Exponential Stabilization of Polynomial Fuzzy Positive Switched Systems with Time Delay Considering MDADT Switching Signal

Xiaomiao Li, Yannan Shan, Hak-Keung Lam, Fellow, IEEE, Zhiyong Bao, and Jundong Zhao

Abstract—Nonlinear switched positive systems with time delay are universally found in practical applications. However, with the consideration of the switching signal, the constraints of positivity and the existence of the time delay make the control of this special system much more complicated. In this paper, a novel exponential stability analysis which can tighten the bounds of switching dwell time is proposed under the switched positive polynomial fuzzy control scheme. Firstly, the polynomial fuzzy model represents the dynamic characteristics of nonlinear switched positive systems with time delay, which can sufficiently promote the approximation capability, thereby simplifying the model structure. Second, the switched polynomial state fuzzy (SPSF) controller is designed based on the principle of the premise variable matching to enhance the design flexibility, and matrix decomposition is proposed to overcome the obstacle caused by the non-convexity conditions both in positive and stable conditions. In addition, as the existing switching control theory is relatively conservative which leads to a high bound of dwell time, a switched fuzzy copositive Lyapunov-Krasovskii function (SFCLKF) with the upper bound of the derivative of the membership function and a Taylor expansion technique are together incorporated in positive switched stability analysis to obtain tight bound of dwell time and relax conditions. At last, two simulation examples are provided to validate proposed methods.

Index Terms—Nonlinear positive switched system, Polynomial fuzzy system, Imperfectly matched premises, Membership-function dependent, Multiple fuzzy co-positive Lyapunov-Krasovskii functional.

I. INTRODUCTION

Positive systems are a particular kind of systems whose state trajectory is confined in the positive orthant if the initial condition is non-negative, such as economic systems, biological systems, and communications systems [1][2][3]. The switched positive system consists of finite positive subsystems with switching signals. It manifests a variety of practical applications, including transmission control protocol-like congestion control [4] and formation flying control [5].

This work was supported in part by the Natural Science Foundation under Project 62103356, the Natural Science Foundation of Hebei Province under Project (F2021203069, F2022203043), the Science and Technology Project of Hebei Education Department under Project BJK2022040 and the Oversea Talents of Hebei Province Foundation under Project C20210319. (Corresponding Author: H.K. Lam)

Xiaomiao Li, Yannan Shan, Zhiyong Bao, and Jundong Zhao are with the Key Lab of Industrial Computer Control Engineering of Hebei Province, Yanshan University, Qinhuangdao 066004, Hebei Province, China. (e-mail: xsm760595299@163.com; syn19980708@163.com; yanshanbzy@163.com; zjddyxcactus@163.com.)

H.K. Lam is with the Department of Engineering, King’s College London, Strand, London WC2R 2LS, United Kingdom. (e-mail: hak-keung.lam@kcl.ac.uk).

Compared with general systems, the dynamic characteristics of this system are more complex as the coupling between positivity of the states and the switching action. Many achievements for switched positive systems have been made in a wider range, such as robust fault detection [6], impulse control [7][8], tracking control [9][10]. The stability analysis of switched positive systems is always a vital research topic because the switching signal can be a primary tool to stabilize the system, though considerable attention has been received [11][12][13], it is still worth further exploitation due to the complexity of theoretical analysis and extensive applications.

It is noteworthy that there are some results served on switched positive linear systems [8][14][15][16][17]. In fact, almost all practical plants have inherent nonlinearity [18]. In addition, the existence of time delay cannot be ignored, which will not only affect the stability of the system but also deteriorate the performance of systems [19]. So the investigation of nonlinear switched positive systems with time delay is a more indispensable task. The Takagi-Sugeno (T-S) fuzzy model is a powerful tool to approximating nonlinear systems through a family of local linear sub-systems and membership functions, such that the stability analysis and control strategies of the T-S fuzzy-mode-based (FMB) control system can be guided by the systematic theory of linear systems [20][21][22]. Research works on nonlinear switched positive systems based on T-S fuzzy model have been published extensively involving observer design [23], L1 gain control [24], filter design [25], etc. However, due to the switched positive system being required to establish a T-S fuzzy model for each positive switched subsystem, a remarkable problem is that its T-S model structure is extremely complicated. In order to solve this problem, the advanced polynomial fuzzy model which can represent the sub-systems by polynomial variables has attracted our attention [26]. Generally speaking, the polynomial fuzzy model can simplify model structure and reduce fuzzy rules extremely by containing polynomial nonlinear terms, which is advantaged to alleviates the analysis complexity [27][28][29]. However, the current control strategy by polynomial fuzzy modelling rarely involves nonlinear switched positive systems, thereby, it is of profound significance to inspire us to study this topic. In this paper, the stability and positivity analysis of the nonlinear switched positive systems with time delay through polynomial fuzzy modelling is investigated. Moreover, owing to the retainment of polynomials, instead of the linear matrix-inequality (LMI) method, the derived stability-positive conditions of are in the form of sum-of-squares (SOS) which
are solved by the SOSTOOLS [30].

Under the assumption that each unstable subsystem is controllable, the typical control strategy for stability of the switched system is introducing the switching signal into the state feedback control. For controller design, the current literature [24] [31] focused on the perfect premise matching technique, which requires the membership function shape of the fuzzy controller to be consistent with the fuzzy model. However, when the number of the fuzzy models’ rules or switching rules is enormous, there is no doubt that the computational burden would be increased since the fuzzy controller design is for each switched subsystem. Therefore, to improve the flexibility of controller design and reduce the implementation cost, the imperfect premise matching (IPM) technique is utilized such that the fuzzy controller’s rule numbers and membership functions can be designed independently by the designer [32] [33]. Furthermore, non-convex positive and stable conditions are inevitable with the introduction of IPM technology. A novel matrix decomposition technique is used to smooth away these difficulties [34]. It is worth mentioning that, thanks to the SOSTOOLS, the switched polynomial state fuzzy (SPSF) controller is adopted instead of the switched linear state feedback controller in our study.

For the design of switched signals, there are three types of general time-dependent switching signals in current literature: switching signals with dwell time (DT) [35], switching signals with average dwell time (ADT) [36], and switching signals with mode-dependent average dwell time (MDADT) [37]. DT and ADT switching signals require the dwell time on each subsystem to be longer than a fixed constant, but with restrictive applicability for systems. The MDADT switching signal relaxes some restrictions on switching rules than other switching methods because the differences between subsystems are considered sufficiently. Some interesting results for the switched positive system based on MDADT switching have been developed sequentially [38] [39] [40]. For the time-dependent switching stabilization design, we will take measures to obtain a tighter bound on the switching signal, which has a good ability to improve system performance by providing additional flexibility to the designer. Considering this point, [41] proposes a novel multiple discontinuous Lyapunov function approach to make the average dwell time get a tighter bound. [42] proposes a weighted mode-dependent average dwell time switched strategy to overcome deficiencies of the existing ADT and MDADT switching techniques. However, existing studies still have paid limited attention to this issue.

The MDADT primarily is contingent on \( \mu_p \) and \( \lambda_p \), where \( \mu_p \) denotes the increasing coefficient in terms of the Lyapunov function at the switching instant and \( \lambda_p \) denotes the decay rate of the Lyapunov function during the time of operation on the \( p^{th} \) activated subsystem. It is feasible to relax the conservativeness of stability conditions and in favour of changing two main parameters, in turn, obtain a tighter bound on MDADT. Based on the above discussion, to deal with the high bound of MDADT issue resulting from the existing conservative stability conditions, efforts are attempted to achieve relaxed stability analysis in two aspects. For the first aspect, as is well known, the fuzzy Lyapunov function approach consists of local sub-Lyapunov function candidates and membership functions. It is contributed to offering greater freedom for conducting stability of the system [43]. In this paper, a switched fuzzy copositive Lyapunov-Krasovskii function (SFCLKF) is proposed to improve the generality of the Lyapunov function form such that more relaxed stability conditions can be derived. Therefore, compared with the general Lyapunov function, it is available that tighter bounds on MDADT can be acquired. In the meantime, the time derivative of membership functions will not be evaded as the boundaries of the derivative of membership functions are implemented to hurdle the existing drawback by the regularization technique. For the second aspect, the membership functions play an important role in relaxing the stability analysis results because critical nonlinear information of the fuzzy system is characterized in the membership function. Most current works conduct the membership function independent stability analysis for switched positive fuzzy systems, the conservativeness of stability analysis is relatively degraded [23] [24]. To solve this problem, the Taylor expansion technique method is brought to introduce membership function information into stable conditions. Then, the stability conditions solved by SOS are aimed at a specific membership function rather than any. Consequently, reducing the conservatism of the stability criterion and improving the lower bound of MDADT, thereby, selection freedom of the switching signal is realized from a new angle.

This paper is dedicated to finding the corresponding SPSF controller and a more admissible set of switching signals, such that the switched positive polynomial fuzzy-model-based (PPFMB) system with time delay is globally uniformly exponentially stable and positive. The contributions of this paper are summarized briefly as follows:

1) The SPSF controller is designed for the switched PPFMB system with time delay under the IMP concept for the first time to reduce the complexity of modelling and implementation costs. The matrix decomposition is used to handle the difficulty of the non-convexity conditions during the process of controller designing.

2) Considering the switched mechanism and positive constraint, a switched copositive Lyapunov-Krasovskii function is constructed to conduct the exponential stability and positivity analysis for the switched closed-loop PPFMB control system with time delay.

3) To deal with the high bound of MDADT issue caused by the current conservative stability-positivity analysis, the SFCLKF candidate with the fuzzy variable is proposed to perform the exponential stability and positivity analysis of the switched PPFMB system with time delay. In addition, the relaxed stability-positivity conditions incorporating membership function information are derived to further reduce the bounds of dwell time.

The organization of this paper is as follows. In Section II, the notations and the closed-loop switched PPFMB system with time delay are described. In Section III, the stability-positivity analysis results for switched PPFMB systems with time delay are carried out based on the membership function independent (MFI) method, membership function independent...
(MFD) method and SFCLKF, respectively. In Section IV, a simulation example is demonstrated to verify the validity of the proposed approach. In Section V, a conclusion is drawn.

II. NOTATIONS AND PRELIMINARIES

A. Notation

The following notations are used throughout the paper.

1) A monomial in \( x(t) = [x_1(t), x_2(t), \ldots, x_n(t)] \) is a function of the form \( x_1^{d_1}(t)x_2^{d_2}(t)\ldots x_n^{d_n}(t) \), where \( d_i \), \( i \in \{1, 2, \ldots, n\} \) is a nonnegative integer. The degree of a monomial is defined as \( d = \sum_{i=1}^{n} d_i \).

2) The notation \( \| . \| \) is used to indicate the Euclidean norm.

3) A polynomial \( p(x(t)) \) can be expressed as a finite linear combination of monomials with real coefficients. When \( q_i(x(t)) \) is a polynomial and \( m \) is a non-zero positive integer, \( p(x(t)) = \sum_{i=1}^{m} q_i(x(t))^2 \) indicates the polynomial \( p(x(t)) \) is an SØS [33], implying \( p(x(t)) \geq 0 \).

4) \( P \geq 0, P > 0, P \leq 0 \) and \( P \prec 0 \) represent that each element of the matrix \( P \) is non-negative, positive, non-positive, and negative, respectively.

5) \( P^{(\alpha, \beta)} \) denotes the \( \alpha \)th row, \( \beta \)th column element of \( P \).\( P^{(\alpha, \cdot)} \) is a vector denoting the \( \alpha \)th row of \( P \), and \( P^{(\cdot, \beta)} \) is a vector as the \( \beta \)th column of \( P \).

B. Switched Positive Polynomial Fuzzy Models With Time Delay

The \( j \)th rule of the \( p \)th switched positive subsystem is:

\[ \text{Rule } R^i_p : \text{IF } f_1(x(t)) \text{ is } M^{i_1}_p \text{ AND \cdots AND } f_\psi(x(t)) \text{ is } M^{i_\psi}_p \text{ THEN} \]

\[ \dot{x}(t) = A^{d}_{pi}(x(t))x(t) + A^{d}_{pi}(x(t))x(t - \tau) + B_{pi}(x(t))u(t), \]

\[ x(t) = \phi(t), t \in [-\tau, 0], \]

where \( M^{i_j}_p \) is the fuzzy set of \( i \) corresponding to premise variable \( f_j(x(t)) \), with \( l = 1, \ldots, \psi; \psi \) is a positive integer; \( i = 1, \ldots, r_p \), \( r_p \) is the number of IF-THEN rules of the \( p \)th switched polynomial fuzzy subsystem; \( p = 1, \ldots, S \), \( S \) is the number of switched subsystems; \( A^{d}_{pi}(x(t)) \) is an \( \mathbb{R}^{n \times n} \) matrix, \( B_{pi}(x(t)) \) is an \( \mathbb{R}^{n \times m} \) matrix, and \( x(t) \in \mathbb{R}^n \) and \( u(t) \in \mathbb{R}^m \) are the state vector and the control input vector; \( \tau \) is the time delay.

The \( p \)th switched polynomial fuzzy subsystem dynamics are described as follows:

\[ \dot{x}(t) = \sum_{i=1}^{r_p} \omega_{pi}(x(t)) \times (A^{d}_{pi}(x(t))x(t) + A^{d}_{pi}(x(t))x(t - \tau)) + B_{pi}(x(t))u(t), \]

\[ x(t) = \phi(t), t \in [-\tau, 0], \]

where \( \omega_{pi}(x(t)) \geq 0, \sum_{i=1}^{r_p} \omega_{pi}(x(t)) = 1, \omega_{pi}(x(t)) = \frac{\prod_{k=1}^{n} \mu_{M^{i_j}_{pl}}(f_k(x(t)))}{\sum_{k=1}^{n} \prod_{l=1}^{n} \mu_{M^{i_l}_{pl}}(f_l(x(t)))} \) for all \( i \) and \( w_{pi}(x(t)) \) is the normalization of the membership grade; \( \mu_{M^{i}_p}(f_k(x(t))), l = 1, \ldots, \psi \) is the membership function corresponding to the fuzzy set \( M^{i}_p \).

Then, the switched PPFMB system can be given as:

\[ \dot{x}(t) = \sum_{p=1}^{S} \delta_{pi}(\sigma(t)) \sum_{i=1}^{r_p} \omega_{pi}(x(t)) \times \left( A_{pi}(x(t))x(t) + A^{d}_{pi}(x(t))x(t - \tau) + B_{pi}(x(t))u(t) \right), \]

where \( \sigma(t) \) is the indication function satisfying

\[ \delta_{pi}(\sigma(t)) = \begin{cases} 1, & \text{if } \sigma(t) = p, \\ 0, & \text{otherwise}. \end{cases} \]

Definition 1 ([28]): It is assumed that a nonlinear system is positive if the corresponding trajectory \( x(t) \geq 0 \) for all \( t \geq 0 \) when the initial condition \( \phi(\cdot) \geq 0 \).

Definition 2 ([33]): A matrix \( M \) is a Metzler matrix if its off-diagonal elements are non-negative: \( m_{rs} \geq 0, r \neq s \).

Definition 3 ([41]): Under certain switching signal \( \sigma(t) \), the switched PPFMB system is globally uniformly exponentially stable (GUES) if there exist constants \( \alpha > 0, \beta > 0 \) such that the solution met the following condition: \( \| x(t) \| \leq e^{-\gamma(t-t_0)}\| x(t_0) \|_{sup}, \forall t \geq t_0, \| x(t_0) \|_{sup} = \sup_{t \in [-\tau, 0]} \| x(t + \theta) \|, x(t_0) = \phi(t_0), t_0 \in [\tau, 0] \).

Definition 4 ([41]): For a switching signal \( \sigma(t) \) and any \( T > t_0 \geq 0, N_{\sigma}(T, t_0) \) and \( T_\sigma(T, t_0) \) are defined as the switching numbers and the total running time of the \( p \)th subsystem being activated respectively over the interval [\( T, t_0 \)], \( p \in S \). If the \( \sigma(t) \) has an average dwell time \( \pi_{\sigma} \), there exist two positive numbers \( N_{\sigma_0} \) and \( \pi_{\sigma_0} \) satisfying

\[ N_{\sigma_0}(T, t_0) \leq N_{\sigma_0} + \frac{T_\sigma(T, t_0)}{\pi_{\sigma_0}}, \forall T \geq t_0 \geq 0, \]

where \( \pi_{\sigma} \) is denoted the mode-dependent average dwell time of the switching signal \( \sigma(t) \).

Lemma 1: System (5) is said to be a switched positive system if \( A_{pi}(x(t)) \) is a Metzler matrix, \( B_{pi}(x(t)) \geq 0, A^{d}_{pi}(x(t)) \geq 0 \).

C. Switched Polynomial State Fuzzy Controller

Based on IPM concept, the SPSF controller for \( p \)th subsystem is described by \( c_p \) rules. The formulation of the \( j \)th rule is given:

\[ \text{Rule } R^j_p : \text{IF } g_1(x(t)) \text{ is } N^{j_1}_{pl} \text{ AND \cdots AND } g_\Omega(x(t)) \text{ is } N^{j_\Omega}_{pl} \text{ THEN} \]

\[ u(t) = G_{\beta j}(x(t))x(t), j = 1, \ldots, c_p, \]

where \( N^{j_\beta}_{pl} \) is a fuzzy set of rule \( j \) corresponding to the function \( g_\beta(x(t)) \), \( \beta = 1, \ldots, \Omega \); \( \Omega \) is a positive integer; \( j = 1, \ldots, c_p, c_p \) is the number of IF-THEN rules of the \( p \)th switched polynomial state fuzzy controllers; \( p = 1, \ldots, S \), \( S \) is the number of switched subsystems; \( G_{\beta j}(x(t)) \in \mathbb{R}^{m \times n} \) are the polynomial feedback gains to be determined.
The SPSF controller for the \( p^{th} \) switched subsystem is expressed as:

\[
\mathbf{u}(t) = \sum_{j=1}^{c_p} \mathbf{m}_{pj}(\mathbf{x}(t)) \mathbf{G}_{pj}(\mathbf{x}(t)) \mathbf{x}(t),
\]

where \( m_{pj}(\mathbf{x}(t)) \geq 0 \), \( c_n \) \( m_{pj}(\mathbf{x}(t)) = 1 \), \( m_{pj}(\mathbf{x}(t)) = \sum_{j=1}^{c_p} \mu_{N_i}(\mathbf{g}_j(\mathbf{x}(t))) \), \( \forall p, j \) \( m_{pj}(\mathbf{x}(t)) \), is the normalized membership grade; \( \mu_{N_i}(\mathbf{g}_j(\mathbf{x}(t))) \), \( \beta = 1, \ldots, \Omega \) is the membership function corresponding to the fuzzy set \( N^\beta_{pj} \).

From the switched PPFMB (5) system and the SPSF controller (8), the switched closed-loop PPFMB control system with time delay is represented as:

\[
\dot{x}(t) = \sum_{p=1}^{S} \sum_{r_p=1}^{r_p} \sum_{c_p} \delta_p(\sigma(t)) \mathbf{w}_{pi}(\mathbf{x}(t)) m_{pj}(\mathbf{x}(t)) \left( (\mathbf{A}_{pi}(\mathbf{x}(t)) \mathbf{B}_{pi}(\mathbf{x}(t)) \mathbf{G}_{pj}(\mathbf{x}(t)) \mathbf{x}(t) + \mathbf{A}^d_{pi}(\mathbf{x}(t)) \mathbf{x}(t - \tau) \right).
\]

**Remark 1:** The main purpose of this paper is to design the SPSF gain \( \mathbf{G}_{\sigma(\cdot)j}(\mathbf{x}(t)) \), \( \sigma(\cdot) \in [s, j] \in c_p \) and an admissible switching signal \( \sigma(\cdot) \) so as to system (9) is GUES and possesses the tighter bounds on the dwell time.

**Remark 2:** The IPM concept offers a higher design flexibility to the switched polynomial fuzzy controller as the number of rules and the premise membership functions can be chosen freely: \( r_p \neq c_p, \mathbf{w}_{pi}(\mathbf{x}(t)) \neq m_{pj}(\mathbf{x}(t)) \) so that the implementation cost can be reduced.

### III. Stability and Positive Analyses

In this section, the basic Lyapunov stability and positivity conditions for the switched PPFMB control system with time delay (9) are formulated. Then, the MFD technique is adopted such that the information of both membership functions and premises variables can be considered in the stability analysis, which is used to obtain a tighter bound of MDADT. At last, the stability analysis based on the SFCLKF and the introduction of derivative information of membership functions are proved to further relax the conservativeness of the stability analysis.

#### A. Positivity Analysis

The one objective is to control the closed-loop switched PPFMB system with time delay positive, i.e., trajectory \( x(t) \geq 0 \) if the initial condition \( \phi(\cdot) \geq 0 \).

To transform non-convex conditions existing in the following positivity and stability analysis into convex conditions, we choose a non-zero constant vector \( \mathbf{v} \in \mathbb{R}_+^N \), then the polynomial fuzzy state feedback gain can be decomposed into the following formula:

\[
\mathbf{G}_{pj}(\mathbf{x}) = \mathbf{v} \mathbf{g}_{pj}(\mathbf{x}),
\]

where \( \mathbf{g}_{pj}(\mathbf{x}) \in \mathbb{R}^{1 \times N} \) is a decision variable. Let us define a new variable:

\[
\chi_{pji}(\mathbf{x}) = \mathbf{v}^T \mathbf{B}_{pi}(\mathbf{x}) \mathbf{G}_{pj}(\mathbf{x}) = \mathbf{v}^T \mathbf{B}_{pi}(\mathbf{x}) \mathbf{v} \mathbf{g}_{pj}(\mathbf{x}),
\]

where \( \mathbf{g}_{pj}(\mathbf{x}) \in \mathbb{R}^{1 \times N} \) is a decision variable. Let us define a new variable:

\[
\chi_{pji}(\mathbf{x}) = \mathbf{v}^T \mathbf{B}_{pi}(\mathbf{x}) \mathbf{G}_{pj}(\mathbf{x}) = \mathbf{v}^T \mathbf{B}_{pi}(\mathbf{x}) \mathbf{v} \mathbf{g}_{pj}(\mathbf{x}),
\]

where \( \mathbf{g}_{pj}(\mathbf{x}) \in \mathbb{R}^{1 \times N} \) is a decision variable. Let us define a new variable:

\[
\chi_{pji}(\mathbf{x}) = \mathbf{v}^T \mathbf{B}_{pi}(\mathbf{x}) \mathbf{G}_{pj}(\mathbf{x}) = \mathbf{v}^T \mathbf{B}_{pi}(\mathbf{x}) \mathbf{v} \mathbf{g}_{pj}(\mathbf{x}),
\]

where \( \mathbf{g}_{pj}(\mathbf{x}) \in \mathbb{R}^{1 \times N} \) is a decision variable. Let us define a new variable:

\[
\chi_{pji}(\mathbf{x}) = \mathbf{v}^T \mathbf{B}_{pi}(\mathbf{x}) \mathbf{G}_{pj}(\mathbf{x}) = \mathbf{v}^T \mathbf{B}_{pi}(\mathbf{x}) \mathbf{v} \mathbf{g}_{pj}(\mathbf{x}),
\]

where \( \mathbf{g}_{pj}(\mathbf{x}) \in \mathbb{R}^{1 \times N} \) is a decision variable. Let us define a new variable:

\[
\chi_{pji}(\mathbf{x}) = \mathbf{v}^T \mathbf{B}_{pi}(\mathbf{x}) \mathbf{G}_{pj}(\mathbf{x}) = \mathbf{v}^T \mathbf{B}_{pi}(\mathbf{x}) \mathbf{v} \mathbf{g}_{pj}(\mathbf{x}),
\]

where \( \mathbf{g}_{pj}(\mathbf{x}) \in \mathbb{R}^{1 \times N} \) is a decision variable. Let us define a new variable:

\[
\chi_{pji}(\mathbf{x}) = \mathbf{v}^T \mathbf{B}_{pi}(\mathbf{x}) \mathbf{G}_{pj}(\mathbf{x}) = \mathbf{v}^T \mathbf{B}_{pi}(\mathbf{x}) \mathbf{v} \mathbf{g}_{pj}(\mathbf{x}),
\]
used to substitute the non-convex term. We define $h_{pij}(x) = \delta_p(x)w_{pi}(x)m_{pj}(x)$. So, the (17) can be written as follows:

$$V_p(x) + \lambda_p V_p(x) = \sum_{i=1}^{S} \sum_{j=1}^{r_p} h_{pij}(x) \Phi_{pij}(x) z,$$

where $\Phi_{pij}(x) = [\Phi_{pij}(x) \Lambda_{pi}(x)]$ and $z = [x(t)^T \lambda(t)^T]$. $\Theta_{pij}(x)$ and $\Lambda_{pi}(x)$ are defined as follows:

$$\Theta_{pij}(x) = \vartheta_p^T \Lambda_{pi}(x) + \lambda_p \vartheta_p^T,$$

$$\Lambda_{pi}(x) = \vartheta_p^T \Lambda_{pi}(x) - e^{-\lambda_p \tau_p} \vartheta_p^T.$$

Under the circumstance of the membership function information is not taken into account, as the positive system $z \geq 0$, thus we can get the following inequality

$$\Phi_{pij}(x) \leq 0,$$

which implies that

$$V_p(x(t)) \leq -\lambda_p V_p(x(t)).$$

In addition, when following inequalities are satisfied

$$\vartheta_p^T - \mu_p \vartheta_q^T \leq 0,$$

$$\psi_p^T - \mu_p \psi_q^T \leq 0,$$

We have

$$V_p(x(t_i)) \leq \mu_p V_q(x(t_i)),$$

where $\mu_p$ is constant and satisfies $\mu_p > 1$.

It follows from (22) and (25) that for any time $t \in [t_j, t_{j+1}),$ we have

$$V_{\sigma(t_j)}(t) \leq \exp \left\{ \sum_{p=1}^{S} N_{op} \ln \mu_p \right\} \times \exp \left\{ \sum_{p=1}^{S} \left( \frac{\ln \mu_p}{\pi_{ap}} - \lambda_p \right) T_p(t_0, t) \right\} V_{\sigma(t_0)}(t_0).$$

Define $\beta_1 = \min_{1 \leq i \leq \alpha} \{ \theta_{\sigma(t_0, y_i)} \}, \beta_2 = \max_{1 \leq i \leq \alpha} \{ \theta_{\sigma(t_0, y_i)} \}, \beta_3 = \max_{1 \leq i \leq \alpha} \{ \psi_{\sigma(t_0, y_i)} \}, \theta_{\sigma(t_0, y_i)}$ and $\psi_{\sigma(t_0, y_i)}$ are the l-th element of vectors $\theta_{\sigma(t_0, y_i)}$ and $\psi_{\sigma(t_0, y_i)}$, respectively, $l \in \{1, \ldots, n\},$ we can obtain that

$$V_{\sigma(t_N, x)}(T) \geq \beta_1 \| x(T) \|,$$

$$V_{\sigma(t_0, x)}(T) \leq \beta_2 \sqrt{n} \| x(t_0) \| + \beta_3 \sqrt{n} \int_{t_0}^{t_N} \| x(s) \| ds.$$

Combining (26), (27) and (28), we obtain

$$\eta \leq \exp \left\{ \sum_{p=1}^{S} N_{op} \ln \mu_p \right\} \left( \frac{\ln \mu_p}{\pi_{ap}} - \lambda_p \right).$$

Finally, it can be concluded that if (21), (23) and (24) hold, system (9) is GUES for any switching signal with

$$\pi_{ap} \geq \pi_{ap}^* = \frac{\ln \mu_p}{\pi_{ap}}, \ p \in S$$

Combining the above analysis, the stability-positivity analysis results of the switched PPFMB control system through the MFI method are displayed in the following theorem.

**Theorem 1:** For the switched closed-loop PPFMB control system with time delay (9), if there exist positive scalars $\lambda_p > 0, \mu_p > 1,$ vectors $\vartheta_p \in \mathbb{R}^n, \psi_p \in \mathbb{R}^n,$ and polytopic vectors $\chi_p(\lambda) \in \mathbb{R}^n, \forall p \in S, i \in t_p, j \in c_p,$ such that the following SOS-based conditions are satisfied:

$$\vartheta_p(0) - \epsilon_1 \leq \mathrm{SOS} \ \forall \alpha, p;$$

$$\psi_p(0) - \epsilon_2 \leq \mathrm{SOS} \ \forall \alpha, p;$$

$$-\left( \Phi_{pij}(x) \right) \leq \mathrm{SOS} \ \forall i, p, i, j;$$

$$-\left( \psi_p(0) - \epsilon_3 \right) \leq \mathrm{SOS} \ \forall \alpha, p, q, p \neq q;$$

$$-\left( \psi_p(0) - \epsilon_4 \right) \leq \mathrm{SOS} \ \forall \alpha, p, q, p \neq q;$$

$$\vartheta_p^T \vartheta_q^T \vartheta_q^T \vartheta_p^T \mathrm{SOS} \ \forall \alpha, \beta, \alpha \neq \beta, p, i, j;$$

$$\Phi_{pij}(x) \leq \mathrm{SOS} \ \forall \alpha, \beta, p, i;$$

$$B_p(0) \leq \mathrm{SOS} \ \forall \alpha, \beta, p, i;$$

where $\epsilon_1 > 0, \epsilon_2 > 0, \epsilon_3 > 0, \epsilon_4 > 0$ and $\epsilon_5 > 0$ are predefined scalar constants; $\alpha, \beta \in n, i \in 2n.$ The feedback gain is calculated as $G_{pi}(x) = v \chi_p(x); \Phi_{pij}(x) = [\Theta_{pij}(x) \Lambda_{pi}(x)], \Theta_{pij}(x) \Lambda_{pi}(x)$ are defined as (19) and (20) respectively. Then, a set of switching signals with MDAT satisfying $\pi_{ap} \geq \ln(\mu_p) / \lambda_p$ and the SPSF controller exist such that the system is GUES and positive.

**Remark 3:** Applying the matrix decomposition technique, a non-zero constant vector $v \in \mathbb{R}^n$ can be chosen by the user, then the polynomial fuzzy state feedback gain can be decomposed into $G_{pi}(x) = v \chi_p(x), \Phi_{pij}(x)$ the non-convex terms $\vartheta_p^T \vartheta_q^T \vartheta_q^T \vartheta_p^T \mathrm{SOS}$ that exist in stability conditions can be replaced by $\gamma_{pij}(x).$ Next, we can unify the variables which are decided in positivity and stability conditions. Then non-convex stability conditions can be successfully solved.

**Remark 4:** From the $\pi_{ap} \geq \ln(\mu_p) / \lambda_p$, the bound of MDAT of switched systems is determined by the $\mu_p$ and $\lambda_p.$ Thereby, the next goal is to obtain tighter dwell time results in incorporating the information contained in membership functions (MF), which degrades the conservativeness of the analysis results in turn cause increasing the parameter $\lambda_p.$

**C. The Membership Function Dependent Stability Analysis**

In this section, Taylor series membership functions (TSMFs) are performed to handle the membership function in (18), which cannot be solved by the SOSTOOLs. TSMFs expand a polynomial function in sample points by using Taylor series expansion technology, thus $h_{pij}(x)$ can be brought into stability conditions. The overall state space $x$ is divided into $s$ connected substate spaces which are denoted as $\psi \xi, \xi = 1, 2, \ldots, s.$ The TSMFs [44] are written as:

$$\tilde{h}_{pij}(x) = \sum_{\xi=1}^{s} \psi(\xi) \tilde{h}_{pij}(x)$$
where \( \zeta_{r,i}(x_r) \) is an interpolation function, which satisfies the properties that \( 0 \leq \zeta_{r,i}(x_r) \leq 1 \) and \( \zeta_{r,i}(x_r) + \zeta_{r,2} \delta(x_r) = 1 \) for \( r \in n, i_r = \{ 1, 2 \} \); \( \delta_{ij,i_12\cdots i_n}(x) \) is the local approximating function at the interpolation point, and \( \delta_{ij,i_12\cdots i_n}(x) = \sum_{k=0}^{r-1} \frac{1}{r!} (x_r - x_{r,i})^k (x_r = x_{r,i}, r = 1, \ldots, n) \); \( s \) is the number of substate spaces \( \psi \) divided by the entire state space \( \psi \); \( \zeta(x) \) is defined as the following function: \( \zeta(x) = \left\{ \begin{array}{ll} 1, & \text{if } x \in \Psi \\
 0, & \text{if } x \notin \Psi \end{array} \right. \\

According to (37), TSFM is composed of Taylor extension \( \delta_{ij,i_12\cdots i_n}(x) \) and interpolation function \( \zeta_{r,i}(x_r) \) by linear combination method. The original membership function can be rewritten by TSFM as:

\[
h_{pij}(x) = \sum_{\omega = 1}^{s} \omega(x)(h_{pij}(x) + \Delta h_{pij}(x)),
\]

where \( \Delta h_{pij}(x) = h_{pij}(x) - \bar{h}_{pij}(x) \) is the approximation error of TSFM, and its lower and upper bounds as \( \gamma_{pij} \) and \( \bar{\gamma}_{pij} \), respectively, satisfying

\[
\gamma_{pij} \leq \Delta h_{pij}(x) \leq \bar{\gamma}_{pij}.
\]

Based on the above properties, we consider introducing the slack scalar \( \Gamma_{pij}(x) \) satisfying \( \Gamma_{pij}(x) \in \mathbb{R}_{>0}^{n} \) and \( \Gamma_{pij}(x) \geq \Phi^{(a)}(x) \), \( \forall p \in S, i \in r_p, j \in c_p \). Thereby the basic stability condition (18) can be relaxed into the following form:

\[
\dot{V}_p(x) + \lambda_p V_p(x) \leq \sum_{\omega = 1}^{s} \omega(x) \sum_{i = 1}^{r_p} \sum_{j = 1}^{c_p} \left( (h_{pij}(x) + \gamma_{pij}) + \Phi_{pij}(x) + (\bar{\gamma}_{pij} - \gamma_{pij}) \Gamma_{pij}(x) \right) z.
\]

To further extract more information of membership functions, the boundary information of the cross terms of TSFM is taken into consideration, which is presented as follows in each subdomain:

\[
h_{pij} \leq \bar{h}_{pij}(x) \leq \tilde{h}_{pij}(x),
\]

where \( h_{pij} \) and \( \bar{h}_{pij} \) are the constant lower bound and upper bound of \( h_{pij}(x) \).

We can introduce the boundary information of TSFM through positive slack scalar \( S_{pij}^{(a)}(x) \in \mathbb{R}_{>0}^{n} \) and \( S_{pij}^{(a)}(x) \in \mathbb{R}_{>0}^{n} \), \( \forall p \in S, i \in r_p, j \in c_p, \zeta \in \zeta \). The following inequalities hold:

\[
(\bar{h}_{pij}(x) - h_{pij})(S_{pij}^{(a)}(x) \geq 0,
\]

\[
(\tilde{h}_{pij}(x) - h_{pij})(S_{pij}^{(a)}(x) \geq 0.
\]

Adding (41) and (42) to the right hand side of (40), we obtain:

\[
\dot{V}_p(x) + \lambda_p V_p(x) \leq \sum_{\omega = 1}^{s} \omega(x) \sum_{i = 1}^{r_p} \sum_{j = 1}^{c_p} \left( (h_{pij}(x) + \gamma_{pij}) + \Phi_{pij}(x) + (\bar{\gamma}_{pij} - \gamma_{pij}) \Gamma_{pij}(x) \right) z.
\]

In addition to the MFs information analyzed above, the premise variable information is also an essential point in relaxing the stability of the system, which restricts the premise variable within a fixed region rather global region. So, we incorporate the upper and lower bound information of the premise variables into the stability conditions. The following formulation can be obtained [32]:

\[
\sum_{r = 1}^{n} (x_r - x_{rc_{\min}}) (x_{r_{\max}} - x_r) R_{pr_c}(x) \geq 0,
\]

where \( x_{rc_{\min}} \) and \( x_{r_{\max}} \) are the minimal and maximal value of the system state \( x_r \) in the \( \zeta \) domain, \( R_{pr_c}(x) \) is positive slack scalar, which need to satisfy the condition \( R_{pr_c}(x) \geq 0 \). The formulation (43) is further dealt as follows:

\[
\dot{V}_p(x) + \lambda_p V_p(x) \leq \sum_{\omega = 1}^{s} \omega(x) \sum_{i = 1}^{r_p} \sum_{j = 1}^{c_p} \left( (h_{pij}(x) + \gamma_{pij}) + \Phi_{pij}(x) + (\bar{\gamma}_{pij} - \gamma_{pij}) \Gamma_{pij}(x) \right) z.
\]
gains are calculated as

\[ G_{\text{stability}} \] based on IMP concept, we innovatively
more freedom in proving stability, it is still conservative in
TSMFs is that the grades of membership are characterized
signal with MDADT property, the MFD method is also valid
systems are uncontrollable, combining the slow-fast switching
exists such that the system is GUES and positive.

**Remark 5:** It is worth mentioning that the switched positive
polynomial fuzzy subsystems are all controllable but originally
unstable. The state feedback controller is designed to make
to all subsystems positive and stable. For case that switched sub-
systems are uncontrollable, the slow-fast switching signal with MDADT property, the MFD method is also valid to
stabilize the switched positive polynomial fuzzy system.

**Remark 6:** TSMFs are a weighted sum of local polynomials
in a favourable form for stability analysis, due to a feature of
TSMFs it is known that the grades of membership are characterized by some sampled points, SOS-based stability conditions are obtained which guarantee the system stability if the fuzzy model-based control system is stable at all chosen Taylor series expansion points. In this way, it is easier to find the
to MDADT for the relaxed stability condition.

**D. Stability Conditions with Switched Fuzzy Copositive Lyapunov-Krasovskii Functional Candidate**

As is well known, the appropriate Lyapunov function form
can derive the relaxed stability conditions of the system. In
the previous section, though multiple Lyapunov functions provide
more freedom in proving stability, it is still customary in
designing a common Lyapunov function for each switched
local fuzzy subsystem. In this section, to further relax the SOS
stability conditions based on IMP concept, we innovatively
construct the switched fuzzy copositive Lyapunov-Krasovskii functional to further reduce the bound of MDADT, which takes
full advantage of the information of fuzzy rules.

The scalar variable \( v_p \) is governed by \( e \) fuzzy rules of the following format:

**Rule \( R_p^e \):** IF \( s_1(x(t)) \) IS \( T_{p_1}^e \) AND \( \cdots \) AND \( s_z(x(t)) \) IS \( T_{p_z}^e \)

THEN

\[ \dot{v}_p(x(t)) = v_p(x(t)), \]

where \( T_{p_z}^e \) is the fuzzy term of rule \( e \) corresponding to
the function \( s_{z}(x(t)), e = 1, ... , e \); \( z \) is a positive integer
; \( v_p(x(t)) \) IS \( \mathbb{R}_+ \), \( e = 1, ... , e \) is positive scalars to be
determined. The inferred \( \dot{v}_p(x(t)) \) can obtain that:

\[ \dot{v}_p(x(t)) = \sum_{e=1}^{e} n_{pe}(x(t)) \dot{v}_p(x(t)), \]

where

\[ n_{pe}(x(t)) = \frac{\prod_{k=1}^e \mu_{pe}^{e}(s_k(x(t))) - \prod_{k=1}^e \mu_{pe}^{e}(s_k(x(t)))}{e \prod_{k=1}^e \mu_{pe}^{e}(s_k(x(t)))}, \]

\( \triangledown_{n_{pe}}(x(t)) = 1, \)

\( n_{pe}(x(t)) \geq 0, \forall p, e; n_{pe}(x(t)) \) is the normalized grade of membership:

\[ \sum_{e=1}^{e} \mu_{pe}^{e}(s_k(x(t))) = 1, \]

where \( \mu_{pe}^{e}(x(t)) \) is the grade of membership corresponding to the fuzzy term \( T_{p_z}^e \).

So, construct the switched fuzzy copositive Lyapunov-Krasovskii functional

\[ V_p(x) = \sum_{e=1}^{e} n_{pe}(x) \dot{v}_p^T(x) + \int_{t-\tau}^{t} e^{\lambda_p(s-t)} \dot{v}_p^T(x(s)) ds. \]

Since \( x \) satisfies \( x \geq 0, \) then we get \( V_p(x(t)) \geq 0 \) and

\[ k_1(||x(t)||) \leq V_p(x(t)) \leq k_2(||x(t)||). \]

Then the \( V_p + \lambda_p V_p \)
can be derived as follows:

\[ \dot{V}_p(x) + \lambda_p V_p(x) \leq \sum_{e=1}^{e} n_{pe}(x) \sum_{i=1}^{r_p} h_{pi}(x) \left( \sum_{e=1}^{e} \hat{n}_{pe}(x) \dot{v}_p^T + \dot{v}_p^T \right) \]

\[ + \chi_{pij}(x) + \psi_p^T + \lambda_p \dot{\psi}_p^T \]

\[ + \left( \dot{v}_p^T \right) \]

According to the properties of membership functions

\[ \sum_{e=1}^{e} n_{pe}(x(t)) = 1, \]

we can obtain the characteristics of derivative of membership functions, it derives

\[ \dot{n}_{pe}(x) = - \sum_{e=1}^{e} n_{pe}(x). \]

So the nonlinear term can be treated as

\[ \sum_{e=1}^{e} \hat{n}_{pe}(x) \dot{v}_p^T x = \sum_{e=1}^{e} n_{pe}(x) \left( \dot{v}_p^T - \dot{\psi}_p^T \right) x. \]

Assuming that the following equation holds:

\[ \dot{v}_p^T - \dot{\psi}_p^T \geq 0, p = 1, ... , S, t = 1, ... , e - 1. \]

Then, one obtains that:

\[ \dot{V}_p(x) + \lambda_p V(x) \leq \sum_{e=1}^{e} \sum_{i=1}^{r_p} \sum_{j=1}^{c_p} f_{pij}(x) \Xi_{pij}(x) z, \]

where \( f_{pij}(x) = n_{pe}(x) h_{pij}(x), p \in S, i \in r_p, j \in c_p, e \in e \Xi_{pij}(x) = \left[ \Phi_{pij}(x) \right] \] and \( z = \left[ x(t) \right] \left[ x(t) - \tau \right]^T \right. \Phi_{pij}(x) \) \) and \( \Omega_{pij}(x) \) are defined as the following equation:

\[ \Phi_{pij}(x) = e^{\lambda_p(t)} \psi_p^T \]

\[ \Omega_{pij}(x) = \dot{v}_p^T \psi_p^T, \]
where \( o_{pt} \) is the upper bound for \( \dot{n}_{pt}(x) \), \( \forall p, \ell \).

Adopting the same approximate technique and relaxation technique as mentioned in Section C, the MFs can be written as follows [44]:

\[
\tilde{n}_{pt}(x) = \sum_{i_1=1}^{s} \omega_{\zeta}(x) \prod_{i_1=1}^{2} \prod_{i_n=1}^{n} \nu_{r_1, \zeta}(x_{r_1}) \bar{\rho}_{i_1 i_2 \cdots i_n}(x),
\]

(69)

\[
\tilde{f}_{pt}(x) = \sum_{i_1=1}^{s} \omega_{\zeta}(x) \prod_{i_1=1}^{2} \prod_{i_n=1}^{n} \nu_{r_1, \zeta}(x_{r_1}) \bar{\eta}_{i_1 i_2 \cdots i_n}(x),
\]

(70)

where \( \nu_{r_1, \zeta}(x_{r_1}) \) is an interpolation function, which satisfies the properties that \( 0 \leq \nu_{r_1, \zeta}(x_{r_1}) \leq 1 \) and \( \nu_{r_1, \zeta}(x_{r_1} + \nu_{r_2, \zeta}(x_{r_2}) = 1 \) for \( r \in n, i_r = \{1, 2\}; \bar{\rho}_{i_1 i_2 \cdots i_n}(x) \) is the local approximating function at the interpolation point, \( \bar{\rho}_{i_1 i_2 \cdots i_n}(x) = \lambda_{i_1} \sum_{r=1}^{\lambda_{i_1}} \prod_{i_1=1}^{2} \prod_{i_n=1}^{n} \frac{x_{r_1} - \rho_{i_1}}{\rho_{i_1}} k \nu_{p}(x) \mid x_{r_1} = x_{r_1} \), \( r = 1, \ldots, n \); \( \bar{\eta}_{i_1 i_2 \cdots i_n}(x) \) is the local approximating function at the interpolation point, \( \bar{\eta}_{i_1 i_2 \cdots i_n}(x) = \lambda_{i_1} \sum_{r=1}^{\lambda_{i_1}} \prod_{i_1=1}^{2} \prod_{i_n=1}^{n} \frac{x_{r_1} - \eta_{i_1}}{\eta_{i_1}} k \nu_{p}(x) \mid x_{r_1} = x_{r_1} \), \( r = 1, \ldots, n \); \( s \) is the number of substate spaces \( \psi \) divided by the entire state space \( \psi \); \( \omega_{\zeta}(x) \) is defined as the following function: \( \omega_{\zeta}(x) = \begin{cases} 1, & \text{if } x_r \in \psi_{\zeta} \\ 0, & \text{if } x_r \notin \psi_{\zeta} \end{cases} \).

Introducing the slack positive polynomial scalars \( Y_{pt}(x), H_{pt}(x), H_{pt}(x), Y_{pr}(x) \) satisfying \( Y_{pt}(x) \geq \Xi_{pt}(x), \forall p \in S, i \in r_p, j \in c_p, \ell \in e, \zeta \in s; \Xi_{pt}(x) \) and \( \bar{\eta}_{pt}(x) \) are the lower and the upper bound of approximation errors of TSFM; \( \tilde{f}_{pt}(x) \) and \( \bar{f}_{pt}(x) \) are the lower bound and upper bound of \( \tilde{f}_{pt}(x) \). On the basis of the above mentioned technique, we have:

\[
\dot{V}_p(x) + \lambda_p V_p(x) \leq \left( \sum_{i_1=1}^{s} \omega_{\zeta}(x) \prod_{i_1=1}^{2} \prod_{i_n=1}^{n} \nu_{r_1, \zeta}(x_{r_1}) \bar{\rho}_{i_1 i_2 \cdots i_n}(x) + \bar{\nu}_{pt}(x) \Xi_{pt}(x) + \bar{\bar{\nu}}_{pt}(x) - \bar{\bar{\rho}}_{pt}(x) \Xi_{pt}(x) + \bar{\bar{\rho}}_{pt}(x) \Xi_{pt}(x) \right) + \left( \sum_{r} (x_{r_1} - x_{r_1}) \left( x_{r_1} - x_{r_1} \right) Y_{pr}(x) \right) \leq 0.
\]

(71)

In addition to, by using (61) and (69), it gives that:

\[
V_p(x(t_i)) - \mu_p V_q(x(t_i)) \leq \sum_{i_1=1}^{s} \omega_{\zeta}(x) \prod_{i_1=1}^{2} \prod_{i_n=1}^{n} \nu_{r_1, \zeta}(x_{r_1}) \bar{\rho}_{i_1 i_2 \cdots i_n}(x) (\theta_{pt} - \mu_p \theta_{pt}^T) x(t_i) + \int_{t_i}^{t} e^{\psi_t (s-t)} (\psi_{pt} - \mu_p \psi_{pt}^T) x(s) ds.
\]

(72)

We obtain that \( V_p(x(t_i)) - \mu_p V_q(x(t_i)) \leq 0 \) and can be ensured by the following conditions:

\[
\sum_{\ell=1}^{c} \bar{\rho}_{i_1 i_2 \cdots i_n}(\theta_{pt}^T - \mu_p \theta_{qt}^T) \leq 0,
\]

(73)

\[
\psi_{pt}^T - \mu_p \psi_{qt}^T \leq 0.
\]

(74)

Define \( c_1 = \min_{1 \leq l \leq n} \left\{ \theta_{\sigma_{t-\ell}}(x_{t_0}) \right\}, c_2 = \max_{1 \leq l \leq n} \left\{ \theta_{\sigma_{t-l}}(x_{t_0}) \right\}, c_3 = \max_{1 \leq l \leq n} \left\{ \bar{\theta}_{\sigma_{t-l}}(x_{t_0}) \right\}, \bar{\theta}_{\sigma_{t-l}}(x_{t_0}) \) and \( \psi_{t-\ell} \) are the l-th element of vectors \( \bar{\theta}_{\sigma_{t-l}}(x_{t_0}) \) and \( \psi_{t-\ell} \), respectively, \( l \in \{1, \ldots, n\} \), it can be noted that:

\[
\forall \sigma_{t-\ell}(x_{t_0}) \geq \sum_{\ell=1}^{c} c_3 \sqrt{t} \| x(t_0) \| + c_4 \sqrt{t} \int_{t_0-t}^{t} \| x(s) \| ds,
\]

(75)

As the proof of (26) in Section III shows, we obtain:

\[
\| x(t) \| \leq \eta e^{-\gamma (t-t_0)} \| x(t_0) \|,
\]

(77)

where \( \eta = \exp \left( \frac{c_1}{c_2} \sum_{\ell=1}^{c} \| x_{t_0} \| \frac{c_3}{c_2} \gamma \right) \), \( \gamma = \max \left( \frac{c_1}{c_2} \right) - \lambda_p \).

Finally, we can draw a conclusion that if (73), (71) and (74) hold, system (9) is GUES for the switching signal with \( \nu_{ap} \geq \nu_{ap}^* = \frac{c_1}{c_2} \lambda_p, \ell \in S \) by Definition 3.

Combining the above analysis, the stability-positivity analysis results of the switched PFMB control system through the switched fuzzy cooperative Lyapunov-Krasovskii functional are displayed in the following theorem.

**Theorem 3:** For the switched closed-loop PFMB control system with time delay (9), if there exist positive scalars \( \lambda_p > 0, \mu_p > 1 \), vectors \( \theta_{pt} \in \mathbb{R}^+, \psi_{pt} \in \mathbb{R}^+ \), polynomial vectors \( \chi_{pt}(x) \in \mathbb{R}^+ \), \( \forall p \in S, i \in r_p, j \in c_p, \ell \in e \) and slack positive polynomial vectors \( Y_{pt}(x), H_{pt}(x), H_{pt}(x), Y_{pr}(x) \) such that the following SOS-based conditions are satisfied:

\[
\theta_{pt}^T - \epsilon_1 \text{ is SOS} \quad \forall \alpha, p, \ell; \]

(78)

\[
\psi_{pt}^T - \epsilon_1 \text{ is SOS} \quad \forall \alpha, p; \]

(79)

\[
\theta_{pt}^T - \psi_{pt}^T - \epsilon_1 \text{ is SOS} \quad \forall \alpha, p, \ell = 1, \ldots, e-1; \]

(80)

\[
\sum_{\ell=1}^{c} \bar{\rho}_{i_1 i_2 \cdots i_n}(\theta_{pt}^T - \mu_p \theta_{qt}^T) + \epsilon_2 \text{ is SOS} \quad \forall \alpha, p, \ell \neq p; \]

(81)

\[
\psi_{pt}^T - \mu_p \psi_{pt}^T + \epsilon_2 \text{ is SOS} \quad \forall \alpha, p, q, p \neq q; \]

(82)

\[
H_{pt}(x) - \epsilon_3 \text{ is SOS} \quad \forall p, i, j, \ell, \zeta, \ell; \]

(83)

\[
H_{pt}(x) - \epsilon_3 \text{ is SOS} \quad \forall p, i, j, \ell, \zeta, \ell; \]

(84)

\[
Y_{pt}(x) - \epsilon_3 \text{ is SOS} \quad \forall p, r, \ell, \zeta, \ell; \]

(85)

\[
Y_{pt}(x) - \epsilon_3 \text{ is SOS} \quad \forall p, i, j, \ell, \zeta, \ell; \]

(86)
fewer fuzzy rules and simple shape of membership function aspects: 1) reducing the number of slack matrices by predefining on the variable, which could enlarge the range of searching find that the fuzzy Lyapunov function relaxes the constraints demonstrate the effectiveness of the proposed analysis results.

In this section, two simulation examples are provided to

\[ x \in [0, 16] \], \( \tau \) is chosen as 0.1. The membership functions of the switched PPFMB system are selected as: \( w_{p1}(x_1) = 1 - \frac{1}{1+e^{-(x_1 - 5)}} \), \( w_{p2}(x_1) = 1 - w_{p1}(x_1) - w_{p3}(x_1) \), \( w_{p3}(x_1) = \frac{1}{1+e^{-(x_1 - 5)}} \), \( \forall x \in S \). Under the IPM concept, a two-rule polynomial fuzzy controller is designed to guarantee the stability of the control system. The membership functions of the fuzzy controller are chosen as \( m_{p1}(x_1) = e^{-(x_1 - 8)^2/10} \) and \( m_{p2}(x_1) = 1 - m_{p1}(x_1) \), \( \forall x \in S \). The bound of MDADT in three scenarios will be compared to demonstrate the effectiveness of the proposed approach.

1) For the purpose of comparison, Theorem1 is applied to investigate the bound of MDADT under MFI, which is the reference for comparing with the results of Theorem2.
2) In accordance with Theorem 2, the TSMFs are employed to process the original MFs for investigating how the MFD method influences the MDADT compared with Theorem 1.
3) Comparing the MDADT obtained by Theorem 3 with one obtained by Theorem 2 to analyze the influence of the SFCLKF. Then analyzing the effect of TSMFs under different highest degrees on the lower bound of MDADT.

In the first scenario, it fails to find controller gains and switching signals to make the system stable on the basis of the stability-positivity conditions in Theorem 1.

In the second scenario, to verify that the MFD conditions can be used to ensure that get the tighter MDADT, TSMFs are applied. Sample points excavate the information of original MFs and the approximation error in each operating domain is exploited to incorporate more membership function information.

We select \( \varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon_4 = \varepsilon_5 = 1 \times 10^{-4} \); \( \chi_{pi}^{(x)}(x) \) and slack scalar \( \Gamma_{pi}^{(x)}(x), \Re_{pi}^{(x)}(x) \) are all of degree of 0 and 2 in \( x_1 \); Slack scalar \( \Sigma_{pi}^{(x)}(x), \bar{S}_{pi}^{(x)}(x) \) are all of degree of 0 in \( x_1 \).

In Case 1, the bounds of MDADT solved by Theorem 2 can be viewed in Table I, the corresponding feedback gains of SPSF controllers are \( \Gamma_{p1}^{(x)}(x_i) = [0.0275x_1 - 0.52597x_1 - 1.64172, 0.00317x_1 + 0.04206x_1, 0.02191], \( \Gamma_{p2}^{(x)}(x_i) = [0.00924x_1 + 0.58999x_1 - 1.83268, 0.00183x_1 - 0.03888x_1 + 0.06075], \Gamma_{p1}^{(x)}(x_i) = [0.01690x_1 - 0.48690x_1 - 1.83268, 0.00183x_1 - 0.03888x_1 + 0.06075]. \)

A. A Numerical Example

A numerical example with system states \( x(t) = [x_1(t) \ x_2(t)] \) are given as:

\[
\begin{bmatrix}
A_{11}(x_1) = & \begin{bmatrix}
-0.01 & -0.01x_1 & 0.34 \\
0.95 & 0.05x_1 & -1.05
\end{bmatrix} \\
A_{12}(x_1) = & \begin{bmatrix}
0.12 & -0.05x_1 & 0.35 \\
0.88 & 0.01x_1 & -1.28
\end{bmatrix} \\
A_{13}(x_1) = & \begin{bmatrix}
0.15 & 0.01x_1 & 0.26 \\
0.68 & -1.02 & 0
\end{bmatrix} \\
B_{11}(x_1) = & \begin{bmatrix}
2.12 \\
1.34
\end{bmatrix}, \ 
B_{12}(x_1) = \begin{bmatrix}
1.73 \\
0.06
\end{bmatrix}
\end{bmatrix}
\]
\[ 1.65992, 0.00318x_1^2 - 0.04263x_1 + 0.02899, \quad G_{22}(x_1) = [0.01001x_1^2 - 0.54443x_1 - 1.86939, 0.00184x_1^2 - 0.03808x_1 + 0.05084]. \]

To guarantee the accuracy of simulation results, the 8 different initial states are chosen as \([4, 4]^T, [6, 8]^T, [10, 4]^T, [14, 10]^T, [4, 16]^T, [8, 12]^T\) and \([16, 10]^T\), the phase plots of the switched PPFMB subsystem 1 and subsystem 2, are shown in Fig. 1.

<table>
<thead>
<tr>
<th>Case</th>
<th>Theorem</th>
<th>Degree</th>
<th>Bounds on MDADT</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>3</td>
<td>[\mu_1 = 2; \mu_2 = 2]  [\lambda_1 = 0.2; \lambda_2 = 0.3]  [\pi_{a1} = 3.466; \pi_{a2} = 2.310]</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>1</td>
<td>[\mu_1 = 2; \mu_2 = 2]  [\lambda_1 = 0.7; \lambda_2 = 0.75]  [\pi_{a1} = 0.990; \pi_{a2} = 0.924]</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>3</td>
<td>[\mu_1 = 2; \mu_2 = 2]  [\lambda_1 = 0.9; \lambda_2 = 1.0]  [\pi_{a1} = 0.770; \pi_{a2} = 0.693]</td>
</tr>
</tbody>
</table>

Fig. 1. Phase plots of the states \(x_1\) and \(x_2\) of the switched PPFMB control subsystem under controllers obtained by Case 1.

It can be found from Fig. 1 that the SPSF controller can stabilize the switching subsystem and make the system state positive under different initial system states. Next, under the initial system state points \([8, 5]^T\) and \([16, 10]^T\), the bounds of MDADT switching signal in Table I and the SPSF controller are selected to control the system. The state response can be viewed in Fig. 2, it is obvious that the switched PPFMB system is positive and asymptotically stable under the proposed control strategy, which demonstrates the effects of the MFD method for reducing the MDADT compared with Theorem 1.

In Fig. 2, the switching signal \(\pi_{a1} = 3.466; \pi_{a2} = 2.310\) and controllers obtained by Case 1.

![Fig. 2. State responses of switched PPFMB system](image1)

Fig. 2. State responses of switched PPFMB system under the switching signal \(\pi_{a1} = 3.466; \pi_{a2} = 2.310\) and controllers obtained by Case 1.

![Fig. 3. Phase plots of the states \(x_1\) and \(x_2\) of the switched PPFMB control subsystem under controllers obtained by Case 2.](image2)

In Case 2, the membership function of the Lyapunov function are chosen as \(n_{p1}(x_1) = e^{-\frac{(x_1 - 8)^2}{10}}, \quad n_{p2}(x_1) = 1 - n_{p1}\). To verify the relaxation effect of the fuzzy Lyapunov function, we select the constant of \(\lambda = 1\) in TSMFs, and choose \(\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon_4 = \varepsilon_5 = 1 \times 10^{-4}\); \(\chi_{p1}(x)\) and \(\chi_{p2}(x)\), \(\chi_{p1}(x)\) are all of degree of 0 and 2 in \(x_1\). Slack scalar \(Y_{p1}(x)\) is degree of 0 and 4 in \(x_1\). The corresponding controllers obtained by Theorem 3 are \(G_{11}(x_1) = [0.01997x_1^2 - 0.63627x_1 + 2.33461, 0.00228x_1^2 - 0.3267x_1 + 0.5013], \quad G_{12}(x_1) = [0.01723x_1^2 - 0.46914x_1 - 1.71076, 0.00164x_1^2 - 0.02921x_1 + 0.02463], \quad G_{21}(x_1) = [0.02037x_1^2 - 0.56307x_1 - 2.32418, 0.00246x_1^2 - 0.03661x_1 + 0.06398], \quad G_{22}(x_1) = [0.01776x_1^2 - 0.52308x_1 + 1.73986, 0.00189x_1 - 0.03390x_1 + 0.04597]\), the bounds of MDADT based on Theorem 3 are shown in Table I. Similarly, we select the same initial system state point as Theorem 2 to verify the stability of the switched PPFMB control subsystem under the bounds of MDADT and the polynomial fuzzy controller solved by Theorem 3, the phase plots of the switched PPFMB positive subsystem are shown in Fig. 3. And under the initial system state points \([8, 5]^T\) and \([16, 10]^T\), the state response is asymptotically stable and positive as shown in Fig. 4.

It can be seen from Table I, Case 2 which employs the TSMF and the SFCLKF can obtain smaller mode average dwell time \(\pi_{a1}\) and \(\pi_{a2}\), respectively. It shows that the SFCLKF is effective for relaxing conservativeness of the bounds of MDADT as Lyapunov variable is extended to the fuzzy ones from the constant ones, which contains more information of membership functions and the derivative of membership functions, so that more relaxed results are derived.

![Fig. 3. Phase plots of the states \(x_1\) and \(x_2\) of the switched PPFMB control subsystem under controllers obtained by Case 2.](image3)

![Fig. 4. The switching signal](image4)
Fig. 4. State responses of switched PPFMB system under the switching signal \( \pi_{s1} = 0.990; \pi_{s2} = 0.924 \) and controllers obtained by Case 2.

Fig. 5. Phase plots of the states \( x_1 \) and \( x_2 \) of the switched PPFMB control subsystem under controllers obtained by Case 3.

Fig. 6. State responses of switched PPFMB system under the switching signal \( \pi_{s1} = 0.770; \pi_{s2} = 0.693 \) and controllers obtained by Case 3.

The relaxed analysis results can be guaranteed with membership functions information explored as much as possible. The TSMFs with higher degrees of the polynomial have a better approximation ability, more information of the original membership functions can be included in the stability conditions, as is shown in Table I, the higher degrees of the approximated polynomial functions are advantageous to decrease the bounds of MDADT.

It can be seen from Fig. 2, Fig. 4 and Fig. 6 that the decay rate of the state is gradually increasing with the relaxation of stability conditions based on the MFD method and the SFCLKF. The reason is that the tighter bounds of MDADT make the decay rate \( \lambda_p \) of Lyapunov function larger, then the exponential convergence rate will increase, such that it improves the decay rate of the system state.

A. A Lotka-Volterra population system

A Lotka-Volterra population model [23] with two modes as follows are given to illustrate the advantages of the proposed theory proposed:

\[
\begin{align*}
\dot{x}_1(t) &= (-1.18 + 0.05\sin^2(x_1(t)))x_1(t) + 1.95x_2(t) + 0.12x_1(t - \tau) + 0.08x_2(t - \tau) + (0.01 + 0.01\sin^2(x_1(t)))u(t), \\
\sum_1 : \dot{x}_2(t) &= 0.95x_1(t) + (-1.60 - 0.08\sin^2(x_1(t)))x_2(t) + (0.05 + 0.10\sin^2(x_1(t)))x_1(t - \tau) + 0.02x_2(t - \tau) + (0.012 + 0.006\sin^2(x_1(t)))u(t), \\
\dot{x}_1(t) &= (-1.50 - 0.18\sin^2(x_1(t)))x_1(t) + 0.70x_2(t) + 0.12x_1(t - \tau) + 0.14x_2(t - \tau) + (0.03 - 0.02\sin^2(x_1(t)))u(t), \\
\sum_2 : \dot{x}_2(t) &= 0.70x_1(t) + (-1.45 + 0.15\sin^2(x_1(t)))x_2(t) + 0.15x_1(t - \tau) + 0.11x_2(t - \tau) + (0.01 + 0.01\sin^2(x_1(t)))u(t).
\end{align*}
\]

The operating domain of system states is chosen as \( x_1(t) \in [0, 3] \), then system is represented by a fuzzy system with the following system matrices:

\[
\begin{align*}
A_{11} &= \begin{bmatrix} -1.18 & 0.95 \\ 0.95 & -1.60 \end{bmatrix}, \quad A_{12}^d(x_1) = \begin{bmatrix} 0.12 & 0.08 \\ 0.05 & 0.10 \end{bmatrix}, \\
A_{12} &= \begin{bmatrix} -1.13 & 0.95 \\ 0.95 & -1.68 \end{bmatrix}, \quad A_{12}^d(x_1) = \begin{bmatrix} 0.12 & 0.08 \\ 0.15 & 0.10 \end{bmatrix}, \\
A_{21} &= \begin{bmatrix} -1.50 & 0.70 \\ 0.70 & -1.45 \end{bmatrix}, \quad A_{21}^d(x_1) = \begin{bmatrix} 0.12 & 0.14 \\ 0.15 & 0.11 \end{bmatrix}, \\
A_{22} &= \begin{bmatrix} -1.68 & 0.70 \\ 0.70 & -1.35 \end{bmatrix}, \quad A_{22}^d(x_1) = \begin{bmatrix} 0.12 & 0.14 \\ 0.15 & 0.11 \end{bmatrix}, \\
B_{11} &= \begin{bmatrix} 0.010 \\ 0.012 \end{bmatrix}, \quad B_{12} = \begin{bmatrix} 0.020 \\ 0.018 \end{bmatrix}, \\
B_{21} &= \begin{bmatrix} 0.03 \\ 0.01 \end{bmatrix}, \quad B_{22} = \begin{bmatrix} 0.01 \\ 0.02 \end{bmatrix},
\end{align*}
\]

with the membership functions of the fuzzy model and the fuzzy controller are given as

\[
\begin{align*}
\omega_1(x_1(t)) &= \sin^2(x_1(t)), \quad \omega_1(x_1(t)) = \omega_2(x_1(t)) = 1 - \omega_1(x_1(t)), \quad m_{11}(x(t)) = m_{21}(x(t)) = \sin^2(x_1(t)), \quad m_{12}(x(t)) = m_{22}(x(t)) = 1 - m_{11}(x(t)).
\end{align*}
\]
Next, Theorems 1 and 3 are used to design the controllers and find the bounds on MDADT, respectively. We choose $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon_4 = \varepsilon_5 = 1 \times 10^{-4}$. $\chi_{pij}(x)$ is of degree 0 and 2 in $x_1$. Besides, the membership function of the Lyapunov function in Theorem 3 are chosen as $n_{p1}(x_1) = e^{-(x_1-1.5)^2/2}$, $n_{p2}(x_1) = 1-n_{p1}$. We select the case of $\lambda = 1$ in TSMFs, and slack scalar $H_{pij}(x)$, $\bar{H}_{pij}(x)$, $\bar{Y}_{pij}(x)$ are all of degree of 0 and 2 in $x_1$; Slack scalar $Y_{pij}(x)$ is degree of 0 and 4 in $x_1$. And the bounds on the MDADT for can be obtained as shown in Table II.

![Table II](image)

Then the obtained controllers and the corresponding MDADT switching signals are used to conduct the state response under the initial condition $[3, 2.5]^T$, as shown in Figs. 7 and 8, respectively. It can be seen from Figs. 7 and 8 that the state response of the system obtained by the stability conditions of Theorem 1 and Theorem 3 is able to be positive and stable. Furthermore, it can be seen from Table II, the bounds of MDADT obtained by Theorem 3 are smaller than the bounds of MDADT obtained by Theorem 1 which shows the merits of the proposed approach for this paper.

Remark 9: In the practical engineering, the nonlinearity of the switched positive system is transformed into membership functions of fuzzy model, and the membership functions of fuzzy controller and fuzzy Lyapunov function can be designed by an engineer considering the comprehensive analysis of actual engineering requirements. The polynomial fuzzy controller and the MDADT switching signal are obtained based the exponential stability-positivity condition including the information of fuzzy rules and then the bound of MDADT is further reduced.

V. CONCLUSION

The exponential stability and positivity analysis of switched PPFMB fuzzy systems with the time delay supported by the IPM concept has been investigated. Also, the bounds on MDADT have been greatly reduced for the MFD technique and the SFCLKD candidate resulted in relaxing the conservativeness of the SOS-based stability conditions. The Lyapunov variable of SFCLKD is designed as a fuzzy variable rather than a constant variable, so the membership functions’ derivative-dependent relaxed conditions can be obtained. Next, according to the Taylor expansion approximation theory, membership functions information is incorporated into the stability conditions for further relaxing the stability analysis, thereby tightening the bounds on MDADT. Finally, two simulation examples have been presented to reveal that the proposed method has the merits of reducing the MDADT. For the development of proposed theory, the following options can be considered: 1) the time-constant delay could be expanded to time-varying delay; 2) the controllable switched sub-systems could be expanded to uncontrollable switched sub-systems.

REFERENCES


