The equivariant Tamagawa Number Conjecture for abelian extensions of imaginary quadratic fields

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We prove the Iwasawa-theoretic version of a conjecture of Mazur–Rubin and Sano in the case of elliptic units. This allows us to derive the p-part of the equivariant Tamagawa Number Conjecture at $s = 0$ for abelian extensions of imaginary quadratic fields in the semi-simple case and, provided that a standard μ-vanishing hypothesis is satisfied, also in the general case.

1 Introduction

The *equivariant Tamagawa Number Conjecture* (eTNC for short) as formulated by Burns and Flach [BF01] (building on earlier work of Kato [Kat93a], [Kat93b] and, independently, Fontaine and Perrin-Riou [FPR94]) is an equivariant refinement of the seminal Tamagawa Number Conjecture of Bloch and Kato [BK90]. It both unifies and refines a great variety of conjectures related to special values of motivic L-functions such as Stark's conjectures, the Birch and Swinnerton-Dyer Conjecture, and the central conjectures of classical Galois module theory (see [Bur07], [Kin11], [Bur01] for more details).
Already Bloch and Kato have deduced cases of their Tamagawa Number Conjecture from the Iwasawa Main Conjecture of Mazur and Wiles [MW84] in the original article [BK90]. However, the necessary descent calculations are particularly involved in cases where the associated p-adic L-function posseses so-called *trivial zeroes*. To handle such cases, Burns and Greither developed in [BG03] a descent machinery that, together with additional arguments for the 2-part by Flach [Fla11], lead to a proof of the eTNC for the Tate motive $(h^0(\operatorname{Spec} K), \mathbb{Z}[\operatorname{Gal}(K/\mathbb{Q})])$ if K is a totally abelian number field (see also the article of Huber and Kings [HK03] for a different proof strategy). The descent formalism of Burns–Greither uses the vanishing of certain Iwasawa μ-invariants, the known validity of the Gross–Kuz'min conjecture in this setting, and a result of Solomon [Sol92] as crucial ingredients. Bley [Ble06] later proved partial results for K an abelian extension of an imaginary quadratic field using the same strategy and an analogue [Ble04] for elliptic units of Solomon's result.
In [BKS17] Burns, Kurihara and Sano showed that an Iwasawa-theoretic version of a conjecture proposed by Mazur–Rubin [MR16] and, independently, Sano [San14] constitutes an appropriate conjectural generalisation of the aforementioned result of Solomon's and therefore allows one to extend the Burns–Greither descent formalism to a general strategy for proving $\mathrm{eTNC}(h^0(\operatorname{Spec} K), \mathbb{Z}[\operatorname{Gal}(K/k)])$ with K/k a finite abelian extension of number fields.

To describe the Iwasawa-theoretic Mazur–Rubin–Sano Conjecture in a little more detail, we let p be a prime number, K/k a finite abelian extension of number fields, and v a p-adic place of k that splits completely in K. If k_∞/k denotes a $\mathbb{Z}_p$-extension in which v is ramified, then the existence of the place v forces (p-truncated) Dirichlet L-series associated with characters of K/k to have a higher order of vanishing at $s = 0$ than L-series of characters that arise further up in the extension Kk_∞/k. As a consequence, the corresponding Rubin–Stark elements can not be related to one another in an immediate fashion. However, the Iwasawa-theoretic Mazur–Rubin–Sano Conjecture, formulated by Burns, Kurihara, and Sano in [BKS17], predicts

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that, after applying a certain 'twisted norm operator' and a regulator map defined in terms of local reciprocity maps, these Rubin–Stark elements satisfy a congruence modulo a power of the augmentation ideal of Kk_∞/K.
These congruences offer a conceptual approach to conjectures concerning trivial zeroes of p-adic L-functions. For example, they recover the Gross–Stark Conjecture resolved by Dasgupta, Kakde, and Ventullo in [DKV18] and also have consequences for the leading term of Katz's p-adic L-function (see [BS22]).

In the present article, we prove the following unconditional result.

Theorem A. *The Iwasawa-theoretic Mazur–Rubin–Sano Conjecture (3.11) holds if k is an imaginary quadratic field.*

As a first consequence of this result, one may extend the validity of the leading term formula for Katz's p-adic L-function that is proved by Büyükboduk and Sakamoto in [BS22, Thm. 1.6] to cases when p divides the class number of the imaginary quadratic base field k (cf. the discussion in Remark 2.10 (ii) of *loc. cit.*). In a similar manner Theorem A has also been used by Maksoud to study leading terms of p-adic L-functions in [Mak21, § 6.5].
As an important step in the proof of Theorem A we first prove the validity of the Iwasawa-theoretic Mazur–Rubin–Sano Conjecture for a wide class of $\mathbb{Z}_p$-extensions by building on results of Bley and the second author in [BH20]. We then combine this intermediate result, stated in Theorem (5.1), with the formalism of Burns, Kurihara and Sano [BKS17] to deduce the following result on the p-part of the eTNC from the relevant equivariant Iwasawa Main Conjecture. Since the eTNC is known to imply the Mazur–Rubin–Sano Conjecture, this then also completes the proof of Theorem A (cf. Corollary (5.9)).

Theorem B. *Let p be a prime number, k an imaginary quadratic field, and K/k a finite abelian extension.*

- *(a) If p is split in k, then* $\mathrm{eTNC}(h^0(\operatorname{Spec} K), \mathbb{Z}_p[\operatorname{Gal}(K/k)])$ *holds.*
- *(b) If p is not split in k, then* $\mathrm{eTNC}(h^0(\operatorname{Spec} K), \mathbb{Z}_p[\operatorname{Gal}(K/k)])$ *holds if $p \nmid [K : k]$ or the classical Iwasawa μ-invariant vanishes (see Theorem (5.8) for a more precise statement).*

In particular, $\mathrm{eTNC}(h^0(\operatorname{Spec}(K)), \mathbb{Z}[\operatorname{Gal}(K/k)])$ *holds if $[K : k]$ is a prime power or every prime factor of $[K : k]$ is split in k (see Corollary (5.10)).*

The first part of Theorem B generalises work of Bley [Ble06] which only covers prime numbers $p > 2$ that do not divide the class number of k. The second part of Theorem B grew out of the second presently named author's thesis [Hof18] and not only settles the descent problem in this previously widely open case but, in combination with the first part of Theorem B, also provides for a large supply of new examples in which the eTNC is valid unconditionally. More precisely, the condition on the class number imposed in [Ble06] meant that the unconditional validity of the eTNC was previously only known in certain cases where k is one of only nine imaginary quadratic fields of class number one. In contrast, Theorem B is devoid of any such restrictive hypotheses on the field k.
The assumption on the vanishing of a certain Iwasawa μ-invariant in Theorem B (b) has been conjectured always to hold by Iwasawa, and we refer the reader to Proposition (5.6) for a discussion of what is currently known towards this.

The proof of Theorem B (b) also requires the validity of an appropriate analogue of the Gross–Kuz'min conjecture which is labelled condition '(F)' in [BKS17, § 5A]. This condition concerns the finiteness of the coinvariants of p-class groups and, as we shall show by extending results known for the cyclotomic $\mathbb{Z}_p$-extension, is in general equivalent to asserting that a certain regulator map has full rank (cf. Theorem (4.6)). Questions regarding the validity of condition (F) are therefore very similar in spirit to Leopoldt's conjecture and can typically only be answered in a small number of classical cases (see Remark (4.7) for an overview of known results in this direction).

In this article, we prove the following result which seems to not have previously appeared in the literature (see Theorems (4.12) (b) and (4.14) for the full statements). We remark that Theorem C (a) is a generalisation of Gross's classical result on the minus part of the Gross–Kuz'min conjecture in the setting of CM extensions of totally real fields (cf. Remark (4.7)).

Theorem C. *Let K/k be an abelian extension of number fields, and let p be a prime number.*

(a) Let k_∞/k be a $\mathbb{Z}_p$-extension in which all infinite places split completely.
Suppose that, for each non-trivial character χ of $\mathrm{Gal}(K/k)$, there is at most one non-archimedean place of k that both ramifies in k_∞ and is such that its decomposition group is contained in the kernel of χ. If there is such a place, assume that the completion of k at this place is $\mathbb{Q}_p$ and that there is at least one archimedean place that splits completely in K/k.
Then the validity of condition (F) for the $\mathbb{Z}_p$-extension Kk_∞ of K is implied by the validity of condition (F) for the $\mathbb{Z}_p$-extension k_∞ of k.

(b) Let k be an imaginary quadratic field such that p is not split in k, and let m be the number of p-adic places of K. Then there are at most m distinct $\mathbb{Z}_p$-extension k_∞ of k such that condition (F) fails for the $\mathbb{Z}_p$-extensions Kk_∞ of K. In particular, there are infinitely many such k_∞ such that condition (F) holds for Kk_∞/k.

While Theorem C (b) proves the validity of condition (F) for all but finitely many $\mathbb{Z}_p$-extensions in the described setting, its method of proof does not imply condition (F) for any concrete extensions. To this end we remark that, after the first version of this article has appeared online, Maksoud has proved the Gross–Kuz'min Conjecture for finite abelian extensions of imaginary quadratic fields in [Mak22, Cor. 1.4]. This shows that, in this case, condition (F) is always satisfied for the cyclotomic $\mathbb{Z}_p$-extension.

The main contents of this article are as follows. In § 2 we recall the definition of Rubin–Stark elements and the Rubin–Stark Conjecture. In § 3 we begin by introducing the objects and notations in Iwasawa theory that will be used throughout most of the article, then discuss a useful family of Iwasawa cohomology complexes, recall the Iwasawa-theoretic Mazur–Rubin–Sano Conjecture, and give a more explicit reformulation of the conjecture. In § 4 we study the Gross–Kuz'min Conjecture and condition (F), and prove Theorem C. In § 5 we finally specialise to imaginary quadratic base fields and prove Theorems A and B.

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Notation *Arithmetic.* For any number field E we write $S_\infty(E)$ for the set of archimedean places of E, and $S_p(E)$ for the set of places of E lying above a fixed rational prime p. Given an extension F/E we write $S_{\mathrm{ram}}(F/E)$ for the places of E that ramify in F. If S is a set of places of E, we denote by S_F the set of places of F that lie above those contained in S. We will however omit the explicit reference to the field in case it is clear from the context. For example, $\mathcal{O}_{F,S}$ shall denote the ring of S_F-integers of F, and $U_{F,S} \coloneqq \mathbb{Z}_p \otimes_{\mathbb{Z}} \mathcal{O}_{F,S}^\times$ the p-completion of its group of units. Note that we will usually use additive notation, including for such a group of units. We also define $\mathcal{Y}_{F,S}$ to be the free abelian group on S_F and set

$$\mathcal{X}_{F,S} \coloneqq \Big\{ \sum_{w\in S_F} a_w w \in \mathcal{Y}_{F,S} \mid \sum_{w\in S_F} a_w = 0\Big\}.$$

The p-completions of $\mathcal{Y}_{F,S}$ and $\mathcal{X}_{F,S}$ will be denoted as $Y_{F,S} \coloneqq \mathbb{Z}_p \otimes_{\mathbb{Z}} \mathcal{Y}_{F,S}$ and $X_{F,S} \coloneqq \mathbb{Z}_p \otimes_{\mathbb{Z}} \mathcal{X}_{F,S}$, respectively. Furthermore, if T is a finite set of finite places disjoint from S, then we let $A_{F,S,T}$ be the p-part of the S_F-ray class group mod T_F, i.e. the p-Sylow subgroup of the quotient of the group of fractional ideals of $\mathcal{O}_{F,S}$ coprime to T_F by the subgroup of principal ideals with a generator congruent to 1 modulo all $w \in T_F$. If $S = S_\infty(E)$ or $T = \varnothing$, then we will suppress the respective set in the notation.
For any place w of F we write $\mathrm{ord}_w\colon F^\times \to \mathbb{Z}$ for the normalised valuation at w. In case of a finite extension H of $\mathbb{Q}_p$ we also write ord_H for the normalised valuation on H. If F/E is abelian and v unramified in F/E, then we let $\mathrm{Frob}_v \in \mathrm{Gal}(F/E)$ be the arithmetic Frobenius at v. If v is a finite place of E, then we denote by $\mathrm{N}v \coloneqq |\mathcal{O}_E/v|$ the norm of v.

Algebra. For an abelian group A we denote by A_{tor} its torsion-subgroup and by $A_{\mathrm{tf}} \coloneqq A/A_{\mathrm{tor}}$ its torsion-free part. If there is no confusion possible, we often shorten the functor $(-)\otimes_{\mathbb{Z}} A$ to just $(-)\cdot A$ (or even $(-)A$) and, if A is also a $\mathbb{Z}_p$-module, similarly for the functor $(-)\otimes_{\mathbb{Z}_p} A$. If A is finite, we denote by $\widehat{A} \coloneqq \mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{C}^\times)$ its character group, and for any $\chi \in \widehat{A}$ we let

$$e_\chi \coloneqq \frac{1}{|A|} \sum_{\sigma\in A} \chi(\sigma)\sigma^{-1} \quad \in \mathbb{C}[A]$$

be the usual primitive orthogonal idempotent associated to χ. Furthermore, $\mathrm{N}_A \coloneqq \sum_{\sigma\in A}\sigma \in \mathbb{Z}[A]$ denotes the norm element of A.

If R is a commutative Noetherian ring, then for any R-module M we write $M^* \coloneqq \mathrm{Hom}_R(M,R)$ for its dual and $\mathrm{Fitt}^0_R(M)$ for its (initial) Fitting ideal. Let $r \geq 0$ be an integer, then the r-th *exterior bidual* of M is defined as

$$\bigcap\nolimits^r_R M \coloneqq \Big(\bigwedge\nolimits^r_R M^*\Big)^*.$$

If $R = \mathbb{Z}[A]$ for a finite abelian group A, then the exterior bidual coincides with the lattice first introduced by Rubin in [Rub96, § 2], see [BS21, Rk. A.9] for the precise relation between these two definitions. The theory of exterior biduals has since seen great development and the reader is invited to consult, for example, [BS21, App. A] or [BD21, § 2] for an overview. At this point we only remark that, if $r \geq 1$, any $f \in M^*$ induces a map

$$\bigcap\nolimits^r_R M \to \bigcap\nolimits^{r-1}_R M$$

which, by abuse of notation, will also be denoted by f, and is defined as the dual of

$$\bigwedge\nolimits^{r-1}_R M^* \to \bigwedge\nolimits^r_R M^*, \quad g \mapsto f\wedge g.$$

Iterating this construction gives, for any $s \leq r$, a homomorphism

$$\bigwedge\nolimits^s_R M^* \to \mathrm{Hom}_R\big(\bigcap\nolimits^r_R M, \bigcap\nolimits^{r-s}_R M\big), \quad f_1\wedge\cdots\wedge f_s \mapsto f_s\circ\cdots\circ f_1. \tag{1}$$

Finally, we write $\mathcal{Q}(R)$ for the total ring of fractions of R.

2 Rubin–Stark elements

Let K/k be a finite abelian extension of number fields with Galois group $\mathcal{G} \coloneqq \mathrm{Gal}(K/k)$ and fix a finite set S of places of k which contains $S_\infty(k)\cup S_{\mathrm{ram}}(K/k)$. Suppose to be given a proper subset $V \subsetneq S$ of places which split completely in K/k and choose an ordering $S = \{v_0,\dots,v_t\}$ such that $V = \{v_1,\dots,v_r\}$. For every $i\in\{0,\dots,t\}$ fix a place $\overline{v_i}$ of the algebraic closure $\overline{\mathbb{Q}}$ of

$\mathbb{Q}$ that extends v_i and write $w_i = w_{K,i}$ for the place of K induced by $\overline{v_i}$.

If $\chi \in \widehat{\mathcal{G}}$ and T is a finite set of places of k disjoint from S, we define the S-truncated T-modified *Dirichlet L-function* associated with χ by means of

$$L_{k,S,T}(\chi, s) = \prod_{v\in T}(1 - \chi(\mathrm{Frob}_v)\mathrm{N}v^{1-s}) \cdot \prod_{v\notin S}(1 - \chi(\mathrm{Frob}_v)\mathrm{N}v^{-s})^{-1}$$

for all complex numbers s of real part strictly greater than 1. It is well-known that this defines a meromorphic function on the complex plane that is holomorphic in $s = 0$ by analytic continuation. By [Tat84, Ch. I, Prop. 3.4], the existence of the set $V \subsetneq S$ implies that the order of vanishing of $L_{K/k,S,T}(\chi, s)$ at $s = 0$ is at least r. This allows us to define the r-th order *Stickelberger element* as

$$\theta^{(r)}_{K/k,S,T}(0) = \sum_{\chi\in\widehat{\mathcal{G}}} e_\chi \cdot \lim_{s\to 0} s^{-r} L_{k,S,T}(\chi^{-1}, s) \quad \in \mathbb{R}[\mathcal{G}].$$

In the sequel we shall use that the *Dirichlet regulator* map

$$\lambda_{K,S} \colon \mathcal{O}^\times_{K,S} \to \mathbb{R}\mathcal{X}_{K,S}, \quad a \mapsto -\sum_{w\in S_K} \log|a|_w \cdot w \tag{2}$$

induces an isomorphism

$$\mathbb{R}\bigwedge\nolimits^r_{\mathbb{Z}[\mathcal{G}]} \mathcal{O}^\times_{K,S} \xrightarrow{\simeq} \mathbb{R}\bigwedge\nolimits^r_{\mathbb{Z}[\mathcal{G}]} \mathcal{X}_{K,S} \tag{3}$$

that will also be denoted as $\lambda_{K,S}$.

(2.1) Definition. *The r-th order* ***Rubin–Stark element*** $\varepsilon^V_{K/k,S,T} \in \mathbb{R}\bigwedge^r_{\mathbb{Z}[\mathcal{G}]} \mathcal{O}^\times_{K,S}$ *is defined to be the preimage of the element* $\theta^{(r)}_{K/k,S,T}(0) \cdot \bigwedge_{1\le i\le r}(w_i - w_0)$ *under the isomorphism (3) induced by the Dirichlet regulator map* $\lambda_{K,S}$*.*

We remark that this definition does not depend on the choice of element $v_0 \in S \setminus V$ (see [San15, Prop. 3.13]).

To state the p-component of the Rubin–Stark Conjecture for a prime number p we now fix an isomorphism $\mathbb{C} \cong \mathbb{C}_p$ that allows us to regard $\varepsilon^V_{K/k,S,T}$ as an element of $\mathbb{C}_p \bigwedge^r_{\mathbb{Z}_p[\mathcal{G}]} U_{K,S}$. Given a finite set T of places of k that is disjoint from S, we also write $U_{K,S,T} \coloneqq \mathbb{Z}_p \otimes_{\mathbb{Z}} \mathcal{O}^\times_{K,S,T}$ for the p-completion of the group of (S_K, T_K)-units which are defined as

$$\mathcal{O}^\times_{K,S,T} \coloneqq \ker\Big\{\mathcal{O}^\times_{K,S} \to \bigoplus_{w\in T_K} (\mathcal{O}_K/w)^\times\Big\}.$$

We will often assume that T is chosen in a way such that $U_{K,S,T}$ is $\mathbb{Z}_p$-torsion free (which is automatically satisfied if, for example, T contains a non-p-adic place).

In [BKS17, Conj. 2.1] Burns, Kurihara, and Sano have proposed a 'p-component version' of the Rubin–Stark Conjecture (from [Rub96]) as follows.

(2.2) Conjecture (p-component of Rubin–Stark)**.** *If $U_{K,S,T}$ is $\mathbb{Z}_p$-torsion free, then $\varepsilon^V_{K/k,S,T}$ belongs to $\bigcap^r_{\mathbb{Z}_p[\mathcal{G}]} U_{K,S,T}$.*

(2.3) Examples. (a) (*cyclotomic units*) Take $k = \mathbb{Q}$, K a finite real abelian extension of $\mathbb{Q}$, $S = S_\infty(\mathbb{Q}) \cup S_{\mathrm{ram}}(K/k)$, and $V = S_\infty(\mathbb{Q}) = \{v_1\}$. If we set $\delta_T \coloneqq \prod_{v\in T}(1 - \mathrm{N}v\mathrm{Frob}_v^{-1})$, then one has

$$\varepsilon^V_{K/\mathbb{Q},S,T} = \delta_T \cdot \big(\frac{1}{2} \otimes N_{\mathbb{Q}(\xi_m)/K}(1 - \xi_m)\big) \quad \in \mathcal{O}^\times_{K,S,T},$$

where $m = m_K$ is the conductor of K and $\xi_m \coloneqq \iota^{-1}(e^{2\pi i/m})$ with $\iota \colon \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ the embedding corresponding to the choice of place $\overline{v_1}$ fixed at the beginning of the section (see [Tat84, Ch. IV, § 5] for a proof). In particular, in this situation Conjecture (2.2) holds for all primes p.

(b) (*Stickelberger elements*) Let k be a totally real field, K a finite abelian CM extension of k, $S = S_\infty(k) \cup S_{\mathrm{ram}}(K/k)$, and $V = \varnothing$. In this setting Conjecture (2.2) holds true for all primes p due to work of Deligne–Ribet [DR80], and the Rubin–Stark element is given by

$$\varepsilon^V_{K/k,S,T} = \theta_{K/k,S,T}(0) \quad \in \mathbb{Z}[\mathcal{G}].$$

(c) (*elliptic units*) Let k be an imaginary quadratic field and $\mathfrak{f} \subseteq \mathcal{O}_k$ a non-zero ideal such that $\mathcal{O}_k^\times \to (\mathcal{O}_k/\mathfrak{f})^\times$ is injective. Take K to be a finite abelian extension of k, $S = S_\infty(k) \cup S_{\mathrm{ram}}(K/k) \cup \{\mathfrak{q} \mid \mathfrak{f}\}$, and $V = S_\infty(k) = \{v_1\}$. Then Conjecture (2.2) holds in this setting, see, for example, [Tat84, Ch. IV, Prop. 3.9]. To describe the Rubin–Stark element in this situation, we write $\mathfrak{m} = \mathfrak{m}_K$ for the conductor of K, let $k(\mathfrak{fm})$ be the ray class field of k modulo $\mathfrak{fm}$, and choose an auxiliary prime ideal $\mathfrak{a} \subsetneq \mathcal{O}_k$ coprime to $6\mathfrak{fm}$. Using the elliptic function ψ introduced by Robert in [Rob92], we set

$$\psi_{\mathfrak{fm},\mathfrak{a}} \coloneqq \iota^{-1}(\psi(1;\mathfrak{fm},\mathfrak{a}^{-1}\mathfrak{fm})) \quad \in \mathcal{O}^\times_{k(\mathfrak{fm}),S}$$

for the embedding $\iota\colon \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ corresponding to $\overline{v_1}$. It then follows from Kronecker's second limit formula, e.g. [Fla09, Lem. 2.2 e)], that

$$\varepsilon^V_{K/k,S,T} = \delta_T \delta^{-1}_{\{\mathfrak{a}\}} \cdot \mathrm{N}_{k(\mathfrak{fm})/K}(\psi_{\mathfrak{fm},\mathfrak{a}}) \in \mathcal{O}^\times_{K,S,T},$$

and this equality does not depend on the choice of $\mathfrak{a}$.

3 Iwasawa-theoretic congruences for Rubin–Stark elements

3.1 The general set-up

In this subsection we describe the Iwasawa-theoretic set-up that will be fixed throughout all of § 3, and also collect a number of elementary facts that will prove useful in the sequel.

Let k be a number field, and fix a rational prime p. We suppose to be given a finite abelian extension K/k and a $\mathbb{Z}_p$-extension k_∞/k. We define $K_\infty \coloneqq Kk_\infty$ and write K_n for the n-th layer of the $\mathbb{Z}_p$-extension K_∞/K (here K_0 means K). We further set $\Gamma_n \coloneqq \mathrm{Gal}(K_n/K)$, $\Gamma^n \coloneqq \mathrm{Gal}(K_\infty/K_n)$, $\mathcal{G}_n \coloneqq \mathrm{Gal}(K/k)$, and $\Lambda \coloneqq \mathbb{Z}_p[\![\mathrm{Gal}(K_\infty/k)]\!]$. If $n = 0$, then we suppress reference to n in the notation and simply write Γ and $\mathcal{G}$ for $\mathrm{Gal}(K_\infty/K)$ and $\mathrm{Gal}(K/k)$, respectively.

Moreover, we introduce the following notations and assumptions:

- S a finite set of places of k which contains $S_\infty(k) \cup S_{\mathrm{ram}}(K/k)$ and is such that not finite place in S splits completely in k_∞/k,
- $\Sigma \coloneqq S \cup S_{\mathrm{ram}}(k_\infty/k)$,
- $V \subsetneq \Sigma$ the subset of places which split completely in K_∞/k (by our assumptions this is a subset of $S_\infty(k)$), and r its cardinality,
- $V' \subsetneq \Sigma$ a set of places which contains V and consists of places that split completely in K/k, and r' its cardinality,
- T a finite set of places of k which is disjoint from Σ, contains only places that do not split completely in k_∞/k, and is such that $U_{K_n,\Sigma,T}$ is $\mathbb{Z}_p$-torsion free for some integer $n \geq 0$. By the general result of [NSW08, Prop. (1.6.12)], this implies that $U_{K_n,\Sigma,T}$ is $\mathbb{Z}_p$-torsion free for all $n \geq 0$.

We fix a labelling $\Sigma = \{v_0, \dots, v_t\}$ such that $V = \{v_1, \dots, v_r\}$ and $V' = \{v_1, \dots, v_{r'}\}$. As at the beginning of § 2, we also choose a place $\overline{v_i}$ of the algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$ that extends v_i and, for every $n \geq 0$, write $w_{n,i} \coloneqq w_{K_n,i}$ for the place of K_n induced by $\overline{v_i}$.

(3.1) Lemma. *For any topological generator γ of Γ, the element $\gamma - 1$ is a non-zero divisor in Λ.*

Proof. Fix a splitting $\mathrm{Gal}(K_\infty/k) \cong \Gamma' \times \Delta$ with $\Gamma' \cong \mathbb{Z}_p$ and Δ finite. Set $L \coloneqq K_\infty^{\Gamma'}$ and write L_n for the n-th layer of the $\mathbb{Z}_p$-extension K_∞/L. By definition we have $K_\infty = \bigcup_{n\geq 0} L_n$, hence there is n such that $K \subseteq L_n$. That is, L_n is an intermediate field of the $\mathbb{Z}_p$-extension K_∞/K and therefore must agree with K_m for some m. If $\gamma_L \in \Gamma'$ denotes a topological generator, then there hence is a unit $a \in \mathbb{Z}_p^\times$ such that $\gamma_L^{ap^n} = \gamma^{p^m}$. The element $\gamma_L^{ap^n} - 1$ is a non-zero divisor in $\mathbb{\Lambda} = \mathbb{Z}_p[\![\Gamma']\!][\Delta]$ because $\mathbb{Z}_p[\![\Gamma']\!]$ is an integral domain. It then follows from

$$\gamma_L^{ap^n} - 1 = \gamma^{p^m} - 1 = (\gamma - 1) \cdot (1 + \gamma + \cdots + \gamma^{p^m - 1})$$

that $\gamma - 1$ must be a non-zero divisor in $\mathbb{\Lambda}$ as well. □

To end this subsection, we recall useful facts concerning the augmentation ideals

$$I(\Gamma_n) \coloneqq \ker\big\{\mathbb{Z}_p[\Gamma_n] \to \mathbb{Z}_p\big\} \quad \text{and} \quad I_{\Gamma_n} \coloneqq \ker\big\{\mathbb{Z}_p[\mathcal{G}_n] \to \mathbb{Z}_p[\mathcal{G}]\big\}.$$

Note that $I_{\Gamma_n} = I(\Gamma_n) \cdot \mathbb{Z}_p[\mathcal{G}_n]$ and hence for any $\mathbb{Z}_p[\mathcal{G}_n]$-module M and $i \in \mathbb{N}$ we have an isomorphism

$$M \otimes_{\mathbb{Z}_p[\mathcal{G}_n]} I_{\Gamma_n}^i \big/ I_{\Gamma_n}^{i+1} \cong M \otimes_{\mathbb{Z}_p} I(\Gamma_n)^i \big/ I(\Gamma_n)^{i+1}. \tag{4}$$

Moreover, we have equalities $I_\Gamma \coloneqq \ker\{\mathbb{\Lambda} \to \mathbb{Z}_p[\mathcal{G}]\} = \varprojlim_n I_{\Gamma_n}$ and $I(\Gamma) \coloneqq \ker\{\mathbb{Z}_p[\![\Gamma]\!] \to \mathbb{Z}_p\} = \varprojlim_n I(\Gamma_n)$. In particular, the latter ideal is generated by $\gamma - 1$ for any topological generator $\gamma \in \Gamma$, and there is an isomorphism

$$I(\Gamma)^i \big/ I(\Gamma)^{i+1} \xrightarrow{\simeq} \Gamma, \quad (\gamma - 1)^i \mapsto \gamma.$$

3.2 Modified Iwasawa cohomology complexes

For any finite abelian extension E/k and finite set of places $M \supseteq S_\infty \cup S_{\mathrm{ram}}(E/k)$ that is disjoint from a second finite set of places Z, Burns, Kurihara, and Sano have constructed in [BKS16, Prop. 2.4] a canonical Z-modified, compactly supported *Weil-étale cohomology complex* $\mathrm{R}\Gamma_{c,Z}((\mathcal{O}_{E,M})_{\mathcal{W}}, \mathbb{Z})$ for the constant sheaf $\mathbb{Z}$. In the sequel we shall need the complex

$$C^\bullet_{E,M,Z} = \mathrm{R}\operatorname{Hom}_{\mathbb{Z}}(\mathrm{R}\Gamma_{c,Z}((\mathcal{O}_{E,M})_{\mathcal{W}}, \mathbb{Z}), \mathbb{Z})[-2]$$

as well as its p-completion $D^\bullet_{E,M,Z} \coloneqq \mathbb{Z}_p \otimes^{\mathbb{L}}_{\mathbb{Z}} C^\bullet_{E,M,Z}$. The essential properties of these complexes are listed in [Bur+23, Prop. 3.1]. For example, there is a canonical isomorphism $H^0(D^\bullet_{E,M,Z}) \cong U_{E,M,Z}$ and a canonical exact sequence

$$0 \longrightarrow A_{E,M,Z} \longrightarrow H^1(D^\bullet_{E,M,Z}) \xrightarrow{\pi_E} X_{E,M} \longrightarrow 0. \tag{5}$$

Furthermore, specialising to our setting, there are natural maps $D^\bullet_{K_{n+1},\Sigma,T} \to D^\bullet_{K_{n+1},\Sigma,T} \otimes^{\mathbb{L}}_{\mathbb{Z}_p[\mathcal{G}_{n+1}]} \mathbb{Z}_p[\mathcal{G}_n] \cong D_{K_n,\Sigma,T}$ in the derived category $D(\mathbb{Z}_p[\mathcal{G}_{n+1}])$ of $\mathbb{Z}_p[\mathcal{G}_{n+1}]$-modules (cf. [Bur+23, Prop. 3.1 (iv)]), which allow us to define the limit complex

$$D^\bullet_{K_\infty,\Sigma,T} = \mathrm{R}\varprojlim_n D^\bullet_{K_n,\Sigma,T}. \tag{6}$$

(3.2) Proposition. *The following claims are valid.*

(a) The complex $D^\bullet_{K_\infty,\Sigma,T}$ is perfect as an element of the derived category $D(\mathbb{\Lambda})$ of $\mathbb{\Lambda}$-modules and acyclic outside degrees zero and one.

(b) There is a canonical isomorphism $H^0(D^\bullet_{K_\infty,\Sigma,T}) \cong U_{K_\infty,\Sigma,T}$ and a canonical exact sequence

$$0 \longrightarrow A_{K_\infty,\Sigma,T} \longrightarrow H^1(D^\bullet_{K_\infty,\Sigma,T}) \xrightarrow{\pi} X_{K_\infty,\Sigma} \longrightarrow 0, \tag{7}$$

where $A_{K_\infty,\Sigma,T} \coloneqq \varprojlim_n A_{K_n,\Sigma,T}$ and $X_{K_\infty,\Sigma} \coloneqq \varprojlim_n X_{K_n,\Sigma}$ with the transition maps taken to be the norm and restriction maps, respectively.

(c) There exists a free $\mathbb{\Lambda}$-module Π_∞ of finite rank $d > |\Sigma|$, a basis $\{b_1, \ldots, b_d\}$ of Π_∞, and an endomorphism $\phi\colon \Pi_\infty \to \Pi_\infty$ with the following properties:

(i) The complex $[\Pi_\infty \xrightarrow{\phi} \Pi_\infty]$ *represents the class of* $D^\bullet_{K_\infty,\Sigma,T}$ *in* $D(\Lambda)$.

(ii) If we set $\Pi_n := \Pi_\infty \otimes_\Lambda \mathbb{Z}_p[\mathcal{G}_n]$ *and write* ϕ_n *for the endomorphism of* Π_n *induced by* ϕ*, then the complex* $[\Pi_n \xrightarrow{\phi_n} \Pi_n]$ *represents the class of* $D^\bullet_{K_n,\Sigma,T}$ *in* $D(\mathbb{Z}_p[\mathcal{G}_n])$.

(iii) If we fix an ordering $\Sigma = \{v_0, \dots, v_t\}$*, then, for any* $i \in \{1, \dots, t\}$*, the composite map* $\Pi_\infty \to H^1(D^\bullet_{K_\infty,\Sigma,T}) \xrightarrow{\pi} X_{K_\infty,\Sigma}$ *sends* b_i *to* $(w_{K_n,i} - w_{K_n,0})_{n\geq 0}$.

Proof. To lighten notation, we set $D^\bullet_n := D^\bullet_{K_n,\Sigma,T}$ and $D^\bullet_\infty := D^\bullet_{K_\infty,\Sigma,T}$ in this proof. Each complex $D^\bullet_n$ is then a complex of finitely generated $\mathbb{Z}_p$-modules, hence of compact Hausdorff spaces. The inverse limit functor therefore commutes with taking cohomology and so claim (b), and the second part of claim (a), follow by taking the limit of the respective statements for the complexes $D^\bullet_n$ as given in [Bur+23, Prop. 3.1 (i)].

Let Π_∞ be a finitely generated free module of finite rank d with basis $\{b_1, \dots, b_d\}$. If d is large enough, we may choose a surjection

$$\mathrm{pr}_\infty \colon \Pi_\infty \twoheadrightarrow H^1(D^\bullet_\infty)$$

with the property that, for any $i \in \{1, \dots, t\}$, the composite $\pi \circ \mathrm{pr}_\infty$ sends b_i to $(w_{K_n,i} - w_{K_n,0})_{n\geq 0}$, as required in condition (iii) of claim (c).

Since, for any integer $m \geq 0$ the complex $D^\bullet_{m+1}$ is acyclic outside degrees 0 and 1, we have a natural identification $H^1(D^\bullet_{m+1}) \otimes_{\mathbb{Z}_p[\mathcal{G}_{m+1}]} \mathbb{Z}_p[\mathcal{G}_m] \cong H^1(D^\bullet_{m+1} \otimes^{\mathbb{L}}_{\mathbb{Z}_p[\mathcal{G}_{m+1}]} \mathbb{Z}_p[\mathcal{G}_m])$. This combines with the natural isomorphism $D^\bullet_{m+1} \otimes^{\mathbb{L}}_{\mathbb{Z}_p[\mathcal{G}_{m+1}]} \mathbb{Z}_p[\mathcal{G}_m] \cong D^\bullet_m$ to imply that each map $H^1(D^\bullet_{m+1}) \to H^1(D^\bullet_m)$ is surjective. In particular, the second arrow in the composite map

$$\Pi_\infty \overset{\mathrm{pr}_\infty}{\twoheadrightarrow} H^1(D^\bullet_\infty) = \varprojlim_{m\geq 0} H^1(D^\bullet_m) \to H^1(D^\bullet_n)$$

is surjective for every integer $n \geq 0$. In addition, the above composite map factors through the natural map $\Pi_\infty \to \Pi_n$ and hence induces a surjection $\mathrm{pr}_n \colon \Pi_n \twoheadrightarrow H^1(D^\bullet_n)$. The method of [BKS16, § 5.4] (see also [BS21, Prop. A.11 (i)]) then allows us to choose a representative of $D^\bullet_n$ that is of the form $[Q_n \to \Pi_n]$ with Q_n a $\mathbb{Z}_p[\mathcal{G}_n]$-projective module. By a standard argument in representation theory (cf. [Tat84, Ch. II, § 4]), the isomorphism $\mathbb{R}\mathcal{O}^\times_{K_n,\Sigma} \cong \mathbb{R}\mathcal{X}_{K_n,\Sigma}$ defined by the Dirichlet regulator (2) induces a *rational* isomorphism $\mathbb{Q}_p H^0(D^\bullet_n) \cong \mathbb{Q}_p H^1(D_n)$, therefore we may identify $Q_n \cong \Pi_n$ by Swan's theorem [CR81, Thm. (32.1)].

We shall now construct, by induction on n, a family of endomorphisms $\phi_n \colon \Pi_n \to \Pi_n$ which are such that, for each n, the map induced by ϕ_{n+1} on Π_n agrees with ϕ_n and, in addition, the induced endomorphism $\phi_\infty := (\phi_n)_n$ of Π_∞ satisfies the conditions (i)-(iii) in claim (c).

To do this, we write $[\Pi_n \xrightarrow{\partial_n} \Pi_n]$ for the representative of $D^\bullet_n$ obtained above. Define $\phi_0 := \partial_0$ and assume that, for some given n, the map ϕ_n has already been constructed. Write γ_n for the isomorphism $D^\bullet_{n+1} \otimes^{\mathbb{L}}_{\mathbb{Z}_p[\mathcal{G}_{n+1}]} \mathbb{Z}_p[\mathcal{G}_n] \cong D^\bullet_n$ in the derived category $D(\mathbb{Z}_p[\mathcal{G}_n])$ and note that, as a morphism between perfect complexes, this map can be represented by a commutative diagram of the form

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & H^0(D^\bullet_{n+1} \otimes^{\mathbb{L}} \mathbb{Z}_p[\mathcal{G}_n]) & \longrightarrow & \Pi_n & \xrightarrow{\overline{\partial_{n+1}}} & \Pi_n & \xrightarrow{\overline{\mathrm{pr}_{n+1}}} & H^1(D^\bullet_{n+1} \otimes^{\mathbb{L}} \mathbb{Z}_p[\mathcal{G}_n]) & \longrightarrow & 0 \\
 & & \simeq\downarrow H^0(\gamma_n) & & \downarrow \gamma^0_n & & \downarrow \gamma^1_n & & \simeq\downarrow H^1(\gamma_n) & & \\
0 & \longrightarrow & H^0(D^\bullet_n) & \longrightarrow & \Pi_n & \xrightarrow{\phi_n} & \Pi_n & \xrightarrow{\mathrm{pr}_n} & H^1(D^\bullet_n) & \longrightarrow & 0.
\end{array} \tag{8}$$

Here $\overline{\partial_{n+1}}$ and $\overline{\mathrm{pr}_{n+1}}$ denote the maps induced by ∂_{n+1} and pr_{n+1}, respectively. By construction, $\mathrm{pr}_n = H^1(\gamma_n) \circ \overline{\mathrm{pr}_{n+1}}$ and so exactness of the bottom line in (8) yields that the image of $\gamma^1_n - \mathrm{id}_{\Pi_n}$ is contained in the image of ϕ_n. Choose a map $h \colon \Pi_n \to \Pi_n$ such that $\phi_n \circ h = \gamma^1_n - \mathrm{id}_{\Pi_n}$ and set $f = \gamma^0_n - h \circ \overline{\partial_{n+1}}$, then h defines a chain homotopy between (γ^0_n, γ^1_n) and (f, id). We may therefore assume that γ^1_n is the identity map. Given this, we can appeal to the Five Lemma to deduce from (8) that γ^0_n is an isomorphism.

By Nakayama's Lemma, we can lift γ_n^0 to an automorphism $\widetilde{\gamma}_n$ of Π_{n+1}. We then set $\phi_{n+1} \coloneqq \partial_{n+1} \circ \widetilde{\gamma}_n^{-1}$. By construction, one now has both that $\overline{\phi_{n+1}} = \phi_n$ and that $D^\bullet_{n+1}$ is represented by $[\Pi_{n+1} \stackrel{\phi_{n+1}}{\to} \Pi_{n+1}]$.

We may therefore pass to the limit to obtain a representative $[\Pi_\infty \xrightarrow{\phi} \Pi_\infty]$ of the complex $D^\bullet_\infty$, as required to prove part (i) of claim (c). In particular, the latter complex is perfect as an element of the derived category $D(\Lambda)$ and this proves claim (a). Moreover, it is clear from the construction that conditions (ii) and (iii) in (c) are satisfied. □

(3.3) Remark. (a) For later purposes we note that, by Proposition (3.2), the complexes $D^\bullet_{K_\infty,\Sigma,T}$ and $D^\bullet_{K_n,\Sigma,T}$ give rise to exact sequences

$$0 \longrightarrow U_{K_\infty,\Sigma,T} \longrightarrow \Pi_\infty \xrightarrow{\phi} \Pi_\infty \longrightarrow H^1(D^\bullet_{K_\infty,\Sigma,T}) \longrightarrow 0 \tag{9}$$

and

$$0 \longrightarrow U_{K_n,\Sigma,T} \longrightarrow \Pi_n \xrightarrow{\phi_n} \Pi_n \longrightarrow H^1(D^\bullet_{K_n,\Sigma,T}) \longrightarrow 0, \tag{10}$$

respectively.

(b) Since, for every $n \geq 0$, the map induced on cohomology in degree zero by the isomorphism $D^\bullet_{K_{n+1},\Sigma,T} \otimes^{\mathbb{L}}_{\mathbb{Z}_p[\mathcal{G}_{n+1}]} \mathbb{Z}_p[\mathcal{G}_n] \cong D^\bullet_{K_n,\Sigma,T}$ in $D(\mathbb{Z}_p[\mathcal{G}_n])$ is the field-theoretic norm map $\mathrm{N}_{K_{n+1}/K_n} \colon U_{K_{n+1},\Sigma,T} \to U_{K_n,\Sigma,T}$, the natural map $\Pi_{n+1} \to \Pi_n$ restricts to N_{K_{n+1}/K_n} on $U_{K_{n+1},\Sigma,T}$.

3.3 Darmon derivatives

For any integer $n \geq 0$, write $\mathrm{N}^r_{K_{n+1}/K_n}$ for the map $\mathbb{Q}_p \bigwedge^r_{\mathbb{Z}_p[\mathcal{G}_{n+1}]} U_{K_{n+1},\Sigma} \to \mathbb{Q}_p \bigwedge^r_{\mathbb{Z}_p[\mathcal{G}_n]} U_{K_n,\Sigma}$ that is induced by the field-theoretic norm map $\mathrm{N}_{K_{n+1}/K_n} \colon K^\times_{n+1} \to K^\times_n$. It is then a consequence of the observation in Remark (3.3) (b) that the map $\mathrm{N}^r_{K_{n+1}/K_n}$ restricts to a morphism $\bigcap^r_{\mathbb{Z}_p[\mathcal{G}_{n+1}]} U_{K_{n+1},\Sigma,T} \to \bigcap^r_{\mathbb{Z}_p[\mathcal{G}_n]} U_{K_n,\Sigma,T}$ (see [BD21, § 3.2] for details of this argument). Using these maps $\mathrm{N}^r_{K_{n+1}/K_n}$, we can therefore define the projective limit

$$\varprojlim_{n\geq 0} \bigcap\nolimits^r_{\mathbb{Z}_p[\mathcal{G}_n]} U_{K_n,\Sigma,T} = \bigcap\nolimits^r_{\Lambda} U_{K_\infty,\Sigma,T},$$

where the identification follows from an application of the general result of Sakamoto in [Sak23, Lem. B.15] to the representative of the complex $D^\bullet_{K_\infty,\Sigma,T}$ constructed in Proposition (3.2).

Suppose the p-part of the Rubin–Stark Conjecture (2.2) holds true for all extensions K_n/k and the data (V, Σ, T). Then [Rub96, Prop. 6.1] (see also [San14, Prop. 3.5]) implies that the family

$$\varepsilon_{K_\infty/k,\Sigma,T} \coloneqq (\varepsilon^V_{K_n/k,\Sigma,T})_{n\geq 0}$$

defines an element of $\varprojlim_{n\geq 0} \bigcap^r_{\mathbb{Z}_p[\mathcal{G}_n]} U_{K_n,\Sigma,T}$, and may hence be regarded as an element of $\bigcap^r_{\Lambda} U_{K_\infty,\Sigma,T}$ by means of the above identification.

Let $W \coloneqq V' \setminus V$ and $e \coloneqq |W|$. We also note that W can only contain finite places.

(3.4) Conjecture. *Assume that, for all $n \geq 0$, the p-part of the Rubin–Stark Conjecture (2.2) holds for the extension K_n/k and the data (V, Σ, T). Then $\varepsilon_{K_\infty/k,\Sigma,T}$ belongs to $I^e_\Gamma \cdot \bigcap^r_{\Lambda} U_{K_\infty,\Sigma,T}$.*

(3.5) Remark. (a) Variants of Conjecture (3.4) have previously appeared in the literature in many places, with its archetypical relative being the 'guess' formulated by Gross for Stickelberger elements [Gro88, top of p. 195]. A version for general Rubin–Stark elements was then formulated by Burns in [Bur07], see also [San14, Conj. 4]. In the form stated above the conjecture has, for example, been studied by Büyükboduk and Sakamoto in [BS22, Conj. 2.7].

(b) Conjecture (3.4) is implied by a (relevant variant of a) Iwasawa Main Conjecture and this is indeed already well-known (see also Remark (3.8) below). A direct proof by analytic means in the case of k totally real and K CM is given by Dasgupta and Spieß in [DS18].

(c) A containment as in the statement of Conjecture (3.4) should be thought of as an order of vanishing statement. In fact, it can be directly linked to the order of vanishing of a p-adic L-function in many cases (see, for example, [BS22, Lem. 4.9]).

Recall that any element $a \in \bigcap_{\mathbb{A}}^r U_{K_\infty,\Sigma,T}$ is by definition a morphism $a\colon \bigwedge_{\mathbb{A}}^r (U_{L_\infty,\Sigma,T})^* \to \mathbb{A}$. In particular, its image $\mathrm{im}(a)$ is a well-defined ideal of $\mathbb{A}$.

(3.6) Proposition. *Assume that, for every $n \geq 0$, the p-part of the Rubin–Stark Conjecture holds for the extension K_n/k and the data (V, Σ, T).*

(a) Conjecture (3.4) holds true if one has an inclusion of reflexive hulls

$$\mathrm{im}(\varepsilon_{K_\infty/k,\Sigma,T})^{**} \subseteq \mathrm{Fitt}^0_{\mathbb{A}}(A_{K_\infty,\Sigma,T})^{**} \cdot \mathrm{Fitt}^0_{\mathbb{A}}(X_{K_\infty,\Sigma\setminus V})^{**}. \qquad (11)$$

(b) If W contains at most one place which ramifies in k_∞/k, then Conjecture (3.4) holds true.

Proof. To prove (a), we consider the surjective composite map

$$X_{K_\infty,\Sigma\setminus V} \twoheadrightarrow Y_{K_\infty,W} \twoheadrightarrow Y_{K,W} = \bigoplus_{v\in W} \mathbb{Z}_p.$$

Here the first arrow is given by sending places above $\Sigma \setminus W$ to zero (which gives a surjective map because of our assumption that V' is a strict subset of Σ), and the second arrow is the natural restriction map. By a standard property of Fitting ideals (see, for example, [MW84, Appendix, property 1]), the above surjection then gives rise to an inclusion $\mathrm{Fitt}^0_{\mathbb{A}}(X_{K_\infty,\Sigma}) \subseteq \mathrm{Fitt}^0_{\mathbb{A}}\left(\bigoplus_{v\in W}\mathbb{Z}_p\right) = I_\Gamma^e$. Taking reflexive hulls, we deduce that $\mathrm{Fitt}^0_{\mathbb{A}}(X_{K_\infty,\Sigma})^{**}$ is contained in $(I_\Gamma^e)^{**} = I_\Gamma^e$.
If, for any $i \in \{1,\dots,d\}$, we write $b_i^*\colon \Pi_\infty^* \to \mathbb{A}$ for the dual basis element of b_i, then the inclusion

$$\left\{ f(\varepsilon^V_{K_\infty/k,\Sigma,T}) \mid f \in \bigwedge\nolimits_{\mathbb{A}}^r \Pi_\infty^* \right\} \subseteq \mathrm{im}(\varepsilon^V_{K_\infty/k,\Sigma,T})$$

combines with the assumption in (a) and the above discussion to imply that $(\bigwedge_{i\in I} b_i^*)(\varepsilon_{K_\infty/k,\Sigma,T})$ belongs to I_Γ^e for any index set $I \subseteq \{1,\dots,d\}$. Since the elements of the form $b_I \coloneqq \bigwedge_{i\in I} b_i$ form a $\mathbb{A}$-basis of $\bigwedge_{\mathbb{A}}^r \Pi_\infty$, this shows that $\varepsilon_{K_\infty/k,\Sigma,T}$ can be written as $\sum_I x_I b_I$ with each coefficient x_I an element of I_Γ^e. We deduce that $\varepsilon_{K_\infty/k,\Sigma,T} \in I_\Gamma^e \cdot \bigwedge_{\mathbb{A}}^r \Pi_\infty$, hence the validity of Conjecture (3.4) now follows from Lemma (3.7) below. This proves claim (a).
By the above observations, it suffices to prove that $\mathrm{im}(\varepsilon^V_{K_\infty/k,\Sigma,T})$ is contained in $\mathrm{Fitt}^0_{\mathbb{A}}(X_{K_\infty,\Sigma\setminus V})$ in order to establish claim (b). Under the stated condition, this follows from the argument of [Bul+21, Lem. 6.5 (a) and (b)]. □

(3.7) Lemma. *Let $u \in \bigcap_{\mathbb{A}}^r U_{K_\infty,\Sigma,T}$ be a norm-coherent sequence. Then*

$$u \in I_\Gamma^e \cdot \bigcap\nolimits_{\mathbb{A}}^r U_{K_\infty,\Sigma,T} \quad \Leftrightarrow \quad u \in I_\Gamma^e \cdot \bigwedge\nolimits_{\mathbb{A}}^r \Pi_\infty.$$

Proof. By the general result of Sakamoto in [Sak23, Lem. B.12], the exact sequence (9) induces an exact sequence

$$0 \longrightarrow \bigcap\nolimits_{\mathbb{A}}^r U_{K_\infty,\Sigma,T} \longrightarrow \bigwedge\nolimits_{\mathbb{A}}^r \Pi_\infty \xrightarrow{\ \phi\ } \Pi_\infty \otimes_{\mathbb{A}} \bigwedge\nolimits_{\mathbb{A}}^{r-1} \Pi_\infty, \qquad (12)$$

whence the implication '⇒' is clear. If we now fix a topological generator $\gamma \in \Gamma$, then $u = (\gamma-1)^e \kappa$ for some $\kappa \in \bigwedge_{\mathbb{A}}^r \Pi_\infty$ implies that

$$0 = \phi(u) = \phi((\gamma-1)^e\kappa) = (\gamma-1)^e \cdot \phi(\kappa),$$

hence $\phi(\kappa) = 0$ since $(\gamma-1)^e \in \mathbb{A}$ is not a zero divisor by Lemma (3.1) and $\Pi_\infty \otimes_{\mathbb{A}} \bigwedge_{\mathbb{A}}^{r-1} \Pi_\infty$ is torsion-free. The exact sequence (12) thus reveals that $\kappa \in \bigcap_{\mathbb{A}}^r U_{K_\infty,\Sigma,T}$. □

(3.8) Remark. The inclusion (11) is one 'divisibility' in an equivariant higher-rank Iwasawa Main Conjecture that is expected to always hold (cf. [BKS17, Conj. 3.14]). In particular, this shows that Conjecture (3.4) would be a consequence of one divisibility in a relevant Iwasawa Main Conjecture.

(3.9) Definition. *Assume Conjecture (3.4) holds. The (Iwasawa-theoretic)* ***Darmon derivative*** *of* $\varepsilon_{K_\infty/k,\Sigma,T}$ *with respect to a topological generator* γ *of* Γ *is the bottom value* κ_0 *of the unique norm-coherent sequence* $\kappa = (\kappa_n)_n$ *in* $\bigcap^r_{\mathbb{A}} U_{K_\infty,\Sigma,T}$ *satisfying* $(\gamma-1)^e \cdot \kappa = \varepsilon_{K_\infty/k,\Sigma,T}$.

(3.10) Remark. (a) If $k = \mathbb{Q}$ (in which case the appearing Rubin–Stark elements are cyclotomic units, see Example (2.3) (a)), the notion of Darmon derivative recovers the element considered by Solomon in [Sol92] (cf. the proof of Proposition (3.13) below).

(b) The question if the Darmon derivative κ_0 vanishes is related to information about class groups. To explain this in a little more detail, we assume 'the higher-rank Iwasawa Main Conjecture' formulated by Burns, Kurihara, and Sano in [BKS17, Conj. 3.1] holds true. One can then show that

$$e_{r'}\mathbb{Q}_p[\mathcal{G}] \otimes_{\mathbb{Z}_p[\mathcal{G}]} (A_{K_\infty,\Sigma,T})^\Gamma = 0 \quad \Leftrightarrow \quad \kappa_0 \neq 0,$$

where $e_{r'} \coloneqq \sum_\chi e_\chi$ is the sum over all primitive orthogonal idempotents e_χ associated with characters $\chi \in \widehat{\mathcal{G}}$ for which one has $e_\chi \varepsilon^{V'}_{K/k,\Sigma,T} \neq 0$. That is, under condition (F) (as formulated in (4.3)) the containment in Conjecture (3.4) is 'optimal'.

(c) Our terminology follows [BKS19] where an element defined via a closely related construction is referred to as the *Iwasawa–Darmon derivative* (see [BKS19, Def. 4.6]). This points to Darmon [Dar95] who first interpreted this construction as a derivative process (see also the discussion in [San14, Rk. 4.8]).

3.4 The conjecture of Mazur–Rubin and Sano

3.4.1 Statement of the conjecture

To state the Iwasawa-theoretic Mazur–Rubin–Sano Conjecture we first need to recall several important constructions.

Since T is chosen such that $U_{K_n,\Sigma,T}$ is $\mathbb{Z}_p$-torsion free for all $n \geq 0$, the general result of [Bul+21, Lem. 2.9] implies that, for every $n \in \mathbb{N}$, there is an isomorphism

$$\bigcap\nolimits^r_{\mathbb{Z}_p[\mathcal{G}]} U_{K,\Sigma,T} \xrightarrow{\simeq} \Big(\bigcap\nolimits^r_{\mathbb{Z}_p[\mathcal{G}_n]} U_{K_n,\Sigma,T}\Big)^{\Gamma_n} \subseteq \bigcap\nolimits^r_{\mathbb{Z}_p[\mathcal{G}_n]} U_{K_n,\Sigma,T}$$

which gives rise to an injection

$$\nu_n\colon \Big(\bigcap\nolimits^r_{\mathbb{Z}_p[\mathcal{G}]} U_{K,\Sigma,T}\Big) \otimes_{\mathbb{Z}_p} \big(I(\Gamma_n)^e/I(\Gamma_n)^{e+1}\big) \hookrightarrow \Big(\bigcap\nolimits^r_{\mathbb{Z}_p[\mathcal{G}_n]} U_{K_n,\Sigma,T}\Big) \otimes_{\mathbb{Z}_p} \big(\mathbb{Z}_p[\Gamma_n]/I(\Gamma_n)^{e+1}\big)$$

that is characterised by

$$\nu_n(\mathrm{N}^r_{K_n/K}(a) \otimes x) = (\mathrm{N}_{\mathrm{Gal}(K_n/K)} a) \otimes x \tag{13}$$

for any $a \in \bigcap^r_{\mathbb{Z}_p[\mathcal{G}_n]} U_{K_n,\Sigma,T}$ and $x \in I(\Gamma_n)^e/I(\Gamma_n)^{e+1}$ (see also [San14, Rk. 2.12]).

Let $v \in W$ and recall that we have fixed a place $w \coloneqq w_K$ of K above v. Denote by Γ_w the decomposition group of w inside Γ and write

$$\mathrm{rec}_w\colon K^\times \hookrightarrow K_w^\times \to \Gamma_w \subseteq \Gamma$$

for the local reciprocity map at w, where K_w denotes the completion of K at w. Consider the map

$$\mathrm{Rec}_v\colon K^\times \to \mathbb{Z}_p[\mathcal{G}] \otimes_{\mathbb{Z}_p} I(\Gamma)/I(\Gamma)^2, \quad a \mapsto \sum_{\sigma\in\mathcal{G}} \sigma^{-1} \otimes (\mathrm{rec}_w(\sigma a) - 1),$$

which induces a map

$$\mathrm{Rec}_W := \bigwedge_{v\in W} \mathrm{Rec}_v \colon \bigcap\nolimits^{r'}_{\mathbb{Z}_p[\mathcal{G}]} U_{K,\Sigma,T} \to \Big(\bigcap\nolimits^{r}_{\mathbb{Z}_p[\mathcal{G}]} U_{K,\Sigma,T}\Big) \otimes_{\mathbb{Z}_p} \big(I(\Gamma)^e/I(\Gamma)^{e+1}\big) \tag{14}$$

(see also [San14, Prop. 2.7]). Finally, we define *Darmon's twisted norm operator* as

$$\mathcal{N}_n \colon \bigcap\nolimits^{r}_{\mathbb{Z}_p[\mathcal{G}]} U_{K_n,\Sigma,T} \to \Big(\bigcap\nolimits^{r}_{\mathbb{Z}_p[\mathcal{G}]} U_{K,\Sigma,T}\Big) \otimes_{\mathbb{Z}_p} \big(\mathbb{Z}_p[\Gamma_n]/I(\Gamma_n)^{e+1}\big), \quad a \mapsto \sum_{\sigma\in\Gamma_n} \sigma a \otimes \sigma^{-1}.$$

(3.11) Conjecture (Iwasawa-theoretic Mazur–Rubin–Sano)**.** *Assume that, for all $n \geq 0$, the p-part of the Rubin–Stark Conjecture (2.2) holds for the extension K_n/k and the data (V, Σ, T). Then, there exists an element*

$$\begin{aligned}\mathfrak{k} = (\mathfrak{k}_n)_{n\geq 0} &\in \Big(\bigcap\nolimits^{r}_{\mathbb{Z}_p[\mathcal{G}]} U_{K,\Sigma,T}\Big) \otimes_{\mathbb{Z}_p} \big(I(\Gamma)^e/I(\Gamma)^{e+1}\big)\\ &= \Big(\bigcap\nolimits^{r}_{\mathbb{Z}_p[\mathcal{G}]} U_{K,\Sigma,T}\Big) \otimes_{\mathbb{Z}_p} \big(\varprojlim_{n\geq 0} I(\Gamma_n)^e/I(\Gamma_n)^{e+1}\big)\end{aligned}$$

such that $\nu_n(\mathfrak{k}_n) = \mathcal{N}_n(\varepsilon^V_{K_n/k,\Sigma,T})$ for all n, and

$$\mathfrak{k} = (-1)^{re} \cdot \mathrm{Rec}_W(\varepsilon^{V'}_{K/k,\Sigma,T}), \tag{15}$$

where the equality takes place in $\mathbb{Q}_p\big(\bigcap^r_{\mathbb{Z}_p[\mathcal{G}]} U_{K,\Sigma,T}\big) \otimes_{\mathbb{Z}_p} (I(\Gamma)^e/I(\Gamma)^{e+1})$.

(3.12) Remark. (a) The above conjecture is taken from [BKS17, Conj. 4.2] and is an Iwasawa-theoretic version of a conjecture that was independently proposed by Mazur–Rubin [MR16] and Sano [San14]. The latter two conjectures, in turn, unify the central conjectures of Burns in [Bur07] and Darmon in [Dar95].

(b) Conjecture (3.11) is known in the following cases:

- $k = \mathbb{Q}$ and K is totally real, in this case the conjecture follows from a classical result of Solomon [Sol92] (see [BKS17, Thm. 4.10]),
- k is totally real and K is CM, in this case the conjecture follows from the validity of the Gross–Stark conjecture that has been settled by Dasgupta, Kakde and Ventullo in [DKV18] (see [BKS17, Thm. 4.9]).

3.4.2 Relation to Darmon derivatives

The following result uses Darmon derivatives to give a more explicit formulation of the Iwasawa-theoretic Mazur–Rubin–Sano Conjecture (3.11), and also explains the relation of the latter conjecture to Conjecture (3.4).

(3.13) Proposition. *The following assertions are equivalent:*

(i) Conjecture (3.11) is valid for the data $(k_\infty/k, K, S, T, V')$,

(ii) Conjecture (3.4) holds for $(k_\infty/k, K, S, T, V')$ and we have an equality

$$\kappa_0 \otimes (\gamma - 1)^e = (-1)^{re} \cdot \mathrm{Rec}_W(\varepsilon^{V'}_{K/k,\Sigma,T}), \tag{16}$$

where κ_0 denotes the Darmon derivative of $\varepsilon_{K_\infty/k,\Sigma,T}$ with respect to a fixed topological generator γ of Γ.

Before turning to the proof of this result, we first record a technical Lemma.

(3.14) Lemma. *Fix a topological generator $\gamma \in \Gamma$ and let $u, \kappa \in \bigcap^r_{\Lambda} U_{K_\infty,\Sigma,T}$ be norm-coherent sequences satisfying $u = (\gamma - 1)^e \kappa$. Then we have*

$$\mathcal{N}_n(u_n) = \nu_n(\kappa_0 \otimes (\gamma - 1)^e) \quad \text{for all } n \in \mathbb{N}.$$

Proof. We calculate:

$$\begin{aligned}\mathcal{N}_n(u_n) &= \mathcal{N}_n((\gamma-1)^e\kappa_n)\\ &= \sum_{\sigma\in\Gamma_n}\sigma(\gamma-1)^e\kappa_n\otimes\sigma^{-1}\\ &= \sum_{\sigma\in\Gamma_n}\sigma\kappa_n\otimes\sigma^{-1}(\gamma-1)^e\\ &= \sum_{\sigma\in\Gamma_n}\sigma\kappa_n\otimes(\gamma-1)^e\\ &= (\mathrm{N}_{\Gamma_n}\kappa_n)\otimes(\gamma-1)^e.\end{aligned}$$

Here the third equality is obtained by reparametrising the sum and the fourth equality follows from

$$\sigma^{-1}(\gamma-1)^e-(\gamma-1)^e=(\sigma^{-1}-1)(\gamma-1)^e\equiv 0 \mod I(\Gamma_n)^{e+1}.$$

The property (13) moreover yields

$$\nu_n(\kappa_0\otimes(\gamma-1)^e)=\nu_n(\mathrm{N}^r_{K_n/K}(\kappa_n)\otimes(\gamma-1)^e)=(\mathrm{N}_{\Gamma_n}\kappa_n)\otimes(\gamma-1)^e.$$

and this then finishes the proof of the Lemma. □

(3.15) Remark. In particular, Lemma (3.14) implies that, for fixed u, the element $\kappa_0\otimes(\gamma-1)^e$ of $(\bigcap^r_{\mathbb{Z}_p[\mathcal{G}]}U_{K,\Sigma,T})\otimes_{\mathbb{Z}_p}(I(\Gamma)^e/I(\Gamma)^{e+1})$ does not depend on the choice of topological generator γ.

We are now in a position to prove Proposition (3.13).

Proof of Proposition (3.13): Let us first assume that statement (ii) holds. In light of Lemma (3.14), the element $\mathfrak{k}=(\mathfrak{k}_n)_n$ given by $\mathfrak{k}_n=\nu_n(\kappa_0\otimes(\gamma-1)^e)\in\big(\bigcap^r_{\mathbb{Z}_p[\mathcal{G}_n]}U_{K_n,\Sigma,T}\big)\otimes_{\mathbb{Z}_p}(I(\Gamma_n)^e/I(\Gamma_n)^{e+1})$ satisfies the requirements of Conjecture (3.11).
Conversely, suppose that Conjecture (3.11) holds true. By assumption $\nu_n(\mathfrak{k}_n)=\mathcal{N}_n(\varepsilon^V_{K_n/k,\Sigma,T})$ for all n, so it follows from [BKS16, Prop. 4.17] that $\varepsilon^V_{K_n/k,\Sigma,T}$ belongs to $I^e_{\Gamma_n}\cdot\bigwedge^r_{\mathbb{Z}_p[\mathcal{G}_n]}\Pi_n$. Thus, we can write $\varepsilon^V_{K_n/k,\Sigma,T}=(\gamma-1)^e x_n$ for some $x_n\in\bigwedge^r_{\mathbb{Z}_p[\mathcal{G}_n]}\Pi_n$.
Let $f_{K_{n+1}/K_n}\colon\bigwedge^r_{\mathbb{Z}_p[\mathcal{G}_{n+1}]}\Pi_{n+1}\to\bigwedge^r_{\mathbb{Z}_p[\mathcal{G}_n]}\Pi_n$ be the map induced by the natural map $\Pi_{n+1}\to\Pi_n$. Since the sequence $(\varepsilon^V_{K_{n+1}/k,\Sigma,T})_{n\geq 0}$ is norm-coherent, it follows from Remark (3.3) (b) that

$$(\gamma-1)^e\cdot(x_n-f_{K_{n+1}/K_n}(x_{n+1}))=\varepsilon^V_{K_n,\Sigma,T}-\mathrm{N}^r_{K_{n+1}/K_n}(\varepsilon^V_{K_{n+1},\Sigma,T})=0,$$

and hence that $x_n-f_{K_{n+1}/K_n}(x_{n+1})$ belongs to $\big(\bigwedge^r_{\mathbb{Z}_p[\mathcal{G}_n]}\Pi_n\big)^{\Gamma_n}$. This shows that the family $(x_n)_{n\geq 0}$ defines an element of $\varprojlim_{n\geq 0}\big(\bigwedge^r_{\mathbb{Z}_p[\mathcal{G}_n]}\Pi_n\big)/\big(\bigwedge^r_{\mathbb{Z}_p[\mathcal{G}_n]}\Pi_n\big)^{\Gamma_n}$ with the limit being taken with respect to the maps f_{K_{n+1}/K_n}.
Observe that in the commutative diagram of transition maps

$$\begin{array}{ccccccccc}0\to&\bigwedge^r_{\mathbb{Z}_p[\mathcal{G}]}\Pi_0&\to&\bigwedge^r_{\mathbb{Z}_p[\mathcal{G}_{n+1}]}\Pi_{n+1}&\to&\Big(\bigwedge^r_{\mathbb{Z}_p[\mathcal{G}_{n+1}]}\Pi_{n+1}\Big)\Big/\Big(\bigwedge^r_{\mathbb{Z}_p[\mathcal{G}_{n+1}]}\Pi_{n+1}\Big)^{\Gamma_{n+1}}&\to 0\\ &\downarrow&&\downarrow{\scriptstyle f_{K_{n+1}/K_n}}&&\downarrow&\\ 0\to&\bigwedge^r_{\mathbb{Z}_p[\mathcal{G}]}\Pi_0&\to&\bigwedge^r_{\mathbb{Z}_p[\mathcal{G}_n]}\Pi_n&\to&\Big(\bigwedge^r_{\mathbb{Z}_p[\mathcal{G}_n]}\Pi_n\Big)\Big/\Big(\bigwedge^r_{\mathbb{Z}_p[\mathcal{G}_n]}\Pi_n\Big)^{\Gamma_n}&\to 0\end{array}$$

the vertical map on the left is multiplication by p. Taking inverse limits (these are all finitely generated $\mathbb{Z}_p$-modules, so compact and therefore taking limits is exact), we get an isomorphism

$$\varprojlim_{n\geq 0}\bigwedge^r_{\mathbb{Z}_p[\mathcal{G}_n]}\Pi_n\cong\varprojlim_{n\geq 0}\big(\bigwedge^r_{\mathbb{Z}_p[\mathcal{G}_n]}\Pi_n\big)\big/\big(\bigwedge^r_{\mathbb{Z}_p[\mathcal{G}_n]}\Pi_n\big)^{\Gamma_n}.$$

Consequently, the family $(x_n)_{n\geq 0}$ can be lifted to an element $(\widetilde{x}_n)_{n\geq 0}$ of $\varprojlim_{n\geq 0} \bigwedge^r_{\mathbb{Z}_p[\mathcal{G}_n]} \Pi_n = \bigwedge^r_{\Lambda} \Pi$. By construction, we have

$$(\gamma - 1)^e \cdot (\widetilde{x}_n)_{n\geq 0} = ((\gamma-1)^e \cdot x_n)_{n\geq 0} = (\varepsilon^V_{K_{n+1}/k,\Sigma,T})_{n\geq 0} \in \bigcap\nolimits^r_{\Lambda} U_{K_\infty,\Sigma,T}$$

and so Lemma (3.7) gives that $(\widetilde{x}_n)_{n\geq 0}$ belongs to $\bigcap^r_{\Lambda} U_{K_\infty,\Sigma,T}$. This shows that Conjecture (3.4) holds, as required to verify the first part of (ii).

For the second part of (ii) we note that Lemma (3.14) implies

$$\nu_n(\widetilde{x}_0 \otimes (\gamma-1)^e) = \mathcal{N}_n(\varepsilon^V_{K_n/k,\Sigma,T}) = \nu_n(\mathfrak{k}_n)$$

for all n. By the injectivity of ν_n, we conclude that $\widetilde{x}_0 \otimes (\gamma-1)^e = \mathfrak{k}_n$ and so the claimed equality (16) follows directly from (15). □

3.4.3 A useful reformulation

In the following, we derive a useful reformulation of the equality (16) that concerns cases where $r \geq e$ and will play a key role in the proof of the Iwasawa-theoretic Mazur–Rubin–Sano Conjecture (3.11) for imaginary quadratic fields k in § 5.1. Note that the verification of Conjecture (3.11) can be reduced to the case $W \subseteq S_{\mathrm{ram}}(k_\infty/k)$ (see [BKS17, Prop. 4.4 (iv)]), and hence it is enough to consider this case.

We will use the map

$$\mathrm{Ord}_W \coloneqq \bigwedge_{v\in W} \mathrm{Ord}_v \colon \bigcap\nolimits^r_{\mathbb{Z}_p[\mathcal{G}]} U_{K,\Sigma,T} \to \bigcap\nolimits^{r-e}_{\mathbb{Z}_p[\mathcal{G}]} U_{K,\Sigma\setminus W,T}$$

with Ord_v defined as $\mathrm{Ord}_v \colon U_{K_n,\Sigma,T} \to \mathbb{Z}_p[\mathcal{G}_n], a \mapsto \sum_{\sigma\in\mathcal{G}_n} \mathrm{ord}_w(\sigma a)\sigma^{-1}$.

(3.16) Proposition. *Assume that $r \geq e$, that $W \subseteq S_{\mathrm{ram}}(k_\infty/k)$, and that Conjecture (3.4) holds for $(k_\infty/k, K, S, T, V')$. The equality (16) holds if and only if the equality*

$$\mathrm{Ord}_W(\kappa_0) \otimes (\gamma-1)^e = (-1)^e \cdot \mathrm{Rec}_W(\varepsilon^V_{K/k,\Sigma\setminus W,T}), \tag{17}$$

holds in $\mathbb{Q}_p\big(\bigcap^{r-e}_{\mathbb{Z}_p[\mathcal{G}]} U_{K,\Sigma\setminus W,T}\big) \otimes_{\mathbb{Z}_p} (I(\Gamma)^e/I(\Gamma)^{e+1})$.

As a key step in the proof of Proposition (3.16) we will show that each side of the conjectural equality (17) can, independently of its validity, be viewed as a 'universal norm'. To explain this, we define the module of *universal norms* to be the submodule of $U_{K,\Sigma,T}$ given by the intersection

$$\mathrm{UN}_{K,\Sigma,T} \coloneqq \bigcap_{n\geq 0} \mathrm{N}_{K_n/K}(U_{K_n,\Sigma,T}).$$

Since the data (K, Σ, T) is fixed throughout this section, we will simply write UN instead of $\mathrm{UN}_{K,\Sigma,T}$. The following technical Lemma describes the properties of UN that will be essential to us.

(3.17) Lemma. *The following claims are valid.*

(a) Let $\chi \in \widehat{\mathcal{G}}$ be a character and write $\mathbb{Q}_p(\chi)$ for the finite extension of $\mathbb{Q}_p$ generated by the values of χ (using the previously fixed isomorphism $\mathbb{C} \cong \mathbb{C}_p$). If we endow $\mathbb{Q}_p(\chi)$ with the natural $\mathcal{G}$-action given by $\sigma \cdot x = \chi(\sigma)x$, then one has

$$\dim_{\mathbb{Q}_p(\chi)}(\mathbb{Q}_p(\chi) \otimes_{\mathbb{Z}_p[\mathcal{G}]} \mathrm{UN}) = |\{v \in S_\infty(k) \mid \chi(\mathcal{G}_v) = 1\}|,$$

where $\mathcal{G}_v$ denotes the decomposition group of v inside $\mathcal{G}$.

(b) If $r \geq e$, then $\bigwedge_{v\in W} \mathrm{Ord}_v$ induces a map

$$\mathbb{Q}_p \bigwedge\nolimits^r_{\mathbb{Z}_p[\mathcal{G}]} \mathrm{UN} \to \mathbb{Q}_p \bigwedge\nolimits^{r-e}_{\mathbb{Z}_p[\mathcal{G}]} U_{K,\Sigma\setminus W,T}$$

which we also denotes as Ord_W. The kernel of this map is then annihilated by the idempotent $e_{S_\infty,r}$ of $\mathbb{Q}_p[\mathcal{G}]$ that is defined as the sum $\sum_\chi e_\chi$ with χ ranging over all characters χ of $\mathcal{G}$ such that $\dim_{\mathbb{Q}_p(\chi)}(\mathbb{Q}_p(\chi) \otimes_{\mathbb{Z}_p[\mathcal{G}]} \mathrm{UN}) = r$.

Proof. To prove (a), we first recall that the natural map $U_{K_\infty,\Sigma,T} \to U_{K,\Sigma,T}$ induces an isomorphism $U_{K_\infty,\Sigma,T} \otimes_{\mathbb{\Lambda}} \mathbb{Z}_p[\mathcal{G}] \cong \mathrm{UN}$ (see, for example, [BD21, Thm. 3.8 (b)] for a proof of this classical result). It follows that there is an isomorphism

$$\mathbb{Q}_p(\chi) \otimes_{\mathbb{Z}_p[\mathcal{G}]} \mathrm{UN} \cong \mathbb{Q}_p(\chi) \otimes_{\mathbb{\Lambda}} U_{K_\infty,\Sigma,T} \cong \mathbb{Q}_p(\chi) \otimes_{\mathbb{\Lambda}_\mathfrak{p}} (U_{K_\infty,\Sigma,T})_\mathfrak{p}, \tag{18}$$

where the subscript $\mathfrak{p}$ denotes localisation at the height-one prime $\mathfrak{p} = \ker\{\mathbb{\Lambda} \xrightarrow{\chi} \mathbb{Q}_p(\chi)\}$ of $\mathbb{\Lambda}$ (note that there is an isomorphism $\mathbb{Q}_p(\chi) \cong \mathbb{\Lambda}_\mathfrak{p}/\mathfrak{p}\mathbb{\Lambda}_\mathfrak{p}$ of $\mathbb{\Lambda}$-modules). The prime $\mathfrak{p}$ does not contain p and hence $\mathbb{\Lambda}_\mathfrak{p}$ is a discrete valuation ring (see [BKS17, § 3C1] for more details). Consequently, $(U_{K_\infty,\Sigma,T})_\mathfrak{p}$ is a free $\mathbb{\Lambda}_\mathfrak{p}$-module and it suffices to calculate its rank in order to prove claim (a). By the exact sequence (9) it is in turn enough to calculate the $\mathbb{\Lambda}_\mathfrak{p}$-rank of the (free part of) the module $H^1(D^\bullet_{K_\infty,\Sigma,T})_\mathfrak{p}$. It is well-known that $A_{K_\infty,\Sigma,T}$ is a $\mathbb{\Lambda}$-torsion module, hence, by the exact sequence (7), we are further reduced to computing the rank of $(X_{K_\infty,\Sigma})_\mathfrak{p}$. Write V_χ for the set $\{v \in S_\infty(k) \mid \chi(\mathcal{G}_v) = 1\}$ that appears in claim (a). Since we assume that no finite place contained in Σ splits completely in k_∞/k, the module $X_{K_\infty,\Sigma\setminus S_\infty(k)}$ is $\mathbb{\Lambda}$-torsion. It then follows from the exact sequence

$$0 \longrightarrow X_{K_\infty,\Sigma\setminus S_\infty(k)} \longrightarrow X_{K_\infty,\Sigma} \longrightarrow Y_{K_\infty,S_\infty(k)} \longrightarrow 0,$$

that $\mathrm{rk}_{\mathbb{\Lambda}_\mathfrak{p}}(X_{K_\infty,\Sigma})_\mathfrak{p} = \mathrm{rk}_{\mathbb{\Lambda}_\mathfrak{p}}(Y_{K_\infty,S_\infty(k)})_\mathfrak{p} = \dim_{\mathbb{Q}_p(\chi)}(\mathbb{Q}_p(\chi) \otimes_{\mathbb{Z}_p[\mathcal{G}]} Y_{K,S_\infty(k)}) = |V_\chi|$, as required to conclude the proof of claim (a).

To prove claim (b), we let χ be a character of $\mathcal{G}$ such that $e_\chi e_{S_\infty,r} = e_\chi$ (in other words, such that $|V_\chi| = r$). By claim (a), the map

$$e_\chi\big(\bigoplus_{v\in W} \mathrm{Ord}_v\big) \colon \mathbb{Q}_p(\chi) \otimes_{\mathbb{Z}_p[\mathcal{G}]} \mathrm{UN} \to \bigoplus_{v\in W} \mathbb{Q}_p(\chi)$$

induced by $\bigoplus_{v\in W} \mathrm{Ord}_v$ is a map on an r-dimensional $\mathbb{Q}_p(\chi)$-vector space. As such, the general result of [BKS16, Lem. 4.2] implies that the injectivity of $e_\chi \mathrm{Ord}_W$ on $\mathbb{Q}_p(\chi) \otimes_{\mathbb{Z}_p[\mathcal{G}]} \bigwedge^r_{\mathbb{Z}_p[\mathcal{G}]} \mathrm{UN}$, and hence claim (b), will follow if we can prove that the map $e_\chi\big(\bigoplus_{v\in W} \mathrm{Ord}_v\big)$ is surjective.

To prove this, we let v be a place in W and choose an integer n such that v is totally ramified in K_∞/K_n. Let $w_n \coloneqq w_{K_n}$ be a place of K_n lying above v and, for $m \geq n$, write w_m for the unique prime of K_m lying above w_n. If h denotes the class number of K_n, then w_n^h is a principal ideal generated by x, say. We then have $\mathrm{N}_{K_m/K_n}(w_m) = w_n$ for all $m \geq n$, hence $\mathrm{N}_{K_n/K}(x) \in \mathrm{N}_{K_m/K}(\mathcal{O}^\times_{K_m,\Sigma})$ for all integers $m \geq 0$. By a standard compactness argument (see, for example, [BD21, Lem. 3.10]) we deduce that $\mathrm{N}_{K_n/K}(x) \in \mathbb{Q}_p\mathrm{UN}$.

By construction, $\mathrm{N}_{K_n/K}(x)$ is a generator of the ideal $(w_n \cap \mathcal{O}_K)^{hf}$, where f is the residual degree of v in K_n/K, and hence is only supported at the prime $w_n \cap \mathcal{O}_K$ above v. In addition, the image y of $\mathrm{N}_{K_n/K}(x)$ under the map $e_\chi(\bigoplus_{v\in W} \mathrm{Ord}_v)$ is non-zero because v is assumed to split completely in K/k. These two facts combine to imply that y generates the copy of $\mathbb{Q}_p(\chi)$ in the codomain of $e_\chi(\bigoplus_{v\in W} \mathrm{Ord}_v)$ that is indexed by v. Since v was chosen to be an arbitrary place in W, this finishes the proof of claim (b). ☐

We now prove Proposition (3.16).

Proof of Proposition (3.16): Let us first show that each side of the conjectural equality (16) belongs to the module

$$e_{S_\infty,r}\mathbb{Q}_p\big(\bigwedge\nolimits^r_{\mathbb{Z}_p[\mathcal{G}]} \mathrm{UN}\big) \otimes_{\mathbb{Z}_p} \big(I(\Gamma)^e / I(\Gamma)^{e+1}\big),$$

with $e_{S_\infty,r}$ as defined in Lemma (3.17) (b). As for the left hand side of (16), it suffices to prove that κ_0 is contained in $e_{S_\infty,r}\mathbb{Q}_p \bigwedge^r_{\mathbb{Z}_p[\mathcal{G}]} \mathrm{UN}$. To this end, we note that, for any character χ of $\mathcal{G}$, the isomorphism (18) induces an isomorphism

$$\mathbb{Q}_p(\chi) \otimes_{\mathbb{\Lambda}_{\mathfrak{p}_\chi}} \big(\bigcap\nolimits^r_{\mathbb{\Lambda}} U_{K_\infty,\Sigma,T}\big)_{\mathfrak{p}_\chi} = \mathbb{Q}_p(\chi) \otimes_{\mathbb{\Lambda}_{\mathfrak{p}_\chi}} \big(\bigwedge\nolimits^r_{\mathbb{\Lambda}} U_{K_\infty,\Sigma,T}\big)_{\mathfrak{p}_\chi} \cong \mathbb{Q}_p(\chi) \otimes_{\mathbb{Z}_p[\mathcal{G}]} \bigwedge\nolimits^r_{\mathbb{Z}_p[\mathcal{G}]} \mathrm{UN}$$

with $\mathfrak{p}_\chi := \ker\{\Lambda \xrightarrow{\chi} \mathbb{Q}_p(\chi)\}$. It therefore suffices to prove that $\kappa = (\kappa_n)_n$ is not supported at $\mathfrak{p}_\chi$ if χ is such that $e_\chi e_{S_\infty,r} = 0$. To do this, we let χ be such a character satisfying $e_\chi e_{S_\infty,r} = 0$. Write K_χ for the kernel field of χ and set $K_{\chi,\infty} := K_\chi k_\infty$, then one has that $\chi(\mathrm{N}_{K_\infty/K_{\infty,\chi}}) \neq 0$ and hence that $\mathrm{N}_{K_\infty/K_{\infty,\chi}}$ is a unit in $\Lambda_{\mathfrak{p}_\chi}$. It follows that

$$\begin{aligned}\Lambda_{\mathfrak{p}_\chi} \cdot \varepsilon_{K_\infty/k,\Sigma,T} = \Lambda_{\mathfrak{p}_\chi} \cdot \mathrm{N}_{K_\infty/K_{\infty,\chi}} \cdot \varepsilon_{K_\infty/k,\Sigma,T} &= \Lambda_{\mathfrak{p}_\chi} \cdot (\mathrm{N}^r_{K_n/K_{\chi,n}}(\varepsilon^V_{K_n/k,\Sigma,T}))_n \\ &= \Lambda_{\mathfrak{p}_\chi} \cdot (\varepsilon^V_{K_{\chi,n}/k,\Sigma,T})_n = 0,\end{aligned}$$

where $K_{\chi,n} := K_\chi k_n$ and the last equality holds because, by assumption on χ, at least $r+1$ infinite places split completely in $K_{\chi,n}/k$. By Lemma (3.1), $\gamma - 1$ is a non-zero divisor in Λ, hence also in $\Lambda_{\mathfrak{p}_\chi}$, and so we deduce that $\Lambda_{\mathfrak{p}_\chi}\kappa = 0$, as desired. This shows the claim for the left hand side of (16).

To investigate the right hand side of (16), we define an idempotent e_r of $\mathbb{Q}_p[\mathcal{G}]$ as the sum of all primitive orthogonal idempotents e_χ with the property that $\dim_{\mathbb{Q}_p(\chi)}(\mathbb{Q}_p(\chi) \otimes_{\mathbb{Z}_p[\mathcal{G}]} U_{K,\Sigma\setminus W}) = r$ or, equivalently, $\dim_{\mathbb{Q}_p(\chi)}(\mathbb{Q}_p(\chi) \otimes_{\mathbb{Z}_p[\mathcal{G}]} U_{K,\Sigma}) = r'$. By [Tat84, Ch. I, Prop. 3.4] one then has $e_r \varepsilon^{V'}_{K/k,\Sigma,T} = \varepsilon^{V'}_{K/k,\Sigma,T}$ and so it suffices to prove that $e_\chi \mathrm{Rec}_W(\varepsilon^{V'}_{K/k,\Sigma,T})$ belongs to the required module for all characters χ of $\mathcal{G}$ satisfying $e_\chi e_r = e_\chi$. If χ is such a character and $e_\chi \mathrm{Rec}_W$ is the zero map, then $e_\chi \mathrm{Rec}_W(\varepsilon^{V'}_{K/k,\Sigma,T})$ vanishes and, in particular, belongs to the required module. If both $e_\chi e_r = e_\chi$ and $e_\chi \mathrm{Rec}_W$ is not the zero map, on the other hand, then [BKS16, Lem. 4.2] shows that the map

$$\begin{aligned}\Big(\bigoplus_{v\in W} \mathrm{Rec}_{v,\chi}\Big)\colon (\mathbb{Q}_p(\chi) \otimes_{\mathbb{Z}_p[\mathcal{G}]} U_{K,\Sigma,T}) &\to \bigoplus_{v\in W} \big(\mathbb{Q}_p(\chi) \otimes_{\mathbb{Z}_p} \big(I(\Gamma)/I(\Gamma)^2\big)\big), \\ a &\mapsto \Big(\sum_{\sigma\in\mathcal{G}} \chi(\sigma) \otimes (\mathrm{rec}_w(\sigma^{-1}a) - 1)\Big)_{v\in W}\end{aligned}$$

must be surjective. Since the codomain of this map has dimension $e = |W|$ as a $\mathbb{Q}_p(\chi)$-vector space, the assumption on χ implies that its kernel is of dimension r. Now, $\mathbb{Q}_p(\chi) \otimes_{\mathbb{Z}_p[\mathcal{G}]} \mathrm{UN}$ is contained in, and hence agrees with by comparing dimensions using Lemma (3.17) (a), the kernel of $\bigoplus_{v\in W} \mathrm{Rec}_{v,\chi}$. By [BKS16, Lem. 4.2] we then have

$$\begin{aligned}&\mathrm{im}\Big\{\mathbb{Q}_p(\chi) \otimes_{\mathbb{Z}_p[\mathcal{G}]} \big(\textstyle\bigwedge^{r'}_{\mathbb{Z}_p[\mathcal{G}]} U_{K,\Sigma,T}\big) \xrightarrow{\mathrm{Rec}_W} \mathbb{Q}_p(\chi) \otimes_{\mathbb{Z}_p[\mathcal{G}]} \big(\textstyle\bigwedge^{r}_{\mathbb{Z}_p[\mathcal{G}]} U_{K,\Sigma,T}\big) \otimes_{\mathbb{Z}_p} \big(I(\Gamma)^e/I(\Gamma)^{e+1}\big)\Big\} \\ &= \mathbb{Q}_p(\chi) \otimes_{\mathbb{Z}_p[\mathcal{G}]} \big(\textstyle\bigwedge^{r}_{\mathbb{Z}_p[\mathcal{G}]} \mathrm{UN}\big) \otimes_{\mathbb{Z}_p} \big(I(\Gamma)^e/I(\Gamma)^{e+1}\big).\end{aligned}$$

This finishes the proof of the claim for the right hand side of (16).

By scalar extension, the map Ord_W induces a map

$$\Big(\mathbb{Q}_p \textstyle\bigwedge^{r}_{\mathbb{Z}_p[\mathcal{G}]} \mathrm{UN}\Big) \otimes_{\mathbb{Z}_p} \big(I(\Gamma)^e/I(\Gamma)^{e+1}\big) \to \Big(\mathbb{Q}_p \textstyle\bigwedge^{r-e}_{\mathbb{Z}_p[\mathcal{G}]} \mathrm{UN}\Big) \otimes_{\mathbb{Z}_p} \big(I(\Gamma)^e/I(\Gamma)^{e+1}\big) \tag{19}$$

which we also denote by Ord_W. Since Ord_W is injective on $e_{S_\infty,r}\mathbb{Q}_p\mathrm{UN}$ by Lemma (3.17) (b) and $I(\Gamma)^e/I(\Gamma)^{e+1} \cong \Gamma$ is isomorphic to $\mathbb{Z}_p$, the kernel of the map (19) is annihilated by the idempotent $e_{S_\infty,r}$ as well. Thus, the equation (16) holds if and only if

$$\mathrm{Ord}_W(\kappa_0) \otimes (\gamma - 1)^e = (-1)^{re} \cdot (\mathrm{Ord}_W \circ \mathrm{Rec}_W)(\varepsilon^{V'}_{K/k,\Sigma,T})$$

holds. Now, by virtue of (1) being a homomorphism, we have

$$\mathrm{Ord}_W \circ \mathrm{Rec}_W = (-1)^{e^2} \cdot (\mathrm{Rec}_W \circ \mathrm{Ord}_W) = (-1)^e \cdot (\mathrm{Rec}_W \circ \mathrm{Ord}_W)$$

and so the Lemma follows by combining this with the fact that

$$\mathrm{Ord}_W(\varepsilon^{V'}_{K/k,\Sigma,T}) = (-1)^{re} \cdot \varepsilon^{V}_{K/k,\Sigma\setminus W,T}, \tag{20}$$

which holds by [San14, Prop. 3.6]. □

3.4.4 A functorial property

Suppose to be given a subextension K' of K/k with the property that all archimedean places split completely in K/K' (which ensures that V is equal to the set of all places of k that split completely in $K'_\infty \coloneqq K'k_\infty$).
The following result describes the functorial behaviour of the Iwasawa-theoretic Mazur–Rubin–Sano Conjecture upon passing from K to K'.

(3.18) Lemma. *If Conjecture (3.11) holds for the data $(k_\infty/k, K, S, T, V')$, then also for $(k_\infty/k, K', S, T, V')$.*

Proof. By Proposition (3.13) it suffices to investigate the functorial behaviour of Conjecture (3.4) and equation (16).
To do this, let us write $\Gamma' \coloneqq \mathrm{Gal}(K'_\infty/K')$, $\Lambda' \coloneqq \mathbb{Z}_p[\![\mathrm{Gal}(K'_\infty/k)]\!]$ etc. for the obvious variants for K' of already introduced notation. Let us write $\mathrm{N}^r_{K_\infty/K'_\infty} \colon \bigcap^r_{\Lambda'} U_{K'_\infty,\Sigma,T} \to \bigcap^r_{\Lambda} U_{K_\infty,\Sigma,T}$ for the map induced by the norm maps $\mathrm{N}^r_{K_n/K'_n}$. If Conjecture (3.4) holds for $(k_\infty/k, K, S, T, V')$, then it follows that

$$\varepsilon_{K'_\infty/k,\Sigma,T} = \mathrm{N}^r_{K_\infty/K'_\infty}(\varepsilon_{K_\infty/k,\Sigma,T}) \in \mathrm{N}^r_{K_\infty/K'_\infty}(I^e_\Gamma \bigcap\nolimits^r_{\Lambda} U_{K_\infty,\Sigma,T}) \subseteq I^e_{\Gamma'} \bigcap\nolimits^r_{\Lambda'} U_{K'_\infty,\Sigma,T},$$

as required to verify this conjecture for $(k_\infty/k, K', S, T, V')$. Assume the validity of the conjecture for the remainder of this proof.
Fix a topological generator γ of Γ. Writing n for the integer satisfying $K \cap K'_\infty = K'_n$, we have an isomorphism $\Gamma \cong (\Gamma')^n$ induced by the restriction map $\mathrm{res}_{K_\infty/K'_\infty} \colon \mathrm{Gal}(K_\infty/k) \to \mathrm{Gal}(K'_\infty/k)$, hence we can find a topological generator γ' of Γ' such that $\mathrm{res}_{K_\infty/K'_\infty}(\gamma) = (\gamma')^{p^n}$. Let κ_0 and κ'_0 the Darmon derivatives of $\varepsilon^V_{K_\infty/k,\Sigma,T}$ and $\varepsilon^V_{K'_\infty/k,\Sigma,T}$ with respect to γ and γ', respectively. By definition, these are the bottom values of norm-coherent sequences $(\kappa_m)_{m\geq 0}$ and $(\kappa'_m)_m$ which satisfy

$$(\gamma - 1)^e \cdot \kappa_m = \varepsilon^V_{K_m,\Sigma,T} \quad \text{and} \quad (\gamma' - 1)^e \cdot \kappa'_m = \varepsilon^V_{K'_m,\Sigma,T}$$

for all integers $m \geq 0$. Define an element of Λ' by $x \coloneqq \sum_{i=0}^{p^n-1} (\gamma')^i$ and note that $x = \frac{(\gamma')^{p^n}-1}{\gamma'-1}$.
We may then calculate that

$$\begin{aligned}(\gamma' - 1)^e x^e \cdot \mathrm{N}^r_{K_m/K'_{n+m}}(\kappa_m) &= (\mathrm{res}_{K_\infty/K'_\infty}(\gamma) - 1)^e \cdot \mathrm{N}^r_{K_m/K'_{n+m}}(\kappa_m) = \mathrm{N}^r_{K_m/K'_{n+m}}((\gamma-1)^e \kappa_m) \\ &= \mathrm{N}^r_{K_m/K'_{n+m}}(\varepsilon^V_{K_m,\Sigma,T}) = \varepsilon^V_{K_{n+m},\Sigma,T},\end{aligned}$$

hence, by uniqueness, it follows that $(\kappa'_{m+n})_{m\geq 0} = (x^e \mathrm{N}^r_{K_m/K'_{n+m}}(\kappa_m))_{m\geq 0}$. In particular,

$$\kappa'_0 = \mathrm{N}^r_{K'_n/K'}(\kappa'_n) = \mathrm{N}^r_{K'_n/K'}(\mathrm{N}^e_{\Gamma'_n} \cdot \mathrm{N}^r_{K/K'_n}(\kappa_n)) = p^{ne} \mathrm{N}^r_{K/K'}(\kappa_n).$$

This implies that

$$\begin{aligned}\kappa'_0 \otimes (\gamma' - 1)^e &= (p^{ne} \mathrm{N}^r_{K/K'}(\kappa_n)) \otimes (\gamma' - 1)^e = \mathrm{N}^r_{K/K'}(\kappa_n) \otimes ((\gamma')^{p^n} - 1)^e \\ &= \mathrm{N}^r_{K/K'}(\kappa_n) \otimes (\mathrm{res}_{K_\infty/K'_\infty}(\gamma) - 1)^e.\end{aligned}$$

For clarity, let us write $\mathrm{Rec}_{W,K/k}$ and $\mathrm{Rec}_{W,K'/k}$ for the instances of Rec associated with K and K', respectively. In order to finish the proof that (16) for K implies that this equation also holds for K', it then suffices to note that, by a straightforward calculation, we have

$$\begin{aligned}((\mathrm{N}^r_{K/K'} \otimes \mathrm{res}_{K_\infty/K'_\infty}) \circ \mathrm{Rec}_{W,K/k})(\varepsilon^{V'}_{K/k,\Sigma,T}) &= (\mathrm{Rec}_{W,K'/k} \circ \mathrm{N}^{r'}_{K/K'})(\varepsilon^{V'}_{K/k,\Sigma,T}) \\ &= \mathrm{Rec}_{W,K'/k}(\varepsilon^{V'}_{K'/k,\Sigma,T}),\end{aligned}$$

where, by slight abuse of notation, we have written $\mathrm{res}_{K_\infty/K'_\infty}$ for the map $I(\Gamma)^e/I(\Gamma)^e \to I(\Gamma')^e/I(\Gamma')^{e+1}$ induced by $\mathrm{res}_{K_\infty/K'_\infty}$. □

4 The Gross–Kuz'min conjecture and condition (F)

In this section we will investigate a conjecture due to Gross [Gro81] and, independently, Kuz'min [Kuz72].

4.1 Coinvariants of class groups

Let k_∞/k denote a $\mathbb{Z}_p$-extension. In this section we do *not* need to assume that no finite place contained in Σ splits completely in k_∞/k. With this exception, we resume the assumptions and notation introduced at the beginning of § 3. In particular, we set $K_\infty \coloneqq Kk_\infty$ and

$$A_{K_\infty,M,T} \coloneqq \varprojlim_{n\geq 0} A_{K_n,M,T} \qquad \text{for any } M \supseteq S_\infty(k).$$

If $M = S_\infty(k)$ or $T = \varnothing$, then we will suppress the respective set in the notation.

(4.1) Conjecture (Gross–Kuz'min)**.** *If $K_\infty^{\mathrm{cyc}}/K$ is the cyclotomic $\mathbb{Z}_p$-extension, then the module of Γ-coinvariants $(A_{K_\infty^{\mathrm{cyc}},\Sigma})_\Gamma \coloneqq A_{K_\infty^{\mathrm{cyc}},\Sigma} \otimes_{\mathbb{Z}_p[\![\Gamma]\!]} \mathbb{Z}_p$ is finite.*

(4.2) Remark. (a) It is necessary to work with Σ-class groups in this context because in general it is *not* true that the Γ-coinvariants of $A_{K_\infty^{\mathrm{cyc}}}$ are finite (see [Kol91, Prop. 1.17] and [Gre73, Prop. 2] for examples). However, it is known to be true if there is only one prime above p in K or if K is totally real and Leopoldt's conjecture holds for K (see [Was97, Prop. 13.22] and [Gre73, Prop. 1], respectively). In particular, Conjecture (4.1) holds in these cases.

(b) We remind the reader that for any finitely generated $\mathbb{Z}_p[\![\Gamma]\!]$-torsion module M the module M_Γ is finite if and only if the module M^Γ of Γ-invariants of M is finite (see, for example, [CS06, App. A.2, Prop. 2]). Conjecture (4.1) can therefore also formulated as the statement that $(A_{K_\infty^{\mathrm{cyc}},\Sigma})^\Gamma$ is finite.

We follow Burns, Kurihara and Sano [BKS17, § 5A] in considering the following condition which, although motivated by the above Conjecture of Gross–Kuz'min, is not a conjecture itself but instead known to not hold in general (see [Kis83, bottom of p. 401], [JS95, Thm. 10] for examples of non-cyclotomic $\mathbb{Z}_p$-extensions that provide counterexamples). The validity of this condition plays a crucial role in the descent formalism of Burns–Kurihara–Sano (see [BKS17, Thm. 5.2]).

(4.3) Condition (F)**.** The $\mathbb{Z}_p$-extension K_∞/K is such that the module of Γ-coinvariants $(A_{K_\infty,\Sigma})_\Gamma$ is finite.

(4.4) Remark. The validity of condition (F) is known in the following important cases (see also [HK22, § 2] for an overview of further results):

(a) If $k = \mathbb{Q}$, then Conjecture (4.1) is (implicitly) proved by Greenberg [Gre73].

(b) If there is exactly one prime of K that ramifies in K_∞/K, then condition (F) is a consequence of Chevalley's ambiguous class number formula (cf. [Kle19, Ex. 2.7]).

(c) If $|S_p(K)| \leq 2$ and K_∞ is the cyclotomic $\mathbb{Z}_p$-extension of K, then the validity of Conjecture (4.1) follows from (b) and a result of Kleine [Kle19, Thm. B].
Said result of Kleine hinges upon the following fact (see the proof of [Kle19, Lem. 3.5]): Let $N/\mathbb{Q}$ be a normal extension and suppose that $x \in \mathcal{O}_{N,S_p(N)}$ is such that we have $\log_p \mathrm{N}_{N_\mathfrak{p}/\mathbb{Q}_p}(x) = 0$ for all $\mathfrak{p} \in S_p(N)$. Then the valuation $\mathrm{ord}_\mathfrak{p}(x)$ is the same for all $\mathfrak{p} \in S_p(N)$.
However, the proof of this assertion given in *loc. cit.* contains an inaccuracy and we therefore take the opportunity to provide a better argument. Let $\mathfrak{p}_0 \in S_p(N)$ be such that $n \coloneqq \mathrm{ord}_{\mathfrak{p}_0}(x)$ is minimal among $\{\mathrm{ord}_\mathfrak{p}(x) \mid \mathfrak{p} \in S_p(N)\}$. Write e_p for the ramification

degree of $N/\mathbb{Q}$ at p, then $x^{e_p}p^{-n}$ is a unit at $\mathfrak{p}_0$ and integral at any other finite place of N. By assumption $\log_p \mathrm{N}_{N_{\mathfrak{p}_0}/\mathbb{Q}_p}(x) = 0$, hence also $\log_p \mathrm{N}_{N_{\mathfrak{p}_0}/\mathbb{Q}_p}(x^{e_p}p^{-n}) = 0$ and we can find an integer $m \geq 0$ such that $\mathrm{N}_{N_{\mathfrak{p}_0}/\mathbb{Q}_p}(x^{e_p}p^{-n})^m = 1$. Let $G_{\mathfrak{p}_0} \subseteq \mathrm{Gal}(N/\mathbb{Q})$ be the decomposition group at $\mathfrak{p}_0$ and set $M = N^{G_{\mathfrak{p}_0}}$. Then we have

$$\mathrm{N}_{N_{\mathfrak{p}_0}/\mathbb{Q}_p}(x^{me_p}p^{-mn}) = \mathrm{N}_{N/M}(x^{me_p}p^{-mn}) = 1,$$

so $x^{me_p}p^{-mn}$ is a unit in $\mathcal{O}_N$ and it follows that

$$\mathrm{ord}_{\mathfrak{p}}(x^{me_p}p^{-mn}) = 0 \quad \Leftrightarrow \quad \mathrm{ord}_{\mathfrak{p}}(x) = n$$

for all $\mathfrak{p} \in S_p(N)$. This finishes the proof of the claim.

(d) Suppose that condition (F) holds for a fixed $\mathbb{Z}_p$-extension K_∞ of K. Kleine has proved in [Kle17, Cor. 3.6] that there exists an integer $n_0 \geq 1$ such that condition (F) also holds for all $\mathbb{Z}_p$-extensions K'_∞ of K with the following property: The n-th layers K'_n and K_n agree for all $n \leq n_0$, and $S_{\mathrm{ram}}(K'_\infty/K) \subseteq S_{\mathrm{ram}}(K_\infty/K)$.

In § 4.4 we will prove condition (F) in new instances. To end this subsection, we record a few elementary functorial properties of condition (F).

(4.5) Lemma. *The following hold:*

(a) $(A_{K_\infty,\Sigma,T})_\Gamma$ is finite if and only if $(A_{K_\infty,\Sigma})_\Gamma$ is finite.

(b) If Σ' is a finite set of places of k which contains $S_\infty(k) \cup S_{\mathrm{ram}}(K_\infty/K)$ and is such that $\Sigma' \subseteq \Sigma$, then $(A_{K_\infty,\Sigma,T})_\Gamma$ is finite as soon as $(A_{K_\infty,\Sigma',T})_\Gamma$ is. If no place in $\Sigma \setminus \Sigma'$ splits completely in K_∞/K, then the converse is true as well.

Proof. For every integer $n \geq 0$, set $\mathbb{F}^{\times,p}_{T_{K_n}} \coloneqq \mathbb{Z}_p \otimes_{\mathbb{Z}} \bigoplus_{w \in T_{K_n}} (\mathcal{O}_{K_n}/w)^\times$. The exact sequence

$$\mathbb{F}^{\times,p}_{T_{K_n}} \longrightarrow A_{K_n,\Sigma,T} \longrightarrow A_{K_n,\Sigma} \longrightarrow 0$$

then implies that it is sufficient to show the module $\varprojlim_{n\geq 0} \mathbb{F}^{\times,p}_{T_{K_n}}$, where the transition maps are induced by the relevant norm maps, has finite Γ-coinvariants in order to prove (a). To do this, we note that $\mathbb{F}^{\times,p}_{T_{K_n}}$ is a cyclic group and that each norm map $\mathbb{F}^{\times,p}_{T_{K_{n+1}}} \to \mathbb{F}^{\times,p}_{T_{K_n}}$ is surjective, which implies that we can find a norm-coherent family $(x_n)_{n\geq 0}$ such that each x_n is a generator of $\mathbb{F}^{\times,p}_{T_{K_n}}$. These choices of generators give rise to exact sequences

$$0 \longrightarrow \bigoplus_{v \in T} \mathbb{Z}_p[\mathcal{G}_{K_n}] \xrightarrow{(1-\mathrm{N}v^{-1}\cdot \mathrm{Frob}_v)_v} \bigoplus_{v \in T} \mathbb{Z}_p[\mathcal{G}_{K_n}] \longrightarrow \mathbb{F}^{\times,p}_{T_{K_n}} \longrightarrow 0, \tag{21}$$

where the third arrow sends 1 to x_n (see [Chi85, (4.16)]). By construction, the exact sequences (21) are compatible and so we may pass to the limit over n to obtain an exact sequence

$$0 \longrightarrow \bigoplus_{v \in T} \Lambda \longrightarrow \bigoplus_{v \in T} \Lambda \longrightarrow \varprojlim_{n\geq 0} \mathbb{F}^{\times,p}_{T_{K_n}} \longrightarrow 0. \tag{22}$$

By taking Γ-coinvariants of (22) we obtain that

$$\Big(\varprojlim_{n\geq 0} \mathbb{F}^{\times,p}_{T_{K_n}}\Big)_\Gamma \cong \mathrm{coker}\Big\{ \bigoplus_{v \in T} \mathbb{Z}_p[\mathcal{G}] \xrightarrow{1-\mathrm{N}v^{-1}\cdot\mathrm{Frob}_v} \bigoplus_{v \in T} \mathbb{Z}_p[\mathcal{G}] \Big\} \overset{(21)}{\cong} \mathbb{F}^{\times,p}_{T_K}$$

is finite, as desired.

The first part of (b) holds because $A_{K_\infty,\Sigma,T}$ is a quotient of $A_{K_\infty,\Sigma',T}$. For the second part we note that, as a consequence of the assumption, any place $v \in \Sigma \setminus \Sigma'$ is inert in K_m/K_n for big enough integers $n, m \geq 0$. It follows that the norm map $A_{K_m,\Sigma',T} \to A_{K_n,\Sigma',T}$ induces multiplication by p^{m-n} on the class of v. Thus, we must have $A_{K_\infty,\Sigma',T} = A_{K_\infty,\Sigma,T}$ and this proves the claim. □

4.2 A criterion for the validity of condition (F)

For any character χ of $\mathcal{G}$, we endow $\mathbb{Z}_p[\operatorname{im}\chi]$ with the natural $\mathcal{G}$-action given by $\sigma \cdot x := \chi(\sigma)x$. If M is a finite set of places of k and $\mathcal{G}_v \subseteq \mathcal{G}$ denotes the decomposition group of any place v in M, then we moreover set

$$r_M(\chi) := \dim_{\mathbb{C}}(e_\chi \mathbb{C} X_{K,M}) = \begin{cases} |\{v \in M \mid \chi(\mathcal{G}_v) = 1\}| & \text{if } \chi \neq 1, \\ |M| - 1 & \text{if } \chi = 1. \end{cases}$$

(4.6) Theorem. *For any character χ of $\mathcal{G}$ with $r_\Sigma(\chi) = r'$, the following assertions are equivalent.*

(i) The module $(A_{K_\infty,\Sigma})_\Gamma \otimes_{\mathbb{Z}_p[\mathcal{G}]} \mathbb{Z}_p[\operatorname{im}\chi]$ is finite.

(ii) The map

$$U_{K,\Sigma} \to \bigoplus_{v \in W} \big(\mathbb{Z}_p[\operatorname{im}\chi] \otimes_{\mathbb{Z}_p} \Gamma\big), \quad a \mapsto \Big(\sum_{\sigma \in \mathcal{G}} \chi(\sigma) \otimes \operatorname{rec}_w(\sigma^{-1}a)\Big)_{v \in W}$$

has finite cokernel.

The proof of this result will be given in § 4.3.

(4.7) Remark. If K_∞/K is the cyclotomic $\mathbb{Z}_p$-extension, then Theorem (4.6) is already known due to [Kol91, Thm. 1.14]. If, in addition, K is a CM extension of a totally real field k and the character χ is totally odd, then Gross has proved in [Gro81, Prop. 1.16] that condition (ii) holds if there is at most one prime $\mathfrak{p}$ of k above p such that $\chi(\mathfrak{p}) = 1$.

We end this subsection by recording the following technical observation that will prove useful in applications of Theorem (4.6).

(4.8) Lemma. *Let $\chi \in \widehat{\mathcal{G}}$ be a character, write K_χ for the subfield of K cut out by the character χ with Galois group $\mathcal{G}_\chi := \operatorname{Gal}(K/K_\chi)$, and denote by $\Gamma_\chi := \operatorname{Gal}(K_{\chi,\infty}/K_\chi)$ the Galois group of the $\mathbb{Z}_p$-extension $K_{\chi,\infty}$ of K_χ.*
The following assertions are equivalent:

(i) $(A_{K_\infty,\Sigma,T})_\Gamma \otimes_{\mathbb{Z}_p[\mathcal{G}]} \mathbb{Z}_p[\operatorname{im}\chi]$ is finite,

(ii) $(A_{K_{\chi,\infty},\Sigma,T})_{\Gamma_\chi} \otimes_{\mathbb{Z}_p[\mathcal{G}_\chi]} \mathbb{Z}_p[\operatorname{im}\chi]$ is finite.

Proof. Let m be such that $K \cap K_{\chi,\infty} = K_{\chi,m}$ and write H for $\operatorname{Gal}(K/K_{\chi,m})$, which we can identify with $\operatorname{Gal}(K_\infty/K_{\chi,\infty})$ and therefore view as a subgroup of $\operatorname{Gal}(K_{\chi,\infty}/k)$. Observe that the norm maps $\mathrm{N}_{K_n/K_{\chi,n+m}} \colon A_{K_n,\Sigma,T} \to A_{K_{\chi,n+m},\Sigma,T}$ induce a map

$$\mathrm{N}_{K_\infty/K_{\chi,\infty}} \colon A_{K_\infty,\Sigma,T} \to A_{K_{\chi,\infty},\Sigma,T}$$

which factors as

$$\begin{array}{ccc} A_{K_\infty,\Sigma,T} & \xrightarrow{\cdot \mathrm{N}_H} & A^H_{K_\infty,\Sigma,T} \\ {\scriptstyle \mathrm{N}_{K_\infty/K_{\chi,\infty}}}\big\downarrow & \nearrow_{i} & \\ A_{K_{\chi,\infty},\Sigma,T} & & \end{array}$$

with i the natural map induced by the inclusions $K_{\chi,n} \subseteq K_n$.
Define a height-one prime ideal of Λ as $\mathfrak{p} = \ker\{\Lambda \xrightarrow{\chi} \mathbb{Z}_p[\operatorname{im}\chi]\}$. We have $\chi(\mathrm{N}_H) = |H| \neq 0$, hence $\mathrm{N}_H \in \Lambda_{\mathfrak{p}}^\times$ and so multiplication by N_H (which is the same as $i \circ \mathrm{N}_{K_\infty/K_{\chi,\infty}}$) becomes an isomorphism. Moreover, the composite $\mathrm{N}_{K_\infty/K_{\chi,\infty}} \circ i$ coincides with multiplication by $|H|$ and is therefore also bijective after localisation at $\mathfrak{p}$. It follows that $\mathrm{N}_{K_\infty/K_{\chi,\infty}}$ induces an isomorphism

$$\big(A_{K_\infty,\Sigma,T}\big)_{\mathfrak{p}} \xrightarrow{\simeq} \big(A_{K_{\chi,\infty},\Sigma,T}\big)_{\mathfrak{p}}.$$

Now, we have an isomorphism of Λ-modules $\Lambda_\mathfrak{p}/\mathfrak{p}\Lambda_\mathfrak{p} \cong \mathbb{Q}_p(\chi)$ and thus obtain

$$\begin{aligned}\mathbb{Q}_p(\chi) \otimes_{\mathbb{Z}_p[\mathcal{G}]} (A_{K_\infty,\Sigma,T})_\Gamma &\cong (\Lambda_\mathfrak{p}/\mathfrak{p}\Lambda_\mathfrak{p}) \otimes_\Lambda A_{K_\infty,\Sigma,T} \\ &\cong (\Lambda_\mathfrak{p}/\mathfrak{p}\Lambda_\mathfrak{p}) \otimes_\Lambda A_{K_{\chi,\infty},\Sigma,T} \\ &\cong (\Lambda_\mathfrak{p}/\mathfrak{p}\Lambda_\mathfrak{p}) \otimes_{\mathbb{Z}_p[\mathcal{G}_\chi]} \big(\mathbb{Z}_p[\mathcal{G}_\chi] \otimes_\Lambda A_{K_{\chi,\infty},\Sigma,T}\big) \\ &\cong \mathbb{Q}_p(\chi) \otimes_{\mathbb{Z}_p[\mathcal{G}_\chi]} (A_{K_{\chi,\infty},\Sigma,T})_{\Gamma_\chi},\end{aligned}$$

thereby proving the Lemma. □

4.3 Computation of Bockstein homomorphisms

To prove Theorem (4.6) we will perform a computation of *Bockstein homomorphisms* similar to [BKS17, § 5B] (see also [Bur07, § 10]). To explain this, let us fix a topological generator γ of Γ and consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Pi_\infty & \xrightarrow{\cdot(\gamma-1)} & \Pi_\infty & \longrightarrow & \Pi_0 & \longrightarrow & 0 \\ & & \big\downarrow \phi & & \big\downarrow \phi & & \big\downarrow \phi_0 & & \\ 0 & \longrightarrow & \Pi_\infty & \xrightarrow{\cdot(\gamma-1)} & \Pi_\infty & \longrightarrow & \Pi_0 & \longrightarrow & 0 \end{array}$$

obtained from the representatives of the complexes $D^\bullet_{K_\infty,\Sigma,T}$ and $D^\bullet_{K,\Sigma,T}$ that are constructed in Proposition (3.2). Applying the Snake Lemma to this diagram then gives a surjective boundary map

$$\delta_\gamma \colon U_{K,\Sigma,T} \to H^1(D^\bullet_{K_\infty,\Sigma,T})^\Gamma.$$

Note that there is an isomorphism $H^1(D^\bullet_{K_\infty,\Sigma,T})_\Gamma \cong H^1(D^\bullet_{K,\Sigma,T})$ obtained by taking Γ-coinvariants of (9) and comparing with (10). Recall that in § 3.1 we have also fixed a labelling on the places in Σ such that $W = \{v_{r+1}, \dots, v_{r'}\}$ as well as a place $w_i \coloneqq w_{K,i}$ of K above v_i for every $i \in \{r+1, \dots, r'\}$. We therefore obtain a composite map

$$\beta_{\infty,\gamma} \colon U_{K,\Sigma,T} \xrightarrow{\delta_\gamma} H^1(D^\bullet_{K_\infty,\Sigma,T})^\Gamma \to H^1(D^\bullet_{K_\infty,\Sigma,T})_\Gamma \cong H^1(D^\bullet_{K,\Sigma,T}) \xrightarrow{\pi_K} X_{K,\Sigma} \xrightarrow{\oplus w_i^*} \mathbb{Z}_p[\mathcal{G}]^{\oplus e},$$

where the second arrow is the restriction of the canonical quotient map $H^1(D^\bullet_{K_\infty,\Sigma,T}) \to H^1(D^\bullet_{K_\infty,\Sigma,T})_\Gamma$, the third arrow is the map π_K from (5), and the last arrow is the sum, over all $i \in \{r+1, \dots, r'\}$, of the $\mathbb{Z}_p[\mathcal{G}]$-linear dual maps w_i^* of the places w_i when considered as an element of $Y_{K,\Sigma}$.
In addition, we define the map

$$\mathrm{rec}_W^\oplus \colon U_{K,\Sigma} \to \big(\mathbb{Z}_p[\mathcal{G}] \otimes_\mathbb{Z} \Gamma\big)^{\oplus e}, \quad a \mapsto \Big(\sum_{\sigma \in \mathcal{G}} \sigma \otimes \mathrm{rec}_{w_i}(\sigma^{-1} a)\Big)_{r < i \leq r'},$$

which coincides with the map in statement (ii) of Theorem (4.6) after tensoring with $\mathbb{Z}_p[\operatorname{im}\chi]$. The following result allows one to compare the maps $\beta_{\infty,\gamma}$ and $\mathrm{rec}_W^\oplus$.

(4.9) Lemma. *Every choice of topological generator γ induces a commutative diagram*

$$\begin{array}{ccc} U_{K,\Sigma} & \xrightarrow{\mathrm{rec}_W^\oplus} & \big(\mathbb{Z}_p[\mathcal{G}] \otimes_{\mathbb{Z}_p} \Gamma\big)^{\oplus e} \\ \subseteq \big\uparrow & & \simeq \big\uparrow \\ U_{K,\Sigma,T} & \xrightarrow{\beta_{\infty,\gamma}} & \mathbb{Z}_p[\mathcal{G}]^{\oplus e}. \end{array}$$

Here the vertical map on the right hand side sends, in each component, any element $x \in \mathbb{Z}_p[\mathcal{G}]$ to $x \otimes \gamma$.

Proof. Let us write $\delta \colon U_{K,\Sigma,T} \to H^1(D^\bullet_{K_\infty,\Sigma,T}) \otimes_\Lambda I_\Gamma$ for the connecting homomorphism that arises from the exact triangle

$$D^\bullet_{K_\infty,\Sigma,T} \otimes^\mathbb{L}_\Lambda I_\Gamma \longrightarrow D^\bullet_{K_\infty,\Sigma,T} \longrightarrow D^\bullet_{K_\infty,\Sigma,T} \otimes^\mathbb{L}_\Lambda \mathbb{Z}_p[\mathcal{G}] \longrightarrow \quad , \tag{23}$$

and write α for the map $H^1(D^\bullet_{K_\infty,\Sigma,T}) \otimes_{\mathbb{\Lambda}} I_\Gamma \to H^1(D^\bullet_{K,\Sigma,T}) \otimes_{\mathbb{Z}_p[\mathcal{G}]} (I_\Gamma/I_\Gamma^2)$ obtained by passing to Γ-coinvariants and identifying $H^1(D^\bullet_{K_\infty,\Sigma,T})_\Gamma$ with $H^1(D^\bullet_{K,\Sigma,T})$. Using the isomorphism $I_\Gamma/I_\Gamma^2 \cong \mathbb{Z}_p[\mathcal{G}] \otimes_{\mathbb{Z}_p} (I(\Gamma)/I(\Gamma)^2) \cong \mathbb{Z}_p[\mathcal{G}] \otimes_{\mathbb{Z}_p} \Gamma$ induced by (4) and the isomorphism $I(\Gamma)/I(\Gamma)^2 \cong \Gamma$ that sends the class of $\gamma - 1$ to γ, we may then define the composite map

$$\beta_\infty \colon U_{K,\Sigma,T} \xrightarrow{\delta} H^1(D^\bullet_{K_\infty,\Sigma,T}) \otimes_{\mathbb{\Lambda}} I_\Gamma \xrightarrow{\alpha} H^1(D^\bullet_{K,\Sigma,T}) \otimes_{\mathbb{Z}_p[\mathcal{G}]} \left(I_\Gamma/I_\Gamma^2\right)$$
$$\xrightarrow{\pi_K} X_{K,\Sigma} \otimes_{\mathbb{Z}_p[\mathcal{G}]} \left(I_\Gamma/I_\Gamma^2\right) \xrightarrow{\oplus w^*} \left(\mathbb{Z}_p[\mathcal{G}] \otimes_{\mathbb{Z}_p} \Gamma\right)^{\oplus e}.$$

By passing to the limit in the result of [BKS16, Lem. 5.21], we find that β_∞ coincides with $\mathrm{rec}^\oplus_W$.

The long exact sequence in cohomology of the triangle (23) shows that the image of δ is given by $H^1(D^\bullet_{K_\infty,\Sigma,T})^\Gamma \otimes_{\mathbb{Z}_p} I(\Gamma)$. Using the representatives of the complexes $D^\bullet_{K_\infty,\Sigma,T}$ and $D^\bullet_{K,\Sigma,T} \cong D^\bullet_{K_\infty,\Sigma,T} \otimes^{\mathbb{L}}_{\mathbb{\Lambda}} \mathbb{Z}_p[\mathcal{G}]$ constructed in Proposition (3.2) to explicitely calculate δ, we find a commutative diagram

$$\begin{array}{ccccc} U_{K,\Sigma,T} & \xrightarrow{\delta} & H^1(D^\bullet_{K_\infty,\Sigma,T})^\Gamma \otimes_{\mathbb{\Lambda}} I_\Gamma & \longrightarrow & \left(\mathbb{Z}_p[\mathcal{G}] \otimes_{\mathbb{Z}_p} \Gamma\right)^{\oplus e} \\ \| & & \simeq\uparrow & & \simeq\uparrow \\ U_{K,\Sigma,T} & \xrightarrow{\delta_\gamma} & H^1(D^\bullet_{K_\infty,\Sigma,T})^\Gamma & \longrightarrow & \mathbb{Z}_p[\mathcal{G}]^{\oplus e} \end{array}$$

in which the composite of the arrows in the first line is β_∞, the composite of the arrows in the second line is $\beta_{\infty,\gamma}$, the first vertical isomorphism sends x to $x \otimes (\gamma - 1)$, and the second vertical isomorphism sends x to $x \otimes \gamma$, as required to prove the Lemma. □

We now give the proof of Theorem (4.6).

Proof (of Theorem (4.6)): To ease exposition in this proof, we define the 'χ-part' of a $\mathbb{Z}_p[\mathcal{G}]$-module M to be $M^\chi := M \otimes_{\mathbb{Z}_p[\mathcal{G}]} \mathbb{Z}_p[\mathrm{im}\,\chi, 1/|\mathcal{G}|]$. Note that $M \mapsto M^\chi$ defines an exact functor in the category of $\mathbb{Z}_p[\mathcal{G}]$-modules.

Taking Γ-invariants and coinvariants of the exact sequence (7) we obtain the exact sequence

$$\begin{array}{l} 0 \longrightarrow A^{\Gamma,\chi}_{K_\infty,\Sigma,T} \longrightarrow H^1(D^\bullet_{K_\infty,\Sigma,T})^{\Gamma,\chi} \xrightarrow{\pi^{\Gamma,\chi}} X^{\Gamma,\chi}_{K_\infty,\Sigma} \\ \quad\hookrightarrow (A_{K_\infty,\Sigma,T})^\chi_\Gamma \longrightarrow H^1(D^\bullet_{K_\infty,\Sigma,T})^\chi_\Gamma \xrightarrow{\pi^\chi_\Gamma} (X_{K_\infty,\Sigma})^\chi_\Gamma \longrightarrow 0. \end{array} \tag{24}$$

To investigate the map labelled π^χ_Γ in this sequence, we write B for its kernel and consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B & \longrightarrow & H^1(D^\bullet_{K_\infty,\Sigma,T})^\chi_\Gamma & \xrightarrow{\pi^\chi_{K,\infty,\Gamma}} & (X_{K_\infty,\Sigma})^\chi_\Gamma & \longrightarrow & 0 \\ & & \downarrow & & \downarrow\simeq & & \downarrow & & \\ 0 & \longrightarrow & A^\chi_{K,\Sigma,T} & \longrightarrow & H^1(D^\bullet_{K,\Sigma,T})^\chi & \xrightarrow{\pi^\chi_K} & X^\chi_{K,\Sigma} & \longrightarrow & 0 \end{array}$$

with bottom line given by (the χ-part of) the exact sequence (5). By Lemma (4.10) (a) below, the vertical arrow on the right is surjective and has finite kernel, hence the Snake Lemma implies that B injects with finite index into $A^\chi_{K,\Sigma,T}$. In particular, B is finite and so the exact sequence (24) shows that the map $\pi^{\Gamma,\chi}$ has finite cokernel if and only if $(A_{K_\infty,\Sigma,T})^\chi_\Gamma$ is finite. By Lemma (4.5) (a) the latter holds if and only if $(A_{K_\infty,\Sigma})^\chi_\Gamma$ or, equivalently, $(A_{K_\infty,\Sigma})_\Gamma \otimes_{\mathbb{Z}_p[\mathcal{G}]} \mathbb{Z}_p[\mathrm{im}\,\chi]$ is finite, so it is enough to prove that statement (ii) in Theorem (4.6) is equivalent to the finiteness of $C := \mathrm{coker}\,\pi^{\Gamma,\chi}$.

To do this, we first note that, since $U_{K,\Sigma,T}$ is a finite-index subgroup of $U_{K,\Sigma,T}$, Lemma (4.9) shows that statement (ii) in Theorem (4.6) is equivalent to the assertion that $\beta_{\infty,\gamma}$ has finite cokernel after passing to χ-parts. As for this latter cokernel, we recall that the map δ_γ is

surjective, and hence that it coincides with the cokernel of the map $f_\chi\colon H^1(D^\bullet_{K_\infty,\Sigma,T})^{\Gamma,\chi} \to H^1(D^\bullet_{K,\Sigma,T})^\chi \to \mathbb{Z}_p[\mathcal{G}]^{\oplus e,\chi}$ that appears in the definition of (the χ-part of) $\beta_{\infty,\gamma}$.
We have a commutative diagram

$$\begin{array}{ccccccc} & & H^1(D^\bullet_{K_\infty,\Sigma,T})^{\Gamma,\chi} & \xrightarrow{\pi^{\Gamma,\chi}} & (X_{K_\infty,\Sigma})^{\Gamma,\chi} & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow f_\chi & & \downarrow g_\chi & & \downarrow & & \\ 0 & \longrightarrow & \mathbb{Z}_p[\mathcal{G}]^{\oplus e,\chi} & = & \mathbb{Z}_p[\mathcal{G}]^{\oplus e,\chi} & \longrightarrow & 0, & & \end{array} \tag{25}$$

where the arrow labelled g_χ is the composite map $(X_{K_\infty,\Sigma})^{\Gamma,\chi} \to X^\chi_{K,\Sigma} \overset{\oplus w^*}{\to} \mathbb{Z}_p[\mathcal{G}]^{\oplus e,\chi}$. Observe that $(X_{K,\Sigma\setminus(W\cup S_\infty(k))})^\chi$ vanishes because $r_\Sigma(\chi) = r'$. By taking χ-parts of the exact sequence

$$0 \longrightarrow X_{K,\Sigma\setminus(W\cup S_\infty(k))} \longrightarrow X_{K,\Sigma\setminus S_\infty(k)} \longrightarrow Y_{K,W} \longrightarrow 0$$

we then obtain an isomorphism $(X_{K,\Sigma\setminus S_\infty(k)})^\chi \cong Y^\chi_{K,W}$. By Lemma (4.10) (b) below, the map g_χ is therefore injective with finite cokernel. An application of the Snake Lemma to (25) now shows that coker f_χ identifies with a finite-index submodule of C. This proves the claim, thereby concluding the proof of Theorem (4.6). □

(4.10) Lemma. *The following claims are valid.*

(a) The natural projection map $(X_{K_\infty,\Sigma})_\Gamma \to X_{K,\Sigma}$ is surjective and has finite kernel.

(b) Let $V'' \subseteq \Sigma$ be the subset of places that split completely in k_∞. The image of the composite map $(X_{K_\infty,\Sigma})^\Gamma \to (X_{K_\infty,\Sigma})_\Gamma \to X_{K,\Sigma}$ is contained in $X_{K,\Sigma\setminus V''}$. The resulting map $(X_{K_\infty,\Sigma})^\Gamma \to X_{K,\Sigma\setminus V''}$ is injective and has finite cokernel.

Proof. From the tautological exact sequence $0 \to X_{K_\infty,\Sigma} \to Y_{K_\infty,\Sigma} \to \mathbb{Z}_p \to 0$ we obtain the exact sequence

$$0 \longrightarrow (X_{K_\infty,\Sigma})^\Gamma \longrightarrow (Y_{K_\infty,\Sigma})^\Gamma \longrightarrow \mathbb{Z}_p \longrightarrow (X_{K_\infty,\Sigma})_\Gamma \longrightarrow (Y_{K_\infty,\Sigma})_\Gamma \longrightarrow \mathbb{Z}_p \longrightarrow 0.$$

Note that $I \coloneqq \mathrm{im}\{(Y_{K_\infty,\Sigma})^\Gamma \to \mathbb{Z}_p\}$ is nontrivial, and hence of finite index in $\mathbb{Z}_p$. The Snake Lemma, applied to the commutative diagram

$$\begin{array}{ccccccccc} \mathbb{Z}_p/I & \longrightarrow & (X_{K_\infty,\Sigma})_\Gamma & \longrightarrow & (Y_{K_\infty,\Sigma})_\Gamma & \longrightarrow & \mathbb{Z}_p & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & X_{K,\Sigma} & \longrightarrow & Y_{K,\Sigma} & \longrightarrow & \mathbb{Z}_p & \longrightarrow & 0, \end{array}$$

therefore implies that it is enough to prove that the natural map $(Y_{K_\infty,\Sigma})_\Gamma \to Y_{K,\Sigma}$ is an isomorphism in order to prove claim (a).
To do this, we write $\mathcal{G}_{v,\infty} \subseteq \mathcal{G}_\infty \coloneqq \mathrm{Gal}(K_\infty/k)$ and $\mathcal{G}_v \subseteq \mathcal{G}$ for the decomposition groups of a place $v \in \Sigma$, and, for any $n \in \mathbb{N}_0$, denote by $w_{K,n}$ our fixed choice of extension of v to K_n. One then has that

$$(Y_{K_\infty,\Sigma})_\Gamma = \Big(\bigoplus_{v\in\Sigma} \mathbb{Z}_p[\![\mathcal{G}_\infty/\mathcal{G}_{v,\infty}]\!] \cdot (w_{K_n})_n\Big) \otimes_{\Lambda} \mathbb{Z}_p[\mathcal{G}] \cong \bigoplus_{v\in\Sigma} \mathbb{Z}_p[\mathcal{G}/\mathcal{G}_v] \cdot w_K = Y_{K,\Sigma}$$

because the natural map $\Lambda \to \mathbb{Z}_p[\mathcal{G}]$ sends each $\mathcal{G}_{v,\infty}$ onto $\mathcal{G}$. This finishes the proof of (a).
To prove claim (b), first observe that $Y_{K_\infty,V''}$ is a projective $\mathbb{Z}_p[\![\Gamma]\!]$-module because, by definition, every place contained in V'' splits completely in k_∞/k. It follows that the Γ-invariants of $Y_{K_\infty,V''}$ vanish and this combines with the exact sequence

$$0 \longrightarrow Y_{K_\infty,\Sigma\setminus V''} \longrightarrow Y_{K_\infty,\Sigma} \longrightarrow Y_{K_\infty,V''} \longrightarrow 0$$

to imply that $(Y_{K_\infty,\Sigma})^\Gamma = (Y_{K_\infty,\Sigma\setminus V''})^\Gamma$. In particular, this shows the first claim in (b).
The argument used in (a) shows that $(X_{K_\infty,\Sigma\setminus V''})_\Gamma$ surjects onto $X_{K,\Sigma\setminus V''}$ with finite kernel.

Setting $C := \ker\{X_{K_\infty,\Sigma\setminus V''} \to X_{K,\Sigma\setminus V''}\}$, we obtain a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I_\Gamma \cdot X_{K_\infty,\Sigma\setminus V''} & \longrightarrow & X_{K_\infty,\Sigma\setminus V''} & \longrightarrow & (X_{K_\infty,\Sigma\setminus V''})_\Gamma & \longrightarrow & 0 \\ & & \downarrow & & \| & & \twoheaddownarrow & & \\ 0 & \longrightarrow & C & \longrightarrow & X_{K_\infty,\Sigma\setminus V''} & \longrightarrow & X_{K,\Sigma\setminus V''} & \longrightarrow & 0. \end{array} \tag{26}$$

Now, because every place in $\Sigma \setminus V''$ splits completely in K_∞/K, we can find an integer m with the property that all such places have full decomposition group in K_∞/K_m. In particular, Γ^m acts trivially on $X_{K_\infty,\Sigma\setminus V''}$ and so the action of Γ on $X_{K_\infty,\Sigma\setminus V''}$ factors through $\Gamma_m = \Gamma/\Gamma^m$. Taking Γ_m-invariants of the second row of (26) then gives that, firstly, the cokernel of the natural map $(X_{K_\infty,\Sigma})^\Gamma \to X_{K,\Sigma\setminus V''}$ identifies with a submodule of $H^1(\Gamma_m, C)$, hence is finite. Secondly, the kernel of this map is given by C^{Γ_m} and, by an application of the Snake Lemma to the diagram (26), contains $\big(I_{\Gamma_m} \cdot X_{K_\infty,\Sigma\setminus V''}\big)^{\Gamma_m}$ as a submodule of finite index. Since N_{Γ_m} acts as multiplication by $p^m = |\Gamma_m|$ on this $\mathbb{Z}_p$-torsion free submodule, and also annihilates it, it must vanish. This shows that C^{Γ_m} is finite, which already forces it to be trivial because $X_{K_\infty,\Sigma\setminus V''}$ is $\mathbb{Z}_p$-torsion free. This completes the proof of claim (b). □

4.4 Proof of condition (F) in special cases

In this section we shall explain how one can prove the equivalent conditions of Theorem (4.6) in special cases, thereby proving a precise version of Theorem C from the introduction. Crucial ingredient in these arguments is the following Lemma, which is a direct consequence of Brumer's p-adic analogue of Baker's Theorem from transcendence theory.

(4.11) Lemma. *Suppose there exists an infinite place of k that splits completely in K. Let w be a p-adic place of K and write $\iota_w \colon K \hookrightarrow K_w$ for the corresponding embedding of K into its completion K_w at w. Then there is $a \in \mathcal{O}_K^\times$ such that*

$$\sum_{\sigma\in\mathcal{G}} \chi(\sigma)\cdot(\log_p \circ \iota_w)(\sigma^{-1}a) \neq 0$$

for all non-trivial characters $\chi \in \widehat{\mathcal{G}}$.

Proof. Let v be an infinite place of k that splits completely in K. The $\mathbb{R}[\mathcal{G}]$-module isomorphism $\mathbb{R}\mathcal{O}_K^\times \cong \mathbb{R}\mathcal{X}_{K,S_\infty(k)}$ induced by the Dirichlet regulator map (2) implies that there is an injective map $\mathcal{X}_{K,\{v\}} \hookrightarrow \mathcal{O}_K^\times$ of $\mathbb{Z}[\mathcal{G}]$-modules. We can therefore find a unit $a \in \mathcal{O}_K^\times$ with the property that the module $\mathbb{Z}[\mathcal{G}]a$ generated by a is isomorphic to the augmentation ideal $\ker\{\mathbb{Z}[\mathcal{G}] \to \mathbb{Z}\}$. Fix a non-trivial character $\chi \in \widehat{\mathcal{G}}$ and suppose that $\sum_{\sigma\in\mathcal{G}} \chi(\sigma)\cdot(\log_p \circ\iota_w)(\sigma^{-1}a)$ vanishes. Since $\sum_{\sigma\in\mathcal{G}} \sigma a = 0$, this is to say that

$$\sum_{\sigma\in\mathcal{G}\setminus\{1\}} (\chi(\sigma)-1)\cdot(\log_p \circ\iota_w)(\sigma^{-1}a) = 0.$$

The elements $\chi(\sigma)$ are algebraic over $\mathbb{Q}$ and not all zero because χ is not the trivial character, hence Brumer's p-adic analogue of Baker's theorem [Bru67] (see also [NSW08, Thm. 10.3.14]) asserts the existence of integers $n_\sigma \in \mathbb{Z}$, not all of them zero, such that

$$(\log_p \circ\iota_w)\Big(\sum_{\sigma\in\mathcal{G}\setminus\{1\}} n_\sigma \sigma a\Big) = 0.$$

After multiplying by a suitable integer if necessary we may therefore assume that $\sum_{\sigma\in\mathcal{G}\setminus\{1\}} n_\sigma\sigma a$ is a power of p, and hence, because a is a unit, is trivial. However, this contradicts the fact that the set $\{\sigma a \mid \sigma \in \mathcal{G}\setminus\{1\}\}$ is $\mathbb{Z}$-linearly independent. □

(4.12) Theorem. *The following claims are valid.*

(a) *Suppose* $S_{\mathrm{ram}}(k_\infty/k)$ *contains at most two primes. If it contains exactly two primes, assume that* k *is not* $\mathbb{Q}$ *or an imaginary quadratic field, and that there is a place* $v \in S_{\mathrm{ram}}(k_\infty/k)$ *such that the completion of* k *at* v *is equal to* $\mathbb{Q}_p$*. Then* $(A_{k_\infty,\Sigma})_{\mathrm{Gal}(k_\infty/k)}$ *is finite.*

(b) *Suppose that, for each non-trivial character* $\chi \in \widehat{\mathcal{G}}$ *there is at most one finite place* $v \in S_{\mathrm{ram}}(k_\infty/k)$ *which is such that* $\chi(\mathcal{G}_v) = 1$*. If there is such a place, assume that it satisfies* $k_v = \mathbb{Q}_p$ *and that there is at least one infinite place of* k *that splits in* K*. Then* $(A_{K_\infty,\Sigma})_\Gamma$ *is finite as soon as* $(A_{k_\infty,\Sigma})_{\mathrm{Gal}(k_\infty/k)}$ *is finite.*

(4.13) Remark. We give two examples of concrete situations in which Theorem (4.12) can be applied.

(a) Suppose that k is an imaginary quadratic field in which p splits completely. If we fix a prime ideal $\mathfrak{p}$ of k above p, then there is a unique $\mathbb{Z}_p$-extension k_∞ of k that is unramified outside $\mathfrak{p}$. Given this, Theorem (4.12) implies that condition (F) holds for all abelian extensions K/k with respect to the $\mathbb{Z}_p$-extension $K_\infty = K \cdot k_\infty$ of K.
We remark, however, that this fact is already known, see the proof of [Rub88, Thm. 1.4] where it is deduced from the known validity of Leopoldt's Conjecture in this setting (the latter is of course also derived from the Theorem of Brumer–Baker, so our proof could maybe be considered more direct).

(b) Suppose that k is a CM field but not imaginary quadratic. Assume that p splits completely in $k/\mathbb{Q}$ and fix a prime $\mathfrak{p}$ of k lying above p. We denote by $\overline{\mathfrak{p}}$ the complex conjugate of $\mathfrak{p}$. By class field theory, there exists a $\mathbb{Z}_p$-extension k_∞ of k that is unramified outside $\{\mathfrak{p}, \overline{\mathfrak{p}}\}$. Theorem (4.12) (a) now implies that $(A_{k_\infty,\Sigma})_{\mathrm{Gal}(k_\infty/k)}$ is finite.

Proof. To prove (a) we first note that we may assume $|S_{\mathrm{ram}}(k_\infty/k)| = 2$ because the case $|S_{\mathrm{ram}}(k_\infty/k)| = 1$ is covered by Remark (4.4) (b). Write v_0 for the unique place in $S_{\mathrm{ram}}(k_\infty/k) \setminus \{v\}$ and set $V' = \Sigma \setminus \{v_0\}$. Taking $K = k$, we have $W = \{v\}$ and $r_\Sigma(1) = |V'|$ in this situation. By Theorem (4.6) (with χ taken to be the trivial character) we are therefore reduced to proving that the map $\mathrm{rec}_W^\oplus$ has finite cokernel in this situation. The codomain of $\mathrm{rec}_W^\oplus$ is a free $\mathbb{Z}_p$-module of rank one, hence it suffices to show that $\mathrm{rec}_W^\oplus$ is not the zero map in order to prove the finiteness of its cokernel.
To do this, we first give a more explicit description of $\mathrm{rec}_W^\oplus$. Let $\Gamma_v \subseteq \mathrm{Gal}(k_\infty/k) = \Gamma$ be the decomposition group at v, which we identify with the Galois group over $\mathbb{Q}_p$ of the completion of k_∞ at our fixed choice of place above v. Consider the composite map

$$f\colon \mathbb{Z}_p \overset{\exp_p}{\cong} 1 + p\mathbb{Z}_p \subseteq \mathbb{Q}_p^{\times,\wedge} \twoheadrightarrow \Gamma_v,$$

where $\mathbb{Q}_p^{\times,\wedge}$ denotes the profinite completion of $\mathbb{Q}_p^\times$, and the last arrow is the local reciprocity map. By local class field theory, the image of f is the inertia subgroup I_v of Γ_v and hence has finite index $d \coloneqq (\Gamma : I_v)$ in Γ. In particular, f has to be injective because it is a non-trivial map between free $\mathbb{Z}_p$-modules of rank one. Composing the inverse of f with the isomorphism $\Gamma \overset{\cdot d}{\to} \Gamma^d = I_w$, we therefore obtain an isomorphism $\Gamma \cong \mathbb{Z}_p$ that we can use to identify $\mathrm{rec}_W^\oplus$ with the map

$$U_{k,\Sigma} \to \mathbb{Z}_p \otimes_{\mathbb{Z}_p} \Gamma \cong \mathbb{Z}_p, \quad a \mapsto f^{-1}(d \cdot \mathrm{rec}_v(a)).$$

If a belongs to $\mathcal{O}_k^\times$, then $f^{-1}(d \cdot \mathrm{rec}_v(a)) = d(\log_p \circ \iota_v)(a)$ by construction of f. Since k is not $\mathbb{Q}$ or an imaginary quadratic field, we can find an element a of $\mathcal{O}_k^\times$ of infinite order. For this element a one then has that $(\log_p \circ \iota_v)(a)$ does not vanish because the kernel of Iwasawa's p-adic logarithm is generated by p and the roots of unity contained in $\mathbb{Q}_p$. This concludes the proof of claim (a).

To prove claim (b), we will show that $(A_{K_\infty,\Sigma})_\Gamma \otimes_{\mathbb{Z}_p[\mathcal{G}]} \mathbb{Z}_p[\mathrm{im}\,\chi]$ is finite for every character χ of $\mathcal{G}$.

By Lemma (4.8), we may assume that K is the field cut out by the character χ for this purpose. Further it suffices to consider non-trivial characters χ because for $\chi = 1$ the claim is the same as the assumption that $(A_{k_\infty,\Sigma})_{\mathrm{Gal}(k_\infty/k)}$ is finite. Due to Lemma (4.5) (b) we may moreover assume that $S = S_\infty(k)$, i.e. $\Sigma = S_\infty(k) \cup S_{\mathrm{ram}}(K_\infty/k)$. Take $V' = \{v \in \Sigma \mid \chi(\mathcal{G}_v) = 1\}$, then we have $r_\Sigma(\chi) = r'$ and so may apply Theorem (4.6). We shall now show that statement (ii) in (4.6), namely that $\mathrm{rec}_W^{\oplus}$ has finite cokernel after passing to χ-parts, holds true in this situation. As W contains only finite places, W must be contained in $S_{\mathrm{ram}}(K_\infty/K)$. If W is empty, then $\mathrm{rec}_W^{\oplus}$ is the zero map and there is nothing to prove. We may therefore assume that $W = \{v\}$ for a single place $v \in S_p(k)$. Write $\mathrm{rec}_W^{\oplus,\chi}$ for the map defined in statement (ii) of (4.6) and note that its codomain is a free $\mathbb{Z}_p[\mathrm{im}\,\chi]$-module of rank one. In particular, the cokernel of $\mathrm{rec}_W^{\oplus,\chi}$ is finite if and only if $\mathrm{rec}_W^{\oplus,\chi}$ is not the zero map.
By the discussion in the proof of (a), we can identify the map $\mathrm{rec}_W^{\oplus,\chi}$ with

$$U_{K,\Sigma} \longrightarrow \mathbb{Z}_p[\mathrm{im}\,\chi] \otimes_{\mathbb{Z}_p} \Gamma \cong \mathbb{Z}_p[\mathrm{im}\,\chi], \quad a \mapsto \sum_{\sigma \in \mathcal{G}} \chi(\sigma) \cdot f^{-1}(d \cdot \mathrm{rec}_w(\sigma^{-1}a)).$$

Now, Lemma (4.11) implies the existence of a unit $a \in \mathcal{O}_K^\times$ such that

$$\sum_{\sigma \in \mathcal{G}} \chi(\sigma) \cdot f^{-1}(d \cdot \mathrm{rec}_w(\sigma^{-1}a)) = d \sum_{\sigma \in \mathcal{G}} \chi(\sigma) \cdot (\log_p \circ \iota_w)(a) \neq 0$$

is nonzero. In particular, the map $\mathrm{rec}_W^{\oplus,\chi}$ is nonzero, as required. □

If p does not split completely in k, the situation is more complicated. We are however able to prove the following result concerning the case of k being an imaginary quadratic field.

(4.14) Theorem. *Let k be an imaginary quadratic field such that p is not split in k.*

- *(a) Let m be the number of p-adic primes of K. Then there are at most m distinct $\mathbb{Z}_p$-extension k_∞ of k such that $(A_{K_\infty,\Sigma})_\Gamma$ is infinite.*
- *(b) There are infinitely many $\mathbb{Z}_p$-extensions k_∞ of k such that the following conditions are satisfied:*
  - *(i) $(A_{K_\infty,\Sigma})_\Gamma$ is finite,*
  - *(ii) at most two finite places of k split completely in k_∞, neither of them contained in Σ.*

Proof. By Lemma (4.8), property (a) is satisfied if, for every character $\chi \in \widehat{\mathcal{G}}$, the module $(A_{K_{\chi,\infty},\Sigma})_{\Gamma_\chi} \otimes_{\mathbb{Z}_p[\mathcal{G}_\chi]} \mathbb{Z}_p[\mathrm{im}\,\chi]$ is finite (with notation as in Lemma (4.8)). By Remark (4.4) (b) this holds for $\chi = 1$ because k contains only one prime above p, so it suffices to consider non-trivial characters. By Lemma (4.5) it is enough to check if $A_{K_{\chi,\infty},\Sigma_\chi} \otimes_{\mathbb{Z}_p[\mathcal{G}_\chi]} \mathbb{Z}_p[\mathrm{im}\,\chi]$ is finite, where $\Sigma_\chi = S_{\mathrm{ram}}(K_{\chi,\infty}/k) \cup S_\infty(k)$. In this situation, we may apply Theorem (4.6) (with K and W taken to be K_χ and $W_\chi \coloneqq \{v \in \Sigma_\chi \setminus S_\infty(k) \mid \chi(v) = 1\}$, respectively) which reduces us to verifying that the map $\mathrm{rec}_{K_\chi/k,W_\chi}^{\oplus,\chi}$ defined in (ii) of Theorem (4.6) as

$$U_{K_\chi,\Sigma_\chi} \to \big(\mathbb{Z}_p[\mathrm{im}\,\chi] \otimes_{\mathbb{Z}_p} \Gamma_\chi\big)^{\oplus|W_\chi|}, \quad a \mapsto \Big(\sum_{\sigma \in \mathcal{G}_\chi} \chi(\sigma) \otimes \mathrm{rec}_w(\sigma^{-1}a)\Big)_{v \in W_\chi}$$

has finite cokernel. Observe that we must have $W_\chi \subseteq S_{\mathrm{ram}}(k_\infty/k) = \{v\}$ for the unique place v of k above p. If $W_\chi = \varnothing$, there is nothing to show. We may therefore assume that $W_\chi = \{v\}$, and we let $\widehat{\mathcal{G}}_W$ be the subset of $\widehat{\mathcal{G}}$ comprising all non-trivial characters χ such that $W_\chi \neq \varnothing$. The basic strategy of the remainder of this proof is now to show that the claim holds if one avoids, if necessary, certain 'bad' $\mathbb{Z}_p$-extensions.
To explain this, let F_∞ be the compositum of all $\mathbb{Z}_p$-extensions of k, which is a $\mathbb{Z}_p^2$-extension as a consequence of the known validity of Leopoldt's Conjecture for this setting. In fact, we know that $\mathrm{Gal}(F_\infty/k) = \mathbb{Z}_p\gamma_{\mathrm{cyc}} \oplus \mathbb{Z}_p\gamma_{\mathrm{anti}}$, where $\gamma_{\mathrm{cyc}}, \gamma_{\mathrm{anti}} \in \mathrm{Gal}(F_\infty/k)$ are such that the fixed

fields $F_\infty^{\langle\gamma_{\mathrm{cyc}}\rangle}$ and $F_\infty^{\langle\gamma_{\mathrm{anti}}\rangle}$ are the cyclotomic and anti-cyclotomic $\mathbb{Z}_p$-extensions of k, respectively. Write $\mathrm{Gal}(F_\infty/k)_v = \mathfrak{g}_w$ for a choice of decomposition group at v inside $\mathrm{Gal}(F_\infty/k)$ and w inside $\mathfrak{g} \coloneqq \mathrm{Gal}(F_\infty K_\chi/K_\chi)$, respectively.
Denote the completion of k at v by k_v, and write $\mathrm{art}_v \colon k_v^{\times,\wedge} \to \mathfrak{g}_w$ for the local reciprocity map, where $k_v^{\times,\wedge}$ is the profinite completion of $k_v^\times$. We then consider the map

$$\mathrm{Art}^{\oplus,\chi}_{K_\chi/k,\{v\}} \colon U_{K_\chi,\Sigma_\chi} \to \mathbb{Z}_p[\operatorname{im}\chi] \otimes_{\mathbb{Z}_p} \mathfrak{g}, \quad a \mapsto \sum_{\sigma\in\mathcal{G}} \chi(\sigma) \otimes (\mathrm{art}_v \circ \iota_w)(\sigma^{-1}a).$$

We shall now first show that $\mathrm{Art}^{\oplus,\chi}_{K/k,\{v\}}$ is not the zero map. For this purpose, we let $\mathfrak{p}_v$ be the maximal ideal of the valuation ring $\mathcal{O}_v$ of k_v, and also choose an integer s large enough such that the p-adic exponential map $\exp_p$ converges on $\mathfrak{p}_v^s$. Given this, we have a composite map

$$\mathfrak{p}_v^s \overset{\exp_p}{\cong} 1 + \mathfrak{p}_v^s \subseteq k_v^{\times,\wedge} \overset{\mathrm{art}_v}{\twoheadrightarrow} \mathfrak{g}_w. \tag{27}$$

By local class field theory, the kernel of art_v contains a uniformiser ϖ of k_v and, because the quotient $k_v^{\times,\wedge}/(\mathbb{Z}_p\varpi)$ is of $\mathbb{Z}_p$-rank two (as is $\mathfrak{g}_w$), said kernel must contain the subgroup $\mathbb{Z}_p\varpi$ with finite index. It follows that the intersection of $1+\mathfrak{p}_v^s$ with the kernel of art_v is trivial. This shows that the map (27) is injective and hence, comparing $\mathbb{Z}_p$-ranks again, must have finite cokernel of cardinality d, say. This way we obtain an injection

$$\omega \colon \mathfrak{g} \cong \mathfrak{g}^d \hookrightarrow \mathfrak{p}_v^s$$

as the composite of multiplication by d and the inverse of the map (27) restricted to its image. To prove that $\mathrm{Art}^{\oplus,\chi}_{K/k,\{v\}}$ is non-zero it now suffices to verify that the composite $(\mathrm{id} \otimes \omega) \circ \mathrm{Art}^{\oplus,\chi}_{K/k,\{v\}}$ is nonzero. To this end, we let $a \in \mathcal{O}_K^\times$ be the unit provided by Lemma (4.11) and choose an integer t that is big enough such that $ta \in 1 + \mathfrak{p}_v^s$. We then have that

$$((\mathrm{id} \otimes \omega) \circ \mathrm{Art}^{\oplus,\chi}_{K_\chi/k,\{v\}})(ta) = td \sum_{\sigma\in\mathcal{G}} \chi(\sigma) \otimes (\log_p \circ \iota_w)(\sigma^{-1}a) \neq 0,$$

as required to prove that the map $\mathrm{Art}^{\oplus,\chi}_{K_\chi/k,\{v\}}$ is nonzero.
To proceed, write $\mathrm{res} \colon \mathfrak{g} \to \Gamma_\chi$ for the canonical restriction map and consider the commutative diagram

$$\begin{array}{ccc} U_{K_\chi,\Sigma_\chi} & \xrightarrow{\mathrm{Art}^{\oplus,\chi}_{K_\chi/k,\{v\}}} & \mathbb{Z}_p[\operatorname{im}\chi] \otimes_{\mathbb{Z}_p} \mathfrak{g} \\ \| & & \downarrow{\scriptstyle \mathrm{id}\otimes\mathrm{res}} \\ U_{K_\chi,\Sigma_\chi} & \xrightarrow{\mathrm{rec}^{\oplus,\chi}_{K_\chi/k,\{v\}}} & \mathbb{Z}_p[\operatorname{im}\chi] \otimes_{\mathbb{Z}_p} \Gamma_\chi. \end{array} \tag{28}$$

Suppose that $\mathrm{rec}^{\oplus,\chi}_{K_\chi/k,\{v\}}$ is the zero map. The above commutative diagram then implies that, firstly, the map $\mathrm{Art}^{\oplus,\chi}_{K_\chi/k,\{v\}}$ cannot be surjective, and hence, being nonzero, its image must be a free $\mathbb{Z}_p[\operatorname{im}\chi]$-module of rank one. Set $\mathscr{G} \coloneqq \mathrm{Gal}(\mathbb{Q}_p(\operatorname{im}\chi)/\mathbb{Q}_p)$ and endow $\mathbb{Z}_p[\operatorname{im}\chi] \otimes_{\mathbb{Z}_p} \mathfrak{g}$ with a $\mathscr{G}$-action by means of $g \cdot (x \otimes y) \coloneqq g(x) \otimes y$. By [Bou03, Ch. V, § 10.4, Prop. 7 (b)], the $\mathbb{Q}_p(\operatorname{im}\chi)$-vector space spanned by the image of $\mathrm{Art}^{\oplus,\chi}_{K_\chi/k,\{v\}}$ then admits a $\mathscr{G}$-invariant $\mathbb{Q}_p(\operatorname{im}\chi)$-basis. In particular, by clearing the denominator if necessary, we can find a nonzero element δ of $\mathfrak{g}$ that belongs to the image of $\mathrm{Art}^{\oplus,\chi}_{K_\chi/k,\{v\}}$. This element δ cuts out an extension $L_\delta \coloneqq (F_\infty K)^{(\mathbb{Z}_p\delta)}$ of k, the Galois group of which over k is isomorphic to the quotient of $\mathfrak{g}$ by $\mathbb{Z}_p\delta$. It follows that L_δ contains a unique $\mathbb{Z}_p$-extension l_∞^χ of k that does not depend on the choice of δ. Indeed, if δ' is another element of $\mathfrak{g}$ that is contained in the image of $\mathrm{Art}^{\oplus,\chi}_{K_\chi/k,\{v\}}$, then δ and δ' are linearly dependent over $\mathbb{Q}_p(\operatorname{im}\chi)$, and hence also over $\mathbb{Q}_p$. Without loss of generality we may therefore assume that $\delta = (\delta')^{p^n x}$ for some integer $n \geq 0$ and unit $x \in \mathbb{Z}_p^\times$. This shows that L_δ is a finite extension of $L_{\delta'}$. In particular, they contain the same unique $\mathbb{Z}_p$-extension. We have therefore proved that l_∞^χ only depends on χ (and the triviality of $\mathrm{rec}^{\oplus,\chi}_{K_\chi/k,\{v\}}$).
Secondly, the diagram (28) gives that the image of $\mathrm{Art}^{\oplus,\chi}_{K_\chi/k,\{v\}}$ must be contained in the kernel

of $\mathrm{id} \otimes \mathrm{res}$, hence be contained in $\mathbb{Z}_p[\mathrm{im}\,\chi] \otimes_{\mathbb{Z}_p} \mathrm{Gal}(F_\infty K_\chi / K_{\chi,\infty})$. In particular, δ belongs to $\mathrm{Gal}(F_\infty K_\chi / K_{\chi,\infty})$, which implies that $K_{\chi,\infty}$ is contained in L_δ. In particular, k_∞ needs to be equal to l_∞^χ.
We have therefore proved that $\mathrm{rec}^{\oplus,\chi}_{K_\chi/k,\{v\}}$ is nonzero if k_∞ does not belong to the set $\{l_\infty^\chi \mid \chi \in \widehat{\mathcal{G}}_W\}$. This proves claim (a) of Theorem (4.14) since $\widehat{\mathcal{G}}_W$ is exactly the character group of $\mathrm{Gal}(K^{(v)}/k)$ with $K^{(v)}$ the decomposition field of v in K/k.

To prove claim (b), we claim that we can choose an integer $N \geq 1$ such that all $\mathbb{Z}_p$-extensions in the infinite set

$$\Omega(N) = \{F_\infty^{(\mathbb{Z}_p\delta)} \mid \delta = \gamma_{\mathrm{anti}}^{p^n} \cdot \gamma_{\mathrm{cyc}} \text{ for some } n \geq N\}$$

satisfy the conditions (i) and (ii) listed in (b). Indeed, every $\mathbb{Z}_p$-extension in $\Omega(N)$ verifies condition (i) as soon as N is chosen big enough such that none of the extensions l_∞^χ for $\chi \in \widehat{\mathcal{G}}_W$ belongs to $\Omega(N)$. Note that $k_\delta \cap k^{\mathrm{cyc}} = k_n^{\mathrm{cyc}}$ if $\delta = \gamma_{\mathrm{anti}}^{p^n} \cdot \gamma_{\mathrm{cyc}}$. Since no finite place splits completely in k^{cyc}/k, we may therefore choose N such that the second part of (ii) is satisfied for each element of $\Omega(N)$. The first part of (ii), in turn, follows from a result of Emsalem [Ems87] which, as a particular case, asserts that in any $\mathbb{Z}_p$-extension of k that is not the anticyclotomic extension at most two finite primes can split completely. □

5 Abelian extensions of imaginary quadratic fields

In this section we specialise to the case where the base field k is imaginary quadratic.

5.1 The conjecture of Mazur–Rubin and Sano for imaginary quadratic base fields

Fix an imaginary quadratic field k and a prime number p. We will often distinguish between the following two cases:

- *(split case)* The rational prime p splits in k. In this case we fix a choice of prime ideal $\mathfrak{p} \subseteq \mathcal{O}_k$ above p, i.e. we then have $p\mathcal{O}_k = \mathfrak{p}\overline{\mathfrak{p}}$ with $\mathfrak{p} \neq \overline{\mathfrak{p}}$.
- *(non-split case)* The prime p is either inert in k, i.e. $p\mathcal{O}_k = \mathfrak{p}$ is prime, or ramified, i.e. $p\mathcal{O}_k = \mathfrak{p}^2$.

We are now able to state the main result of this subsection.

(5.1) Theorem. *The Iwasawa-theoretic Mazur–Rubin–Sano Conjecture (3.11) holds if one takes k_∞ to be*

- *the unique $\mathbb{Z}_p$-extension of k unramified outside $\mathfrak{p}$, in the split case,*
- *any $\mathbb{Z}_p$-extension of k, in the non-split case.*

Extending the methods employed in the proof of Theorem (5.1) to directly prove Conjecture (3.11) for more general $\mathbb{Z}_p$-extensions in the split case seems to be difficult. This is because for such $\mathbb{Z}_p$-extensions one must also consider 'rank-two' cases in which W contains two p-adic places. However, the 'rank-one' cases of Conjecture (3.11) covered by Theorem (5.1) are sufficient to deduce, in § 5.3, the relevant instance of the eTNC as stated in Theorem B. Since the eTNC is known to imply Conjecture (3.11) (cf. [BKS17, Thm. 5.16]), we will in Corollary (5.9) obtain the validity of Conjecture (3.11) also for the aforementioned 'rank-two' cases of Conjecture (3.11). This will thereby complete the proof of Theorem A in the split case.

The proof of Theorem (5.1) occupies the remainder of this subsection. Our argument crucially relies on the use of relative Lubin–Tate theory, the essential features of which we now first recall. For more details on this theory the reader is referred to [Sha87, Ch. I].

Let H be a finite extension of $\mathbb{Q}_p$ and denote the cardinality of its residue field $\mathcal{O}_H/\mathfrak{p}_H$ by q.

We fix an integer $d > 0$ and let H' be the unramified extension of H of degree d. We write $\varphi \in \mathrm{Gal}(H'/H)$ for the arithmetic Frobenius automorphism.
Fix an element $\xi \in H^\times$ such that $\mathrm{ord}_H(\xi) = d$. For each power series f satisfying 'Frobenius-like' properties with respect to ξ (see [Sha87, Ch. I, § 1.2] for the precise definition) there exists a unique one-dimensional commutative formal group law $F_f \in \mathcal{O}_{H'}[\![X, Y]\!]$ satisfying $F_f^\varphi \circ f = f \circ F_f$ called a 'relative Lubin–Tate group' (relative to the extension H'/H). Given an integer $n \geq 0$, we let W_f^n be the group of 'division points of level n' of F_f and set $\widetilde{W_f^{n+1}} := W_f^{n+1} \setminus W_f^n$. Then $H'_n := H'(W_f^{n+1})$ is a totally ramified extension of H' of degree $q^n(q-1)$ that does not depend on the choice of f and is abelian over H. It follows that also $H'_\infty := \bigcup_{n \in \mathbb{N}} H'_n$ is a totally ramified extension of H' that is abelian over H.
For each integer $n \geq 0$, fix $\omega_n \in \widetilde{W}^n_{\varphi^{-n}(f)}$ such that $(\varphi^{-n} f)(\omega_n) = \omega_{n-1}$. The family $(\omega_n)_{n \geq 0}$ is then called a 'generator for the Tate module of F_f'. Let $u \in \varprojlim_n (H'_n)^\times$ be a norm-coherent sequence and note that there is a unique integer $\nu(u)$ such that $u_n \mathcal{O}_{H'_n} = \mathfrak{p}_{H'_n}^{\nu(u)}$ for all $n \geq 0$. By [Sha87, Ch. I, Thm. 2.2] there is a unique power series $\mathrm{Col}_u \in t^{\nu(u)} \mathcal{O}_{H'}[\![t]\!]^\times$ such that

$$(\varphi^{-(i+1)} \mathrm{Col}_u)(\omega_{i+1}) = u_i$$

for all $i \geq 0$. This power series Col_u is called the *Coleman power series* associated to u.

Let $\rho\colon \mathrm{Gal}(H'_\infty/H) \to \mathbb{Q}/\mathbb{Z}$ be a character of finite order. Write $H_\rho = (H'_\infty)^{\ker \rho}$ for the field cut out by ρ and choose m minimal with the property that $H_\rho \subseteq H'_m$.
If $u \in \varprojlim_n \mathcal{O}_{H'_n}^\times$ is a norm-coherent sequence, then $\mathrm{N}_{H'_0/H}(u_0) = 1$ because $\mathrm{N}_{H'_n/H}({H'_n}^\times) = \xi^{\mathbb{Z}} \cdot (1 + \mathfrak{p}_H^{n+1})$ for all $n \geq 0$ (cf. [Sha87, p. 11]). Hilbert's Theorem 90 therefore ensures the existence of an element $\beta_{\sigma,\rho} \in H_\rho^\times$ satisfying $(\sigma - 1) \cdot \beta_{\sigma,\rho} = \mathrm{N}_{H'_m/H_\rho}(u_m)$, where σ denotes a generator of $\mathrm{Gal}(H_\rho/H)$.

The following is proved in [BH20, Cor. 3.17].

(5.2) Proposition. *Using the notation introduced above, assume that* $\rho(\sigma) = \frac{1}{[H_\rho:H]} + \mathbb{Z}$. *If* $\mathrm{Col}_u(0)$ *belongs to* $\mathcal{O}_H^\times$, *then we have that*

$$\frac{\mathrm{ord}_{H_\rho}(\beta_{\sigma,\rho})}{e_{H_\rho/H}} = -\rho(\mathrm{rec}_H(\mathrm{N}_{H'/H}(\mathrm{Col}_u(0)))) \quad \text{in } \mathbb{Q}/\mathbb{Z},$$

where we write $e_{H_\rho/H}$ *for the ramification degree of the extension* H_ρ/H *and* rec_H *denotes the local reciprocity map* $H^\times \to \mathrm{Gal}(H'_\infty/H)$.

We now give the proof of Theorem (5.1).

Proof of Theorem (5.1): At the outset we recall that Conjecture (3.11) is formulated for a finite abelian extension K of k with the property that no finite place of k that ramifies in K splits completely in k_∞. Note that, in the split case, this latter condition is automatically satisfied because no finite place splits completely in k_∞/k, see [Sha87, Ch. II, Prop. 1.9].
Fix such K. As in § 3.1, we then set set $K_\infty = K \cdot k_\infty$, write K_n for the n-th layer of the $\mathbb{Z}_p$-extension K_∞/K, and use the notations $\mathcal{G}, \mathcal{G}_n, \Gamma_n, \Gamma^n$ and Λ etc.
In the notation of § 3.1, we take $V = S_\infty(k)$ and S any finite set of places which contains $S_\infty(k) \cup S_{\mathrm{ram}}(K/k)$ but does not contain any finite place that splits completely in k_∞/k. Recall that in § 3.1 we have also fixed a proper subset $V' \subsetneq \Sigma$ consisting of places that split completely in K/k, and have set e to be the size of $W = V' \setminus V$. Finally, we fix a finite set T of places of k that is disjoint from $\Sigma := S \cup S_{\mathrm{ram}}(k_\infty/k)$ and has the property that $U_{K,\Sigma,T}$ is $\mathbb{Z}_p$-torsion free. One then has that $U_{E,\Sigma,T}$ is $\mathbb{Z}_p$-torsion free for every subfield E of K_∞/k.

We now first observe that Conjecture (3.11) is independent of the choice of the auxiliary set T. Indeed, if $T' \supseteq T$ is another such set, then one has $\varepsilon^V_{K_n/k,\Sigma,T'} = \delta_{T' \setminus T} \cdot \varepsilon^V_{K_n/k,\Sigma,T}$ and

$\varepsilon^V_{K/k,\Sigma\setminus W,T'} = \delta_{T'\setminus T} \cdot \varepsilon^V_{K/k,\Sigma\setminus W,T}$, where $\delta_{T'\setminus T} = \prod_{v\in T'\setminus T}(1 - \mathrm{Frob}_v^{-1}\mathrm{N}v)$. Since $\delta_{T'\setminus T}$ is a non-zero divisor in $\mathbb{Z}[\mathcal{G}]$, we therefore conclude that the claim holds for T if and only if it does for T'.

Given this, we may assume that $T = \{\mathfrak{a}\}$ for a prime ideal $\mathfrak{a} \subsetneq \mathcal{O}_k$ that is coprime to $6\mathfrak{p}\mathfrak{m}$, where $\mathfrak{m} = \mathfrak{m}_K$ denotes the conductor of K.

Observe that by [BKS17, Prop. 4.4 (iv)] we may reduce to the case $W \subseteq S_{\mathrm{ram}}(K_\infty/K) = \{\mathfrak{p}\}$. Since (29) is trivially satisfied if $W = \varnothing$, we may assume that $W = \{\mathfrak{p}\}$. In particular, $\mathfrak{p} \nmid \mathfrak{m}$ and hence, because V' is a proper subset of Σ, there must be a finite place $\mathfrak{q} \in \Sigma$ that is different from $\mathfrak{p}$. Observe that $\mathcal{O}_k^\times \to (\mathcal{O}_k/\mathfrak{q}^l)^\times$ is injective if we choose l big enough (this is because the group of units of $\mathcal{O}_{k_\mathfrak{q}}^\times$ that are congruent to $1 \mod \mathfrak{q}^l$ will be torsion-free for l big enough). We may therefore take the ideal $\mathfrak{f}$ appearing in Example (2.3) (c) to be an appropriate power of $\mathfrak{q}$. Given this, we have that, for any $n \in \mathbb{N}_0$,

$$\varepsilon^V_{K_n/k,\Sigma,T} = \mathrm{N}_{k(\mathfrak{f}\mathfrak{m}\mathfrak{p}^{n+1})/K_n}(\psi_{\mathfrak{f}\mathfrak{m}\mathfrak{p}^{n+1},\mathfrak{a}})$$

is the elliptic unit defined in Example (2.3) (c).

Note that, by Proposition (3.6) (b), Conjecture (3.4) holds in this situation. In particular, for any choice of topological generator γ of Γ the Darmon derivative of κ_0 of $\varepsilon_{K_\infty/k,\Sigma,T}$ exists. By Proposition (3.16), it then suffices to prove that

$$\mathrm{Ord}_W(\kappa_0) \otimes (\gamma - 1) = -\mathrm{Rec}_W(\varepsilon^V_{K/k,\Sigma\setminus W,T}). \tag{29}$$

in order to establish Conjecture (3.11).

To verify (29), we may assume that our fixed place $w \coloneqq w_K$ of K above $\mathfrak{p}$ has full decomposition group both in $k(\mathfrak{f}\mathfrak{m})/K$ and K_∞/K because, by Lemma (3.18), we may replace K by the decomposition field of $\mathfrak{p}$ in $k(\mathfrak{f}\mathfrak{m})k_\infty/k$ if necessary.

Recall that we have previously fixed an embedding $\iota_w \colon \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}_p}$ that restricts to w on K. In the following, we will denote the completion of a finite abelian extension field F of k at the place induced by ι_w by $\widetilde{F}$. Put $H = \widetilde{K}$ and $H' = \widetilde{k(\mathfrak{f}\mathfrak{m})}$.

Choose an elliptic curve E as in [BH20, § 4], then its formal group $\widehat{E}$ is a relative Lubin–Tate group with respect to the unramified extension H'/H. Moreover, with these definitions, one has that $H'_n = \widetilde{k(\mathfrak{f}\mathfrak{m}\mathfrak{p}^{n+1})}$, hence we can define a norm-coherent sequence $u = (u_n)_n \in \varprojlim_n (H'_n)^\times$ by setting

$$u_n = \iota_w(\psi_{\mathfrak{f}\mathfrak{m}\mathfrak{p}^{n+1},\mathfrak{a}}) \quad \text{for all } n > 0.$$

Note that, because $\mathfrak{f}\mathfrak{m}\mathfrak{p}^{n+1}$ has at least two distinct prime factors, $\psi_{\mathfrak{f}\mathfrak{m}\mathfrak{p}^{n+1},\mathfrak{a}}$ is a (global) unit (cf. [Ble04, Thm 2.4]) and hence $u_n \in \mathcal{O}_{H'_n}^\times$ for all $n \geq 0$.

(5.3) Lemma. *There is a generator $(\omega_n)_{n\geq 0}$ of the Tate module of $\widehat{E}$ such that*

$$\mathrm{Col}_u(0) = \iota_w(\psi_{\mathfrak{f}\mathfrak{m},\mathfrak{a}}).$$

In particular, $\mathrm{Col}_u(0)$ *belongs to* $\mathcal{O}_H^\times$.

Proof. In the split case the first claim is proved in [Sha87, Chp. II, Sec 4.9, Prop.] combined with the evaluation of the power series at zero and an application of the monogeneity relation of Robert's ψ-function. A more detailed proof of the split case is given in [OV16, Prop. 4.5] following the same strategy as [Sha87].

We claim that essentially the same proof works in the non-split case. First observe that in the proof of part (i) of the cited Proposition in [Sha87] the fact that the prime is split in k is not used. In part (ii) the condition that p is split is used to obtain a certain generator of the Tate module $(\omega_n)_{n\geq 0}$ of the underlying formal group $\widehat{E}$ (because in this case the formal group $\widehat{E}$ is isomorphic to $\widehat{\mathbb{G}}_m$ and hence of height one). It is then shown that there exist torsion points denoted u_n in *loc. cit.* which can be used to give an explicit description of the elements ω_n at each level [Sha87, Chp. II, Sec. 4.4, (12)]. In the non-split case one can now invert the strategy:

Indeed, it is easy to see that there exist torsion points u_n such that the explicit description given in (12) is a generator of the Tate module of $\widehat{E}$. Using this as the definition of $(\omega_n)_{n\geq 0}$, the remaining steps in the proof are exactly as in the split case.
To justify the final claim of the Lemma, it suffices to note that $\mathfrak{p}$ does not divide $\mathfrak{fm}$, which implies that $\psi_{\mathfrak{fm},\mathfrak{a}} = \varepsilon^V_{k(\mathfrak{fm}),\Sigma\setminus W,T}$ is integral at $\mathfrak{p}$ (cf. also [Ble04, Thm. 2.4]). □

Having fixed a topological generator γ of Γ, we may define the isomorphisms

$$s_\gamma \colon \Gamma \longrightarrow \mathbb{Z}_p, \quad \gamma^a \longmapsto a \qquad\qquad s_{\gamma,n} \colon \Gamma_n \longrightarrow \mathbb{Z}/p^n\mathbb{Z}, \quad \gamma^a \longmapsto a \bmod p^n\mathbb{Z}$$

and the character

$$\rho_{\gamma,n} \colon \mathrm{Gal}(H'_\infty/H) \xrightarrow{\mathrm{res}_{H'_\infty/\widetilde{K_n}}} \mathrm{Gal}(\widetilde{K_n}/H) \xrightarrow{s_{\gamma,n}} \mathbb{Z}/p^n\mathbb{Z} \cong (\tfrac{1}{p^n}\mathbb{Z})/\mathbb{Z}, \tag{30}$$

where $\mathrm{res}_{H'_\infty/\widetilde{K_n}}$ is the natural restriction map. By definition, $\rho_{\gamma,n}$ is a character of finite order with kernel $\mathrm{Gal}(H'_\infty/\widetilde{K_n})$, hence Proposition (5.2) combines with Lemma (5.3) to reveal that

$$\frac{\mathrm{ord}_{\widetilde{K_n}}(\beta_{\gamma,\rho_{\gamma,n}})}{e_{\widetilde{K_n}/H}} \equiv -\frac{s_{\gamma,n}(\mathrm{res}_{H'_\infty/\widetilde{K_n}}(\mathrm{rec}_H(\mathrm{N}_{H'/H}(\psi_{\mathfrak{fm},\mathfrak{a}}))))}{p^n} \bmod \mathbb{Z} \tag{31}$$

for all $n \geq 0$. By definition the Darmon derivative κ_0 of $\varepsilon_{K_\infty/k,\Sigma,T}$ with respect to γ is the bottom value of a norm-coherent sequence $\kappa = (\kappa_n)_n \in \varprojlim_n U_{K_n,\Sigma,T}$ that satisfies

$$(\gamma - 1)\iota_w(\kappa_n) = \iota_w(\varepsilon^V_{K_n/k,\Sigma,T}) = \mathrm{N}_{H'_m/H_{\rho_{\gamma,n}}}(u_m),$$

thus we may take $\beta_{\gamma,\rho_{\gamma,n}} \equiv \iota_w(\kappa_n) \mod \widetilde{K}^\times$. We now obtain from (31) that

$$\begin{aligned}\mathrm{ord}_H(\iota_w(\kappa_0)) &= \mathrm{ord}_H(\mathrm{N}_{\widetilde{K_n}/H}(\iota_w(\kappa_n))) = \frac{p^n}{e_{\widetilde{K_n}/H}} \cdot \mathrm{ord}_{\widetilde{K_n}}(\iota_w(\kappa_n)) \\ &\equiv -s_{\gamma,n}(\mathrm{res}_{H'_\infty/\widetilde{K_n}}(\mathrm{rec}_H(\mathrm{N}_{H'/H}(\psi_{\mathfrak{fm},\mathfrak{a}}))) \mod p^n\mathbb{Z}.\end{aligned}$$

Note that $\mathrm{N}_{H'/H}(\psi_{\mathfrak{fm},\mathfrak{a}})) = \iota_w(\mathrm{N}_{k(\mathfrak{fm})/K}(\psi_{\mathfrak{fm},\mathfrak{a}}))) = \iota_w(\varepsilon^V_{K/k,\Sigma\setminus W,T})$ because, by assumption, $\mathfrak{p}$ has full decomposition group in $k(\mathfrak{fm})/K$. Taking the limit over n of the last displayed congruence therefore gives

$$\mathrm{ord}_w(\kappa_0) = -s_\gamma(\mathrm{rec}_w(\varepsilon^V_{K/k,\Sigma\setminus W,T})) \tag{32}$$

as an equality in $\mathbb{Z}_p$. By repeating the argument we also obtain equation (32) for the places σw, where $\sigma \in \mathcal{G}$. Collating these equations, we find that

$$\begin{aligned}\mathrm{Ord}_W(\kappa_0) \otimes (\gamma - 1) &= \sum_{\sigma\in\mathcal{G}} \mathrm{ord}_{\sigma w}(\kappa_0)\sigma \otimes (\gamma - 1) \\ &= -\sum_{\sigma\in\mathcal{G}} s_\gamma(\mathrm{rec}_{\sigma w}(\varepsilon^V_{K/k,\Sigma\setminus W,T}))\sigma \otimes (\gamma - 1) \\ &= -\mathrm{Rec}_W(\varepsilon^V_{K/k,\Sigma\setminus W,T}).\end{aligned}$$

This proves (29), as required to conclude the proof of Theorem (5.1). □

5.2 The equivariant Iwasawa Main Conjecture

In this section we prove a suitable variant of the equivariant Iwasawa Main Conjecture for abelian extensions of imaginary quadratic fields. In this setting, numerous results on the Iwasawa Main Conjecture have already appeared in the literature, both in classical and equivariant formulations (cf. [Rub88], [Rub91], [Rub94], [Ble06], [Fla09], [JLK11], [Vig13]). However, we require a result that is slightly more general and, to some extent, also of a different shape than is available in the literature thus far.

Suppose to be given an abelian extension L_∞/k such that $\mathrm{Gal}(L_\infty/k) \cong \Gamma \times \Delta$, where Δ is a finite abelian group and $\Gamma \cong \mathbb{Z}_p^d$ for an integer $d > 0$. Note that $d \in \{1, 2\}$ as a consequence of the known validity of Leopoldt's Conjecture for the imaginary quadratic field k.

We also fix a finite set Σ of places of k that contains $S_\infty(k) \cup S_p(k)$ and a finite set T of places of k that is disjoint from Σ. Assume that no finite place contained in $\Sigma \cup T$ splits completely in the $\mathbb{Z}_p^d$-extension L_∞^Δ/k.

As before we write $\mathbb{A} = \mathbb{Z}_p[\Delta][\![\Gamma]\!]$ for the relevant equivariant Iwasawa algebra and denote its total field of fractions by $\mathcal{Q}(\mathbb{A})$. One can then define a perfect complex $D^\bullet_{L_\infty,\Sigma,T}$ as in (6), and construct a map

$$\begin{aligned} \mathrm{Det}_{\mathbb{A}}(D^\bullet_{L_\infty,\Sigma,T}) &\hookrightarrow \mathcal{Q}(\mathbb{A}) \otimes_{\mathbb{A}} \mathrm{Det}_{\mathbb{A}}(D^\bullet_{L_\infty,\Sigma,T}) \\ &\cong \mathrm{Det}_{\mathcal{Q}(\mathbb{A})}(\mathcal{Q}(\mathbb{A}) \otimes^{\mathbb{L}}_{\mathbb{A}} D^\bullet_{L_\infty,\Sigma,T}) \\ &\cong \big(\mathcal{Q}(\mathbb{A}) \otimes_{\mathbb{A}} U_{L_\infty,\Sigma,T}\big) \otimes_{\mathcal{Q}(\mathbb{A})} \big(\mathcal{Q}(\mathbb{A}) \otimes_{\mathbb{A}} Y_{L_\infty,S_\infty(k)}\big)^* \\ &\cong \mathcal{Q}(\mathbb{A}) \otimes_{\mathbb{A}} U_{L_\infty,\Sigma,T}, \end{aligned} \tag{33}$$

where the first isomorphism follows from a well-known property of the determinant functor, the second isomorphism is the natural 'passage-to-cohomology' map, and the last isomorphism is due to the isomorphism $Y_{L_\infty,S_\infty(k)} \cong \mathbb{A}$ obtained from our fixed choice of extension of the unique infinite place of k to L_∞.

The map (33) then restricts to a map

$$\Theta^1_{L_\infty/k,\Sigma,T} \colon \mathrm{Det}_{\mathbb{A}}(D^\bullet_{L_\infty,\Sigma,T}) \hookrightarrow U^{**}_{L_\infty,\Sigma,T} \cong U_{L_\infty,\Sigma,T},$$

see [BD21, Lem. 3.12] for more details.

We now recall the (higher-rank) equivariant Iwasawa Main Conjecture in this setting as proposed in [BKS17, Conj. 3.1 and Rk. 3.3].

(5.4) Conjecture. *There exists a $\mathbb{A}$-basis $\mathcal{L}_{L_\infty/k,\Sigma,T}$ of $\mathrm{Det}_{\mathbb{A}}(D^\bullet_{L_\infty,\Sigma,T})$ such that*

$$\Theta^1_{L_\infty/k,\Sigma,T}(\mathcal{L}_{L_\infty/k,\Sigma,T}) = \varepsilon_{L_\infty/k,\Sigma,T}.$$

Fix a prime ideal $\mathfrak{p}$ of k above p as in § 5.1. The main result of this is subsection is as follows.

(5.5) Theorem. *Let K/k be an abelian extension and put $L_\infty = Kl_\infty$, where l_∞ is the maximal $\mathbb{Z}_p$-power extension of k unramified outside $\mathfrak{p}$. Assume the following condition:*

- *(∗) $\mathrm{Gal}(L_\infty/k)$ is p-torsion free or the Iwasawa μ-invariant of $A_{L_\infty,\Sigma}$ (as a $\mathbb{Z}_p[\![\Gamma]\!]$-module) vanishes.*

Then Conjecture (5.4) holds for $(L_\infty/k, \Sigma, T)$. In particular, Conjecture (5.4) holds for $(K_\infty/k, \Sigma, T)$ with $K_\infty = Kk_\infty$ if one takes k_∞/k to be any of the $\mathbb{Z}_p$-extensions described at the beginning of § 5.1.

Proof. Let us first prove that it is indeed enough to prove Conjecture (5.4) for L_∞/k. If p is split in $k/\mathbb{Q}$, then l_∞ and k_∞ agree and so the claim is clear in this case. In the non-split case, l_∞ is the maximal $\mathbb{Z}_p$-power extension of k and hence K_∞ is contained in L_∞. We then have a commutative diagram

$$\begin{array}{ccc} \mathrm{Det}_{\mathbb{A}'}(D^\bullet_{L_\infty,\Sigma,T}) & \xrightarrow{\Theta^1_{L_\infty/k,\Sigma,T}} & U_{L_\infty,\Sigma,T} \\ {\scriptstyle \varpi_{L_\infty/K_\infty}}\downarrow & & \downarrow{\scriptstyle \mathrm{N}_{L_\infty/K_\infty}} \\ \mathrm{Det}_{\mathbb{A}}(D^\bullet_{K_\infty,\Sigma,T}) & \xrightarrow{\Theta^1_{K_\infty/k,\Sigma,T}} & U_{K_\infty,\Sigma,T}, \end{array} \tag{34}$$

where the left hand vertical map $\varpi_{L_\infty/K_\infty}$ is induced by the isomorphism (cf. Proposition (3.2) (c) (ii))

$$D^\bullet_{L_\infty,\Sigma,T} \otimes^{\mathbb{L}}_{\mathbb{Z}_p[\![\mathrm{Gal}(L_\infty/k)]\!]} \mathbb{Z}_p[\![\mathrm{Gal}(K_\infty/k)]\!] \cong D^\bullet_{K_\infty,\Sigma,T}.$$

The claim now follows directly from the above commutative diagram (34).

To prove Conjecture (5.4) for L_∞/k, we first note that the explicit condition $(*)$ ensures that $A_{L_\infty,\Sigma}$ has projective dimension at most one after localising at any height-one prime $\mathfrak{p}$ of $\mathbb{A}$. By [Bul+21, Prop. 6.4 (b) (ii)] it is therefore enough to show that one has an inclusion

$$\mathrm{im}(\varepsilon_{L_\infty/k,\Sigma,T})^{**} \subseteq \mathrm{Fitt}^0_{\mathbb{A}}(A_{L_\infty,\Sigma,T})^{**} \cdot \mathrm{Fitt}^0_{\mathbb{A}}(X_{L_\infty,\Sigma\setminus S_\infty(k)})^{**}$$

and this is a well-known consequence of the theory of Euler systems (see the proof of [Bul+21, Thm. 6.12 (b)], where the above inclusion, which agrees with (46) of *loc. cit.*, is verified). □

To end this subsection we clarify the nature of condition $(*)$.

(5.6) Proposition. *Let K/k be an abelian extension and put $K_\infty = K \cdot l_\infty$, where l_∞ is the maximal $\mathbb{Z}_p$-power extension of k unramified outside $\mathfrak{p}$. The μ-invariant of $A_{L_\infty,\Sigma}$ (as a $\mathbb{Z}_p[\![\Gamma]\!]$-module) vanishes in each of the following cases:*

(a) The prime number p splits in $k/\mathbb{Q}$,

(b) the degree $[K : k]$ is a power of p,

(c) there is a $\mathbb{Z}_p$-extension F_∞ of K contained in L_∞ in which no prime above p splits completely and which is such that the μ-invariant of A_{F_∞} (as a $\mathbb{Z}_p[\![\mathrm{Gal}(F_\infty/K)]\!]$-module) vanishes,

(d) A_{K,S_p} vanishes and $|S_p(K)| = 1$.

(5.7) Remark. Iwasawa has conjectured that statement (c) in Proposition (5.6) is always satisfied if one takes F_∞ to be the cyclotomic $\mathbb{Z}_p$-extension of K.

Proof. It is well-known that the Iwasawa μ-invariants of $A_{L_\infty,\Sigma}$ and A_{L_∞} agree because, by [Sha87, Ch. II, § 1.9, Prop.], no finite prime splits completely in L_∞. It therefore suffices to discuss the vanishing of the latter.

In the situation of (a) the required vanishing follows from the main results of Gillard in [Gil85] (for $p > 3$) and Oukhaba–Viguié in [OV16] (for $p \in \{2, 3\}$).

Let k_∞^{cyc} and K_∞^{cyc} be the cyclotomic $\mathbb{Z}_p$-extensions of k and K, respectively. From [Iwa73, Thm. 2] we know that the vanishing of the μ-invariant of $A_{K_\infty^{\mathrm{cyc}}}$ is implied by the vanishing of the μ-invariant of $A_{K^P\cdot k_\infty^{\mathrm{cyc}}}$, where K^P denotes the fixed field of the p-Sylow subgroup P of $\mathcal{G} = \mathrm{Gal}(K/k)$. This combines with the Theorem of Ferrero–Washington [FW79] to imply the claim for (b) once we have verified that it is valid if (c) holds.

To do this, we may assume that p is not split in $k/\mathbb{Q}$ because we have already dealt with split primes in (a). In this case, K_∞/K is a $\mathbb{Z}_p^2$-extension in which all primes above p are finitely decomposed. Given this, statement (c) implies the claim by [Cuo80, Prop. 4.1 and Cor. 4.8].

Finally, as is well-known, (d) follows from Nakayama's Lemma using that $(A_{K_\infty,S_p})_\Gamma \cong A_{K,S_p}$ if $|S_p(K)| = 1$ (cf. [Was97, Prop. 13.22]). □

5.3 Proofs of Theorems A and B

We are finally in a position to prove Theorem B from the introduction.

(5.8) Theorem. *Let p be a prime number, k an imaginary quadratic field, and K/k a finite abelian Galois extension with Galois group $\mathcal{G}$.*

(a) If p splits in k, then $\mathrm{eTNC}(h^0(\mathrm{Spec}(K)), \mathbb{Z}_p[\mathcal{G}])$ holds.

(b) If p does not split in k, then $\mathrm{eTNC}(h^0(\mathrm{Spec}(K)), \mathbb{Z}_p[\mathcal{G}])$ holds if the following condition is satisfied:

$(*)$ Let l_∞ be the maximal $\mathbb{Z}_p$-power extension of k and put $L_\infty = K \cdot l_\infty$. The Iwasawa μ-invariant of $A_{L_\infty,\Sigma}$ (as a $\mathbb{Z}_p[\![\mathrm{Gal}(L_\infty/K)]\!]$-module) vanishes or $\mathrm{Gal}(L_\infty/k)$ is p-torsion free.

Proof. This will follow from the equivariant Iwasawa Main Conjecture proved in Theorem (5.5) and the descent argument of Burns, Kurihara and Sano in [BKS17, Thm. 5.2].

To do this, we first need to introduce some notation. In the split case, we take k_∞ to be the unique $\mathbb{Z}_p$-extension of k that is unramified outside $\mathfrak{p}$. In the non-split case, we take k_∞ to be one of the $\mathbb{Z}_p$-extensions of k provided by Theorem (4.14) (b).

For any character $\chi \in \widehat{\mathcal{G}}$ we moreover introduce the following notation:

- $K_\chi = K^{\ker \chi}$ the field cut out by the character χ, and $\mathcal{G}_\chi = \mathrm{Gal}(K_\chi/k)$ its Galois group,
- $K_{\chi,\infty} = K_\chi \cdot k_\infty$ the composite of K_χ with k_∞, and $\Gamma_\chi = \mathrm{Gal}(K_{\chi,\infty}/K_\chi)$.

Fix a finite set S of places of k containing $S_\infty(k) \cup S_{\mathrm{ram}}(K_\infty/k)$ and a finite set T of places of k with the property that $U_{K,S,T}$ is $\mathbb{Z}_p$-torsion free. In addition, we write $S_{\mathrm{split}}(K_\chi/k)$ for the set of places of k that split completely in K_χ and define

$$V'_\chi = \begin{cases} S_{\mathrm{split}}(K_\chi/k) \cap S & \text{if } \chi \neq 1, \\ S \setminus \{\mathfrak{p}\} & \text{if } \chi = 1. \end{cases}$$

Furthermore, we set $V = S_\infty(k)$ and $W_\chi = V'_\chi \setminus V$. By enlarging S if necessary we may assume that $S_p(k) \subseteq S$ and that V'_χ is a proper subset of S for all $\chi \in \widehat{\mathcal{G}}$.

Let us now address each condition required to apply the general result [BKS17, Thm. 5.2] separately:

- Under assumption $(*)$ in the non-split case, the equivariant Iwasawa Main Conjecture holds for $(K_\infty/k, S, T)$ by Theorem (5.5).
- The Iwasawa-theoretic Mazur–Rubin–Sano Conjecture (in the formulation [BKS17, Conj. 4.2]) holds for the data $(K_{\chi,\infty}/k, K_\chi, S, T, V'_\chi)$: If χ is non-trivial, then this is proved in Theorem (5.1). For the trivial character the set $V'_\chi = S \setminus \{\mathfrak{p}\}$ consists only of places unramified in k_∞/k, hence in this case the conjecture holds as a consequence of [BKS17, Prop. 4.4 (iv)].
- Condition (F) (as stated in (4.3)) for K_∞/K is valid: In the split case this is Remark (4.13) (b), in the non-split case this is Theorem (4.14)).

This concludes the proof of Theorem (5.8). □

As a first consequence of Theorem (5.8) we are now able to complete the proof of Theorem A.

(5.9) Corollary. *If k is an imaginary quadratic field, then the Iwasawa-theoretic Mazur–Rubin–Sano Conjecture (3.11) holds.*

Proof. In the non-split case we already proved Conjecture (3.11) in Theorem (5.1). In the split case, Conjecture (3.11) holds as a consequence of the fact that, by Theorem (5.8), the eTNC is valid in this case. Details for the deduction of Conjecture (3.11) from the eTNC can be found in [BKS17, Lem. 5.17] (cf. also [BKS16, Thm. 5.16]). □

From Theorem (5.8) and Proposition (5.6) (b) we also immediately obtain the following unconditional result towards the integral eTNC.

(5.10) Corollary. *If all prime factors of $[K:k]$ are split in k or $[K:k]$ is a prime power, then* $\mathrm{eTNC}(h^0(\mathrm{Spec}(K)), \mathbb{Z}[\mathcal{G}])$ *holds.*

(5.11) Remark. (a) If $p \nmid h_k[K:k]$, where h_k denotes the class number of k, then the validity of $\mathrm{eTNC}(h^0(\mathrm{Spec}(K)), \mathbb{Z}_p[\mathcal{G}])$ also follows from unpublished work of Bley [Ble98, Part II, Thm. 1.1] on the Strong Stark Conjecture. It should be straightforward to strengthen said result to cover all primes $p \nmid [K:k]$ by taking into account the improvements of [Rub91, § 3] provided in [Rub94]. We remark that even this expected strengthening is covered by Theorem (5.8).

(b) As illustrated by Corollary (5.10), the validity of the p-part of the eTNC for split primes $p \mid 2h_k$ allows for a significant improvement on results towards the integral eTNC in this setting. Previously, one had to restrict to cases where k is one of only nine imaginary quadratic fields of class number one and all prime factors of $[K : k]$ are split in k to obtain unconditional results towards the validity of the eTNC for the pair $(h^0(\mathrm{Spec}(K)), \mathbb{Z}[\mathcal{G}])$.

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