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# A switching asynchronous control approach for Takagi-Sugeno fuzzy Markov jump systems with time-varying delay

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***Abstract*—Till now, most of the results on T-S (Takagi-Sugeno) fuzzy Markov jump systems are synchronous and independent of the analysis of membership functions (MFs), which leads to conservatism. Therefore, it is significant to study the MFs-dependent Lyapunov-Krasovskii functional (LKF) while considering asynchrony. This article investigates the $H_\infty$ control issue of T-S fuzzy Markov jump systems with time-varying delay. First, delay-product-type (DPT) terms and MFs are introduced to design a new LKF, which fully utilizes the information of time-varying delay and MFs simultaneously. Moreover, based on a hidden Markov model, the controller mode is assumed to operate asynchronously with the system mode, and an asynchronous switching fuzzy controller is proposed by employing the switching approach. Thus a more practical and less conservative stability criterion is obtained to ensure the closed-loop system is stochastically stable with $H_\infty$ performance. Finally, three examples are provided to illustrate the superiority and practicability of the presented method.**

***Index Terms*—Asynchronous control, fuzzy Markov jump systems, membership functions, hidden Markov model.**

## I. Introduction

With the progress of technology, the complexity of actual control systems increases, including more uncertainties and nonlinearities. As a powerful tool to approximate complicated nonlinear systems (NSs), fuzzy logic control technique becomes popular. Especially, based on the T-S fuzzy model, the complex NSs can be decomposed into the summation of local linear subsystems with fuzzy rules and sets [1]. Hence, abundant considerations have been attracted to the study of T-S fuzzy system, to name a few, $H_\infty$ control [2], [3], finite-time control [4], event-triggered control [5]–[7], stability analysis [8]–[11], and fault detection filter design [12]–[15].

On the other hand, due to component failures, operation point shifting, abrupt environmental disturbances and other reasons, the structure and parameters of practical systems often change randomly. As a specific class of hybrid systems, Markov jump systems (MJSs) can appropriately depict these abrupt changes, and have wide applications in biochemical processes, electric power and aircraft control [16]–[19]. Considering the random variation in T-S fuzzy systems, fuzzy MJSs (FMJSs) are obtained. In recent years, the research on FMJSs is a hot topic and has achieved extensive results [20]–[27]. For example, the issue of finite-time control for FMJSs with partly uncertain transition probabilities (TPs) was investigated in [20]. The stability problem for FMJSs with stochastic sampling was discussed in [21], [23]. The reachable set estimation issue for FMJSs with bounded external disturbances was discussed in [24]. The dissipativity-based control for nonhomogeneous FMJSs with mismatched fuzzy-basis functions was addressed in [25].

However, the LKFs constructed in the above studies are independent of MFs, that inevitably leads to conservatism. Therefore, the design of MFs-dependent LKF has attracted plentiful attention. In [28]–[30], the MFs-dependent LKF was developed, but the time derivative of MFs needs to be bounded, which is hard to get. Recently, a switching approach was presented in [31], [32] to deal with the time derivative of MFs, and it was verified to be effective for reducing conservatism. Nevertheless, these works do not fully consider the information of time-varying delay, and the switching approach is based on the assumption that the number of switching is finite. Hence, it is necessary to design a LKF containing both time-varying delay and MFs information. Furthermore, extending the results to FMJS is also a challenging work, which is the first motivation of this paper.

On the other hand, for MJSs, a universal assumption is the mode of system requires to be obtained accurately by controller in real time. However, due to information loss (e.g. the system modes), delay and quantization in the process of network transmission, it is difficult to keep the controller mode persistent with the system mode in practical control systems, which is not considered in many existing works. Fortunately, the irrationality of this assumption has been realized by researchers, and the issue of asynchrony has received increased attention [33]–[35]. In recent years, relevant results have been extended to FMJSs [36]–[38]. For instance, based on the hidden Markov model (HMM), the problem of asynchronous dissipative control for FMJSs was considered in [36], [37]. The non-fragile sampled-data control issue for FMJSs with actuator failures and gain fluctuations was investigated in [38]. Despite all this, the study on asynchronous $H_\infty$ control of FMJS under switching strategy is still a gap to be stuffed, which is another motivation of this paper.

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Inspired by the discussion mentioned above, we further investigate the asynchronous $H_\infty$ control issue for FMJSs via the switching approach. The parallel distributed compensation (PDC) scheme is utilized to design the state-feedback controller. The main contributions lie in the following:

1) The switching method is applied to deal with the fuzzy systems whose system states are constrained in an area, so the invariant set has to be estimated.
2) Two DPT terms are introduced in the MFs-dependent LKF. Due to the inclusion of more time-varying delay information and the relaxation of some location restrictions, the presented criterion is less conservative and more general than the existing ones.
3) Different from the synchronous phenomenon described in the past results, in this paper, the mismatched property of jump modes is fully taken into account on the basis of HMM. In addition, a more general switching fuzzy controller is presented in asynchronous framework.

**Notation:** $R^n$ denotes the $n$-dimensional Euclidean space; $E\{\cdot\}$ represents the mathematic expectation; $\mathcal{F}$ is the weak infinitesimal operator; $H<0$ ($\leq 0$) means $H$ is symmetric negative-definite (semi-negative-definite); "$*$" represents the term caused by symmetry in LMIs; $diag\{\cdots\}$ is a block-diagonal matrix; $sym(A)$ stands for $A+A^T$; and the block-column vector is presented by $col\{\cdots\}$. Unless otherwise stated, all matrices are hypothesized to have compatible dimensions.

## II. System description and preliminaries

Consider the following FMJSs with time-varying delay:
*Plant Rule* $q$: IF $\theta_1(t)$ is $M_{q1}$, ..., $\theta_p(t)$ is $M_{qp}$, then

$$\begin{cases} \dot{x}(t)=A_{r(t)q}x(t)+A_{dr(t)q}x(t-d(t))+B_{1r(t)q}u(t) \\ \qquad +D_{r(t)q}w(t) \\ z(t)=C_{r(t)q}x(t)+C_{dr(t)q}x(t-d(t))+B_{2r(t)q}u(t) \\ \qquad +E_{r(t)q}w(t) \\ x(t)=\phi(t), t\in[-d_2,0], \end{cases} \tag{1}$$

where $x(t)\in R^n$, $u(t)\in R^l$, $w(t)\in R^m$, and $z(t)\in R^s$ denote the state vector, controlled input, deterministic disturbance input, and measured output, respectively. $\theta_j(t)$ is the premise variable bounded and smooth in compact set $\mathbb{C}$, $M_{qj}(q=1,2,\ldots,r, j=1,2,\ldots,p)$ represents the fuzzy set with $r$ rules. $A_{r(t)q}$, $A_{dr(t)q}$, $B_{1r(t)q}$, $D_{r(t)q}$, $C_{r(t)q}$, $C_{dr(t)q}$, $B_{2r(t)q}$, $E_{r(t)q}$ are known system matrices. $d(t)$ and $\dot{d}(t)$ respectively represent the time-varying delay and the delay change rate, satisfying

$$0\leq d_1\leq d(t)\leq d_2, \kappa_1\leq \dot{d}(t)\leq \kappa_2, \forall t\geq 0, \tag{2}$$

in which $d_1$, $d_2$, $\kappa_1$, $\kappa_2$ are given constant bounds.

$\{r(t),\forall t\geq 0\}\in\mathcal{N}=\{1,2,\ldots,N\}$ represents the continuous Markov process, and the TP matrix $\Pi=[\pi_{ij}]$ is described as:

$$\Pr\{r_{t+\Delta t}=j\,|r_t=i\}=\begin{cases} \pi_{ij}\Delta t+o(\Delta t), & i\neq j, \\ 1+\pi_{ii}\Delta t+o(\Delta t), & i=j, \end{cases} \tag{3}$$

with $\Delta t>0$ and $\lim\limits_{\Delta t\to 0}\frac{o(\Delta t)}{\Delta t}=0$, $\pi_{ij}$ satisfies $\pi_{ij}\geq 0$ with $i\neq j$ and $\pi_{ii}=-\sum_{j=1,j\neq i}^{N}\pi_{ij}$.

Employing the product-fuzzy inference with $r(t)=i$ (for brevity), the FMJSs (1) can be presented as

$$\begin{cases} \dot{x}(t)=A_{ih}x(t)+A_{dih}x(t-d(t))+B_{1ih}u(t)+D_{ih}w(t) \\ z(t)=C_{ih}x(t)+C_{dih}x(t-d(t))+B_{2ih}u(t)+E_{ih}w(t) \\ x(t)=\phi(t), t\in[-d_2,0], \end{cases} \tag{4}$$

where

$$A_{ih}=\sum_{q=1}^{r}h_q(\theta(t))A_{iq}, A_{dih}=\sum_{q=1}^{r}h_q(\theta(t))A_{diq},$$
$$B_{1ih}=\sum_{q=1}^{r}h_q(\theta(t))B_{1iq}, B_{2ih}=\sum_{q=1}^{r}h_q(\theta(t))B_{2iq},$$
$$D_{ih}=\sum_{q=1}^{r}h_q(\theta(t))D_{iq}, C_{ih}=\sum_{q=1}^{r}h_q(\theta(t))C_{iq},$$
$$C_{dih}=\sum_{q=1}^{r}h_q(\theta(t))C_{diq}, E_{ih}=\sum_{q=1}^{r}h_q(\theta(t))E_{iq},$$

$h_q(\theta(t))$ is the membership function denoted by $h_q(\theta(t))=\frac{\prod_{j=1}^{p}\mu_{qj}(\theta_j(t))}{\sum_{q=1}^{r}\prod_{j=1}^{p}\mu_{qj}(\theta_j(t))}$, and $\mu_{qj}(\theta_j(t))$ is the grade of membership of $\theta_j(t)$ in $\mu_{qj}$. It is clear that $h_q(\theta(t))\geq 0$ and $\sum_{q=1}^{r}h_q(\theta(t))=1$ for $t>0$.

For simplicity, we use $u$ to supersede $u(t)$, and the single sums are expressed as $P_{ih}=\sum_{q=1}^{r}h_qP_{iq}$ with $h_q=h_q(\theta(t))$.

In real systems, the system mode information observed by the controller is usually inaccurate since quantization and data dropouts, which leads to the asynchrony of modes between the controller and the plant. Furthermore, due to the tight coupling between the controller and the system, the controller needs to take into account the same delay as the system to effectively compensate for it. Therefore, the following asynchronous PDC controller with time-varying delay is applied:

$$u=K_{\partial(t)h}x(t)+K_{d\partial(t)h}x(t-d(t)), \tag{5}$$

in which the variable $\partial(t)$ is adopted to express the mode of the controller, taking values in another finite set $\mathcal{M}=\{1,2,\ldots,M\}$, and is constrained by conditional probability (CP) matrix $\Theta=[\lambda_{i\mu}]$, satisfying

$$\Pr\{\partial(t)=\mu\,|r(t)=i\}=\lambda_{i\mu}. \tag{6}$$

It is clear that $\lambda_{i\mu}$ pertains to the interval $[0,1]$, and $\sum_{\mu=1}^{M}\lambda_{i\mu}=1$.

To analyze conveniently, $K_{\partial(t)h}$, $K_{d\partial(t)h}$ are described by $K_{\mu h}$, $K_{d\mu h}$, respectively. From (4) and (5), we get the following closed-loop system:

$$\begin{cases} \dot{x}(t)=A_{i\mu hh}x(t)+A_{di\mu hh}x(t-d(t))+D_{ih}w(t) \\ z(t)=C_{i\mu hh}x(t)+C_{di\mu hh}x(t-d(t))+E_{ih}w(t) \\ x(t)=\phi(t), t\in[-d_2,0], \end{cases} \tag{7}$$

with

$$A_{i\mu hh}=\sum_{q=1}^{r}\sum_{p=1}^{r}h_qh_pA_{i\mu qp}, A_{di\mu hh}=\sum_{q=1}^{r}\sum_{p=1}^{r}h_qh_pA_{di\mu qp},$$

$$C_{i\mu hh}=\sum_{q=1}^{r}\sum_{p=1}^{r}h_qh_pC_{i\mu qp}, C_{di\mu hh}=\sum_{q=1}^{r}\sum_{p=1}^{r}h_qh_pC_{di\mu qp},$$
$$A_{i\mu qp}=A_{iq}+B_{1iq}K_{\mu p}, A_{di\mu qp}=A_{diq}+B_{1iq}K_{d\mu p},$$
$$C_{i\mu qp}=C_{iq}+B_{2iq}K_{\mu p}, C_{di\mu qp}=C_{diq}+B_{2iq}K_{d\mu p}.$$

*Remark 1:* In this paper, the unavailability of jump modes is considered. First, a random variable $\partial(t)$ is introduced to represent the controller mode, which is indirectly affected by Markov process $r(t)$ (subject to the TP matrix $\Pi$) through the CP matrix $\Theta$ as shown in (6). Therefore, a HMM $(r(t), \partial(t), \Pi, \Theta)$ is formed to describe the asynchrony between the system and the controller. In addition, the designed controller (5) can be reduced to other types as follows:

(i) It is clustering situation when the Markov parameter $r(t)$ is grouped into multiple clusters. Within each cluster, the CP of $\partial(t)$ only depends on $r(t)$ within that particular cluster. A special case arises when there is only one cluster, that is, the CP matrix has identical rows.

(ii) A mode-independent controller is designed when $\mathcal{M}=\{1\}$ and $\Theta=[1\cdots 1]^T$. This situation is equivalent to the one mentioned in (i) where there is only one cluster.

(iii) A synchronous controller is constructed when $\mathcal{M}=\mathcal{N}$ and $\lambda_{ii}=1$.

Consequently, the asynchronous controller (5) presented in this article contains the previous ones as special cases.

*Lemma 1:* *[31]* For $P_{ih}=\sum_{q=1}^{r}h_qP_{iq}$, we have $\dot{P}_{ih}\le 0$ if the following switching rules hold

$$\begin{cases}\text{if } \dot{h}_k<0, \text{ then } P_{ik}-P_{ir}>0,\\ \text{if } \dot{h}_k\ge 0, \text{ then } P_{ik}-P_{ir}\le 0,\end{cases} \tag{8}$$

where $k=1,2,\ldots,r-1$.

*Remark 2:* In [31], $\dot{P}_{ih}\le 0$ is ensured by (8) based on the assumption that the number of switching is finite. In this paper, the disturbance should be energy bounded, otherwise, the assumption will not be satisfied. On the other hand, (8) can be ensured by applying average dwell time method.

*Definition 1:* *[16]* The system (7) with $w(t)=0$ is stochastically stable (SS) if the following condition holds

$$E\left\{\int_0^{\infty}\|x(t)\|^2dt\,|(x(t)=\phi(t), r(0))\right\}<\infty.$$

## III. Main results

For simplicity, we predefine some relevant notations as:

$$\xi_1(t)=\frac{1}{d_1}\int_{t-d_1}^{t}x(s)ds, \xi_2(t)=\frac{1}{d_1(t)}\int_{t-d(t)}^{t-d_1}x(s)ds,$$
$$\xi_3(t)=\frac{1}{d_2(t)}\int_{t-d_2}^{t-d(t)}x(s)ds, \xi_4(t)=\frac{1}{d(t)}\int_{t-d(t)}^{t}x(s)ds,$$
$$\xi_5(t)=\frac{1}{d_1^2}\int_{t-d_1}^{t}\int_{\theta}^{t}x(s)dsd\theta, \xi_6(t)=\frac{1}{d_2}\int_{t-d_2}^{t}x(s)ds,$$
$$\xi_7(t)=\frac{1}{d_2^2}\int_{t-d_2}^{t}\int_{\theta}^{t}x(s)dsd\theta, \kappa t=1-\dot{d}(t),$$
$$d_1(t)=d(t)-d_1, d_2(t)=d_2-d(t), d_{12}=d_2-d_1,$$
$$d_1(0)=d(0)-d_1, d_2(0)=d_2-d(0),$$
$$\upsilon_1(t)=[x(t), x(t-d(t)), x(t-d_1), x(t-d_2), \dot{x}(t)],$$
$$\xi(t)=[\xi_1(t),\xi_2(t),\xi_3(t),\xi_4(t),\xi_5(t),\xi_6(t),\xi_7(t)],$$
$$\upsilon(t)=col\{\upsilon_1(t),\xi(t),w(t)\},$$
$$\eta_0(t)=col\{x(t), d_1\xi_1(t)\},$$
$$\eta_1(t)=col\{x(t), d_1\xi_1(t), \xi_2(t)\},$$
$$\eta_2(t)=col\{x(t), d_1\xi_1(t), \xi_3(t)\}.$$

*Theorem 1:* For given scalars $0\le d_1\le d_2$, $\kappa_1$, $\kappa_2$, and $\gamma>0$, the system (7) is SS with $H_\infty$ performance level $\gamma$ in $\mathcal{D}$

$$\mathcal{D}:=\left\{\phi(t):\sum_{\ell=1}^{5}V_\ell(x_t)\,|_{t=0}\le 1\right\}, \tag{9}$$

$$\begin{aligned}
V_1(x_t)\,|_{t=0} &= \eta_0^T(0)P_{ih(0)}\eta_0(0),\\
V_2(x_t)\,|_{t=0} &= \int_{-d_1}^{0}\phi^T(s)R_{1h(0)}\phi(s)ds+\int_{-d(0)}^{-d_1}\phi^T(s)R_{2h(0)}\\
&\quad\times\phi(s)ds+\int_{-d_2}^{-d(0)}\phi^T(s)R_{3h(0)}\phi(s)ds,\\
V_3(x_t)\,|_{t=0} &= d_2\int_{-d_2}^{0}\int_{\theta}^{0}\dot{\phi}^T(s)S_{1h(0)}\dot{\phi}(s)dsd\theta\\
&\quad+d_{12}\int_{-d_2}^{-d_1}\int_{\theta}^{0}\dot{\phi}^T(s)S_{2h(0)}\dot{\phi}(s)dsd\theta,\\
V_4(x_t)\,|_{t=0} &= d_1(0)\eta_1^T(0)Q_{1ih(0)}\eta_1(0)+d_2(0)\eta_2^T(0)\\
&\quad\times\ Q_{2ih(0)}\eta_2(0),\\
V_5(x_t)\,|_{t=0} &= \int_{-d_1}^{0}\int_{u}^{0}\int_{\theta}^{0}\dot{\phi}^T(s)U_{1h(0)}\dot{\phi}(s)dsd\theta du\\
&\quad+\int_{-d_2}^{0}\int_{u}^{0}\int_{\theta}^{0}\dot{\phi}^T(s)U_{2h(0)}\dot{\phi}(s)dsd\theta du,
\end{aligned}$$

$$\eta_0(0)=col\left\{\phi(0),\int_{-d_1}^{0}\phi(s)ds\right\},$$
$$\eta_1(0)=col\left\{\phi(0),\int_{-d_1}^{0}\phi(s)ds,\frac{1}{d_1(0)}\int_{-d(0)}^{-d_1}\phi(s)ds\right\},$$
$$\eta_2(0)=col\left\{\phi(0),\int_{-d_1}^{0}\phi(s)ds,\frac{1}{d_2(0)}\int_{-d_2}^{-d(0)}\phi(s)ds\right\},$$
$$P_{ih(0)}=\begin{bmatrix}P_{1ih(0)} & P_{2ih(0)}\\ * & P_{4ih(0)}\end{bmatrix},$$
$$Q_{gih(0)}=\begin{bmatrix}Q_{gih(0)}^1 & Q_{gih(0)}^2 & Q_{gih(0)}^3\\ * & Q_{gih(0)}^4 & Q_{gih(0)}^5\\ * & * & Q_{gih(0)}^6\end{bmatrix},$$

if there exist symmetric positive-definite matrices $P_{iq}\in R^{2n\times 2n}$, $Q_{giq}\in R^{3n\times 3n}$, $R_{fq}, S_{gq}, U_{gq}\in R^{n\times n}$, and any matrices $X_{gq}^c, V\in R^{n\times n}$ such that (8) and the following inequalities hold for $i\in\mathcal{N}$, $\mu\in\mathcal{M}$, $p,q=1,2,\ldots,r$, $g=1,2$, $f=1,2,3$, $c=1,2,3,4$

$$\begin{cases}\Omega_{i\mu pp}<0,\\ \Omega_{i\mu pq}+\Omega_{i\mu qp}<0, \quad p<q,\end{cases} \tag{10}$$

$$\psi_{gq} \geq 0, \quad (11)$$

where

$$\psi_{gq} = \begin{bmatrix} \bar{S}_{gq} & \bar{X}_{gq} \\ * & \bar{S}_{gq} \end{bmatrix}, \bar{X}_{gq} = \begin{bmatrix} X_{gq}^1 & X_{gq}^2 \\ X_{gq}^3 & X_{gq}^4 \end{bmatrix},$$

$$\Omega_{i\mu pq} = \Omega^*_{i\mu pq} + \Omega^0_{i\mu pq},$$
$$\Omega^*_{i\mu pq} = \Omega^1_{iq} + \Omega^2_{iq} + \Omega^3_{q} + \Omega^4_{i\mu pq},$$
$$\Omega^0_{i\mu pq} = z^T_{i\mu pq} z_{i\mu pq} - \gamma^2 e^T_{13} e_{13},$$
$$\Omega^1_{iq} = sym(\Pi_1^T P_{iq} \Pi_2) + \Pi_1^T \sum_{j=1}^{N} \pi_{ij} P_{jq} \Pi_1,$$
$$\begin{aligned}\Omega^2_{iq} = {} & e_1^T R_{1q} e_1 - e_3^T (R_{1q} - R_{2q}) e_3 - \kappa t e_2^T (R_{2q} \\ & - R_{3q}) e_2 - e_4^T R_{3q} e_4 + e_5^T (d_2^2 S_{1q} \\ & + d_{12}^2 S_{2q}) e_5 + \dot{d}(t)(\Pi_3^T Q_{1iq} \Pi_3 - \Pi_4^T Q_{2iq} \Pi_4) \\ & + d_1(t) \Pi_3^T \sum_{j=1}^{N} \pi_{ij} Q_{1jq} \Pi_3 + d_2(t) \Pi_4^T \sum_{j=1}^{N} \pi_{ij} \\ & \times Q_{2jq} \Pi_4 + 2\Pi_3^T Q_{1iq} \Pi_5 + 2\Pi_4^T Q_{2iq} \Pi_6,\end{aligned}$$
$$\begin{aligned}\Omega^3_{q} = {} & e_5^T (\frac{1}{2} d_1^2 U_{1q} + \frac{1}{2} d_2^2 U_{2q}) e_5 - \Gamma_1^T \psi_{1q} \Gamma_1 \\ & - \Gamma_2^T \psi_{2q} \Gamma_2 - \Pi_{11}^T \bar{U}_{1q} \Pi_{11} - \Pi_{12}^T \bar{U}_{2q} \Pi_{12},\end{aligned}$$
$$\Omega^4_{i\mu pq} = sym(V_0^T \bar{A}_{i\mu pq}), V_0 = V^T e_5 + V^T e_1 + V^T e_2,$$
$$\bar{A}_{i\mu pq} = \sum_{\mu=1}^{M} \lambda_{i\mu} A_{i\mu pq} e_1 + \sum_{\mu=1}^{M} \lambda_{i\mu} A_{di\mu pq} e_2 + D_{ip} e_{13} - e_5,$$
$$z_{i\mu pq} = \sum_{\mu=1}^{M} \lambda_{i\mu} C_{i\mu pq} e_1 + \sum_{\mu=1}^{M} \lambda_{i\mu} C_{di\mu pq} e_2 + D_{ip} e_{13},$$
$$A_{i\mu pq} = A_{ip} + B_{1ip} K_{\mu q}, A_{di\mu pq} = A_{dip} + B_{1ip} K_{d\mu q},$$
$$C_{i\mu pq} = C_{ip} + B_{2ip} K_{\mu q}, C_{di\mu pq} = C_{dip} + B_{2ip} K_{d\mu q},$$
$$\bar{S}_{gq} = diag\{S_{gq}, 3S_{gq}\}, \bar{U}_{gq} = diag\{2U_{gq}, 4U_{gq}\},$$
$$\Gamma_1 = col\{\Pi_7, \Pi_8\}, \Gamma_2 = col\{\Pi_8, \Pi_9\},$$
$$\Pi_1 = col\{e_1, d_1 e_6\}, \Pi_2 = col\{e_5, e_1 - e_3\},$$
$$\Pi_3 = col\{e_1, d_1 e_6, e_7\}, \Pi_4 = col\{e_1, d_1 e_6, e_8\},$$
$$\Pi_5 = col\left\{d_1(t) e_5, d_1(t)(e_1 - e_3), e_3 - \kappa t e_2 - \dot{d}(t) e_7\right\},$$
$$\Pi_6 = col\left\{d_2(t) e_5, d_2(t)(e_1 - e_3), \kappa t e_2 - e_4 + \dot{d}(t) e_8\right\},$$
$$\Pi_7 = col\{r_1, r_2\}, \Pi_8 = col\{r_3, r_4\},$$
$$\Pi_9 = col\{r_5, r_6\}, \Pi_{10} = col\{r_7, r_8\},$$
$$\Pi_{11} = col\{r_9, r_{10}\}, r_1 = e_1 - e_2, r_2 = e_1 + e_2 - 2e_9,$$
$$r_3 = e_2 - e_4, r_4 = e_2 + e_4 - 2e_8, r_5 = e_3 - e_2,$$
$$r_6 = e_3 + e_2 - 2e_7, r_7 = e_1 - e_6, r_8 = e_1 + 2e_6 - 6e_{10},$$
$$r_9 = e_1 - e_{11}, r_{10} = e_1 + 2e_{11} - 6e_{12},$$
$$e_k = \left[0_{n\times(k-1)n}, I_n, 0_{n\times((12-k)n+m)}\right],$$
$$e_{13} = \left[0_{m\times 12n}, I_m\right], k = 1, 2, \ldots, 12.$$

*Proof:* Construct the MFs-dependent LKF as

$$V(x_t) = \sum_{\ell=1}^{5} V_\ell(x_t), \quad (12)$$

where

$$V_1(x_t) = \eta_0^T(t) P_{ih} \eta_0(t),$$
$$\begin{aligned}V_2(x_t) = {} & \int_{t-d_1}^{t} x^T(s) R_{1h} x(s) ds + \int_{t-d(t)}^{t-d_1} x^T(s) R_{2h} x(s) ds \\ & + \int_{t-d_2}^{t-d(t)} x^T(s) R_{3h} x(s) ds,\end{aligned}$$
$$\begin{aligned}V_3(x_t) = {} & d_2 \int_{-d_2}^{0} \int_{t+\theta}^{t} \dot{x}^T(s) S_{1h} \dot{x}(s) ds d\theta \\ & + d_{12} \int_{-d_2}^{-d_1} \int_{t+\theta}^{t} \dot{x}^T(s) S_{2h} \dot{x}(s) ds d\theta,\end{aligned}$$
$$V_4(x_t) = d_1(t) \eta_1^T(t) Q_{1ih} \eta_1(t) + d_2(t) \eta_2^T(t) Q_{2ih} \eta_2(t),$$
$$\begin{aligned}V_5(x_t) = {} & \int_{t-d_1}^{t} \int_{u}^{t} \int_{\theta}^{t} \dot{x}^T(s) U_{1h} \dot{x}(s) ds d\theta du \\ & + \int_{t-d_2}^{t} \int_{u}^{t} \int_{\theta}^{t} \dot{x}^T(s) U_{2h} \dot{x}(s) ds d\theta du,\end{aligned}$$
$$P_{ih} = \begin{bmatrix} P_{1ih} & P_{2ih} \\ * & P_{4ih} \end{bmatrix}, Q_{gih} = \begin{bmatrix} Q_{gih}^1 & Q_{gih}^2 & Q_{gih}^3 \\ * & Q_{gih}^4 & Q_{gih}^5 \\ * & * & Q_{gih}^6 \end{bmatrix}.$$

Denoting $\mathcal{F}$ as the weak infinitesimal operator, it follows that

$$\begin{aligned}\mathcal{F}V_1(x_t) = {} & 2\upsilon^T(t) \Pi_1^T P_{ih} \Pi_2 \upsilon(t) + \eta_0^T(t) \dot{P}_{ih} \eta_0(t) \\ & + \upsilon^T(t) \Pi_1^T \sum_{j=1}^{N} \pi_{ij} P_{jh} \Pi_1 \upsilon(t),\end{aligned}$$
$$\begin{aligned}\mathcal{F}V_2(x_t) = {} & \upsilon^T(t) \left\{e_1^T R_{1h} e_1 - e_3^T (R_{1h} - R_{2h}) e_3 \right. \\ & \left. -\kappa t e_2^T (R_{2h} - R_{3h}) e_2 - e_4^T R_{3h} e_4\right\} \upsilon(t) \\ & + \int_{t-d_1}^{t} x^T(s) \dot{R}_{1h} x(s) ds + \int_{t-d(t)}^{t-d_1} x^T(s) \dot{R}_{2h} \\ & \times x(s) ds + \int_{t-d_2}^{t-d(t)} x^T(s) \dot{R}_{3h} x(s) ds,\end{aligned}$$
$$\begin{aligned}\mathcal{F}V_3(x_t) = {} & \upsilon^T(t) \left\{e_5^T (d_2^2 S_{1h} + d_{12}^2 S_{2h}) e_5\right\} \upsilon(t) \\ & - d_2 \sum_{q=1}^{r} h_q \int_{t-d_2}^{t} \dot{x}^T(s) S_{1q} \dot{x}(s) ds \\ & - d_{12} \sum_{q=1}^{r} h_q \int_{t-d_2}^{t-d_1} \dot{x}^T(s) S_{2q} \dot{x}(s) ds \\ & + d_2 \int_{-d_2}^{0} \int_{t+\theta}^{t} \dot{x}^T(s) \dot{S}_{1h} \dot{x}(s) ds d\theta \\ & + d_{12} \int_{-d_2}^{-d_1} \int_{t+\theta}^{t} \dot{x}^T(s) \dot{S}_{2h} \dot{x}(s) ds d\theta,\end{aligned}$$
$$\begin{aligned}\mathcal{F}V_4(x_t) = {} & \upsilon^T(t) \left\{\dot{d}(t)(\Pi_3^T Q_{1ih} \Pi_3 - \Pi_4^T Q_{2ih} \Pi_4) + d_1(t) \right. \\ & \times \Pi_3^T \sum_{j=1}^{N} \pi_{ij} Q_{1jh} \Pi_3 + d_2(t) \Pi_4^T \sum_{j=1}^{N} \pi_{ij} Q_{2jh} \\ & \left. \times \Pi_4 + 2\Pi_3^T Q_{1ih} \Pi_5 + 2\Pi_4^T Q_{2ih} \Pi_6\right\} \upsilon(t) \\ & + d_1(t) \eta_1^T(t) \dot{Q}_{1ih} \eta_1(t) + d_2(t) \eta_2^T(t) \dot{Q}_{2ih} \eta_2(t),\end{aligned}$$
$$\mathcal{F}V_5(x_t) = \upsilon^T(t) \left\{e_5^T (\frac{1}{2} d_1^2 U_{1h} + \frac{1}{2} d_2^2 U_{2h}) e_5\right\} \upsilon(t)$$

$$
\begin{aligned}
&-\sum_{q=1}^{r}h_q\int_{t-d_1}^{t}\int_{\theta}^{t}\dot{x}^T(s)U_{1q}\dot{x}(s)dsd\theta\\
&-\sum_{q=1}^{r}h_q\int_{t-d_2}^{t}\int_{\theta}^{t}\dot{x}^T(s)U_{2q}\dot{x}(s)dsd\theta\\
&+\int_{t-d_1}^{t}\int_{u}^{t}\int_{\theta}^{t}\dot{x}^T(s)\dot{U}_{1h}\dot{x}(s)dsd\theta du\\
&+\int_{t-d_2}^{t}\int_{u}^{t}\int_{\theta}^{t}\dot{x}^T(s)\dot{U}_{2h}\dot{x}(s)dsd\theta du.
\end{aligned}
$$

Applying the methods in [39] and [40], one has

$$
\begin{aligned}
&-d_2\int_{t-d_2}^{t}\dot{x}^T(s)S_{1q}\dot{x}(s)ds\le -\upsilon^T(t)\Gamma_1^T\psi_{1q}\Gamma_1\upsilon(t),\\
&-d_{12}\int_{t-d_2}^{t-d_1}\dot{x}^T(s)S_{2q}\dot{x}(s)ds\le -\upsilon^T(t)\Gamma_2^T\psi_{2q}\Gamma_2\upsilon(t),\\
&-\int_{t-d_1}^{t}\int_{\theta}^{t}\dot{x}^T(s)U_{1q}\dot{x}(s)dsd\theta\le -\upsilon^T(t)\Pi_{10}^T\bar{U}_{1q}\Pi_{10}\upsilon(t),\\
&-\int_{t-d_2}^{t}\int_{\theta}^{t}\dot{x}^T(s)U_{2q}\dot{x}(s)dsd\theta\le -\upsilon^T(t)\Pi_{11}^T\bar{U}_{2q}\Pi_{11}\upsilon(t).
\end{aligned}
$$

For any invertible matrix $V\in R^{n\times n}$, based on system (7), it follows

$$
\begin{aligned}
0=&\,2\upsilon^T(t)\left\{(e_1^TV+e_2^TV+e_5^TV)(\sum_{\mu=1}^{M}\lambda_{i\mu}A_{i\mu hh}e_1\right.\\
&\left.+\sum_{\mu=1}^{M}\lambda_{i\mu}A_{di\mu hh}e_2+D_{ih}e_{13}-e_5)\right\}\upsilon(t)\\
=&\,\upsilon^T(t)\sum_{p=1}^{r}\sum_{q=1}^{r}h_ph_q sym(V_0^T\bar{A}_{i\mu pq})\upsilon(t).
\end{aligned}
$$

Therefore, we can obtain

$$
\begin{aligned}
\mathcal{F}V(x_t)\le&\,\upsilon^T(t)\sum_{p=1}^{r}\sum_{q=1}^{r}h_ph_q\Omega^*_{i\mu pq}\upsilon(t)+\eta_0^T(t)\dot{P}_{ih}\\
&\times\eta_0(t)+d_1(t)\eta_1^T(t)\dot{Q}_{1ih}\eta_1(t)+d_2(t)\eta_2^T(t)\\
&\times\dot{Q}_{2ih}\eta_2(t)+\int_{t-d_1}^{t}x^T(s)\dot{R}_{1h}x(s)ds\\
&+\int_{t-d(t)}^{t-d_1}x^T(s)\dot{R}_{2h}x(s)ds+\int_{t-d_2}^{t-d(t)}x^T(s)\\
&\times\dot{R}_{3h}x(s)ds+d_2\int_{-d_2}^{0}\int_{t+\theta}^{t}\dot{x}^T(s)\dot{S}_{1h}\dot{x}(s)dsd\theta\\
&+d_{12}\int_{-d_2}^{-d_1}\int_{t+\theta}^{t}\dot{x}^T(s)\dot{S}_{2h}\dot{x}(s)dsd\theta\\
&+\int_{t-d_1}^{t}\int_{u}^{t}\int_{\theta}^{t}\dot{x}^T(s)\dot{U}_{1h}\dot{x}(s)dsd\theta du\\
&+\int_{t-d_2}^{t}\int_{u}^{t}\int_{\theta}^{t}\dot{x}^T(s)\dot{U}_{2h}\dot{x}(s)dsd\theta du.
\end{aligned}
$$

Due to $\sum_{q=1}^{r}\dot{h}_q=0$, the time derivative of MFs-dependent terms is considered as follows

$$
\dot{P}_{ih}=\sum_{q=1}^{r}\dot{h}_qP_{iq}=\sum_{k=1}^{r-1}\dot{h}_k(P_{ik}-P_{ir}),\tag{13}
$$

$$
\begin{aligned}
\dot{Q}_{gih}&=\sum_{q=1}^{r}\dot{h}_qQ_{giq}=\sum_{k=1}^{r-1}\dot{h}_k(Q_{gik}-Q_{gir}),\\
\dot{R}_{fh}&=\sum_{q=1}^{r}\dot{h}_qR_{fq}=\sum_{k=1}^{r-1}\dot{h}_k(R_{fk}-R_{fr}),\\
\dot{S}_{gh}&=\sum_{q=1}^{r}\dot{h}_qS_{gq}=\sum_{k=1}^{r-1}\dot{h}_k(S_{gk}-S_{gr}),\\
\dot{U}_{gh}&=\sum_{q=1}^{r}\dot{h}_qU_{gq}=\sum_{k=1}^{r-1}\dot{h}_k(U_{gk}-U_{gr}),
\end{aligned}
$$

where $g=1,2$, $f=1,2,3$, $\dot{h}_k$ is the time derivative of the $k$th rule of MFs, which is negative or positive. In order to ensure the negativity of $\dot{P}_{ih}$, $\dot{Q}_{gih}$, $\dot{R}_{fh}$, $\dot{S}_{gh}$, and $\dot{U}_{gh}$, according to Lemma 1, we design the following switching approach

$$
\begin{cases}
\text{if } \dot{h}_k<0,\ \text{then } P_{ik}-P_{ir}>0, Q_{gik}-Q_{gir}>0,\\
\qquad\qquad\qquad R_{fk}-R_{fr}>0, S_{gk}-S_{gr}>0,\\
\qquad\qquad\qquad U_{gk}-U_{gr}>0,\\
\text{if } \dot{h}_k\ge 0,\ \text{then } P_{ik}-P_{ir}\le 0, Q_{gik}-Q_{gir}\le 0,\\
\qquad\qquad\qquad R_{fk}-R_{fr}\le 0, S_{gk}-S_{gr}\le 0,\\
\qquad\qquad\qquad U_{gk}-U_{gr}\le 0.
\end{cases}\tag{14}
$$

It is found that there are $2^{r-1}$ possible cases in (14). Denote $\mathcal{H}_\aleph=\left\{\aleph:\text{The possible permutations of } \dot{h}_k\right\}$, $\mathcal{G}_\aleph=\{\aleph:\text{The possible constraints of } P_{iq},\ Q_{giq},\ R_{fq},\ S_{gq},\ U_{gq}\}$ with $\aleph=1,2,\ldots,2^{r-1}$, then (14) can be presented as

$$
\text{if } \mathcal{H}_\aleph, \text{then } \mathcal{G}_\aleph.\tag{15}
$$

Thus, in the light of (15), one has

$$
\mathcal{F}V(x_t)\le\upsilon^T(t)\sum_{p=1}^{r}\sum_{q=1}^{r}h_ph_q\Omega^*_{i\mu pq}\upsilon(t).\tag{16}
$$

On account of Schur complement, from (10), we have $\sum_{p=1}^{r}\sum_{q=1}^{r}h_ph_q\Omega^*_{i\mu pq}<0$, which implies

$$
\mathcal{F}V(x_t)\le-\varepsilon x^T(t)x(t).\tag{17}
$$

Applying Dynkins formula to (17) from $[0,T]$ and letting $T\to\infty$, it follows that

$$
E\left\{\int_0^\infty\|x(t)\|^2dt\,|(x(0),r(0))\right\}\le\frac{1}{\varepsilon}V(x(0),r(0))<\infty.\tag{18}
$$

As a result, on the basis of Definition 1, system (7) is SS. In addition, the $H_\infty$ performance function $\mathcal{J}$ can be written as follows

$$
\begin{aligned}
\mathcal{J}=&\int_0^\infty z^T(t)z(t)-\gamma^2w^T(t)w(t)dt\\
\le&\int_0^\infty\upsilon^T(t)\sum_{p=1}^{r}\sum_{q=1}^{r}h_ph_q(z^T_{i\mu pq}z_{i\mu pq}\\
&-\gamma^2e_{13}^Te_{13})\upsilon(t)+\mathcal{F}V(x_t)dt\\
\le&\int_0^\infty\upsilon^T(t)\sum_{p=1}^{r}\sum_{q=1}^{r}h_ph_q\Omega_{i\mu pq}\upsilon(t)dt.
\end{aligned}\tag{19}
$$

According to (10), we have $\mathcal{J}<0$. Therefore, system (7) is SS with $H_\infty$ performance level $\gamma$. ■

*Remark 3:* An augmented LKF which consists of more integral terms and time-varying delay information is constructed in this paper to reduce conservatism. In addition, unlike the existing results in [8]–[11], the LKF depends on MFs, which is more consistent with the FMJSs model and yields less conservative results. And the time derivative of MFs is dealt with by a switching method, which depends on the switching rule (15). It should be indicated that when the designed matrices in LKF are associated with Markov process $r(t)$ (e.g. $P_{ih}$), each possible case in (15) will be converted to corresponding $n$ modes, that is, there are $2^{r-1}$ cases for each mode. Then (15) can be expressed as: if $\mathcal{H}_{\aleph}$, then $\mathcal{G}_{\aleph i}$.

*Remark 4:* In the process of stability analysis, a LKF is usually constructed with the following form:

$$V(x_t) = \wp_1^T(t)R\wp_1(t) + \int_a^b \wp_2^T(s)S\wp_2(s)ds + \int_a^b \int_{t+\theta}^t \wp_3^T(s)U\wp_3(s)dsd\theta. \tag{20}$$

Generally speaking, (20) can be improved by three ways: 1) dividing the delay interval $[a, b]$ into several equal or unequal intervals; 2) adding extra multiple integral terms to the designed LKF; 3) augmenting state vectors $\wp_1(t)$, $\wp_2(s)$, and $\wp_3(s)$. Unlike the above methods, two DPT terms (i.e. $V_4(x_t)$) are introduced in this paper. And the advantages of this method are summarized in the following two aspects:

1) Consider all non-integral terms in LKF, $V_1(x_t) + V_4(x_t)$ has more common form than $V_1(x_t)$, because some location restrictions are relaxed. In other words, the stability criterion obtained in this paper has a more general form.
2) The derivative of $V_4(x_t)$ includes more information of FMJSs and time-varying delay, thus a less conservative stability criterion is obtained.

*Theorem 2:* For given scalars $0 \le d_1 \le d_2$, $\kappa_1$, $\kappa_2$, and $\gamma > 0$, the system (7) is SS with $H_\infty$ performance level $\gamma$ in $\mathcal{D}$, if there exist symmetric positive-definite matrices $\tilde{P}_{iq} \in R^{2n\times 2n}$, $\tilde{Q}_{giq} \in R^{3n\times 3n}$, $\tilde{R}_{fq}, \tilde{S}_{gq}, \tilde{U}_{gq} \in R^{n\times n}$, and any matrices $\tilde{X}^c_{gq}, Z \in R^{n\times n}$, $G_{\mu q}, G_{d\mu q} \in R^{l\times n}$ such that (15) and the following inequalities hold for $i \in \mathcal{N}$, $\mu \in \mathcal{M}$, $p, q = 1, 2, \ldots, r$, $g = 1, 2$, $f = 1, 2, 3$, $c = 1, 2, 3, 4$

$$\begin{cases} \begin{bmatrix} \tilde{\Omega}^*_{i\mu pp} & \tilde{z}^T_{i\mu pp} \\ * & -I \end{bmatrix} < 0, \\ \begin{bmatrix} \tilde{\Omega}^*_{i\mu pq} & \tilde{z}^T_{i\mu pq} \\ * & -I \end{bmatrix} + \begin{bmatrix} \tilde{\Omega}^*_{i\mu qp} & \tilde{z}^T_{i\mu qp} \\ * & -I \end{bmatrix} < 0, \, p < q, \end{cases} \tag{21}$$

$$\tilde{\psi}_{gq} \ge 0, \tag{22}$$

where

$$\tilde{\psi}_{gq} = \begin{bmatrix} \hat{S}_{gq} & \tilde{X}_{gq} \\ * & \hat{S}_{gq} \end{bmatrix}, \tilde{X}_{gq} = \begin{bmatrix} \tilde{X}^1_{gq} & \tilde{X}^2_{gq} \\ \tilde{X}^3_{gq} & \tilde{X}^4_{gq} \end{bmatrix},$$

$$\tilde{\Omega}^*_{i\mu pq} = \tilde{\Omega}^1_{iq} + \tilde{\Omega}^2_{iq} + \tilde{\Omega}^3_{q} + \tilde{\Omega}^4_{i\mu pq},$$

$$\tilde{\Omega}^1_{iq} = sym(\Pi_1^T \tilde{P}_{iq}\Pi_2) + \Pi_1^T \sum_{j=1}^N \pi_{ij}\tilde{P}_{jq}\Pi_1,$$

$$\begin{aligned}\tilde{\Omega}^2_{iq} &= e_1^T \tilde{R}_{1q} e_1 - e_3^T(\tilde{R}_{1q} - \tilde{R}_{2q})e_3 - \kappa t e_2^T(\tilde{R}_{2q} \\ &\quad - \tilde{R}_{3q})e_2 - e_4^T \tilde{R}_{3q} e_4 + e_5^T(d_2^2 \tilde{S}_{1q} + d_{12}^2 \\ &\quad \times \tilde{S}_{2q})e_5 + \dot{d}(t)(\Pi_3^T \tilde{Q}_{1iq}\Pi_3 - \Pi_4^T \tilde{Q}_{2iq}\Pi_4) \\ &\quad + d_1(t)\Pi_3^T \sum_{j=1}^N \pi_{ij}\tilde{Q}_{1jq}\Pi_3 + d_2(t)\Pi_4^T \sum_{j=1}^N \pi_{ij} \\ &\quad \times \tilde{Q}_{2jq}\Pi_4 + 2\Pi_3^T \tilde{Q}_{1iq}\Pi_5 + 2\Pi_4^T \tilde{Q}_{2iq}\Pi_6,\end{aligned}$$

$$\begin{aligned}\tilde{\Omega}^3_{q} &= e_5^T(\frac{1}{2}d_1^2\tilde{U}_{1q} + \frac{1}{2}d_2^2\tilde{U}_{2q})e_5 - \Gamma_1^T\tilde{\psi}_{1q}\Gamma_1 - \Gamma_2^T\tilde{\psi}_{2q} \\ &\quad \times \Gamma_2 - \Pi_{11}^T\hat{U}_{1q}\Pi_{11} - \Pi_{12}^T\hat{U}_{2q}\Pi_{12} - \gamma^2 e_{13}^T e_{13},\end{aligned}$$

$$\tilde{\Omega}^4_{i\mu pq} = sym(\tilde{V}_0^T \tilde{A}_{i\mu pq}), \tilde{V}_0 = e_5 + e_1 + e_2,$$

$$\begin{aligned}\tilde{A}_{i\mu pq} &= (\sum_{\mu=1}^M \lambda_{i\mu} B_{1ip}G_{\mu q} + A_{ip}Z^T)e_1 + (\sum_{\mu=1}^M \lambda_{i\mu} \\ &\quad \times B_{1ip}G_{d\mu q} + A_{dip}Z^T)e_2 + D_{ip}e_{13} - Z^T e_5,\end{aligned}$$

$$\begin{aligned}\tilde{z}_{i\mu pq} &= (\sum_{\mu=1}^M \lambda_{i\mu} B_{2ip}G_{\mu q} + C_{ip}Z^T)e_1 + (\sum_{\mu=1}^M \lambda_{i\mu} \\ &\quad \times B_{2ip}G_{d\mu q} + C_{dip}Z^T)e_2 + E_{ip}e_{13},\end{aligned}$$

$$\hat{S}_{gq} = diag\left\{\tilde{S}_{gq}, 3\tilde{S}_{gq}\right\}, \hat{U}_{gq} = diag\left\{2\tilde{U}_{gq}, 4\tilde{U}_{gq}\right\},$$

$$\tilde{P}_{iq} = \bar{Z}P_{iq}\bar{Z}^T, \tilde{R}_{fq} = ZR_{fq}Z^T, \tilde{S}_{gq} = ZS_{gq}Z^T,$$

$$\tilde{Q}_{giq} = \hat{Z}Q_{giq}\hat{Z}^T, \tilde{U}_{gq} = ZU_{gq}Z^T, \tilde{X}^c_{gq} = ZX^c_{gq}Z^T,$$

and the controller gains are

$$K_{\mu q} = G_{\mu q}Z^{-T}, K_{d\mu q} = G_{d\mu q}Z^{-T}. \tag{23}$$

*Proof:* Recalling Theorem 1, applying Schur complement, we have

$$\begin{bmatrix} \Omega^*_{i\mu pq} - \gamma^2 e_{13}^T e_{13} & z^T_{i\mu pq} \\ * & -I \end{bmatrix} < 0. \tag{24}$$

Denote $\mathcal{Z} = diag\left\{\tilde{Z}, \hat{Z}, \hat{Z}, \bar{Z}, I, I\right\}$, pre- and post- multiply (24) by $\mathcal{Z}$ and $\mathcal{Z}^T$ with $\tilde{Z} = diag\{Z, Z, Z, Z\}$, $\hat{Z} = diag\{Z, Z, Z\}$, $\bar{Z} = diag\{Z, Z\}$, and $Z = V^{-1}$, we get (21). Similarly, pre- and post- multiply (11) by $\tilde{Z}$ and $\tilde{Z}^T$, we have (22). ∎

Solving inequalities (21) and (22) with $\mathcal{G}_{\aleph}$ in Theorem 2, we get a maximum delay $d_{2\aleph}$, and the final maximum delay is $d_2 = \min_{\aleph=1,2,\ldots,2^{r-1}} d_{2\aleph}$. In addition, the number of decision variables in Theorem 2 is $nr(11ni + 11.5n + 2l\mu + 4i + 3.5) + n^2$. It is not difficult to get the complexity of the computational depends on the number of system orders $n$, controller orders $l$, fuzzy rules $r$, system modes $i$, and controller modes $\mu$. Hence, the size of $n$, $l$, $r$, $i$ or $\mu$ will affect the time spent to resolve the SS conditions directly.

*Remark 5:* Based on (15), an asynchronous switching controller is firstly proposed in this paper, which can be represented as:

$$u^{\aleph} = K^{\aleph}_{\mu h}x(t) + K^{\aleph}_{d\mu h}x(t - d(t)), \tag{25}$$

with $K^{\aleph}_{\mu h} = \sum_{q=1}^r h_q K^{\aleph}_{\mu q}$, $K^{\aleph}_{d\mu h} = \sum_{q=1}^r h_q K^{\aleph}_{d\mu q}$, $\aleph = 1, 2, \ldots, 2^{r-1}$. The controller can be regarded as a fuzzy composition of each controller of Markov jump subsystem, and it switches according to different $\mathcal{H}_{\aleph}$. Therefore, the controller (25) is matched, which is more precise and general. In addition, if the designed matrices in LKF are independent of the MFs (i.e. $P_i$, $R_f$, $S_g$, $Q_{gi}$, $U_g$), the switching fuzzy controller (25) will be reduced to the non-switching version presented in the existing results.

*Remark 6:* Due to the existence of integral terms and the derivative of the initial state (i.e. $\dot{\phi}(t)$), $\mathcal{D}$ is very complicated and difficult to be measured accurately. In this paper, the Lyapunov level set is used to get an estimate of $\mathcal{D}$. Assume $\phi(t)$ and $\dot{\phi}(t)$ are smooth in $[-d_2, 0]$ and $\dot{\phi}(t)$ can be bounded as $\phi(t_1) \leq \dot{\phi}(t) \leq \phi(t_2)$ for $t_1, t_2 \in [-d_2, 0]$. Hence, $\dot{\phi}(t)$ can be represented as $\dot{\phi}(t) = \sum_{w=1}^{2} \rho_w(t)\phi(t_w)$ with $0 \leq \rho_w(t) \leq 1$, $\sum_{w=1}^{2} \rho_w(t) = 1$ for $t \in [-d_2, 0]$, then it has

$$\begin{aligned}
&\dot{\phi}^T(s)U_{gh(0)}\dot{\phi}(s)\\
&= \sum_{w=1}^{2} \rho_w^2(s)\phi_{ww}^g + \rho_1(s)\rho_2(s)(\phi_{12}^g + \phi_{21}^g)\\
&\leq \sum_{w=1}^{2} \rho_w^2(s)\phi_{ww}^g + \rho_1(s)\rho_2(s)(\phi_{11}^g + \phi_{22}^g)\\
&= \sum_{w=1}^{2} \rho_w(s)\phi_{ww}^g,\\
\phi_{ww}^g &= \phi^T(t_w)U_{gh(0)}\phi(t_w), w = 1, 2, \ g = 1, 2.
\end{aligned}$$

Let $\phi^T(y)U_{gh(0)}\phi(y) = \bar{U}_g$, we have

$$\begin{aligned}
V_5(x_t)\,|_{t=0} \leq& \int_{-d_1}^{0}\int_{u}^{0}\int_{\theta}^{0} \sum_{w=1}^{2} \rho_w(s)\phi_{ww}^1 ds d\theta du\\
&+ \int_{-d_2}^{0}\int_{u}^{0}\int_{\theta}^{0} \sum_{w=1}^{2} \rho_w(s)\phi_{ww}^2 ds d\theta du\\
\leq& \frac{1}{6}d_1^3 \max_{-d_2\leq y\leq 0}(\bar{U}_1) + \frac{1}{6}d_2^3 \max_{-d_2\leq y\leq 0}(\bar{U}_2).
\end{aligned} \tag{26}$$

Similarly, let $\phi^T(y)R_{fh(0)}\phi(y) = \bar{R}_f$, $\phi^T(y)S_{gh(0)}\phi(y) = \bar{S}_g$ for $f = 1, 2, 3$, $g = 1, 2$, we have

$$\begin{aligned}
V_2(x_t)\,|_{t=0} \leq& d_1 \max_{-d_2\leq y\leq 0}(\bar{R}_1) + d_{12} \max_{-d_2\leq y\leq 0}(\bar{R}_2)\\
&+ d_{12} \max_{-d_2\leq y\leq 0}(\bar{R}_3),
\end{aligned} \tag{27}$$

$$\begin{aligned}
V_3(x_t)\,|_{t=0} \leq& \frac{1}{2}d_2^3 \max_{-d_2\leq y\leq 0}(\bar{S}_1) + \frac{1}{2}d_{12}^2(d_1 + d_2)\\
&\times \max_{-d_2\leq y\leq 0}(\bar{S}_2).
\end{aligned} \tag{28}$$

On the other hand, supposing $P_{2ih(0)} = P_{2ih(0)}^T \geq 0$ and let $F_1 = \int_{-d_1}^{0} \phi(s)ds$, it follows

$$\begin{aligned}
V_1(x_t)\,|_{t=0} \leq& \phi^T(0)P_{1ih(0)}\phi(0) + F_1^T P_{4ih(0)}F_1\\
&+ \phi^T(0)P_{2ih(0)}\phi(0) + F_1^T P_{2ih(0)}F_1\\
\leq& \max_{-d_2\leq y\leq 0}(J_1) + d_1^2 \max_{-d_2\leq y\leq 0}(J_2),\\
J_1 =& \phi^T(y)(P_{1ih(0)} + P_{2ih(0)})\phi(y),\\
J_2 =& \phi^T(y)(P_{4ih(0)} + P_{2ih(0)})\phi(y).
\end{aligned} \tag{29}$$

Similarly, supposing $Q_{gih(0)}^2 = Q_{gih(0)}^{2T} \geq 0$, $Q_{gih(0)}^3 = Q_{gih(0)}^{3T} \geq 0$, $Q_{gih(0)}^5 = Q_{gih(0)}^{5T} \geq 0$, and let $F_2 = \frac{1}{d_1(0)}\int_{-d(0)}^{-d_1} \phi(s)ds$, $F_3 = \frac{1}{d_2(0)}\int_{-d_2}^{-d(0)} \phi(s)ds$, we can obtain

$$\begin{aligned}
&V_4(x_t)\,|_{t=0}\\
&\leq d_{12}\Bigg\{\phi^T(0)(\sum_{g=1}^{2} Q_{gih(0)}^1 + \sum_{g=1}^{2} Q_{gih(0)}^2 + \sum_{g=1}^{2} Q_{gih(0)}^3)\\
&\quad\times \phi(0) + F_1^T(\sum_{g=1}^{2} Q_{gih(0)}^2 + \sum_{g=1}^{2} Q_{gih(0)}^4\\
&\quad+ \sum_{g=1}^{2} Q_{gih(0)}^5)F_1 + F_2^T(Q_{1ih(0)}^3 + Q_{1ih(0)}^5\\
&\quad+ Q_{1ih(0)}^6)F_2 + F_3^T(Q_{2ih(0)}^3 + Q_{2ih(0)}^5 + Q_{2ih(0)}^6)F_3\Bigg\}\\
&\leq d_{12} \max_{-d_2\leq y\leq 0}(H_1) + d_1^2 d_{12} \max_{-d_2\leq y\leq 0}(H_2)\\
&\quad+ d_{12} \max_{-d_2\leq y\leq 0}(H_3) + d_{12} \max_{-d_2\leq y\leq 0}(H_4),\\
&\quad H_1 = \phi^T(y)(\sum_{g=1}^{2} Q_{gih(0)}^1 + \sum_{g=1}^{2} Q_{gih(0)}^2 + \sum_{g=1}^{2} Q_{gih(0)}^3)\phi(y),\\
&\quad H_2 = \phi^T(y)(\sum_{g=1}^{2} Q_{gih(0)}^2 + \sum_{g=1}^{2} Q_{gih(0)}^4 + \sum_{g=1}^{2} Q_{gih(0)}^5)\phi(y),\\
&\quad H_3 = \phi^T(y)(Q_{1ih(0)}^3 + Q_{1ih(0)}^5 + Q_{1ih(0)}^6)\phi(y),\\
&\quad H_4 = \phi^T(y)(Q_{2ih(0)}^3 + Q_{2ih(0)}^5 + Q_{2ih(0)}^6)\phi(y).
\end{aligned} \tag{30}$$

There, an intermediate variable $M_{iq}$ is introduced to unify the results of each part, combining (26)-(30), consider the constraints

$$\begin{cases}
\frac{1}{6}d_1^3 U_{1q} \leq M_{iq}, \frac{1}{6}d_2^3 U_{2q} \leq M_{iq}, \frac{1}{2}d_2^3 S_{1q} \leq M_{iq},\\
\frac{1}{2}d_{12}^2(d_1 + d_2)S_{2q} \leq M_{iq}, d_1 R_{1q} \leq M_{iq}, d_{12}R_{2q} \leq M_{iq},\\
d_{12}R_{3q} \leq M_{iq}, P_{1iq} + P_{2iq} \leq M_{iq}, d_1^2(P_{4iq} + P_{2iq}) \leq M_{iq},\\
d_{12}(\sum_{g=1}^{2} Q_{giq}^1 + \sum_{g=1}^{2} Q_{giq}^2 + \sum_{g=1}^{2} Q_{giq}^3) \leq M_{iq},\\
d_1^2 d_{12}(\sum_{g=1}^{2} Q_{giq}^2 + \sum_{g=1}^{2} Q_{giq}^4 + \sum_{g=1}^{2} Q_{giq}^5) \leq M_{iq},\\
d_{12}(Q_{1iq}^3 + Q_{1iq}^5 + Q_{1iq}^6) \leq M_{iq},\\
d_{12}(Q_{2iq}^3 + Q_{2iq}^5 + Q_{2iq}^6) \leq M_{iq}, 13M_{iq} \leq L_{iq},
\end{cases}$$

we get an estimation of $\mathcal{D}$

$$\bar{\mathcal{D}} := \left\{\phi(t) : \phi^T(y)L_{ih(0)}\phi(y) \leq 1, \forall y \in [-d_2, 0]\right\}. \tag{31}$$

Note that each of the switches corresponds to a set $\bar{\mathcal{D}}$, so there are $2^{r-1}$ sets. And we ultimately need to guarantee the stability in the intersection of all sets.

*Remark 7:* In general, the compact set $\mathbb{C}$ is expressed with the following form:

$$\mathbb{C} = \bigcap_{v} \{x : |l_v x| \leq \varepsilon_v\}, v = 1, 2, ..., n,$$

where $l_v \in R^{1\times n}$ (only the element in $(1, v)$ is 1 and the others are 0). It is naturally that $\bar{\mathcal{D}}$ must satisfy $\bar{\mathcal{D}} \subset \mathbb{C}$, so that the system is SS no matter how the controller switches. Applying the Lagrange multiplier approach, $\bar{\mathcal{D}} \subset \mathbb{C}$ means, any $y \in$

$[-d_2, 0]$ satisfying $l_v\phi(y) = \pm\varepsilon_v$, we have $\phi^T(y)L_{iq}\phi(y) \geq 1$, that is

$$\min\left\{\phi^T(y)L_{iq}\phi(y)\,|l_v\phi(y) = \pm\varepsilon_v\right\} \geq 1. \tag{32}$$

Define the Lagrange function $L(\phi(y)) = \phi^T(y)L_{iq}\phi(y) + \lambda(l_v\phi(y) \mp \varepsilon_v)$ with Lagrange factor $\lambda$, it follows

$$\begin{cases} \frac{\partial L(\phi(y))}{\partial \phi(y)} = 2\phi^T(y)L_{iq} + \lambda l_v = 0, \\ l_v\phi(y) \mp \varepsilon_v = 0. \end{cases} \tag{33}$$

Solving (33), we have $\lambda^* = \mp 2\varepsilon_v(l_v L_{iq}^{-1} l_v^T)^{-1}$, $\phi^*(y) = \pm\varepsilon_v L_{iq}^{-1} l_v^T (l_v L_{iq}^{-1} l_v^T)^{-1}$. Substituting $\phi^*(y)$ into (32) and applying Schur complement, we can obtain

$$\begin{bmatrix} \varepsilon_v^2 & l_v \\ * & L_{iq} \end{bmatrix} \geq 0. \tag{34}$$

## IV. Numerical Example

*Example 1:* The first numerical example illustrates that the asynchronous switching control approach presented in this paper is effective and has practical value by considering the following mass-spring-damper mechanical system [17], [41], [42]:

$$M\ddot{y}(t) + D\dot{y}(t) + Ky(t) = (1 + c_3\dot{y}^3(t))u(t) + w(t),$$

where $M$ denotes the mass, $D$ is the viscous damping, $K = c(t) \in [c_1, c_2]$ represents the uncertain linear stiffness. $y(t)$, $u(t)$, and $w(t)$ are the position, force, and disturbance, respectively. Here, we choose $M = 1$, $D = 1$, $c_1 = 0.5$, $c_2 = 1.81$, $c_3 = 0.13$, $x(t) = \begin{bmatrix} \dot{y}^T(t) & y^T(t) \end{bmatrix}^T$, $z(t) = \begin{bmatrix} y^T(t) & u^T(t) \end{bmatrix}^T$.

Due to the random changes of structure or parameter in the system, two jump modes are assumed to occur, and the TP matrix is as follows:

$$\Pi = \begin{bmatrix} -3 & 3 \\ 4 & -4 \end{bmatrix}.$$

According to [17] and [41], the original NS can be accurately expressed by the following T-S fuzzy model:

$$\dot{x}(t) = \sum_{q=1}^{2} h_q(\theta(t))[A_{iq}x(t) + B_{1iq}u(t) + D_{iq}w(t)],$$

$$z(t) = \sum_{q=1}^{2} h_q(\theta(t))[C_{iq}x(t) + B_{2iq}u(t)],$$

where

$$A_{11} = \begin{bmatrix} -1 & -1.155 \\ 1 & 0 \end{bmatrix}, A_{12} = \begin{bmatrix} -1 & -1.155 \\ 1 & 0 \end{bmatrix},$$
$$A_{21} = \begin{bmatrix} -1 & -2.210 \\ 1 & 0 \end{bmatrix}, A_{22} = \begin{bmatrix} -1 & -2.210 \\ 1 & 0 \end{bmatrix},$$
$$B_{111} = \begin{bmatrix} 1.4387 \\ 0 \end{bmatrix}, B_{112} = \begin{bmatrix} 0.5613 \\ 0 \end{bmatrix},$$
$$B_{121} = \begin{bmatrix} 0.5755 \\ 0 \end{bmatrix}, B_{122} = \begin{bmatrix} 0.2245 \\ 0 \end{bmatrix},$$
$$B_{211} = B_{212} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, B_{221} = B_{222} = \begin{bmatrix} 0 \\ 0.8 \end{bmatrix},$$

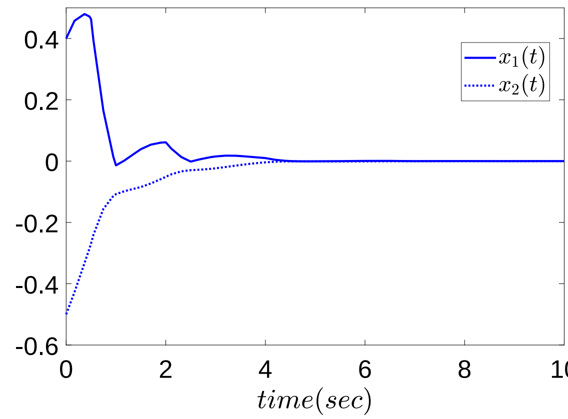


Fig. 1. The state responses (Example 1).

$$D_{11} = D_{12} = D_{21} = D_{22} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$
$$C_{11} = C_{12} = C_{21} = C_{22} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

To consider the asynchronous control problem of FMJSs (1), we take the other matrices as follows:

$$A_{d11} = \begin{bmatrix} -0.1 & 0.2 \\ 0.1 & -0.2 \end{bmatrix}, A_{d12} = \begin{bmatrix} -0.2 & 0.1 \\ 0.1 & -0.3 \end{bmatrix},$$
$$A_{d21} = \begin{bmatrix} -0.1 & 0.1 \\ 0.1 & -0.2 \end{bmatrix}, A_{d22} = \begin{bmatrix} 0 & 0.1 \\ 0.2 & 0 \end{bmatrix},$$
$$C_{d11} = \begin{bmatrix} 0.4 & 0 \\ 0 & 0.1 \end{bmatrix}, C_{d12} = \begin{bmatrix} 0.2 & 0.1 \\ 0.2 & 0 \end{bmatrix},$$
$$C_{d21} = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.1 \end{bmatrix}, C_{d22} = \begin{bmatrix} 0 & 0.4 \\ 0.3 & 0 \end{bmatrix},$$
$$E_{11} = E_{12} = E_{21} = E_{22} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

with $h_1 = 0.5 + \frac{x_1^3(t)}{6.75}$, $h_2 = 1 - h_1$, and the subsequent CP matrix

$$\Theta = \begin{bmatrix} 0.2 & 0.8 \\ 0.4 & 0.6 \end{bmatrix}.$$

Moreover, the corresponding constraints are: if $\dot{h}_1 < 0$, we have

$$\mathcal{G}_1 = \left\{ \begin{array}{l} P_{11} > P_{12}, P_{21} > P_{22}, Q_{111} > Q_{112}, \\ Q_{121} > Q_{122}, Q_{211} > Q_{212}, Q_{221} > Q_{222}, \\ R_{11} > R_{12}, R_{21} > R_{22}, R_{31} > R_{32}, S_{11} > S_{12}, \\ S_{21} > S_{22}, U_{11} > U_{12}, U_{21} > U_{22}. \end{array} \right\},$$

if $\dot{h}_1 \geq 0$, we have

$$\mathcal{G}_2 = \left\{ \begin{array}{l} P_{11} \leq P_{12}, P_{21} \leq P_{22}, Q_{111} \leq Q_{112}, \\ Q_{121} \leq Q_{122}, Q_{211} \leq Q_{212}, Q_{221} \leq Q_{222}, \\ R_{11} \leq R_{12}, R_{21} \leq R_{22}, R_{31} \leq R_{32}, S_{11} \leq S_{12}, \\ S_{21} \leq S_{22}, U_{11} \leq U_{12}, U_{21} \leq U_{22}. \end{array} \right\}.$$

By calculating the LMI-based inequalities in Theorem 2 with $d_1 = 0.3$, $d_2 = 0.5$, $-\kappa_1 = \kappa_2 = 0.1$, the minimum $H_\infty$ performance index is $\gamma_1 = 1.2074$ under $\mathcal{G}_1$ and $\gamma_2 = 1.2074$

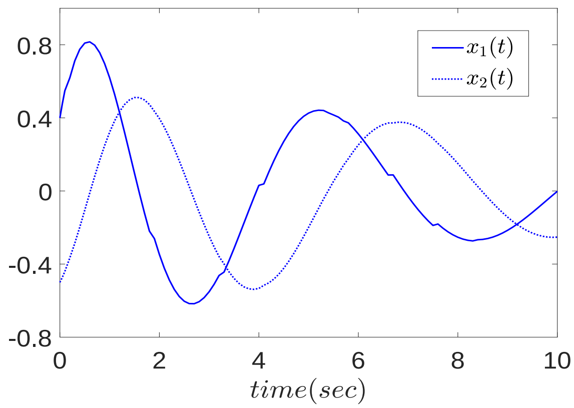


Fig. 2. The state responses of the open-loop system (Example 1).

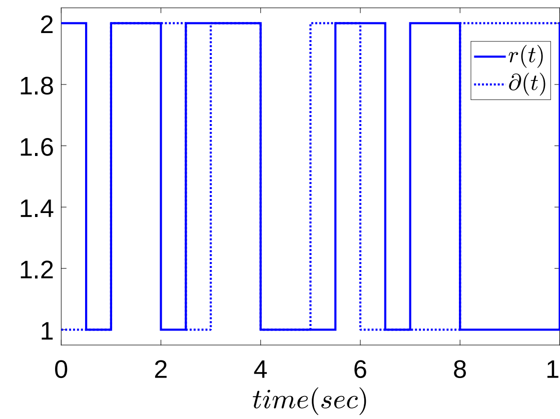


Fig. 3. Jumping between $r(t)$ and $\partial(t)$ (Example 1).

under $\mathcal{G}_2$. Therefore, the final minimum $H_\infty$ performance index obtained by Theorem 2 is $\gamma_{\min} = \max\{\gamma_1, \gamma_2\} = 1.2074$, and the corresponding controller matrices are

$$\begin{aligned}
K_{11}^1 &= \begin{bmatrix} -1.37 & 1.62 \end{bmatrix}, K_{12}^1 = \begin{bmatrix} -0.20 & 0.71 \end{bmatrix},\\
K_{21}^1 &= \begin{bmatrix} -0.91 & -0.72 \end{bmatrix}, K_{22}^1 = \begin{bmatrix} -0.76 & -0.33 \end{bmatrix},\\
K_{d11}^1 &= \begin{bmatrix} -0.27 & 1.07 \end{bmatrix}, K_{d12}^1 = \begin{bmatrix} -0.95 & 0.12 \end{bmatrix},\\
K_{d21}^1 &= \begin{bmatrix} -0.01 & -0.31 \end{bmatrix}, K_{d22}^1 = \begin{bmatrix} -0.08 & -0.08 \end{bmatrix},\\
K_{11}^2 &= \begin{bmatrix} -0.88 & 1.52 \end{bmatrix}, K_{12}^2 = \begin{bmatrix} -0.10 & 0.91 \end{bmatrix},\\
K_{21}^2 &= \begin{bmatrix} -0.66 & -0.52 \end{bmatrix}, K_{22}^2 = \begin{bmatrix} -0.78 & -0.37 \end{bmatrix},\\
K_{d11}^2 &= \begin{bmatrix} -0.19 & -0.23 \end{bmatrix}, K_{d12}^2 = \begin{bmatrix} -1.00 & -0.06 \end{bmatrix},\\
K_{d21}^2 &= \begin{bmatrix} -0.19 & -0.15 \end{bmatrix}, K_{d22}^2 = \begin{bmatrix} -0.07 & -0.04 \end{bmatrix}.
\end{aligned}$$

In order to prove the effectiveness of the proposed switching method in asynchronous framework, the non-switching case is taken into account (e.g. $P_{11} = P_{12}, P_{21} = P_{22}$). And the corresponding minimum $H_\infty$ performance index is 1.2353, which is larger than that obtained by the switching method.

For simulation, assume $w(t) = 0.1e^{-5t}\cos(t)$, $d(t) = 0.4 + 0.1\cos(t)$, under initial conditions $x(0) = \begin{bmatrix} 0.4 & -0.5 \end{bmatrix}^T$, the state responses, the state responses of the open-loop system, and asynchronous Markov stochastic processes are shown in Figure 1-Figure 3, which show that the proposed design approach of asynchronous controller is effective. Meanwhile, Figure 4 plots the evolution of $dh_1/dt$ and control input $u$, where $\mathrm{T}_1\,(t = 0.3680)$, $\mathrm{T}_2\,(t = 1)$ are the switching points, and $\dot{h}_1(0.3680) = 7.4853e-04$, $\dot{h}_1(1) = 6.6857e-06$. It can be seen that the controller is $u_2$ in the time range $[0, \mathrm{T}_1]$, then switches to $u_1$ in the time range $[\mathrm{T}_1, \mathrm{T}_2]$, and finally switches to $u_2$ in the time range $[\mathrm{T}_2, +\infty]$.

Especially, based on Remark 6 and Remark 7 with $d_1 = 0$, $d_2 = 0.7$, $\gamma = 2.7$, $\varepsilon_1 = \varepsilon_2 = 1.5$, $l_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}$, $l_2 = \begin{bmatrix} 0 & 1 \end{bmatrix}$, Figure 5 illustrates the set of $\bar{\mathcal{D}}$, which is the intersection of sets under $\dot{h}_1 < 0$ and $\dot{h}_1 \geq 0$. It can be seen that the set is contained in $\mathbb{C}$, so the system is SS no matter how the controller switches. Furthermore, the trajectories of three initial states starting on the boundary are also plotted in Figure 5, which shows that the states are all stabilized by the controller.

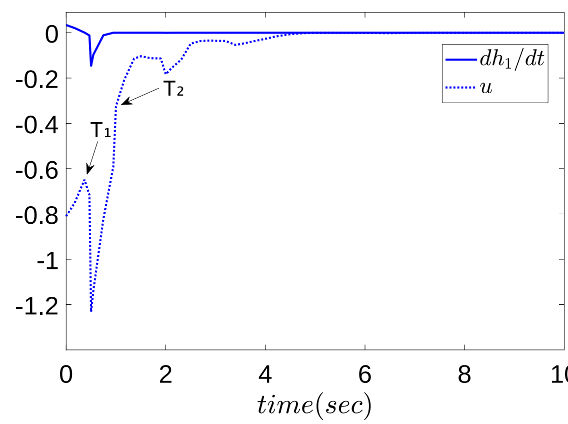


Fig. 4. Evolution of $dh_1/dt$ and control input $u$ (Example 1).

*Example 2:* This example considers the special case when $i \in \mathcal{N} = \{1\}$, by comparing the maximum delay bounds, the superiority of the method in this paper is proved. Consider the nominal fuzzy time-delay system as follows:

$$\dot{x}(t) = \sum_{q=1}^{2} h_q(\theta(t))[A_q x(t) + A_{dq} x(t - d(t))],$$

where

$$\begin{aligned}
A_1 &= \begin{bmatrix} -2.1 & 0.1 \\ -0.2 & -0.9 \end{bmatrix}, A_{d1} = \begin{bmatrix} -1.1 & 0.1 \\ -0.8 & -0.9 \end{bmatrix},\\
A_2 &= \begin{bmatrix} -1.9 & 0 \\ -0.2 & -1.1 \end{bmatrix}, A_{d2} = \begin{bmatrix} -0.9 & 0 \\ -1.1 & -1.2 \end{bmatrix},\\
h_1 &= \frac{1}{1 + e^{-2x_1(t)}}, h_2 = 1 - h_1.
\end{aligned}$$

This non-feedback fuzzy time-delay system has been extensively studied in numerous literatures with the aim of determining the maximum time delay that the system can tolerate while still maintaining stability. In this paper, with $d_1 = 0$, $-\kappa_1 = \kappa_2 = 0.1$, several cases are discussed:

(I) MFs-dependent LKF with DPT. $P_{ih}$, $R_{1h}$, $R_{2h}$, $R_{3h}$, $S_{1h}$, $S_{2h}$, $Q_{1ih}$, $Q_{2ih}$, $U_{1h}$, $U_{2h}$.

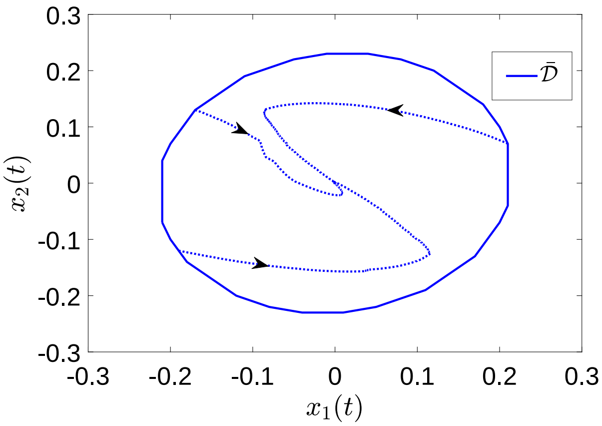


Fig. 5. The set $\bar{\mathcal{D}}$ and three trajectories starting on the boundary (Example 1).

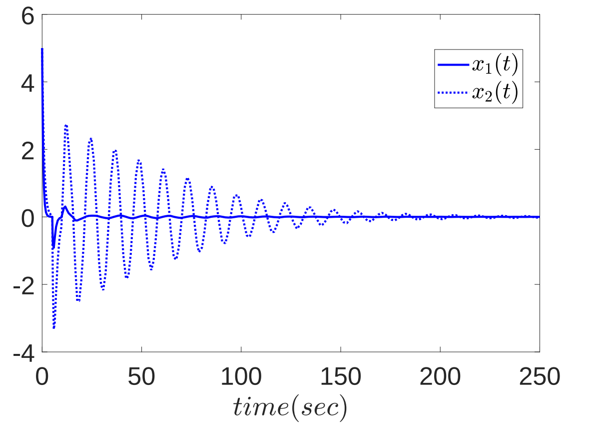


Fig. 6. The state responses (Example 2).

(II) MFs-dependent LKF without DPT. $Q_{1ih}=0$, $Q_{2ih}=0$, $P_{ih}$, $R_{1h}$, $R_{2h}$, $R_{3h}$, $S_{1h}$, $S_{2h}$, $U_{1h}$, $U_{2h}$.

(III) MFs-independent LKF with DPT. $P_i$, $R_1$, $R_2$, $R_3$, $S_1$, $S_2$, $Q_{1i}$, $Q_{2i}$, $U_1$, $U_2$.

The maximum delay bounds $d_2$ obtained by different methods are shown in Table 1, which shows that the proposed approach in this paper (i.e. case (I)) is less conservative than those in [8]–[11]. In addition, comparing the results of case (I) and case (II), the effectiveness of the DPT terms can be certified; comparing the results of case (I) and case (III), the superiority of introducing MFs-dependent terms can be verified. Assume $d(t) = 5.091 + 0.1\cos(t)$, under initial condition $x(0) = \begin{bmatrix} 5 & 5 \end{bmatrix}^T$, the state trajectories are shown in Figure 6, which shows that the considered system is SS for $d_2 = 5.191$.

TABLE I
THE UPPER BOUND $d_2$ OBTAINED BY DIFFERENT METHODS (EXAMPLE 2)

| Method | Max $d_2$ |
|---|---|
| [9] | 3.42 |
| [8] | 4.2044 |
| [10] | 4.324 |
| [11] $(K=2)$ | 4.91 |
| Theorem 2 with (I) | 5.191 |
| Theorem 2 with (II) | 4.622 |
| Theorem 2 with (III) | 3.477 |

*Example 3:* In order to demonstrate that the switching method presented in this paper has better $H_\infty$ performance, this example takes into account a two-rule fuzzy control system with the following parameters, which is borrowed from [43]:

$$A_1 = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}, A_2 = \begin{bmatrix} -1 & 0.5 \\ 0 & -1 \end{bmatrix},$$
$$B_{11} = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}, B_{12} = \begin{bmatrix} -1 & 0 \\ 0.1 & -1 \end{bmatrix},$$
$$B_{21} = \begin{bmatrix} 0.4 & 0.2 \\ 0.1 & 0.7 \end{bmatrix}, B_{22} = \begin{bmatrix} 0.3 & 0.5 \\ 0.2 & 0.6 \end{bmatrix},$$
$$D_1 = \begin{bmatrix} 0.5 & 0.4 \\ 0.2 & 0.3 \end{bmatrix}, D_2 = \begin{bmatrix} 0.1 & 0.6 \\ 0.8 & 0.7 \end{bmatrix},$$
$$C_1 = \begin{bmatrix} 0.1 & 0.3 \\ 0.4 & 0.6 \end{bmatrix}, C_2 = \begin{bmatrix} 0.5 & 0.1 \\ 0.4 & 0.1 \end{bmatrix}.$$

For this example, the time-delay dependent terms are set to zero, and the minimum $H_\infty$ performance is considered. In this paper, the minimum $H_\infty$ performance $\gamma_{\min} = \max\{\gamma_1, \gamma_2\} = \max\{0.4025, 0.4396\} = 0.4396$. In [43], the minimum $H_\infty$ performance $\gamma$ under the same parameters is $0.8984\,(\sigma=0)$, which is larger than that obtained by Theorem 2. Hence, the approach proposed in this paper is less conservative than [43].

## V. CONCLUSION

This paper has studied the asynchronous $H_\infty$ control issue for FMJSs with time-varying delay. A new MFs-dependent LKF with DPT terms has been designed to reduce conservatism. Furthermore, utilizing the time derivative information of MFs, an asynchronous switching fuzzy controller has been proposed according to their sign. Meanwhile, the Lyapunov level set has been utilized to estimate the invariant set, ensuring that the system is stable no matter how the controller switches within the set. Finally, the effectiveness and practicability of the presented results have been certified by three examples. In the future, it is necessary to investigate a method that can reduce conservatism with less computational complexity. Besides, extending the theoretical results to event-triggered control and adapt control is also the challenging research topic.

## REFERENCES


[1] T. Takagi and M. Sugeno, Fuzzy identification of systems and its applications to modeling and control. IEEE Transactions on Systems, Man, and Cybernetics: Systems 15 (1985) 116-132.

[2] Y. Li, L. Liu, and G. Feng, Finite-time $H_\infty$ controller synthesis of T-S fuzzy systems. IEEE Transactions on Systems, Man, and Cybernetics: Systems 50 (2020) 1956-1963.

[3] Y. X. Guo and J. M. Li, Network-based quantized $H_\infty$ control for T-S fuzzy singularly perturbed systems with persistent dwell-time switching mechanism and packet dropouts. Nonlinear Analysis: Hybrid Systems 42 (2021) 101060.

[4] B. Kaviarasan, O. M. Kwon, M. J. Park, and R. Sakthivel, Input-output finite-time stabilization of T-S fuzzy systems through quantized control strategy. IEEE Transactions on Fuzzy Systems 30 (2022) 3589-3600.

[5] H. Shen, M. S. Chen, Z.-G. Wu, J. D. Cao, and J. H. Park, Reliable event-triggered asynchronous extended passive control for semi-Markov jump fuzzy systems and its application. IEEE Transactions on Fuzzy Systems 28 (2020) 1708-1722.

[6] X. Y. Zheng, H. Zhang, Z. P. Wang, C. Z. Zhang, and H. C. Yan, Finite-time dynamic event-triggered distributed $H_\infty$ filtering for T-S fuzzy systems. IEEE Transactions on Fuzzy Systems 30 (2022) 2476-2486.

[7] S. Shi, Z. Y. Fei, H. R. Karimi, and H.-K. Lam, Event-triggered control for switched T-S fuzzy systems with general asynchronism. IEEE Transactions on Fuzzy Systems 30 (2022) 27-38.

[8] O. M. Kwon, M. J. Park, J. H. Park, and S. M. Lee, Stability and stabilization of T-S fuzzy systems with time-varying delays via augmented Lyapunov-Krasovskii functionals. Information Sciences 372 (2016) 1-15.

[9] Z. Lian, Y. He, C.-K. Zhang, and M. Wu, Further robust stability analysis for uncertain Takagi-Sugeno fuzzy systems with time-varying delay via relaxed integral inequality. Information Sciences 409-410 (2017) 139-150.

[10] J. Y. Tan, S. Y. Dian, and T. Zhao, Further studies on stability and stabilization of T-S fuzzy systems with time-varying delays via fuzzy Lyapunov-Krasovskii functional method. Asian Journal of Control 20 (2018) 2207-2222.

[11] G. Wang, L. Jia, and H. G. Zhang, Stability and stabilization of T-S fuzzy time-delay system via relaxed integral inequality and dynamic delay partition. IEEE Transactions on Fuzzy Systems 29 (2021) 2829-2843.

[12] X. L. Xu, H. C. Yan, H. Zhang, and F. W. Yang, $H_\infty$ filtering for T-S fuzzy networked systems with stochastic multiple delays and sensor faults. Neurocomputing 207 (2016) 590-598.

[13] A. Chibani, M. Chadli, P. Shi, and N. B. Braiek, Fuzzy fault detection filter design for T-S fuzzy systems in the finite-frequency domain. IEEE Transactions on Fuzzy Systems 25 (2017) 1051-1061.

[14] M. Wang, G. Feng, J. B Qiu, H. C. Yan, and H. Zhang, Fault detection filtering design for discrete-time interval type-2 T-S fuzzy systems in finite frequency domain. IEEE Transactions on Fuzzy Systems 29 (2021) 213-225.

[15] Q. Zhang, H. C. Yan, M. Wang, Z. C. Li, and Y. F. Chang, Asynchronous fault detection filter design for T-S fuzzy singular systems via dynamic event-triggered scheme. IEEE Transactions on Fuzzy Systems 31 (2023) 970-981.

[16] Z.-G. Wu, P. Shi, H. Y. Su, and J. Chu, Stochastic synchronization of Markovian jump neural networks with time-varying delay using sampled data. IEEE Transactions on Cybernetics 43 (2013) 1796-1806.

[17] S. L. Dong, Z.-G. Wu, H. Y. Su, P. Shi, and H. R. Karimi, Asynchronous control of continuous-time nonlinear Markov jump systems subject to strict dissipativity. IEEE Transactions on Automatic Control 64 (2019) 1250-1256.

[18] P. P. Zhang, Y. G. Kao, J. Hu, and B. Niu, Robust observer-based sliding mode $H_\infty$ control for stochastic Markovian jump systems subject to packet losses. Automatica 130 (2021) 109665.

[19] F. B. Li, Z. Wu, C. H. Yang, Y. Shi, T. W. Huang, and W. H. Gui, A novel learning-based asynchronous sliding mode control for discrete-time semi-Markov jump systems. Automatica 143 (2022) 110428.

[20] J. Cheng, J. H. Park, Y. J. Liu, Z. J. Liu, and L. M. Tang, Finite-time $H_\infty$ fuzzy control of nonlinear Markovian jump delayed systems with partly uncertain transition descriptions. Fuzzy Sets and Systems 314 (2017) 99-115.

[21] X. Y. Liang, J. W. Xia, G. L. Chen, H. S. Zhang, and Z. Wang, Dissipativity-based sampled-data control for fuzzy Markovian jump systems. Applied Mathematics and Computation 361 (2019) 552-564.

[22] R. Kavikumar, R. Sakthivel, O. M. Kwon, and B. Kaviarasan, Reliable non-fragile memory state feedback controller design for fuzzy Markov jump systems. Nonlinear Analysis: Hybrid Systems 35 (2020) 100828.

[23] R. Kavikumar, R. Sakthivel, and Y. R. Liu, Design of $H_\infty$-based sampled-data control for fuzzy Markov jump systems with stochastic sampling. Nonlinear Analysis: Hybrid Systems 41 (2021) 101041.

[24] B. Visakamoorthi, P. Muthukumar, and H. Trinh, Reachable set estimation for T-S fuzzy Markov jump systems with time-varying delays via membership function dependent $H_\infty$ performance. IEEE Transactions on Fuzzy Systems 30 (2022) 4980-4990.

[25] K. H. Nguyen and S. H. Kim, Improved dissipativity-based sampled-data control synthesis of nonhomogeneous Markovian jump fuzzy systems against mismatched fuzzy-basis functions. Information Sciences 607 (2022) 1439-1464.

[26] W. Q. Xie, S. K. Nguang, H. Zhu, Y. P. Zhang, and K. B. Shi, A novel event-triggered asynchronous $H_\infty$ control for T-S fuzzy Markov jump systems under hidden Markov switching topologies. Fuzzy Sets and Systems 443 (2022) 258-282.

[27] G. T. Ran, C. J. Li, R. Sakthivel, C. S. Han, B. H. Wang, and J. Liu, Adaptive event-triggered asynchronous control for interval type-2 fuzzy Markov jump systems with cyberattacks. IEEE Transactions on Control of Network Systems 9 (2022) 88-99.

[28] Y. L. Wei, J. B. Qiu, P. Shi, and H.-K. Lam, A new design of H-infinity piecewise filtering for discrete-time nonlinear time-varying delay systems via T-S fuzzy affine models. IEEE Transactions on Systems, Man, and Cybernetics: Systems 47 (2017) 2034-2047.

[29] X. Z. Yang, H.-K. Lam, and L. G. Wu, Membership-dependent stability conditions for type-1 and interval type-2 T-S fuzzy systems. Fuzzy Sets and Systems 356 (2019) 44-62.

[30] M. Wang, G. Feng, H. C. Yan, J. B. Qiu, and H. Zhang, Membership-function-dependent fault detection filtering design for interval type-2 T-S fuzzy systems in finite frequency domain. IEEE Transactions on Fuzzy Systems 29 (2021) 2760-2773.

[31] L. Wang and H.-K. Lam, A new approach to stability and stabilization analysis for continuous-time Takagi-Sugeno fuzzy systems with time delay. IEEE Transactions on Fuzzy Systems 26 (2018) 2460-2465.

[32] L. Wang and H.-K. Lam, New stability criterion for continuous-time Takagi-Sugeno fuzzy systems with time-varying delay. IEEE Transactions on Cybernetics 49 (2019) 1551-1556.

[33] B. P. Jiang, Z. T. Wu, and H. R. Karimi, A distributed dynamic event-triggered mechanism to HMM-based observer design for $H_\infty$ sliding mode control of Markov jump systems. Automatica 142 (2022) 110357.

[34] S. Wang and Z.-G. Wu, Asynchronous control of uncertain Markov jump systems with actuator saturation. IEEE Transactions on Circuits and Systems II: Express Briefs 69 (2022) 3269-3273.

[35] X. X. Wang and Y. C. Ma, Observer-based finite-time asynchronous sliding mode control for Markov jump systems with time-varying delay. Journal of the Franklin Institute 359 (2022) 5488-5511.

[36] Z.-G. Wu, S. L. Dong, H. Y. Su, and C. D. Li, Asynchronous dissipative control for fuzzy Markov jump systems. IEEE Transactions on Cybernetics 48 (2018) 2426-2436.

[37] S. L. Dong, C. L. P. Chen, M. Fang, and Z.-G. Wu, Dissipativity-based asynchronous fuzzy sliding mode control for T-S fuzzy hidden Markov jump systems. IEEE Transactions on Cybernetics 50 (2020) 4020-4030.

[38] B. Visakamoorthi, K. Subramanian, and P. Muthukumar, Hidden Markov model based non-fragile sampled-data control design for mode-dependent fuzzy systems with actuator faults. Applied Mathematics and Computation 435 (2022) 127454.

[39] A. Seuret and F. Gouaisbaut, Wirtinger-based integral inequality: application to time-delay systems. Automatica 49 (2013) 2860-2866.

[40] P. G. Park, W. I. Lee, and S. Y. Lee, Auxiliary function-based integral inequalities for quadratic functions and their applications to time-delay systems. Journal of the Franklin Institute 352 (2015) 1378-1396.

[41] K. Tanaka, T. Ikeda, and H. O. Wang, Robust stabilization of a class of uncertain nonlinear systems via fuzzy control: quadratic stabilizability, $H_\infty$ control theory, and linear matrix inequalities. IEEE Transactions on Fuzzy Systems 4 (1996) 1-13.

[42] H. N. Wu, Reliable robust $H_\infty$ fuzzy control for uncertain nonlinear systems with Markovian jumping actuator faults. Journal of Dynamic Systems, Measurement, and Control 129 (2007) 252-261.

[43] H. Shen, F. Li, H. C. Yan, H. R. Karimi, and H.-K. Lam, Finite-time event-triggered $H_\infty$ control for T-S fuzzy Markov jump systems. IEEE Transactions on Fuzzy Systems 26 (2018) 3122-3135.