{\square}_{\overline{\rho}}$$

for $r := \dim\ker(Z^1(\mathrm{tr}) : Z^1(G_K,\mathrm{ad}\,\overline{\rho}) \to Z^1(G_K,k))$ *(where* G_K *acts trivially on* k*), and* $t := \dim_k H^2(G_K,\mathrm{ad}^0\,\overline{\rho})$ *such that* $(f_1,\ldots,f_t)$ *forms a regular sequence in* $R_{\det\overline{\rho}}[\![x_1,\ldots,x_r]\!]$*. Moreover,* $r-t=(d^2-1)([K:\mathbb{Q}_p]+1)$*.*

Proof. In this proof, the maximal ideal of a ring decorated by sub- or superscripts or a tilde is decorated in the same way. This argument is a minor modification of Kisin's method of presenting global deformation rings over local ones. Kisin's is an important refinement of a similar argument of Mazur.

The map $Z^1(\mathrm{tr})$ is the induced map on Zariski tangent spaces of the map of deformation rings $R_{\det\overline{\rho}}\to R^{\square}_{\overline{\rho}}$, and lifts to a surjection

$$\widetilde{\phi}:\widetilde{R}:=R_{\det\overline{\rho}}[\![x_1,\ldots,x_r]\!]\twoheadrightarrow R^{\square}_{\overline{\rho}}.$$

We set $J:=\ker\widetilde{\phi}$. By Nakayama's lemma, we need to show that $\dim_k J/\widetilde{\mathfrak{m}}J\le t$.

The module $J/\widetilde{\mathfrak{m}}J$ appears as the kernel in the sequence

$$0\to J/\widetilde{\mathfrak{m}}J\to\widetilde{R}/\widetilde{\mathfrak{m}}J\to\widetilde{R}/J\cong R^{\square}_{\overline{\rho}}\to 0. \tag{4.21}$$

In view of the above sequence, we shall construct a homomorphism

$$\alpha:\mathrm{Hom}(J/\widetilde{\mathfrak{m}}J,k)\to\ker\left(H^2(\mathrm{tr})\colon H^2(G_K,\mathrm{ad}\,\overline{\rho})\to H^2(G_K,k)\right)$$

and show that the kernel of α injects into $\operatorname{coker}(H^1(\mathrm{tr}))$. This will imply the existence of the presentation in the statement of the Proposition, since then

$$\dim_k J/\widetilde{\mathfrak{m}}J \leq \dim_k \ker(H^2(\mathrm{tr})) + \dim_k \operatorname{coker}(H^1(\mathrm{tr})) = \dim_k H^2(G_K, \operatorname{ad}^0 \overline{\rho}),$$

where the last equality comes from the long exact cohomology sequence that arises from $0 \to \operatorname{ad}^0 \overline{\rho} \to \operatorname{ad} \overline{\rho} \to k \to 0$.

Fix $u \in \operatorname{Hom}_k(J/\widetilde{\mathfrak{m}}J, k)$. The pushout under u of the sequence (4.21) yields

$$0 \to I_u \to R_u \xrightarrow{\phi_u} R_{\overline{\rho}}^{\square} \to 0,$$

where $I_u = k$. The surjection of profinite groups $\mathrm{GL}_d(R_u) \twoheadrightarrow \mathrm{GL}_d(R_{\overline{\rho}}^{\square})$ has a continuous section by [RZ10, Proposition 2.2.2]. Thus there is a continuous map $\widetilde{\rho}_u : G_K \to \mathrm{GL}_d(R_u)$ such that the diagram

$$\begin{array}{ccc} G_K & \xrightarrow{\widetilde{\rho}_u} & \mathrm{GL}_d(R_u) \\ & {\scriptstyle \rho^{\square}} \searrow & \downarrow {\scriptstyle \mathrm{GL}_d(\phi_u)} \\ & & \mathrm{GL}_d(R_{\overline{\rho}}^{\square}) \end{array}$$

commutes. The kernel $1 + M_d(I_u)$ of $\mathrm{GL}_d(\phi_u)$ can be identified with $\operatorname{ad} \overline{\rho} \otimes_k I_u$, and so the set-theoretic lift yields a continuous 2-cocycle

$$c_u \in Z^2(G_K, \operatorname{ad} \overline{\rho}) \otimes_k I_u$$

given by $1 + c_u(g_1, g_2) = \widetilde{\rho}_u(g_1 g_2)\widetilde{\rho}_u(g_2)^{-1}\widetilde{\rho}_u(g_1)^{-1}$. The class

$$[c_u] \in H^2(G_K, \operatorname{ad} \overline{\rho}) \otimes_k I_u$$

is independent of the chosen lifting. The representation $\rho_{\overline{\rho}}^{\square}$ can be lifted to a homomorphism

$G_K \to \mathrm{GL}_d(R_u)$ precisely if $[c_u] = 0$. The existence of the homomorphisms $R_{\det \overline{\rho}} \to R_u \to R^{\square}_{\overline{\rho}}$ together with the universality of $R_{\det \overline{\rho}}$ imply that the image of $[c_u]$ in $H^2(G_K, k)$ is zero. We define α as the homomorphism $u \mapsto [c_u]$.

To analyze the kernel of α, let u be such that $[c_u] = 0$, so that $\rho^{\square}$ can be lifted to R_u. By the universality of $R^{\square}_{\overline{\rho}}$ we obtain a splitting s_u of ϕ_u. One deduces that the map from I_u to the kernel of the surjective map

$$t_u : \mathfrak{m}_{R_u}/(\mathfrak{m}^2_{R_u} + \varpi R_u) \to \mathfrak{m}^{\square}/((\mathfrak{m}^{\square})^2 + \varpi R^{\square}_{\overline{\rho}})$$

of mod ϖ cotangent spaces is an isomorphism.

The map t_u in turn is induced from the homomorphism $\widetilde{R}/(J\widetilde{\mathfrak{m}}) \to R^{\square}_{\overline{\rho}}$ by pushout and from the analogous surjection

$$\widetilde{t} : \widetilde{\mathfrak{m}}/(\widetilde{\mathfrak{m}}^2 + \varpi \widetilde{R}) \to \mathfrak{m}^{\square}/((\mathfrak{m}^{\square})^2 + \varpi R^{\square}_{\overline{\rho}})$$

Via our identification $I_u \cong \ker t_u$, the pushout along u induces a surjective homomorphism $\gamma_u : \ker(\widetilde{t}) \to I_u \cong k$ of k-vector spaces. One easily verifies that $u \mapsto \gamma_u$ induces an injective k-linear map

$$\ker(\alpha) \hookrightarrow \mathrm{Hom}_k(\ker(\widetilde{t}), k)$$

Upon identifying $\ker(\widetilde{t})^*$ with $\mathrm{coker}(H^1(\mathrm{tr}))$, the proof of the bound is complete.

It remains to show that $f_1, \ldots, f_t$ is a regular sequence. We may write $\mathcal{O}[\mu] = \mathcal{O}[\![z]\!]/((1+z)^m - 1)$, where m is the order of μ. By Lemma 4.3.0.1, we get a presentation

$$\frac{\mathcal{O}[\![z, y_1, \ldots, y_{[K:\mathbb{Q}_p]+1}, x_1, \ldots, x_r]\!]}{((1+z)^m - 1, f_1, \ldots, f_t)} \xrightarrow{\cong} R^{\square}_{\overline{\rho}}.$$

We claim that

$$\dim R^{\square}_{\overline{\rho}} = [K:\mathbb{Q}_p] + 2 + r - t. \tag{4.22}$$

The claim implies that $((1+z)^m-1, f_1, \ldots, f_t)$ can be extended to a system of parameters in a regular ring $S := \mathcal{O}[\![z, x_1, \ldots, x_{[K:\mathbb{Q}_p]+1}, y_1, \ldots, y_r]\!]$. Thus $((1+z)^m-1, f_1, \ldots, f_t)$ is a regular sequence in S and so $(f_1, \ldots, f_t)$ is a regular sequence in $R_{\det \overline{\rho}}[\![x_1, \ldots, x_r]\!] = S/((1+z)^m-1)$.

By Corollary 4.2.5.7 the relative dimension of $R^{\square}_{\overline{\rho}}$ over $\mathcal{O}$ is $\dim Z^1(G_K, \operatorname{ad}\overline{\rho}) - \dim_k H^2(G_K, \operatorname{ad}\overline{\rho})$, so to verify the claim it is enough to show that

$$r + [K:\mathbb{Q}_p] + 1 - t = \dim Z^1(G_K, \operatorname{ad}\overline{\rho}) - \dim_k H^2(G_K, \operatorname{ad}\overline{\rho}) \overset{(4.17)}{=} d^2 + d^2[K:\mathbb{Q}_p]. \tag{4.23}$$

We now deduce (4.23) in a straightforward manner:

$$\begin{aligned}
&r + [K:\mathbb{Q}_p] + 1 - t \\
&\overset{(+)}{=} \quad [K:\mathbb{Q}_p] + 1 + \dim\ker(Z^1(\mathrm{tr})) - \dim\operatorname{coker}(Z^1(\mathrm{tr})) - \dim\ker(H^2(\mathrm{tr})) \\
&\overset{(\mathrm{EP})}{=} \quad \dim_k H^1(G_K, k)\dim_k H^2(G_K, k) + \dim Z^1(G_K, \operatorname{ad}\overline{\rho}) - \dim_k H^1(G_K, k) - \dim\ker(H^2(\mathrm{tr})) \\
&= \quad \dim Z^1(G_K, \operatorname{ad}\overline{\rho}) - \dim\ker(H^2(\mathrm{tr})) - \dim_k H^2(G_K, k) \\
&\overset{(+)}{=} \quad \dim Z^1(G_K, \operatorname{ad}\overline{\rho}) - \dim_k H^2(G_K, \operatorname{ad}\overline{\rho}),
\end{aligned}$$

where (EP) stands for the Euler-Poincaré formula for k, and where (+) arises from counting dimensions in the exact sequences

$$0 \to \ker(Z^1(\mathrm{tr})) \to Z^1(G_K, \operatorname{ad}\overline{\rho}) \xrightarrow{Z^1(\mathrm{tr})} Z^1(G_K, k) \to \operatorname{coker}(Z^1(\mathrm{tr})) \to 0$$

and

$$0 \to \ker(H^2(\mathrm{tr})) \to H^2(G_K, \operatorname{ad}\overline{\rho}) \xrightarrow{H^2(\mathrm{tr})} H^2(G_K, k) \to 0.$$

It follows from (4.23) that $r - t = (d^2-1)([K:\mathbb{Q}_p]+1)$. □

Remark 4.3.1.2. The Proposition also holds for continuous representations $\overline{\rho} : G_K \to$

$\mathrm{GL}_d(\kappa)$, where κ is a local field. Showing that the 2-cocycle is a continuous becomes a bit more subtle, but this is well explained in [Con10, Lecture 6].

As a consequence, we get the complete intersection property when we specialize to a character of μ.

Corollary 4.3.1.3. *For each character $\chi : \mu_{p^\infty}(K) \to \mathcal{O}^\times$ and each closed point $x \in X^{\mathrm{gen}}$ above $\mathfrak{m}_{R^{\mathrm{ps}}}$ the following hold:*

1. *$R^{\square,\chi}_{\rho_x}$ is $\mathcal{O}$-torsion free of dimension $1 + d^2 + d^2[K : \mathbb{Q}_p]$ and is complete intersection;*
2. *$R^{\square,\chi}_{\rho_x}/\varpi$ is complete intersection of dimension $d^2 + d^2[K : \mathbb{Q}_p]$.*

Proof. Without loss of generality we may assume that the residue field of x is equal to k. Proposition 4.3.1.1 gives the presentation

$$R^{\chi}_{\det \rho_x}[\![x_1, \dots, x_r]\!]/(f_1, \dots, f_t) \xrightarrow{\cong} R^{\square,\chi}_{\rho_x},$$

where $R^{\chi}_{\det \rho_x} := R_{\det \rho_x} \otimes_{\mathcal{O}[\mu],\chi} \mathcal{O}$. Since $R^{\chi}_{\det \rho_x}$ is formally smooth over $\mathcal{O}$ of dimension $[K : \mathbb{Q}_p] + 2$ by Lemma 4.3.0.1, it is enough to show that

$$\dim R^{\square,\chi}_{\rho_x}/\varpi \leq [K : \mathbb{Q}_p] + 1 + r - t.$$

Then the same argument as in the proof of Proposition 4.3.1.1 shows that the sequence $\varpi, f_1, \dots, f_t$ is regular in $R^{\chi}_{\det \rho_x}[\![x_1, \dots, x_r]\!]$. Since $R^{\square,\chi}_{\rho_x}$ is a quotient of $R^{\square}_{\rho_x}$ and $R^{\square}_{\rho_x}$ is $\mathcal{O}$-torsion free by Corollary 4.2.5.7, we have $\dim R^{\square,\chi}_{\rho_x}/\varpi \leq \dim R^{\square}_{\rho_x}/\varpi = \dim R^{\square}_{\rho_x} - 1$ and the desired inequality follows from (4.22). ∎

The restriction of a pseudorepresentation $D : A[G_K] \to A$ to G defines a continuous group homomorphism $D|_{G_K} : G_K \to A^\times$, see [BJ19, Definition 4.1.5]. Moreover, if D is associated to a representation $\rho : G_K \to \mathrm{GL}_d(A)$ then $D|_{G_K} = \det \rho$. This induces a map of deformation rings $R_{\det \overline{\rho}} \xrightarrow{\sim} R_{\overline{D}|_{G_K}} \to R^{\mathrm{ps}}$ and makes R^{ps} into an $\mathcal{O}[\mu]$-algebra.

Since A^{gen} is an R^{ps}-algebra, we may define

$$A^{\mathrm{gen},\chi} := A^{\mathrm{gen}} \otimes_{\mathcal{O}[\mu],\chi} \mathcal{O}, \quad X^{\mathrm{gen},\chi} := \mathrm{Spec}\, A^{\mathrm{gen},\chi}$$

and we let $\overline{X}^{\mathrm{gen},\chi}$ denote its special fibre. Note that since a character of G_K^{ab} valued in a characteristic p field is trivial after pulling back to $\mu_{p^\infty}(K)$, we have that $\overline{X}^{\mathrm{gen},\chi} = \overline{X}^{\mathrm{gen},\mathbf{1}}$ for all χ, where $\mathbf{1}$ is the trivial character. Moreover, the reduced subschemes of $\overline{X}^{\mathrm{gen}}$ and $\overline{X}^{\mathrm{gen},\chi}$ coincide and so

$$\dim \overline{X}^{\mathrm{gen},\chi} = \dim \overline{X}^{\mathrm{gen}} = d^2 + d^2[K : \mathbb{Q}_p],$$

where the last equality is given by Corollary 4.2.5.9.

Corollary 4.3.1.4. *For each character $\chi : \mu_{p^\infty}(K) \to \mathcal{O}^\times$ the following hold:*

1. *$A^{\mathrm{gen},\chi}$ is $\mathcal{O}$-torsion free, equi-dimensional of dimension $1 + d^2 + d^2[K : \mathbb{Q}_p]$ and is locally complete intersection;*
2. *$A^{\mathrm{gen},\chi}/\varpi$ is equi-dimensional of dimension $d^2 + d^2[K : \mathbb{Q}_p]$ and is locally complete intersection.*

Proof. We claim that the local rings at closed points of $X^{\mathrm{gen},\chi}$ are $\mathcal{O}$-torsion free and complete intersection. Given the claim the proof is the same as in Corollary 4.2.5.9.

We will prove the claim using the strategy outlined in Remark 4.2.5.12. We already know from Corollary 4.3.1.3 that $R^{\square,\chi}_{\rho_x}$ is $\mathcal{O}$-torsion free and complete intersection of dimension $d^2 + d^2[K : \mathbb{Q}_p] + 1$ whenever $x \in X^{\mathrm{gen},\chi}$ is a closed point with $\kappa(x)/k$ a finite extension. By applying $\otimes_{\mathcal{O}[\mu],\chi}\mathcal{O}$ we obtain the χ-versions of Proposition 4.2.5.3 and Proposition 4.2.5.10 and Corollary 4.2.5.11.

Let x be a closed point of $X^{\mathrm{gen},\chi}$. If $\kappa(x)$ is a finite extension of k then $\widehat{\mathcal{O}}_{X^{\mathrm{gen},\chi},x} \cong R^{\square,\chi}_{\rho_x}$ by Proposition 4.2.5.3 and hence $\mathcal{O}_{X^{\mathrm{gen},\chi},x}$ is complete intersection. Otherwise, let x' and z be as in Corollary 4.2.5.11. In particular, z is a closed point of $X^{\mathrm{gen},\chi}$ and $\kappa(z)$ is a finite extension of k. It follows from the argument explained in Remark 4.2.5.12 that if $\kappa(x)$ is a local field of characteristic p then

$$\widehat{\mathcal{O}}_{X^{\mathrm{gen},\chi},x}[\![T]\!] \cong R^{\square,\chi}_{\rho_x} \cong R^{\square,\chi}_{\rho_{x'}} \cong \widehat{(R^{\square,\chi}_{\rho_z})}_{x'}[\![T]\!],$$

and if $\kappa(x)$ is a finite extension of L then

$$\widehat{\mathcal{O}}_{X^{\mathrm{gen},\chi},x} \cong R^{\square,\chi}_{\rho_x} \cong R^{\square,\chi}_{\rho_{x'}} \cong \widehat{(R^{\square,\chi}_{\rho_z})}_{x'}.$$

Since $R^{\square,\chi}_{\rho_z}$ is complete intersection, it follows from [Stacks, Tag 09Q4] that the local ring $(R^{\square,\chi}_{\rho_z})_{x'}$ (and hence its completion) is also complete intersection. The isomorphisms above imply that $\widehat{\mathcal{O}}_{X^{\mathrm{gen},\chi},x}$ is complete intersection. Hence, $\mathcal{O}_{X^{\mathrm{gen},\chi},x}$ is complete intersection, see [Stacks, Tag 09Q3]. □

Remark 4.3.1.5. Alternatively, one could first prove a version of Proposition 4.3.1.1 for deformation rings of $\overline{\rho} : G_K \to \mathrm{GL}_d(\kappa(x))$ to Artinian Λ-algebra as in Section 4.2.5 for any closed point of $x \in X^{\mathrm{gen}}$, by changing $\mathcal{O}$ to Λ and k to $\kappa(x)$ everywhere. The Euler characteristic formula still holds in this setting, see [BJ19, Theorem 3.4.1]. Then deduce Corollary 4.3.1.3 in this more general setting using the same proof and then obtain Corollary 4.3.1.4 by repeating verbatim the proof of Corollary 4.2.5.9.

4.3.2 Bounding singular loci

Our next aim is to bound the singular locus in $\overline{X}^{\mathrm{gen},\chi}$ and $X^{\mathrm{gen},\chi}[1/p]$. This will be used in the next section to deduce normality.

In the Lemmas below, κ is either a finite field extension of k, or of L or a local field of characteristic p containing k, and Λ is defined in Section 4.2.5. If $\mathrm{char}(\kappa) = 0$ then $\Lambda = \kappa$, if $\mathrm{char}(\kappa) = p$ then Λ is an $\mathcal{O}$-algebra, which is a complete DVR with uniformiser ϖ and residue field κ. As in Section 4.2.5 we consider deformation problems of $\rho : G_K \to \mathrm{GL}_d(\kappa)$ to local Artinian Λ-algebras with residue field κ.

Lemma 4.3.2.1. *Let κ be a finite or local field of characteristic p or a finite extension of L and let $\rho : G_K \to \mathrm{GL}_d(\kappa)$ be a representation, such that $H^2(G_K, \mathrm{ad}^0 \rho) = 0$, where $\mathrm{ad}^0 \rho$ is the kernel of the trace map. Then for all characters $\chi : \mu_{p^\infty}(K) \to \mathcal{O}^\times$ the ring $R^{\square,\chi}_\rho$ is formally smooth over Λ.*

Proof. It follows from the proof of [BJ19, Lemma 3.4.2], where an analogous statement is proved for the deformation functors without the framing and for Artinian κ-algebras, that the map

$$R_{\det \rho} \to R^{\square}_\rho,$$

induced by sending a deformation of ρ to an Artinian Λ-algebra to its determinant, is formally smooth. By applying $\otimes_{\mathcal{O}[\mu],\chi}\mathcal{O}$ we deduce that the map

$$R^{\chi}_{\det \rho} \to R^{\square,\chi}_\rho$$

is formally smooth.

Since the group $G_K^{\mathrm{ab}}/\mathsf{Art}_K(\mu_{p^\infty}(K))$ is p-torsion free, the ring $R^{\chi}_{\det \rho}$ is formally smooth over Λ. Hence, $R^{\square,\chi}_\rho$ is formally smooth over Λ. (Alternatively, one could prove Proposition 4.3.1.1 for ρ, see Remark 4.3.1.2, and then obtain the Lemma as a Corollary.) □

Recall that in Section 4.2.7 we have defined an open subscheme U^{nspcl} of $\overline{X}^{\mathrm{ps}} \setminus \{\mathfrak{m}_{R^{\mathrm{ps}}}\}$ and defined V^{nspcl} to be a preimage of U^{nspcl} in $\overline{X}^{\mathrm{gen}}$. We will refer to V^{nspcl} as the *absolutely irreducible non-special locus.*

Proposition 4.3.2.2. *For each character $\chi : \mu_{p^\infty}(K) \to \mathcal{O}^\times$ the absolutely irreducible non-special locus in $\overline{X}^{\mathrm{gen},\chi}$ is regular.*

Proof. It is enough to show that localization of $A^{\mathrm{gen},\chi}/\varpi$ at x is a regular ring for every closed point x in $V^{\mathrm{nspcl}} \cap \overline{X}^{\mathrm{gen},\chi}$. It follows from Lemma 4.2.5.4 applied with $R = R^{\mathrm{ps},\chi}/\varpi$ and $A = A^{\mathrm{gen},\chi}/\varpi$ that it is enough to show that the completion of $\kappa(x) \otimes_{\mathcal{O}} A^{\mathrm{gen},\chi}$ at the kernel of the map of $\kappa(x)$-algebras $\kappa(x) \otimes_{\mathcal{O}} A^{\mathrm{gen},\chi} \to \kappa(x)$ is regular. Proposition 4.2.5.3 implies that we may identify this ring with deformation ring $R^{\square,\chi}_{\rho_x}/\varpi$. If $\zeta_p \in K$ then since x is non-special $H^2(G_K, \mathrm{ad}^0 \rho_x) = 0$, see [BJ19, Lemma 5.1.1], Lemma 4.3.2.1 implies that $R^{\square,\chi}_{\rho_x}/\varpi$ is formally smooth over $\kappa(x)$. If $\zeta_p \notin K$ then μ is trivial, so that $R^{\square,\chi}_{\rho_x} = R^{\square}_{\rho_x}$, and $H^2(G_K, \mathrm{ad}\, \rho_x) = 0$, see [BJ19, Lemma 5.1.1]. It follows from (4.16) that $R^{\square}_{\rho_x}/\varpi$ is formally smooth over $\kappa(x)$. □

Proposition 4.3.2.3. *For each character $\chi : \mu_{p^\infty}(K) \to \mathcal{O}^\times$ the absolutely irreducible locus in $X^{\mathrm{gen},\chi}[1/p]$ is regular.*

Proof. Let x be a closed point in $X^{\mathrm{gen},\chi}[1/p]$ and let $\rho_x : G_K \to \mathrm{GL}_d(\kappa(x))$ be the corresponding Galois representation. We claim that if ρ_x is absolutely irreducible then $H^2(G_K, \mathrm{ad}^0 \rho_x) = 0$. Since $\kappa(x)$ is a finite extension of L, $\mathrm{ad}^0 \rho_x$ is a direct summand of $\mathrm{ad}\, \rho_x$, and thus it is enough to show that $H^2(G_K, \mathrm{ad}\, \rho_x) = 0$. By local Tate duality, it is enough to show that $H^0(G_K, \mathrm{ad}\, \rho_x(1)) = 0$. Since ρ_x is absolutely irreducible, non-vanishing of this group is equivalent to $\rho_x \cong \rho_x(1)$. By considering determinants we would obtain that the d-th power of the cyclotomic character is trivial, yielding a contradiction.

Given the claim the rest of the proof is the same as the proof of Proposition 4.3.2.2, since Lemma 4.2.5.6 implies that $\widehat{\mathcal{O}}_{X^{\mathrm{gen},\chi},x} \cong R^{\square,\chi}_{\rho_x}$. □

Lemma 4.3.2.4. *Assume that $K = \mathbb{Q}_p$ and $d = 2$. Let κ be either a finite extension of L or a finite or local field of characteristic p, in which case we further assume that $p > 2$. Let*

$\rho : G_{\mathbb{Q}_p} \to \mathrm{GL}_2(\kappa)$ *be a continuous representation, which is an extension of characters* ψ_1 *and* ψ_2*, such that* $\psi_1 \neq \psi_2(1)$ *and* $\psi_2 \neq \psi_1(1)$*. Then* $H^2(G_{\mathbb{Q}_p}, \mathrm{ad}\,\rho) = H^2(G_K, \mathrm{ad}^0\,\rho) = 0$*. In particular, the ring* $R_\rho^{\square,\chi}$ *is formally smooth over* Λ*.*

Proof. Since $\mathrm{char}(\kappa) \neq 2$, $\mathrm{ad}^0\,\rho$ is a direct summand $\mathrm{ad}\,\rho$, and thus it is enough to show that $H^2(G_{\mathbb{Q}_p}, \mathrm{ad}\,\rho) = 0$. By local Tate duality, see [BJ19, Section 3.4], it is enough to show that $H^0(G_{\mathbb{Q}_p}, \mathrm{ad}\,\rho(1)) = 0$. Non-vanishing of this group would imply that $\psi_i\psi_j^{-1}(1)$ is a trivial character for some $i, j \in \{1, 2\}$. If $i = j$ then this would imply $\chi_{\mathrm{cyc}} \otimes_{\mathbb{Z}_p} \kappa$ is trivial, which is not the case as $\mathrm{char}(\kappa) \neq 2$. If $i \neq j$ then this does not hold by assumption.

The last assertion follows from Lemma 4.3.2.1. □

Lemma 4.3.2.5. *Assume that* $p = 2$*,* $K = \mathbb{Q}_2$ *and* $d = 2$*. Let* κ *be a finite or local field of characteristic* 2 *and let* $\rho : G_{\mathbb{Q}_2} \to \mathrm{GL}_2(\kappa)$ *be a continuous representation, which is a non-split extension of distinct characters. Then* $H^2(G_{\mathbb{Q}_2}, \mathrm{ad}^0\,\rho) = 0$*. In particular, the ring* $R_\rho^{\square,\chi}$ *is formally smooth over* Λ*.*

Proof. After twisting we may assume that we can choose a basis of the underlying vector space of ρ, such that with respect to that basis

$$\rho(g) = \begin{pmatrix} 1 & b(g) \\ 0 & \psi(g) \end{pmatrix}, \quad \forall g \in G_{\mathbb{Q}_2}.$$

We use the same basis to identify $\mathrm{ad}\,\rho$ with $M_2(k)$ with the $G_{\mathbb{Q}_2}$-action given by

$$g \cdot M := \rho(g) M \rho(g)^{-1}.$$

For $i, j \in \{1, 2\}$ let $e_{ij} \in M_2(k)$ be the matrix with the ij-entry equal to 1 and all the other entries equal to zero. Let $\overline{\mathrm{ad}}\rho$ be the quotient $\mathrm{ad}\,\rho$ by the scalar matrices and let $\bar{e}_{ij}$ be the image of e_{ij} in $\overline{\mathrm{ad}}\rho$. A direct computation shows that

$$g \cdot \overline{e}_{12} = \psi(g)^{-1}\overline{e}_{12}, \quad g \cdot \overline{e}_{11} = \overline{e}_{11} - \psi(g)^{-1}b(g)\overline{e}_{12}, \quad g \cdot \overline{e}_{21} = \psi(g)\overline{e}_{21} - \psi(g)^{-1}b(g)^2\overline{e}_{12}.$$

Since ρ is non-split, $b(g) \neq 0$ for some $g \in G_{\mathbb{Q}_2}$. Thus $\kappa\overline{e}_{12}$ is the unique irreducible subrepresentation of $\overline{\mathrm{ad}}\rho$. Since $G_{\mathbb{Q}_2}$ acts on $\overline{e}_{12}$ by a non-trivial character, we deduce that $H^0(G_{\mathbb{Q}_2}, \overline{\mathrm{ad}}\rho) = 0$.

It follows from local Tate duality, see [BJ19, Section 3.4], that $H^2(G_{\mathbb{Q}_2}, \mathrm{ad}^0\rho) = 0$. Note that the cyclotomic character is trivial modulo 2.

The last assertion follows from Lemma 4.3.2.1. ☐

Proposition 4.3.2.6. *There is an open subscheme $V^{0,\chi} \subset \overline{X}^{\mathrm{gen},\chi}$ such that*

1. $H^2(G_K, \mathrm{ad}^0\rho_x) = 0$ *for all closed points* $x \in V^{0,\chi}$*;*
2. $\dim \overline{X}^{\mathrm{gen},\chi} - \dim(\overline{X}^{\mathrm{gen},\chi} \setminus V^{0,\chi}) \geq 2$.

In particular, $\overline{X}^{\mathrm{gen},\chi}$ *is regular in codimension* 1.

Proof. We first note that if $V \subset \overline{X}^{\mathrm{gen},\chi}$ is open and satisfies part (1) then V is regular by the argument explained in the proof of Proposition 4.3.2.2. Thus if (1) and (2) hold then $\overline{X}^{\mathrm{gen},\chi}$ is regular in codimension 1. We also note that Lemma 4.A.0.2 implies that part (1) holds for $V^{\mathrm{Kirr},\chi} := V^{\mathrm{Kirr}} \cap \overline{X}^{\mathrm{gen},\chi}$. We consider three separate cases.

Case 1: $d > 2$ or $K \neq \mathbb{Q}_p$ or $\overline{D}$ is (absolutely) irreducible. These three conditions correspond to parts (1), (2), and (3) of Proposition 4.2.7.3 respectively, and indeed Proposition 4.2.7.3 implies that the complement of $V^{\mathrm{Kirr},\chi}$ in $\overline{X}^{\mathrm{gen},\chi}$ has dimension at most $\dim \overline{X}^{\mathrm{gen},\chi} - 2$. Hence, we may take $V^{0,\chi} = V^{\mathrm{Kirr},\chi}$.

Case 2: $d = 2$ and $K = \mathbb{Q}_p$ and $p > 2$ and $\overline{D}$ is reducible. In this case, $\mu = \{1\}$ so $\chi = \mathbf{1}$ and thus $\overline{X}^{\mathrm{gen},\mathbf{1}} = \overline{X}^{\mathrm{gen}}$. It follows from Proposition 4.2.4.3, Lemma 4.2.4.7, Lemma 4.2.7.2 that

$$V^{0,\chi} := \overline{X}^{\mathrm{gen}} \setminus (Y \cup Z^{12}_{\mathcal{P}_{\max}} \cup Z^{21}_{\mathcal{P}_{\max}} \cup Z^{\mathrm{Kred}})$$

satisfies part (2). We may also write $V^{0,\chi} = V^{\mathrm{Kirr}} \cup V'_{\mathcal{P}_{\max}}$, where we use the notation introduced in the proof of Proposition 4.2.4.4. Since part (1) holds for V^{Kirr} it is enough to consider closed points $x \in V'_{\mathcal{P}_{\max}}$. The definition of $V'_{\mathcal{P}_{\max}}$ implies firstly that ρ_x is reducible and secondly that if we let ψ_1 and ψ_2 denote its irreducible Jordan-Hölder constituents then $\psi_1 \neq \psi_2(1)$ and $\psi_2 \neq \psi_1(1)$. Therefore, $H^2(G_K, \mathrm{ad}^0 \rho_x) = 0$ by Lemma 4.3.2.4.

Case 3: $d = 2$ and $K = \mathbb{Q}_2$ and $\overline{D}$ is reducible. The proof is the same as in Case 2, using Lemma 4.3.2.5 instead of Lemma 4.3.2.4. However, one additionally has to remove the maximal reducible semi-simple locus in $\overline{X}^{\mathrm{gen},\chi}$. Its dimension is at most $4 + 2 = 6$ by Corollary 4.2.6.4 and the dimension of $\overline{X}^{\mathrm{gen},\chi}$ is 8. Thus the codimension is at least 2. □

4.3.3 Normality

We now combine the results of the previous section with Corollary 4.3.1.4 and apply Serre's criterion to deduce normality of the generic and special fibers. Finally we combine them in Corollary 4.3.3.5, which says that $X^{\mathrm{gen},\chi}$ is normal. It follows immediately afterwards that $R^{\square,\chi}_{\overline{\rho}}$ is normal

Proposition 4.3.3.1. *$\overline{X}^{\mathrm{gen},\chi}$ is normal.*

Proof. Since $\overline{X}^{\mathrm{gen},\chi}$ is a local complete intersection by Corollary 4.3.1.4, it is Cohen–Macaulay and satisfies property (S2), and Proposition 4.3.2.6 says that it satisfies property (R1). Hence, $\overline{X}^{\mathrm{gen},\chi}$ is normal by Serre's criterion for normality. □

Corollary 4.3.3.2. *For each character $\chi : \mu_{p^\infty}(K) \to \mathcal{O}^\times$ and $\overline{\rho} : G_K \to \mathrm{GL}_d(k)$ the ring*

$R_{\overline{\rho}}^{\square,\chi}/\varpi$ *is a normal integral domain.*

Proof. Since $\overline{X}^{\mathrm{gen},\chi}$ is normal and excellent the completions of its local rings are normal, [Mat89, Theorem 32.2 (i)]. Thus by formally completing along the maximal ideal corresponding to $\overline{\rho}$ we see that $R_{\overline{\rho}}^{\square,\chi}/\varpi$ is normal, and thus an integral domain since it is a local ring. □

Lemma 4.3.3.3. *Let $\hat{Y}$ be the preimage of $\mathfrak{m}_{R^{\mathrm{ps}}}$ in* $\operatorname{Spec} R_{\overline{\rho}}^{\square,\chi}/\varpi$*. Let W be a closed subscheme of* $\operatorname{Spec} R_{\overline{\rho}}^{\square,\chi}/\varpi$ *such that $H^2(G_K, \mathrm{ad}^0 \rho_x) \neq 0$ for all closed points $x \in W \setminus \hat{Y}$. Then* $\dim R_{\overline{\rho}}^{\square,\chi}/\varpi - \dim W \geq 2$.

Proof. The assumptions imply that W is contained in $\hat{Z} \cup \hat{Y}$, where $Z = \overline{X}^{\mathrm{gen},\chi} \setminus V^{0,\chi}$ and $\hat{Z}$ is a formal completion of Z at the point corresponding to $\overline{\rho}$. In terms of commutative algebra the ring of functions of $\hat{Z}$ corresponds to the completion of the ring of functions of Z with respect to the maximal ideal corresponding to $\overline{\rho}$. Hence, $\dim \hat{Z} = \dim Z$, and Proposition 4.3.2.6 implies that $\hat{Z}$ has codimension at least 2 in $\operatorname{Spec} R_{\overline{\rho}}^{\square,\chi}/\varpi$. Similarly $\hat{Y}$ is a formal completion of Y at the point corresponding to $\overline{\rho}$, and using Lemma 4.2.4.7 we conclude that $\hat{Y}$ also has codimension of at least 2 in $\operatorname{Spec} R_{\overline{\rho}}^{\square,\chi}/\varpi$. □

Proposition 4.3.3.4. $X^{\mathrm{gen},\chi}[1/p]$ *is normal.*

Proof. The proof is essentially the same as the proof of Proposition 4.3.3.1. It follows from Corollary 4.3.1.4 that $X^{\mathrm{gen},\chi}[1/p]$ is Cohen–Macaulay and we have to check that the codimension of the singular locus is at least 2. Since $X^{\mathrm{gen},\chi}[1/p]$ is a preimage of $\operatorname{Spec} R^{\mathrm{ps},\chi}[1/p]$ in X^{gen}, Lemma 4.2.3.2 implies that $X^{\mathrm{gen},\chi}[1/p]$ is Jacobson and we may argue with closed points.

We have already shown in Proposition 4.3.2.3 that the absolutely irreducible locus $V^{\mathrm{irr},\chi}$ in

$X^{\mathrm{gen},\chi}[1/p]$ is regular. Thus the singular locus is contained in

$$\bigcup_{\mathcal{P}_{\min}<\mathcal{P}} X_{\mathcal{P}}^{\mathrm{gen},\chi}[1/p],$$

where $X_{\mathcal{P}}^{\mathrm{gen},\chi} := X^{\mathrm{gen},\chi} \cap X_{\mathcal{P}}^{\mathrm{gen}}$.

If either $\overline{\rho}$ is absolutely irreducible, $K \neq \mathbb{Q}_p$ or $d > 2$ then it follows from Proposition 4.2.7.8 that $X^{\mathrm{gen},\chi}[1/p]$ is regular in codimension 1.

If $\overline{\rho}$ is reducible, $K = \mathbb{Q}_p$ and $d = 2$ then there are two partitions $\mathcal{P}_{\min}$ and $\mathcal{P}_{\max}$ and $\dim X^{\mathrm{gen},\chi}[1/p] - \dim X_{\mathcal{P}_{\max}}^{\mathrm{gen},\chi}[1/p] = 1$, so the previous argument does not work. If $x \in X^{\mathrm{gen},\chi}[1/p]$ is a closed singular point then it follows from Proposition 4.3.2.3 and Lemma 4.3.2.4 that ρ_x is reducible and its semi-simplification has the form $\psi \oplus \psi(1)$ for some character $\psi : G_K \to \kappa(x)^\times$. It follows from Lemma 4.2.3.7 that the dimension of this locus is at most $\dim Z_{\mathcal{P}_{\max}}^{12}$, which is at most 6 by Proposition 4.2.4.3. It follows from Corollary 4.3.1.4 that $\dim X^{\mathrm{gen},\chi}[1/p] = \dim \overline{X}^{\mathrm{gen},\chi} = 8$. Thus the codimension of the singular locus in $X^{\mathrm{gen},\chi}[1/p]$ is at least 2. □

Corollary 4.3.3.5. *$X^{\mathrm{gen},\chi}$ is normal.*

Proof. Since $A^{\mathrm{gen},\chi}$ is $\mathcal{O}$-torsion free by Corollary 4.3.1.4, the map $\mathcal{O} \to A^{\mathrm{gen},\chi}$ is flat. We have shown in Proposition 4.3.3.1 and Proposition 4.3.3.4 that the fibre rings $L \otimes_{\mathcal{O}} A^{\mathrm{gen},\chi}$ and $k \otimes_{\mathcal{O}} A^{\mathrm{gen},\chi}$ are normal. Since $\mathcal{O}$ is a regular ring [BH93, Corollary 2.2.23] implies that $A^{\mathrm{gen},\chi}$ is normal. □

Corollary 4.3.3.6. *For each character $\chi : \mu_{p^\infty}(K) \to \mathcal{O}^\times$ and $\overline{\rho} : G_K \to \mathrm{GL}_d(k)$ the ring $R_{\overline{\rho}}^{\square,\chi}$ is a normal integral domain.*

Proof. The proof is the same as the proof of Corollary 4.3.3.2 using Corollary 4.3.3.5. □

Lemma 4.3.3.7. *Let W be a closed subscheme of* $\operatorname{Spec} R^{\square,\chi}_{\overline{\rho}}[1/p]$ *with the property that* $H^2(G_K, \operatorname{ad}^0 \rho_x) \neq 0$ *for all closed points $x \in W$. Then* $\dim R^{\square,\chi}_{\overline{\rho}}[1/p] - \dim W \geq 2$.

Proof. Since in characteristic zero $\operatorname{ad}^0 \rho_x$ is a direct summand of $\operatorname{ad} \rho_x$ we obtain that $H^2(G_K, \operatorname{ad} \rho_x) \neq 0$ for all $x \in W$. This implies that W is contained in the singular locus of $\operatorname{Spec} R^{\square,\chi}_{\overline{\rho}}[1/p]$. Since $R^{\square,\chi}_{\overline{\rho}}[1/p]$ is normal, the singular locus has codimension of at least 2. □

The next result answers affirmatively a question raised by Böckle–Juschka in [BJ15].

Corollary 4.3.3.8. *The map $R_{\det \overline{\rho}} \to R^{\square}_{\overline{\rho}}$ induces a bijection between the sets of irreducible components.*

Proof. Since $R^{\square}_{\overline{\rho}}$ is $\mathcal{O}$-torsion free by Corollary 4.2.5.7, the irreducible components of $R^{\square}_{\overline{\rho}}$ and $R^{\square}_{\overline{\rho}}[1/p]$ coincide. Since the algebra $\mathcal{O}[\mu_{p^\infty}(K)][1/p]$ is semi-simple, we have

$$R^{\square}_{\overline{\rho}}[1/p] \cong \prod_{\chi:\mu_{p^\infty}(K)\to\mathcal{O}^\times} R^{\square,\chi}_{\overline{\rho}}[1/p]. \tag{4.24}$$

It follows from Corollary 4.3.1.3 and Corollary 4.3.3.6 that $R^{\square,\chi}_{\overline{\rho}}$ is $\mathcal{O}$-torsion free integral domain. We note that the special fibres of these rings are non-zero, thus the rings themselves are non-zero. Hence, the localization $R^{\square,\chi}_{\overline{\rho}}[1/p]$ is non-zero and is an integral domain. □

Corollary 4.3.3.9. $R^{\square}_{\overline{\rho}}[1/p]$ *is normal.*

Proof. This follows from (4.24) and Corollary 4.3.3.6. □

4.4 Deformation rings with fixed determinant

Let $\overline{\rho} : G_K \to \mathrm{GL}_d(k)$ be a representation with pseudorepresentation $\overline{D}$ and let $\psi : G_K \to \mathcal{O}^\times$ be a character lifting $\det \overline{\rho} = \det \overline{D}$. Let

$$R^{\square,\psi}_{\overline{\rho}} := R^{\square}_{\overline{\rho}} \otimes_{R_{\det \overline{\rho},\psi}} \mathcal{O}.$$

Let $\mu := \mu_{p^\infty}(K)$ and let $\chi : \mu \to \mathcal{O}^\times$ be a character such that the restriction of ψ to μ under the Artin map $\mu \to G_K^{\mathrm{ab}}$ from local class field theory is equal to χ. Then $R^{\square,\psi}_{\overline{\rho}}$ is a quotient of the ring $R^{\square,\chi}_{\overline{\rho}}$ considered in the previous section. We let $X^{\square,\chi} = \operatorname{Spec} R^{\square,\chi}_{\overline{\rho}}$, $X^{\square,\psi} = \operatorname{Spec} R^{\square,\psi}_{\overline{\rho}}$ and denote by $\overline{X}^{\square,\chi}$ and $\overline{X}^{\square,\psi}$ their special fibres.

Let $\mathcal{X} : \mathsf{Art}_{\mathcal{O}} \to \mathsf{Set}$ be the functor, which sends $(A, \mathfrak{m}_A)$ to the group $\mathcal{X}(A)$ of continuous characters $\theta : G_K \to 1 + \mathfrak{m}_A$ whose restriction to μ under the Artin map is trivial. It follows from Lemma 4.3.0.1 that the functor $\mathcal{X}$ is pro-represented by

$$\mathcal{O}(\mathcal{X}) \cong R_{\mathbf{1}} \otimes_{\mathcal{O}[\mu]} \mathcal{O} \cong \mathcal{O}[\![y_1, \dots, y_{[K:\mathbb{Q}_p]+1}]\!]. \tag{4.25}$$

For $e \in \mathbb{N}$ let $\varphi_e : \mathcal{X} \to \mathcal{X}$ be the natural transformation that sends $\theta \in \mathcal{X}(A)$ to θ^e. We also write φ_e for the induced maps $\mathcal{O}(\mathcal{X}) \to \mathcal{O}(\mathcal{X})$ and $\operatorname{Spec} \mathcal{O}(\mathcal{X}) \to \operatorname{Spec} \mathcal{O}(\mathcal{X})$. The natural transformation $D^{\square,\chi}_{\overline{\rho}} \to \mathcal{X}$, $\rho \mapsto (\det \rho)\psi^{-1}$ induces a homomorphism of local $\mathcal{O}$-algebras $\mathcal{O}(\mathcal{X}) \to R^{\square,\chi}_{\overline{\rho}}$; we will consider $R^{\square,\chi}_{\overline{\rho}}$ as $\mathcal{O}(\mathcal{X})$-algebra via this map in the statements below.

Proposition 4.4.0.1. *One has a natural isomorphism of functors*

$$D^{\square,\chi}_{\overline{\rho}} \times_{\mathcal{X},\varphi_d} \mathcal{X} \cong D^{\square,\psi}_{\overline{\rho}} \times \mathcal{X}.$$

Proof. Let $(A, \mathfrak{m}_A)$ be in $\mathsf{Art}_{\mathcal{O}}$. An element in $(D^{\square,\chi}_{\overline{\rho}} \times_{\mathcal{X},\varphi_d} \mathcal{X})(A)$ is a pair (ρ, θ) such that

$\theta : G_K \to 1 + \mathfrak{m}_A$ is a continuous homomorphism that is trivial on μ, $\rho : G_K \to \mathrm{GL}_d(A)$ is a continuous homomorphism such that $\det \rho$ and χ agree when restricted to μ, and one has $(\det \rho)\psi^{-1} = \theta^d$. An element in $(D_{\overline{\rho}}^{\square,\psi} \times \mathcal{X})(A)$ is a pair (ρ_1, θ_1) where $\theta_1 : G_K \to 1 + \mathfrak{m}_A$ is a continuous homomorphism that is trivial on μ and $\rho_1 : G_K \to \mathrm{GL}_d(A)$ is a continuous homomorphism such that $\det \rho_1 = \psi$. One verifies that the map

$$(\rho, \theta) \mapsto (\rho \cdot \theta^{-1}, \theta)$$

defines a bijection that is natural in A. □

Corollary 4.4.0.2. *Proposition 4.4.0.1 induces a natural isomorphism*

$$R_{\overline{\rho}}^{\square,\chi} \otimes_{\mathcal{O}(\mathcal{X}),\varphi_d} \mathcal{O}(\mathcal{X}) \cong R_{\overline{\rho}}^{\square,\psi} \widehat{\otimes}_{\mathcal{O}} \mathcal{O}(\mathcal{X}).$$

We now clarify some properties of the map $\varphi_d : \mathcal{O}(\mathcal{X}) \to \mathcal{O}(\mathcal{X})$.

Lemma 4.4.0.3. *The map φ_d is finite and flat and becomes étale after inverting p. Moreover, it induces a universal homeomorphism on the special fibres.*

Proof. We may write $d = ep^m$, such that p does not divide e. Then $\varphi_d = \varphi_{p^m} \circ \varphi_e$. Since e is prime to p, elements in $1 + \mathfrak{m}_A$ for $(A, \mathfrak{m}_A)$ in $\mathsf{Art}_{\mathcal{O}}$ possess a unique e-th root in $1 + \mathfrak{m}_A$ by the binomial theorem, and it follows that φ_e is an isomorphism. We thus may assume that d is a power of p.

The map $\varphi_d : \mathcal{O}(\mathcal{X}) \to \mathcal{O}(\mathcal{X})$ sends y_i to $(1 + y_i)^d - 1$. One checks that the monomials $\prod_{i=1}^{[K:\mathbb{Q}_p]+1} y_i^{m_i}$ with $0 \leq m_i \leq d-1$ form a basis of $\mathcal{O}(\mathcal{X})$ as $\mathcal{O}(\mathcal{X})$-module via φ_d, by checking the assertion modulo ϖ and using Nakayama's lemma. A (standard) calculation shows that the discriminant is a power of p up to a sign. Thus φ_d becomes étale after inverting p.

The map $\overline{\varphi}_d : \mathcal{O}(\mathcal{X})/\varpi \to \mathcal{O}(\mathcal{X})/\varpi$ is a power of the relative Frobenius of $\operatorname{Spec}(\mathcal{O}(\mathcal{X})/\varpi)/\operatorname{Spec} k$. The last assertion follows from [Stacks, Tag 0CCB]. $\square$

In the following results we deduce properties of the ring $R_{\overline{\rho}}^{\square,\psi}$.

Corollary 4.4.0.4. *The following hold:*

1. *$R_{\overline{\rho}}^{\square,\psi}$ is a local complete intersection, flat over $\mathcal{O}$ and of relative dimension $(d^2-1)([K:\mathbb{Q}_p]+1)$.*
2. *$R_{\overline{\rho}}^{\square,\psi}/\varpi$ is a local complete intersection of dimension $(d^2-1)([K:\mathbb{Q}_p]+1)$.*

Proof. The pushout of the isomorphism from Proposition 4.3.1.1 under $R_{\det\overline{\rho}} \to \mathcal{O}$, which corresponds to ψ, gives an isomorphism

$$\mathcal{O}[\![x_1,\dots,x_r]\!]/(f_1,\dots,f_t) \xrightarrow{\simeq} R_{\overline{\rho}}^{\square,\psi}$$

with $r-t=(d^2-1)([K:\mathbb{Q}_p]+1)$. To prove (1) and (2) it thus suffices to show that the dimension of $R_{\overline{\rho}}^{\square,\psi}/\varpi$ is at most $(d^2-1)([K:\mathbb{Q}_p]+1)$, or equivalently, see (4.25), it suffices to show that

$$\dim\Big((R_{\overline{\rho}}^{\square,\psi}\,\widehat{\otimes}_{\mathcal{O}}\,\mathcal{O}(\mathcal{X}))/\varpi\Big) \le d^2([K:\mathbb{Q}_p]+1). \tag{4.26}$$

Let us write $\overline{\mathcal{X}} := \operatorname{Spec}\mathcal{O}(\mathcal{X})/\varpi$. Since $\overline{\varphi}_d : \overline{\mathcal{X}} \to \overline{\mathcal{X}}$ is a universal homeomorphism by Lemma 4.4.0.3 the map

$$\overline{X}^{\square,\chi} \times_{\overline{\mathcal{X}},\overline{\varphi}_d} \overline{\mathcal{X}} \to \overline{X}^{\square,\chi} \tag{4.27}$$

is a homeomorphism. In particular, the spaces have the same dimension, which is equal to $d^2([K:\mathbb{Q}_p]+1)$ by Corollary 4.3.1.3. We conclude using Corollary 4.4.0.2 that (4.26) is an equality. $\square$

Lemma 4.4.0.5. *Let κ be either a finite or local field of characteristic p or a finite extension* *of L and let $\rho : G_K \to \mathrm{GL}_d(\kappa)$ be a representation, such that $\det \rho = \psi$ and $H^2(G_K, \mathrm{ad}^0 \rho) = 0$, where $\mathrm{ad}^0 \rho$ is the kernel of the trace map. Then the ring $R_\rho^{\square,\psi}$ is formally smooth over Λ with Λ as in Section 4.2.5.*

Proof. This is the same proof as the proof of Lemma 4.3.2.1. □

Theorem 4.4.0.6. *The rings $R_{\overline{\rho}}^{\square,\psi}$ and $R_{\overline{\rho}}^{\square,\psi}/\varpi$ are normal integral domains.*

Proof. We will first prove that $R_{\overline{\rho}}^{\square,\psi}/\varpi$ is normal. Since $R_{\overline{\rho}}^{\square,\psi}/\varpi$ is complete intersection by Corollary 4.4.0.4, it suffices to show that $R_{\overline{\rho}}^{\square,\psi}/\varpi$ satisfies Serre's condition (R1). Let $\mathfrak{p} \in \overline{X}^{\square,\psi} := \operatorname{Spec} R_{\overline{\rho}}^{\square,\psi}/\varpi$ be a point of height at most 1 and assume that the local ring at $\mathfrak{p}$ is not regular. Then by Lemma 4.4.0.5 there is a closed irreducible subset Z of $\overline{X}^{\square,\psi}$ of codimension at most 1, the closure of $\mathfrak{p}$, such that for all $z \in Z$ with finite or local residue field the space $H^2(G_K, \mathrm{ad}^0 \rho_z)$ is non-zero. Using the explicit bijection from the proof of Proposition 4.4.0.1, and the isomorphism of Corollary 4.4.0.2 modulo ϖ it follows that there is a closed irreducible subset $W \subset \overline{X}^{\square,\chi} \times_{\overline{\mathcal{X}},\overline{\varphi}_d} \overline{\mathcal{X}}$ of codimension at most 1, such that for all $w \in W$ with finite or local residue field the space $H^2(G_K, \mathrm{ad}^0 \rho_w)$ is non-zero, where, as in the proof of Proposition 4.4.0.1, the point w corresponds to a pair (ρ_w, θ_w). Since the map (4.27) is a homeomorphism and sends (ρ_w, θ_w) to ρ_w, the image of W in $\overline{X}^{\square,\chi}$, which we denote by W', is closed irreducible of codimension at most 1 in $\overline{X}^{\square,\chi}$ and all $x \in W'$ with finite or local residue field have non-vanishing $H^2(G_K, \mathrm{ad}^0 \rho_x)$. Lemma 4.3.3.3 implies that the codimension of W' is at least 2 yielding a contradiction.

Let us prove that $R_{\overline{\rho}}^{\square,\psi}$ is normal. Since $R_{\overline{\rho}}^{\square,\psi}$ is $\mathcal{O}$-torsion free by Corollary 4.4.0.4 and we know that the special fibre is normal, it is enough to prove that $R_{\overline{\rho}}^{\square,\psi}[1/p]$ is normal, see the proof of Corollary 4.3.3.5. Lemma 4.4.0.3 implies that the map

$$X^{\square,\chi}[1/p] \times_{\mathcal{X}[1/p],\varphi_d} \mathcal{X}[1/p] \to X^{\square,\chi}[1/p] \tag{4.28}$$

is finite étale. We proceed exactly as in the proof for the special fibre, using (4.28) instead of (4.27) and Lemma 4.3.3.7 instead of Lemma 4.3.3.3. ☐

Corollary 4.4.0.7. *The absolutely irreducible locus is dense in* $\operatorname{Spec} R^{\square,\psi}_{\overline{\rho}}[1/p]$ *and the Kummer-irreducible locus is dense in* $\operatorname{Spec} R^{\square,\psi}_{\overline{\rho}}/\varpi$*.*

Proof. By Proposition 4.2.7.5 and Corollary 4.2.7.9 the absolutely irreducible locus is dense open in $\operatorname{Spec} R^{\square,\chi}_{\overline{\rho}}/\varpi$ and in $\operatorname{Spec} R^{\square,\chi}_{\overline{\rho}}[1/p]$. Arguing as in the proof of Theorem 4.4.0.6 one deduces that the absolutely irreducible locus is dense open in the spaces $\operatorname{Spec} R^{\square,\psi}_{\overline{\rho}}/\varpi$ and $\operatorname{Spec} R^{\square,\psi}_{\overline{\rho}}[1/p]$. For absolutely irreducible $x \in \operatorname{Spec} R^{\square,\chi}_{\overline{\rho}}/\varpi$ Kummer-irreducibility implies $H^2(G_K, \operatorname{ad}^0 \rho_x) = 0$, so the assertion on the density of the Kummer-irreducible locus in $\operatorname{Spec} R^{\square,\chi}_{\overline{\rho}}/\varpi$ follows from the proof of Theorem 4.4.0.6. ☐

As explained in Section 4.3 both R^{ps} and A^{gen} are naturally $R_{\det \overline{D}}$-algebras. Moreover, $\det \overline{D} = \det \overline{\rho}$. We let

$$R^{\mathrm{ps},\psi} := R^{\mathrm{ps}} \otimes_{R_{\det \overline{D}},\psi} \mathcal{O}, \quad A^{\mathrm{gen},\psi} := A^{\mathrm{gen}} \otimes_{R_{\det \overline{D}},\psi} \mathcal{O}.$$

Corollary 4.4.0.8. *The following hold:*

1. $A^{\mathrm{gen},\psi}$ *is* $\mathcal{O}$*-flat, equi-dimensional of dimension* $1+(d^2-1)([K:\mathbb{Q}_p]+1)$*, normal and is locally complete intersection;*

2. $A^{\mathrm{gen},\psi}/\varpi$ *is equi-dimensional of dimension* $(d^2-1)([K:\mathbb{Q}_p]+1)$*, normal, and is locally complete intersection.*

Proof. The ring A^{gen} is excellent, since it is a finitely generated over a complete local Noetherian ring. Thus its local rings are also excellent. An excellent local ring is normal if and only if its completion with respect to the maximal ideal is normal, [Mat89, Theorem 32.2 (i)]. Given this the proof is the same as the proof of Corollary 4.3.1.4 using Theorem 4.4.0.6. □

Proposition 4.4.0.9. *The map*

$$R_{\det \overline{\rho}} \to R_{\overline{\rho}}^{\square} \tag{4.29}$$

is flat.

Proof. As in the proof of Proposition 4.3.1.1 let $S := \mathcal{O}[\![z, x_1, \ldots, x_{1+[K:\mathbb{Q}_p]}]\!]$ then we may choose presentations

$$R_{\det \overline{\rho}} \cong S/((1+z)^m - 1), \quad R_{\overline{\rho}}^{\square} \cong S[\![y_1, \ldots, y_r]\!]/((1+z)^m - 1, f_1, \ldots, f_t),$$

such that (4.29) is a map of S-algebras and $(1+z)^m - 1, f_1, \ldots, f_t$ is a regular sequence in $S[\![y_1, \ldots, y_r]\!]$. Let $S' := S[\![y_1, \ldots, y_r]\!]/(f_1, \ldots, f_t)$. Then S' is complete intersection, and hence Cohen–Macaulay, and the fibre ring $k \otimes_S S'$ is isomorphic to $R_{\overline{\rho}}^{\square,\psi}/\varpi$ (with ψ the trivial character), which has dimension equal to $\dim R_{\overline{\rho}}^{\square} - \dim R_{\det \rho} = \dim S' - \dim S$, by Corollary 4.4.0.4. Since S is regular, the fibre-wise criterion for flatness, [Mat89, Theorem 23.1], implies that S' is flat over S. Hence, $R_{\overline{\rho}}^{\square} \cong S'/((1+z)^m - 1)$ is flat over $R_{\det \overline{\rho}} \cong S/((1+z)^m - 1)$. □

4.A Kummer-irreducible points

The purpose of the appendix is to slightly generalize the notion of non-special points in $\overline{X}^{\mathrm{ps}} = \operatorname{Spec} R^{\mathrm{ps}}/\varpi$ in [BJ19, Definition 5.1.2]. We use the notation of the main text. If $x \in \overline{X}^{\mathrm{ps}}$ then we let $D_x = D^u \otimes_{R^{\mathrm{ps}}} \overline{\kappa(x)}$, where $\overline{\kappa(x)}$ is an algebraic closure of the residue field at x, and we let $\rho_x : G_K \to \mathrm{GL}_d(\overline{\kappa(x)})$ be the semisimple representation whose determinant is D_x.

Definition 4.A.0.1. We say that $x \in P_1(R^{\mathrm{ps}}/\varpi)$ is *Kummer-irreducible* if the restriction of D_x to $G_{K'}$ is absolutely irreducible for all degree p Galois extensions K' of $K(\zeta_p)$. Otherwise, we say that x is *Kummer-reducible.*

Thus x is Kummer-irreducible if and only if $\rho_x|_{G_{K(\zeta_p)}}$ is non-special in the sense [BJ19, Definition 5.1.2]. In particular, if $\zeta_p \in K$ then both notions coincide. Our main interest in this notion is the following Lemma.

Lemma 4.A.0.2. *If x is Kummer-irreducible then $H^2(G_K, \mathrm{ad}^0 \rho_x) = 0$.*

Proof. Since the order of $\mathrm{Gal}(K(\zeta_p)/K)$ is prime to p we have

$$H^2(G_K, \mathrm{ad}^0 \rho_x) \cong H^2(G_{K(\zeta_p)}, \mathrm{ad}^0 \rho_x)^{\mathrm{Gal}(K(\zeta_p)/K)}.$$

Since x is Kummer-irreducible, the restriction of ρ_x to $G_{K(\zeta_p)}$ is non-special, and it follows from [BJ19, Lemma 5.1.1] that $H^2(G_{K(\zeta_p)}, \mathrm{ad}^0 \rho_x) = 0$. $\square$

We need the following result from Clifford theory.

Lemma 4.A.0.3. *Let G be a group and $H \subset G$ a normal subgroup such that G/H is cyclic of order m. Let κ be an algebraically closed field with $\mathrm{char}(\kappa)$ not dividing m. Let V be a finite dimensional irreducible representation of G over κ. Suppose the restriction $\mathrm{Res}_H^G V$ of V to H is reducible. Then there exists a proper subgroup H^* of G that contains H and an irreducible representation W of H^* such that $V \cong \mathrm{Ind}_{H^*}^G W$.*

Proof. Let X be the group of characters $\chi : G/H \to \kappa^\times$. We have

$$\mathrm{Ind}_H^G \mathrm{Res}_H^G V \cong V \otimes_\kappa \mathrm{Ind}_H^G \mathbf{1} \cong \bigoplus_{\chi \in X} V \otimes \chi.$$

Since $\mathrm{Res}_H^G V$ is semi-simple by [BJ19, Lemma 2.1.4 (d)] and reducible by assumption

$$\dim_\kappa \mathrm{Hom}_H(\mathrm{Res}_H^G V, \mathrm{Res}_H^G V) > 1.$$

Frobenius reciprocity implies that $V \cong V \otimes \chi$ for some non-trivial $\chi \in X$. The lemma now follows from [BJ19, Theorem 2.2.1] with $H^* = \ker(\chi)$. □

Let $\overline{X}^{\mathrm{ps,irr}} \subset \overline{X}^{\mathrm{ps}} \setminus \{\mathfrak{m}_{R^{\mathrm{ps}}}\}$ be the absolutely irreducible locus; it is denoted by $U_{\mathcal{P}_{\min}}$ in Section 4.2.7.

Lemma 4.A.0.4. *Let $X^{\mathrm{Kred}}_{K(\zeta_p)} \subset \overline{X}^{\mathrm{ps,irr}}$ be the closure of all $x \in \overline{X}^{\mathrm{ps,irr}} \cap P_1(R^{\mathrm{ps}}/\varpi)$ such that $D_x|_{G_{K(\zeta_p)}}$ is reducible. Then $\dim \overline{X}^{\mathrm{ps,irr}} - \dim X^{\mathrm{Kred}}_{K(\zeta_p)} \geq \frac{1}{2}d^2[K:\mathbb{Q}_p] \geq 2$.*

Proof. Let $y \in X^{\mathrm{Kred}}_{K(\zeta_p)}$ and let $\mathfrak{p}_y$ be the kernel of $R^{\mathrm{ps}} \to \kappa(y)$. As a byproduct of the proof of Lemma 4.2.4.1 we obtain a description of the universal object of over $\overline{X}^{\mathrm{ps}}_{\mathcal{P}}$, which shows that $D_y|_{G_{K(\zeta_p)}}$ is reducible. Since y lies in the absolutely irreducible locus D_y and ρ_y are irreducible. Now Lemma 4.A.0.3 yields a subextension K' of $K(\zeta_p)/K$ such that ρ_y is induced from $G_{K'}$ and $[K':K] > 1$. It follows from [BJ19, Lemma 5.3.2] and the proof of [BJ19, Theorem 5.3.1] that

$$\dim R^{\mathrm{ps}}/\mathfrak{p}_y \leq \frac{1}{[K':K]} d^2[K:\mathbb{Q}_p] + 1.$$

This yields $\dim X^{\mathrm{Kred}}_{K(\zeta_p)} \leq d^2[K:\mathbb{Q}_p]$ by applying the above to the generic points y of $X^{\mathrm{Kred}}_{K(\zeta_p)}$ and observing that $\mathfrak{m}_{R^{\mathrm{ps}}} \notin X^{\mathrm{Kred}}_{K(\zeta_p)}$. We also have $\dim \overline{X}^{\mathrm{ps,irr}} = [K:\mathbb{Q}_p]d^2$ by [BJ19, Theorem 5.4.1]. Since $[K':K] \geq 2$ we obtain the assertion. □

Remark 4.A.0.5. Let $K' \supset K$ be finite. Arguing as in the proof of [BJ19, Lemma 5.1.3], one finds that $Z_{K'} = \{x \in \overline{X}^{\mathrm{ps,irr}} : D_x|_{G_{K'}} \text{ is reducible}\}$ is a closed subset of $\overline{X}^{\mathrm{ps,irr}}$. As $\overline{X}^{\mathrm{ps}} \setminus \{\mathfrak{m}_{R^{\mathrm{ps}}}\}$ is Jacobson, one has $X^{\mathrm{Kred}}_{K(\zeta_p)} = Z_{K(\zeta_p)}$. This gives a direct definition of $X^{\mathrm{Kred}}_{K(\zeta_p)}$

avoiding a closure operation.

Lemma 4.A.0.6. *Let $E \subset \overline{K}$ be a finite extension of K, denote by R_E^{ps} the universal ring for deformations of the pseudorepresentation $\overline{D}|_{G_E}$, and let $r : R_E^{\mathrm{ps}} \to R^{\mathrm{ps}}$ be the ring map induced from the morphism of deformation functors that restricts a pseudorepresentation of G_K to one of G_E. Then $r : R_E^{\mathrm{ps}} \to R^{\mathrm{ps}}$ is finite.*

Proof. The map r is a local homomorphism of local rings with residue field k. So we need to show that $S = R^{\mathrm{ps}}/\mathfrak{m}_{R_E^{\mathrm{ps}}} R^{\mathrm{ps}}$ has finite k-dimension, which amounts to showing $\operatorname{Spec} S = \{\mathfrak{m}_S\}$. We note that S represents the functor of deformations $D : G_K \to A$ of $\overline{D}$ to k-algebras A such that $D|_{G_E} = \overline{D}|_{G_E} \otimes_k A$.

Let y be any point of $\operatorname{Spec} S$ with associated pseudorepresentation D_y and semisimple representation $\rho_y : G_K \to \mathrm{GL}_d(\overline{\kappa(y)})$. The restriction $\rho_y|_{G_E}$ is semisimple, cf. [BJ19, Lemma 2.1.4], and its associated pseudorepresentation is $\overline{D}|_{G_E} \otimes_k \overline{\kappa(y)}$, so that $\rho_y(G_E)$ is finite. Hence $\rho_y(G_K)$ is finite, and therefore D_y is defined over a finite field $k' \supset k$. This shows that the corresponding ring map $S \to \kappa(y)$ factors via k', and thus its kernel y is the maximal ideal $\mathfrak{m}_S$. □

Lemma 4.A.0.7. *For a degree p Galois extension K' of $K(\zeta_p)$ let $X_{K'}^{\mathrm{Kred}} \subset \overline{X}^{\mathrm{ps,irr}}$ be the closure of the set of $x \in \overline{X}^{\mathrm{ps,irr}} \cap P_1(R^{\mathrm{ps}}/\varpi)$ such that $D_x|_{G_{K'}}$ is reducible. Then*

$$\dim \overline{X}^{\mathrm{ps,irr}} - \dim \Big(\bigcup\nolimits_{K'} X_{K'}^{\mathrm{Kred}} \Big) \geq d[K : \mathbb{Q}_p] \geq 2,$$

where the union is over all degree p Galois extension K' of $K(\zeta_p)$.

Proof. Let $r : R_{K(\zeta_p)}^{\mathrm{ps}} \to R^{\mathrm{ps}}$ be the ring map from Lemma 4.A.0.6 with $E = K(\zeta_p)$ and denote by $\overline{X}_{K(\zeta_p)}^{\mathrm{spcl}} \subset \operatorname{Spec}\big(R_{K(\zeta_p)}^{\mathrm{ps}}/\varpi\big) \setminus \{\mathfrak{m}_{R_{K(\zeta_p)}^{\mathrm{ps}}}\}$ the locus of special points as defined in

[BJ19, Definition 5.1.2]. Then by Lemma 4.A.0.6 the morphism

$$\operatorname{Spec}(r/\varpi) : \operatorname{Spec} R^{\mathrm{ps}}/\varpi \to \operatorname{Spec} R^{\mathrm{ps}}_{K(\zeta_p)}/\varpi$$

is finite and hence so is the induced morphism $\left(\bigcup_{K'} X^{\mathrm{Kred}}_{K'}\right) \setminus X^{\mathrm{Kred}}_{K(\zeta_p)} \to \overline{X}^{\mathrm{spcl}}_{K(\zeta_p)}$. We deduce

$$\dim\left(\left(\bigcup_{K'} X^{\mathrm{Kred}}_{K'}\right) \setminus X^{\mathrm{Kred}}_{K(\zeta_p)}\right) \leq \dim \overline{X}^{\mathrm{spcl}}_{K(\zeta_p)},$$

from [Stacks, Tag 01WG]. Moreover by the proof of [BJ19, Theorem 5.3.1] we have

$$\dim \overline{X}^{\mathrm{spcl}}_{K(\zeta_p)} \leq (d/p)^2[K' : \mathbb{Q}_p] = \frac{[K(\zeta_p) : K]}{p} d^2[K : \mathbb{Q}_p],$$

and [BJ19, Theorem 5.4.1] gives $\dim \overline{X}^{\mathrm{ps,irr}} = d^2[K : \mathbb{Q}_p]$. Since p divides d and $[K(\zeta_p) : K] \leq p-1$, using the bound from Lemma 4.A.0.4 we conclude that

$$\dim \overline{X}^{\mathrm{ps,irr}} - \dim\left(\bigcup_{K'} X^{\mathrm{Kred}}_{K'}\right) \geq \frac{d}{p} d[K : \mathbb{Q}_p] \geq d[K : \mathbb{Q}_p] \geq 2.$$

□

Proposition 4.A.0.8. *There exists an open dense subscheme $U^{\mathrm{Kirr}} \subset \overline{X}^{\mathrm{ps,irr}}$ such that $x \in P_1(R^{\mathrm{ps}}/\varpi)$ is Kummer-irreducible if and only if x is a closed point in U^{Kirr}. Moreover, $\dim \overline{X}^{\mathrm{ps,irr}} - \dim(\overline{X}^{\mathrm{ps,irr}} \setminus U^{\mathrm{Kirr}}) \geq d[K : \mathbb{Q}_p] \geq 2$.*

Proof. Let $U^{\mathrm{Kirr}} = \overline{X}^{\mathrm{ps,irr}} \setminus (\bigcup_{K'} X^{\mathrm{Kred}}_{K'})$, where the union is taken over all degree p Galois extensions K' of $K(\zeta_p)$. Since there are only finitely many such extensions, U^{Kirr} is open. Remark 4.A.0.5 implies that its closed points are precisely the Kummer-irreducible points. Its complement in $\overline{X}^{\mathrm{ps,irr}}$ has codimension at least $d[K : \mathbb{Q}_p]$ by Lemma 4.A.0.7. Since $\overline{X}^{\mathrm{ps,irr}}$ is equi-dimensional by [BJ19, Theorem 5.4.1] this implies density. □

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