Fuzzy Adaptive Containment Control for Nonlinear Multi-Manipulator Systems with Actuator Faults and Predefined Accuracy

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Abstract—Compensating for the infinite number of actuator faults considered in nonlinear multi-manipulator systems is a significant, yet a crucial issue for control problems. In view of this, a novel fuzzy adaptive fault-tolerant control algorithm is proposed for nonlinear multi-manipulator systems to eliminate the influence of actuator faults and improve accuracy essentially, as well as achieve the objective of containment. First, a Lyapunov function is constructed by replacing the square term of the error with a series of smooth functions. Moreover, a predefined accuracy function is introduced and a new fuzzy adaptive fault-tolerant containment control scheme is designed. This overcomes the problem that existing methods cannot guarantee the negative-definite property of time derivative of Lyapunov function between two adjacent actuator faults. And it also ensures the boundedness of containment errors. Furthermore, projection-based adaptive parameters and the turning function are fused to compensate the influence of actuator faults. Finally, the effectiveness of the designed control scheme is verified by a simulation example of multiple single-link manipulator systems.

Index Terms—Actuator faults, multi-manipulator systems, predefined accuracy function, containment control, fault-tolerant control.

I. INTRODUCTION

The cooperative control of multiagent systems (MASs) has attracted great attention among researchers in recent decades [1]–[6], and has been applied to various engineering problems. At first, the consensus problem was concentrated on the consensus criterion, i.e., the leaderless consensus problem [7]–[9]. Then, in order to obtain the desired reference trajectory for a group of agents, the concept of consensus tracking was proposed [10]–[12], from which it is noted that only one leader is involved. When there are multiple leaders in MASs, the consensus tracking problem is called the containment control problem. One of the purposes in designing containment control is to prevent followers from entering dangerous areas. Therefore, containment control is widely used for the cooperative control of multi-manipulator systems (MMSs) in recent years. For example, based on the backstepping technique, the fuzzy adaptive containment control method was proposed for uncertain nonlinear MMSs [13]. Zakerimanesh et al. [14] designed an adaptive containment control algorithm to ensure that the follower robot’s end-effector asymptotically converges to the convex hull formed by the leaders’ traversed trajectories.

The above-mentioned works do not take actuator faults occurring in the control systems into account. However, the actuator of the connection plant may suddenly become stuck or lose partial effectiveness in practical systems. These faults always possess negative effects on system performance and damage the stability of systems. In recent years, with the continuous improvement of the safety and stability requirements in practical systems (such as unmanned aircraft systems and MMSs), the research on actuator fault compensation technology has been stimulated via several different approaches [15]–[18]. For example, based on neural networks (NNs), a fault-tolerant control scheme for stochastic nonlinear systems was proposed to ensure the prescribed transient performance in the case of actuator failure and hysteresis [16]. In [17], for a class of single-input and single-output (SISO) systems with actuator faults, an adaptive fuzzy fault-tolerant optimal control method was developed to solve the actuator fault compensation problem. The above results have been used to solve the case with a finite number of actuator faults, that is, there are no further failures/faults after a finite time. However, these results cannot handle the infinite number of actuator faults. The main reason is that the superposition of an infinite number of actuator faults generally leads to the continuous increase of the Lyapunov function, which cannot be bounded actually. This motivates studies in adaptive compensation control for an infinite number of actuator faults. In [19], based on the backstepping technique and fuzzy logic systems (FLSs), a fault-tolerant control scheme for stochastic nonlinear systems was proposed to ensure a prescribed transient performance in the case of actuator failure and hysteresis. By using the projection adaptation technique, an adaptive fuzzy control strategy was proposed to solve the infinite number of actuator faults, which ensures the system stability and security [20]. Due to the advantages of the adaptive compensation technique, the control scheme in [20] was extended to MASs, and some remarkable results were obtained. For example, a cooperative
fault-tolerant control method was developed to compensate for multiple heterogeneous actuator faults [21]. By establishing a conditional inequality, an adaptive asymptotic cooperative control scheme was proposed to improve the robustness of the MASs with actuator faults [22]. Although the existing results can guarantee the system performance with an infinite number of actuator faults, the boundedness of Lyapunov function at the fault instant is ignored. Because there are some jumps in the actuator parameters, and the jumping size is bounded at the fault instant. When the number of faults is infinite, the accumulation of infinite jumps will make the Lyapunov function lose boundedness. Therefore, how to prove the boundedness of the Lyapunov function and establish an adaptive fault compensation strategy of MMSs with an infinite number of actuator faults are the research motivation of this paper.

Besides, for traditional fuzzy adaptive control algorithms, many fuzzy rules are used to decrease the approximating error, which generates enormous estimated parameters and increases the computation burden of the designed controller. In [23], the aforementioned problem was well solved by estimating the square of the norm of unknown fuzzy weight vectors. Due to its advantage of reducing computation, this method was expanded to MASs [24]–[26]. However, although this algorithm can ensure that the synchronization error converges to a small neighbourhood of the origin, it still cannot reach the condition of the Lyapunov stability $\dot{V} \leq 0$, with $V$ as a Lyapunov function. The main reason is that some additional constant terms are generated when dealing with the squared norm of the fuzzy weight vector in the control design process. To overcome this difficulty, an adaptive strategy based on the predefined accuracy function was proposed in [27]. However, most of the existing research results [15]–[20] are aimed at simple SISO systems, but the research on MMSs still lacks and is challenging. Hence, this paper aims to shorten this gap.

Through the above discussion, a fault-tolerant adaptive containment control scheme for MMSs with infinite actuator faults is proposed, and it ensures the boundedness of all signals during the minimum fault interval. The main innovations of this paper are outlined as follows.

1) For [24]–[26], it is difficult to prove that the Lyapunov function is non-increasing in the interval of two adjacent faults due to some constant terms in the algorithm design process. To address this issue, according to properties of the predefined accuracy function, a novel Lyapunov function is designed by replacing the square error term with a series of smooth functions. Then, based on the new Lyapunov function, a class of performance-oriented controllers is designed, which realizes that constant terms are not generated in the design process.

2) When the actuator fault occurs, the Lyapunov function as a function of time has a bounded jump at this instant due to the change of fault-related parameters. If the number of jumps increases, the size of Lyapunov function will also increase. Therefore, an adaptive fuzzy fault-tolerant control algorithm based on the projection operator is proposed. At each fault time, the control algorithm guarantees the boundedness of the Lyapunov function and eliminates the increase in its size. Moreover, the boundedness of Lyapunov function can still be maintained even if the number of actuator faults is infinite.

3) In the traditional adaptive control methods [28]–[31], FLs or NNs will produce approximation errors when approximating nonlinear functions, and have a negative impact on the system stability. In order to solve this problem, the FLs weight vector, fuzzy basis function, and approximation error are lumped together, and combined with the designed control algorithm, the approximation error will no longer have a negative effect on the system stability.

II. PRELIMINARIES AND PROBLEM FORMULATION

A. Graph Theory

To describe the information exchange between agents more clearly, the graph theory is introduced here. The directed topological graph is denoted as $G = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, where $\mathcal{V} = \{1, \ldots, N\}$ is defined as the set of nodes, $\mathcal{E} = \{e_i \times e_j \in \mathcal{V} \times \mathcal{V}\}$ is a set of edges, where $(e_i, e_j) \in \mathcal{E}$ means that the agent $i$ can obtain information from the agent $j$. The adjacency matrix is denoted as $A = \{a_{ij}\} \in \mathbb{R}^{N \times N}$, and we define $a_{ij}$ as:

$$ a_{ij} = \begin{cases} 1 & (e_i, e_j) \in \mathcal{E} \\ 0 & (e_i, e_j) \notin \mathcal{E} \end{cases} $$

$N_i = \{|e_i| : (e_i, e_j) \in \mathcal{E}\}$ denotes the set of neighbor nodes of the $i$-th agent. The followers are labeled $1$ to $N$ and the leaders $N + 1$ to $N + M$. The Laplacian matrix $L$ of the directed graph $G$ is defined as $L = \{l_{ij}\}_{i=1}^{N+M}$, where $l_{ij} = -a_{ij}$ if $i \neq j$; otherwise $l_{ij} = \sum_{j \in N_i} a_{ij}$. $L = D - A$ expresses the Laplacian matrix. The in-degree matrix associated with $G$ is expressed as $D = \text{diag}\{d_1, \ldots, d_N\}$, where $d_i = \sum_{j \in N_i} a_{ij}$.

Then, one can obtain

$$ L = \begin{bmatrix} L_1 & L_2 \\ L_2 & 0 \end{bmatrix} \in \mathbb{R}^{M \times N} \quad \mathbb{R}^{N \times N} \in \mathbb{R}^{N \times M} $$

where $L_1 \in \mathbb{R}^{M \times N}$, $L_2 \in \mathbb{R}^{N \times M}$. This paper assumes that a directed graph contains a directed spanning tree, that is, at least one agent (called root node) has a directed path to all other agents. Hence, each entry of $-L_1^{-1}L_2$ is non-negative, and each row of $-L_1^{-1}L_2$ has a sum equal to 1. Let $y_d(t) = [y_{d(1)}(t), \ldots, y_{d(N+M)}(t)]^T$, then combined with matrix $-L_1^{-1}L_2$, the convex hull formed by multiple leaders can be expressed as $y_d(t) = -L_1^{-1}L_2y_d(t)$.

Definition 1: [32] $C$ is defined as a set in a real vector space $\mathcal{V} \subseteq \mathbb{R}^n$. The set $C$ is called a convex set if the point $(1 - \beta_1)x + \beta_1y$ exists in $C$ for any $\beta_1 \in [0, 1]$ and $(x, y) \in C$. The convex hull of a finite set of points $\mathcal{X} = \{x_1, \ldots, x_n\}$ in $\mathcal{V}$ is the minimal convex set containing all points in $\mathcal{X}$. Define $\text{Co}\{\mathcal{X}\} = \{\sum_{i=1}^n \beta_i x_i | x_i \in \mathcal{X}, \beta_i \in \mathbb{R} \geq 0, \sum_{i=1}^n \beta_i = 1\}$. When $\mathcal{V} \subseteq \mathbb{R}$, $\text{Co}\{\mathcal{X}\} = \{x | x \in [\min x_i, \max x_i]\}$ can be obtained.

B. Problem Statement

We consider a class of multiple single-link manipulator systems with actuator faults, which consists of leaders and
followers. Each follower dynamic is expressed as:

\[
\begin{align*}
H\ddot{q}_i + D\dot{q}_i + N\sin(q_i) &= I_{Ai} + F_{id} \\
L_i\dot{I}_{Ai} &= V_{oi} - R_{L}I_{Ai} - K_B\dot{q}_i \quad (i = 1, \ldots, N)
\end{align*}
\]

(2)

where \(H = \frac{2M_0R_0^2}{K_r} + \frac{M_rL_0^2}{K_r} + \frac{m_0L_0G}{K_r} + \frac{mL_0G}{K_r} \), \(D = \frac{B_r}{K_r} \), \(M_0 \) and \(m_0 \) are masses of the load and link, respectively, \(B_r, K_r, K_B \) and \(G \) are coefficients of the viscous friction at the joint, back-emf, torque and gravity. \(L_0 \) is the link length, \(J \) the rotor inertia, \(L_i \) armature inductance, \(R \) the protective resistance, and \(R_0 \) the load radius. \(q_i, \dot{q}_i, \ddot{q}_i \) are the position, angular velocity and acceleration. \(I_{Ai} \) is the motor current. \(V_{oi} = \sum_{j=1}^{N_m} \bar{g}_{ij}\bar{u}_{ij}(\bar{x}_{i3})u_{ij} \) is the input control voltage, where \(\bar{g}_{ij} \) is assumed to be a bounded unknown constant. \(F_{id} \) is an unknown external disturbance.

For the convenience of calculation, we define \(\bar{q}_{i1} = x_{i1}, \dot{\bar{q}}_{i1} = x_{i2}, I_{Ai}/H = x_{i3} \) and \(V_{oi}/L_iH = \sum_{j=1}^{N_m} g_{ij}\bar{u}_{ij}(\bar{x}_{i3})u_{ij} \).

By deforming model (2), the dynamics of the follower can be transformed into

\[
\begin{align*}
\dot{x}_{i1} &= x_{i2} \\
\dot{x}_{i2} &= x_{i3} - \frac{N}{H}\sin(x_{i1}) - \frac{D}{H}x_{i2} + \frac{F_{id}}{H} \\
\dot{x}_{i3} &= -\frac{K_B}{L_iH}x_{i2} + \sum_{j=1}^{N_m} g_{ij}\bar{u}_{ij}(\bar{x}_{i3})u_{ij} - \frac{R}{L_i}x_{i3} \\
y_i &= x_{i1} \quad (i = 1, \ldots, N).
\end{align*}
\]

The state vector and output signal of the system are represented by \(\bar{x}_i = [x_{i1}, x_{i2}, x_{i3}]^T \in \mathbb{R}^3 \) and \(y_i \in \mathbb{R}, \) respectively. \(u_{ij} (j = 1, \ldots, N_m) \) is the input of the \(i\)-th system. \(\bar{u}_{ij} \) is a known smooth function. \(g_{ij} (g_{ij} = \frac{\bar{g}_{ij}}{L_iH}) \) is a bounded unknown constant.

C. Actuator Faults Model

When the actuator suffers faults, its model can be defined as follows:

\[
u_{ij} = \bar{\eta}_{ij}u_{adj} + u_{ij} \quad t \in [t_\tau, t_{\tau+1}) \quad \text{and} \quad \tau = 1, 2, \ldots \]

(4)

where \(u_{adj} \) denotes the input signal of the \(j\)-th actuator of the \(i\)-th follower, \(t_\tau \) expresses the fault time instant and \(\bar{\eta}_{ij} \in [0, 1] \) is called the actuator efficiency factor of control signal \(u_{adj} \).

At each time interval \( [t_\tau, t_{\tau+1}) \), \(\bar{\eta}_{ij} \) and \(u_{adj} \) are unknown constants. Model (4) covers normal conditions as well as typical actuator faults.

1) Case 1: When \(\bar{\eta}_{ij} = 1 \) and \(u_{adj} = 0 \), the \(j\)-th actuator is working normally, and free of failures and faults.

2) Case 2: When \(\bar{\eta}_{ij} = 0 \), the \(j\)-th actuator completely failed, which means that the \(j\)-th actuator has no contribution to the overall control input \(u_{adj} \).

3) Case 3: When \(\bar{\eta}_{ij} \in (0, 1) \), \(u_{adj} = 0 \), there is partial loss of effectiveness of the \(i\)-th actuator.

The aim of this paper is to design a fault-tolerant containment control algorithm to eliminate the negative effects of infinite actuator faults on the system (3) and ensure that all signals are bounded. At the same time, it is guaranteed that the output signal of each manipulator is driven into the convex hull composed of multiple leaders. To obtain control objectives, the following assumptions are proposed for MMSs.

**Assumption 1:** The output signal of the leader is represented as \(y_{id}(l = N + 1, \ldots, N + M) \), which is a known smooth function and 3-th order differentiable.

**Assumption 2:** Up to \(N_m - 1 \) actuators suffer from stuck faults, and the remaining actuators can still achieve the desired control objective.

**Assumption 3:** The unknown constant \(g_{ij}\) satisfies \(0 < g_{i0} \leq g_{ij} \leq g_{iM} \), the parameters of faults patterns satisfy \(0 \leq \bar{\eta}_{i0} \leq \bar{\eta}_{ij} \leq 1 \) and \(|u_{ij}| \leq u_{ij,M} \), \(g_{i0}, g_{iM}, \bar{\eta}_{i0} \) and \(u_{ij,M} \) are known positive constants.

**Remark 1:** According to the characteristics of the backstepping method, we need to ensure that the order of the leaders is the same as the order of the system, which can further ensure the continuity of controller with leader related terms. Therefore, under the framework of backstepping, Assumption 1 is essential. In a manipulator with only one actuator, when the complete failure of this actuator occurs, the desired control objective will not be achieved. Hence, it is necessary to reasonably configure multiple actuators on the manipulator to prevent the failure, and similar operations have been widely used in various practical environments. Therefore, Assumption 2 is proposed in line with practical applications, and implies that there exists a solution for fault-tolerant control, where the excess control input is used not only to achieve the control objective, but also to optimize the total control effort. And even if there exists the total failure for one or more (Up to \(N_m - 1 \) actuators), the control objective can still be achieved. In practical applications, the direction of the control signal is not known all the way. Therefore, we assume an unknown parameter \(g_{ij} \) to describe the unknown control direction. Because the system energy is finite, we need to further assume that the parameter \(g_{ij} \) is bounded. Therefore, the proposal of Assumption 3 is reasonable.

D. FLSs and Function Approximation

The nonlinear function of MMSs will be approximated by FLSs method. The \(\eta\)-th fuzzy rule is represented as follows.

\[R^\eta: \text{If } \eta_1 \text{ is } F_1^\eta \text{ and } \eta_2 \text{ is } F_2^\eta, \ldots, \text{and } \eta_n \text{ is } F_n^\eta, \text{ Then: } y(\eta) = G^\eta,\]

where \(\eta = [\eta_1, \ldots, \eta_n]^T \in \mathbb{R}^n \) and \(y(\eta) \) are denoted input and output of FLSs, respectively. \(F_n^\eta \) and \(G^\eta \) are denoted as fuzzy sets. \(\mu_{F_n^\eta}(\eta_i) \) and \(\mu_{G^\eta}(y) \) are the membership functions.
associated with $F_h^*$ and $G^*$, respectively. The output $y(\eta)$ of FLSs can be represented as:

$$y(\eta) = \frac{\sum_{s=1}^{m} \sum_{h=1}^{N} \mu_{F_h}(\eta_h)}{\sum_{s=1}^{m} \sum_{h=1}^{N} \mu_{F_h}(\eta_h)}$$

where $m$ is the total number of fuzzy rules. The point $\mu_{G^*}(w_s) = 1$ is represented by $W_s$. The Gaussian function $\mu_{F_h}(\eta_h)$ is chosen as follows:

$$\mu_{F_h}(\eta_h) = \exp\left[-\frac{(\eta_h - \bar{\eta}_h)^2}{\bar{\epsilon}_h^2}\right]$$

where $\bar{\eta}_h$ and $\bar{\epsilon}_h$ denote the center and width, respectively.

The fuzzy basis function is selected as follows:

$$\varphi_s(\eta) = \frac{\sum_{h=1}^{N} \mu_{F_h}(\eta_h)}{\sum_{s=1}^{m} \sum_{h=1}^{N} \mu_{F_h}(\eta_h)}.$$ (7)

Then FLSs are redefined as $y(\eta) = W^T \varphi(\eta)$, where $W = [W_1, W_2, \ldots, W_m]^T$ and $\varphi(\eta) = [\varphi_1(\eta), \varphi_2(\eta), \ldots, \varphi_m(\eta)]^T$.

Lemma 1: [24] Define $f(\eta)$ to be a smooth function on a compact set $\Omega$. Then, for an arbitrarily small constant $\vartheta > 0$, there exists a FLS such that

$$\sup_{\eta \in \Omega} |f(\eta) - y(\eta)| \leq \vartheta.$$ (8)

### III. MAIN RESULTS

In this section, a new fault-tolerant containment control strategy is developed to compensate the effect of infinite actuator faults and guarantee that the containment error of the MMSs can converge within a predefined interval. To explain the algorithm principle more clearly, Fig. 1 shows the block diagram of the algorithm framework, and the detailed design process is described as follows.

**Fault-Tolerant Adaptive Fuzzy Control Design**

We define the containment error as

$$\epsilon_i = \sum_{j=1}^{N} a_{ij}(y_i - y_j) + \sum_{l=N+1}^{N+M} a_{il}(y_i - y_{l:d}).$$ (9)

Then the error variable $\epsilon_{ij}$ is defined as

$$\epsilon_{ij} = x_{ij} - \alpha_{ij-1} \quad (j = 2, 3)$$

where $\alpha_{ij-1}$ is the intermediate control signal. To ensure the control performance of MMSs, this paper uses the predefined accuracy functions [33]

$$sg_{ij}(\epsilon_{ij}) = \begin{cases} \frac{\epsilon_{ij}}{|\epsilon_{ij}|} & \text{if } |\epsilon_{ij}| \geq \kappa_{ij} \\ \frac{(\kappa_{ij}^2 - \epsilon_{ij}^2)^2 + |\epsilon_{ij}|}{|\epsilon_{ij}|} & \text{if } |\epsilon_{ij}| < \kappa_{ij} \end{cases}$$

and a class of turning functions

$$\bar{\varphi}_{ij}(\epsilon_{ij}) = \begin{cases} 1 & |\epsilon_{ij}| \geq \kappa_{ij} \\ 0 & |\epsilon_{ij}| < \kappa_{ij} \end{cases}$$

where $\kappa_{ij}$ $(j = 1, 2, 3)$ are predefined interval parameters. With (11) and (12), we can get a function

$$sg_{ij}(\epsilon_{ij})\bar{\varphi}_{ij}(\epsilon_{ij}) = \begin{cases} \frac{\epsilon_{ij}}{|\epsilon_{ij}|} & \text{if } |\epsilon_{ij}| \geq \kappa_{ij} \\ \frac{(\kappa_{ij}^2 - \epsilon_{ij}^2)^2 + |\epsilon_{ij}|}{|\epsilon_{ij}|} & \text{if } |\epsilon_{ij}| < \kappa_{ij} \end{cases}$$ (13)

Remark 2: Compared with the existing fault-tolerant control strategies [15]–[17], this paper proposes a fault-tolerant containment control to improve system resilience regardless of whether the actuator faults are finite or infinite. In addition, the range of error convergence is defined by the predefined accuracy function to ensure that the containment error converges to a predefined interval.

In the existing adaptive control technologies [6], [34]–[38], the Lyapunov function is usually constructed by the square term of the error, that is, $V = \sum_{i=1}^{n} (1/2)\epsilon_i^2$. Thus, $\dot{V} \leq -a_i\epsilon_i + b_i$ can be deduced, where $a_i > 0$ and $b_i > 0$ are set parameters. Therefore, it can only show that the signals are bounded, and the condition for system stability cannot be directly satisfied, that is, $\dot{V} \leq 0$. To obtain asymptotic stability, Lu et al. [33] designed the Lyapunov function as follows.

$$V = \sum_{i=1}^{n} \frac{(|\epsilon_i| - \kappa_i)^{n-i+2}\bar{\varphi}_i}{n - i + 2}$$

where $\epsilon_i^2$ is replaced by $(|\epsilon_i| - \kappa_i)^{n-i+2}\bar{\varphi}_i$. By using this equation, we can ensure $V \leq 0$. Additionally, in order to simplify the design process, we replace $(|\epsilon_i| - \kappa_i)^{n-i+2}\bar{\varphi}_i$ with $(|\epsilon_i| - \kappa_i)^2\bar{\varphi}_i$.

In the following description, for the transformed system (3), we construct the fault-tolerant controller and adaptive laws by the backstepping technique, and the desired control performance can be obtained.

Lemma 2: [24] Define $\epsilon_{iN+1} = [\epsilon_{i1}, \epsilon_{i2}, \ldots, \epsilon_{iN}]^T$ and $y_i = [y_1, y_2, \ldots, y_N]$, the containment error satisfies

$$||\dot{y}_i + L_2\epsilon_{i1}|| \leq ||\epsilon_{i1}||/||\eta||$$

where $||\eta||$ is the minimum singular value of $L_2$.

Step 1: Combining the system (3) and (9), $\dot{\epsilon}_{i1}$ is calculated as

$$\dot{\epsilon}_{i1} = \sum_{j=1}^{N} a_{ij}(\dot{y}_i - \dot{y}_j) + \sum_{l=N+1}^{N+M} a_{il}(\dot{y}_i - \dot{y}_{l:d})$$

$$= -\sum_{j=1}^{N} a_{ij}\dot{x}_{ij} - \sum_{l=N+1}^{N+M} a_{il}\dot{y}_{l:d} + d_i\dot{x}_{i2}. $$ (14)

Consider the Lyapunov function candidate as follows:

$$V_{i1} = \frac{1}{2}(|\epsilon_{i1}| - \kappa_{i1})^2\bar{\varphi}_{i1}.$$ (15)

Combined with (11)-(14), the derivative of $V_{i1}$ is calculated as

$$\dot{V}_{i1} = (|\epsilon_{i1}| - \kappa_{i1})\bar{\varphi}_{i1}sg_{i1}(\epsilon_{i1})[d_1\dot{\epsilon}_{i2}$$

$$+ d_i\alpha_{i1} + g_{i1}(\chi_{i1})]$$

where $\chi_{i1} = [x_{i2}, \dot{y}_d]^T$ and

$$g_{i1}(\chi_{i1}) = -\sum_{l=N+1}^{N+M} a_{il}\dot{y}_{l:d} - \sum_{j=1}^{N} a_{ij}\dot{x}_{i2}.$$
The intermediate control signal $\alpha_{i1}^*$ is chosen as follows:

$$\alpha_{i1}^* = -\frac{1}{d_i}\left\{\left[\lambda_{i1} + \frac{d_i}{4}\right](\epsilon_{i1} - \kappa_{i1})sg_{i1}(\epsilon_{i1}) + d_i(\kappa_{i2} + 1)sg_{i1}(\epsilon_{i1}) + d_i g_{i1}(\chi_{i2})\right\}$$

(17)

where $\lambda_{i1} (\lambda_{i1} > 0)$ is constant.

Substituting (17) into (16), it yields that

$$\dot{\chi}_{i1} \leq d_i(\epsilon_{i1} - \kappa_{i1})\bar{\varphi}_{i1}(\epsilon_{i2} - \kappa_{i2} - 1) - (\lambda_{i1} + \frac{d_i}{4})(\epsilon_{i1} - \kappa_{i1})^2\bar{\varphi}_{i1}.$$  

(18)

**Step 2:** From (3) and (10), the time derivative of $\epsilon_{i2}$ is given

$$\dot{\epsilon}_{i2} = x_{i3} - \frac{N}{H}\sin(x_{i1}) - \frac{D}{H}x_{i2} + \frac{F_{id}}{H} - \sum_{j=0}^{N+M} A_{ij}^{(2)}y_{i(j+1)} - \sum_{j\in N_i} A_{ij}^{(1)}j_{j1}.$$  

(19)

Consider the following Lyapunov function candidate

$$V_{i2} = \frac{1}{2}(\epsilon_{i2} - \kappa_{i2})^2\varphi_{i2} + V_{i1}.$$  

(20)

Invoking (19), $\dot{V}_{i2}$ is calculated as

$$\dot{V}_{i2} = (\epsilon_{i2} - \kappa_{i2})\bar{\varphi}_{i2}g_{i2}(\epsilon_{i2})$$

$$\times \left[\epsilon_{i3} + \alpha_{i2}^* + g_{i2}(\chi_{i2})\right] + \dot{V}_{i1}.$$  

(21)

where

$$g_{i2}(\chi_{i2}) = -\frac{N}{H}\sin(x_{i1}) - \frac{D}{H}x_{i2} - \frac{\partial G_{i1}^{(2)}}{\partial x_{i2}}x_{i2} + \frac{F_{id}}{H} - \sum_{j=0}^{N+M} A_{ij}^{(2)}y_{i(j+1)} - \sum_{j\in N_i} A_{ij}^{(1)}j_{j1}.$$  

(22)

By using FLSs to approximate nonlinear functions, we can get

$$g_{i2}(\chi_{i2}) = W_{i2}^{T}\varphi_{i2}(\chi_{i2}) + g_{i2}(\chi_{i2})(\chi_{i2})$$  

(23)

where $\chi_{i2} = [x_{i1}, x_{i2}, x_{i1}^2, y_{i2}, y_{i2}^2, \ldots, y_{iN+M}, y_{iN+M}^2, \ldots]$  

Then, this paper lumps $\varpi_i = [W_{i2}^{T}, \theta_{i2}(\chi_{i2})]^T$ and $\vartheta_{i2}(\chi_{i2})(\xi = 2)$ into the following vectors:

$$\varpi_i = [W_{i2}^{T}, \ldots, W_{iM}^{T}, \theta_{i1}(\chi_{i1}), \ldots, \theta_{i2}(\chi_{i2}), \ldots, \theta_{iM}(\chi_{iM})]^T.$$  

(24)

$$\vartheta_{i2}(\chi_{i2}) = \frac{[0, \ldots, 0, \varphi_{i2}^{\xi}(\chi_{i2}), 0, \ldots, 0, 0, \ldots, 0]}{\xi - 1}.$$  

(25)

**Remark 3:** In fuzzy adaptive control, a large number of fuzzy rules need to be used for improving the ability of FLSs to approach nonlinear terms, which leads to the increase of computation burden. Even if the element $W_{ij}^{T}$ in (24) is a constant term, it will still increase the calculation burden during the recursive process. The main reason is that the element $W_{ij}^{T}$ needs to be estimated in real-time. Hence, we design a new fuzzy adaptive containment control method to solve this issue well, as shown in the following derivation.

The function $g_{i2}(\chi_{i2})$ is rewritten as

$$g_{i2}(\chi_{i2}) = W_{i2}^{T}\varphi_{i2}(\chi_{i2}) + g_{i2}(\chi_{i2}).$$  

(26)

Substituting (26) into (21), it yields that

$$\dot{V}_{i2} = (\epsilon_{i2} - \kappa_{i2})\bar{\varphi}_{i2}g_{i2}(\epsilon_{i2})$$

$$\times g_{i2}(\chi_{i2}) + \dot{V}_{i1}.$$  

(27)

where $\omega_i$ and $\mu_i$ are defined as follows

$$\omega_i = \sqrt{\max(\varpi_i^T \varpi_i) + \vartheta_i^T \vartheta_i}$$

$$\mu_i = g_{i2}(\epsilon_{i2})\sqrt{\vartheta_i^T \vartheta_i} + |\epsilon_{i2}| + |\epsilon_{i2}|^2\varphi_{i2}.$$  

(28)

**Remark 4:** As explained in Remark 2, although selecting a large number of fuzzy logic rules can improve the accuracy of FLSs to approximate the nonlinear term, it will increase the dimension of $\varpi_i$. In this paper, we lump all fuzzy weight vectors and fuzzy approximation errors together. Then the approximation errors are integrated into $\omega_i$, which not only reduces the computational complexity, but also avoids approximation errors as in existing literatures when dealing with nonlinear terms.

Design the intermediate control signal $\alpha_{i2}^*$ as

$$\alpha_{i2}^* = -(\lambda_{i2} + d_i + \frac{1}{4})(\epsilon_{i2} - \kappa_{i2})sg_{i2}(\epsilon_{i2}) - (\kappa_{i3} + 1)sg_{i2}(\epsilon_{i2}) - \varrho_i^2 \mu_{i2}$$  

(29)

where $\varrho_i$ is the estimate of $\omega_i$, $r_{i2}(\epsilon_{i2}) > 0$ and $\lambda_{i2}(\lambda_{i2} > 0)$ are constants. $\varrho_i$ is the turning function and designed as follows:

$$\varrho_i = \begin{cases} 
(\epsilon_{i2} - \kappa_{i2})\varphi_{i2}g_{i2}(\epsilon_{i2})/2 & |\epsilon_{i2}| \geq \kappa_{i2} \\
0 & |\epsilon_{i2}| < \kappa_{i2}.
\end{cases}$$  

Combining (28) with (21), one has

$$\dot{V}_{i2} \leq -\lambda_{i1}(\epsilon_{i1} - \kappa_{i1})^2\varphi_{i1} - (\lambda_{i2} + \frac{1}{4})(\epsilon_{i2} - \kappa_{i2})^2\varphi_{i2}$$

$$+ (\epsilon_{i2} - \kappa_{i2})\varphi_{i2}(\epsilon_{i2}) - \lambda_{i2}^2 + \varrho_i^2 \mu_{i2}^2$$  

(30)

where

$$\lambda_{i2} = d_i\left[ -\frac{1}{4}\left(\epsilon_{i1} - \kappa_{i1}\right)^2\varphi_{i1} - (\epsilon_{i2} - \kappa_{i2})^2\varphi_{i2} + (\epsilon_{i2} - \kappa_{i2})\varphi_{i2}(\epsilon_{i2}) - \lambda_{i2} - 1\right].$$

In particular, $\lambda_{i2} \leq 0$ is always satisfied. The detailed proof is shown below:

**Case 1:** If $|\epsilon_{i2}| \leq \kappa_{i2} + 1$, $(\epsilon_{i1} - \kappa_{i1})\varphi_{i1}(\epsilon_{i2} - \kappa_{i2} - 1) \leq 0$ and $\lambda_{i2} \leq 0$ can be obtained.
Case 2: If $|\epsilon_2| > \kappa_2 + 1$, we can get
\[
\dot{\lambda}_2 \leq d_1 \left[ -\frac{1}{4}(|\epsilon_1| - |\kappa_1|)^2 \dot{\varphi}_{i1} + \frac{1}{4}(|\epsilon_1| - |\kappa_1|)^2 \dot{\varphi}_{i1} - (|\epsilon_2| - \kappa_2)^2 \dot{\varphi}_{i2} + (|\epsilon_2| - \kappa_2 - 1)^2 \right] \leq 0.
\]

Remark 5: It should be noted that the adaptive control strategies [38]-[40] based on FLSs or NNS have some shortcomings, which can only guarantee $\dot{V} \leq -a_{i0}V + b_{i0}$, but cannot ensure $\dot{V} \leq 0$. That is because 1) FLSs or NNS will generate approximation errors in the approximation process of nonlinear functions, which are merged into $b_{i0}$, especially in backstepping. 2) When using Young's inequality, it may bring constant terms, which may have a negative impact on the control performance, and the time derivative of Lyapunov function is difficult to ensure the negative-definite property. To solve this problem, we construct the Lyapunov function (15), and design a new adaptive fuzzy control method based on the predefined accuracy function. Hence, $\dot{V} \leq 0$ can be directly obtained, and the negative influence of positive constants on system stability is eliminated.

Define the Lyapunov function as $V_h = (|\epsilon_h| - \kappa_h)^2 \dot{\varphi}_h / 2$. When $\epsilon_h = \kappa_h$ and $\epsilon_h = -\kappa_h$, we can get $V_h = 0$. For the differentiability of function $V_h$, this paper gives the following detailed explanation.

When $|\epsilon_h| < |\kappa_h|$, we have
\[
\lim_{\epsilon_h \to -\kappa_h} \frac{V_h(\epsilon_h) - V_h(-\kappa_h)}{\epsilon_h - (-\kappa_h)} = \lim_{\epsilon_h \to -\kappa_h} \frac{0 - 0}{\epsilon_h - (-\kappa_h)} = 0
\]
and
\[
\lim_{\epsilon_h \to -\kappa_h} \frac{V_h(\epsilon_h) - V_h(-\kappa_h)}{\epsilon_h - (-\kappa_h)} = \lim_{\epsilon_h \to -\kappa_h} \frac{0 - 0}{\epsilon_h - (-\kappa_h)} = 0.
\]

When $|\epsilon_h| > |\kappa_h|$, it can be obtained that
\[
\lim_{\epsilon_h \to \kappa_h} \frac{V_h(\epsilon_h) - V_h(\kappa_h)}{\epsilon_h - \kappa_h} = \lim_{\epsilon_h \to \kappa_h} \frac{\frac{1}{2}(|\epsilon_h| - \kappa_h)^2 - 0}{\epsilon_h - \kappa_h} = 0
\]
and
\[
\lim_{\epsilon_h \to \kappa_h} \frac{V_h(\epsilon_h) - V_h(\kappa_h)}{\epsilon_h - \kappa_h} = \lim_{\epsilon_h \to \kappa_h} \frac{\frac{1}{2}(|\epsilon_h| - \kappa_h)^2 - 0}{\epsilon_h - (-\kappa_h)} = 0.
\]

Then we can get
\[
\lim_{\epsilon_h \to \kappa_h} \frac{V_h(\epsilon_h) - V_h(\kappa_h)}{\epsilon_h - \kappa_h} = \lim_{\epsilon_h \to \kappa_h} \frac{V_h(\epsilon_h) - V_h(\kappa_h)}{\epsilon_h - (-\kappa_h)} = \lim_{\epsilon_h \to \kappa_h} \frac{V_h(\epsilon_h) - V_h(-\kappa_h)}{\epsilon_h - (-\kappa_h)}.
\]

Hence, the Lyapunov function $V_h$ is differentiable at two user-defined points $\epsilon_h = \kappa_h$ and $\epsilon_h = -\kappa_h$. We can further deduce that $V_h$ is continuous.

Step 3: Since $\epsilon_{i3} = x_{i3} - \alpha_{i3}^*$, it yields that
\[
\dot{\epsilon}_{i3} = \dot{x}_{i3} - \dot{\alpha}_{i3} = -\frac{K_B}{\lambda_j \hat{H}} x_{i2} - \frac{R}{\lambda_j} x_{i3} - \frac{\partial \alpha_{i3}^*}{\partial \hat{\omega}_i} \dot{\hat{\omega}}_i + \sum_{j=1}^{N_m} \gamma_{ij} \tilde{m}_{ij} (\tilde{x}_{i3}) u_{ij} - \frac{2}{\lambda_j} \dot{x}_{i3} \dot{\hat{\omega}}_i
\]
\[
\quad \dot{\hat{\omega}}_i = \sum_{j=0}^{N_M} \frac{\partial \alpha_{i3}^*}{\partial \hat{\psi}_{ij}} (j+1) y_{id} + \frac{2}{\lambda_j} \dot{x}_{i3} \dot{\hat{\omega}}_i \quad (31)
\]

Let
\[
g_{i3} (\chi_{i3}) = -\frac{K_B}{\lambda_j \hat{H}} x_{i2} - \frac{R}{\lambda_j} x_{i3} - \frac{\partial \alpha_{i3}^*}{\partial \hat{\psi}_{ij}} (j+1) y_{id} + \frac{2}{\lambda_j} \dot{x}_{i3} \dot{\hat{\omega}}_i \quad (32)
\]

The time derivative of $V_{i3}$ is computed as:
\[
\dot{V}_{i3} \leq \dot{V}_{i2} + (|\epsilon_{i3}| - |\kappa_{i3}|) \tilde{\varphi}_{i3} s_{i3} g_{i3} (\epsilon_{i3}) \left[ \alpha_{i3} - \alpha_{i3}^* \right] + \sum_{j=1}^{N_m} g_{ij} \tilde{m}_{ij} u_{ij} (\tilde{x}_{i3}) - \frac{\partial \alpha_{i3}^*}{\partial \hat{\omega}_i} \dot{\hat{\omega}}_i + g_{i3} (\chi_{i3}) \quad (33)
\]

where $\alpha_{i3}^*$ is the auxiliary signal. The auxiliary signal $\alpha_{i3}^*$ and control signal $u_{ipj}$ are chosen as follows:
\[
\alpha_{i3}^* = \left( \lambda_{i3} + 1 \right) (|\epsilon_{i3}| - |\kappa_{i3}|) s_{i3} g_{i3} (\epsilon_{i3}) - g_{i3} (\chi_{i3})
\]
\[
= \left( |\epsilon_{i3}| - |\kappa_{i3}| \right) s_{i3} g_{i3} (\epsilon_{i3}) \left( \frac{\partial \alpha_{i3}^*}{\partial \hat{\omega}_i} \right)^2 \quad (34)
\]

and
\[
\quad u_{ipj} = \text{sign}(g_{ij}) \tilde{L}_{i3} T \pi_i \quad (35)
\]

where $\pi_i = [\alpha_{i3}^*, \tilde{m}_{i1} (\tilde{x}_i), \ldots, \tilde{m}_{iM_m} (\tilde{x}_i)]^T$. $\lambda_{i3}$ is the designed positive parameter. $\bar{L}_i$ is denoted as the estimation of $L_i$, and
\[
\bar{L}_i = \left[ \begin{array}{c}
1
-\frac{g_{i3} u_{if} f_{i3}}{|g_{ij}| \bar{y}_{ij}}
-\frac{g_{i3} N_m u_{if} f_{i3}}{|g_{ij}| \bar{y}_{ij}}
\end{array} \right]^T \quad (36)
\]

In order to obtain the control objectives, the adaptive parameters are designed as
\[
\dot{\hat{\omega}}_i = \text{Proj}(\alpha_{i3} P_{i2}) \quad \hat{\omega}_i (0) \in \Pi_{\omega_i} \quad (37)
\]
\[
\dot{\bar{L}}_i = \text{Proj}(-\Lambda_{L_i} (|\epsilon_{i3}| - |\kappa_{i3}|) \tilde{\varphi}_{i3} s_{i3} g_{i3} (\epsilon_{i3}) \pi_i) \quad \bar{L}_i (0) \in \Pi_{\bar{L}_i} \quad (38)
\]

where $\alpha_i$ expresses a parameter, and $\Lambda_{L_i}$ is an asymmetric positive-definite matrix. $\Pi_{\omega_i}$ and $\Pi_{\bar{L}_i}$ are compact sets of $\hat{\omega}_i (0)$ and $\bar{L}_i (0)$, respectively. $\Pi_{\omega_i}$ is determined by the norm
of the optimal weight vector \( W \) and maximum approximation error \( g_{1M} \). \( \Pi_{L_1} \) is given as follows:

\[
\Pi_{L_1} = \left\{ \left[ \kappa_{01}, \kappa_{11}, \ldots, \kappa_{iN_M} \right]^T \in \mathbb{R}^{N_{m+1}} \mid j = 1, \ldots, N_m \right. \\
\frac{1}{N_m g_{1M}} \leq \kappa_{01} \leq \frac{1}{g_{01} \gamma_{10}} \quad \text{and} \quad |\kappa_{ij}| \leq \frac{g_{1M} \mu_{1fM}}{g_{01} \gamma_{10}} \right\}.
\]

Note that \((L_i, \tilde{L}_i) \in \Pi_{L_1}\). Then \(\|L_i - \tilde{L}_i\| \leq L_{1M}\), where

\[
L_{1M} = \left[ \left( \frac{1}{g_{01} \gamma_{10}} - \frac{1}{N_m g_{1M}} \right)^2 + \frac{4N_m g_{1M} \mu_{1fM}^2}{g_{01} \gamma_{10}} \right]^{\frac{1}{2}}. \quad (39)
\]

**Remark 6:** Based on the projection operator, an adaptive law \( \dot{\omega}_i = \text{Proj}(\alpha_i \rho_i) \) is designed. It always leads to the discontinuous term \((|\epsilon_3| - \kappa_3)\varphi_i g_{i3}(\epsilon_3)\frac{\partial \alpha_i}{\partial \omega_i} \dot{\omega}_i \) exists, which causes the virtual control discontinuous. This problem is effectively resolved in our work by skillfully introducing \(-r_{1i}^2 \rho_i \kappa_2 \) and \(-\kappa_3 g_{i3}(\epsilon_3)\frac{\partial \alpha_i}{\partial \omega_i}^2 \) to virtual controllers \( \alpha_i \). As a result, we can conclude that \( \alpha_2 \) and \( \alpha_3 \) are continuous.

### IV. Stability Analysis

Note that the actuator faults mode does not change in the time interval \([t_r, t_{r+1}])\). During the interval \([t_r, t_{r+1})\), we choose the following Lyapunov function:

\[
V = \sum_{i=1}^{N} \left\{ \sum_{j=1}^{N} \frac{g_{ij}^2}{2} \tilde{L}_i^T \Lambda_{L_i}^{-1} \tilde{L}_i + \frac{1}{2} (|\epsilon_3| - \kappa_3)^2 \varphi_i^2 - \frac{1}{2} \dot{\omega}_i^2 \right\} \quad (40)
\]

where \( \tilde{L}_i (\tilde{L}_i = L_i - \tilde{L}_i) \) expresses the estimation errors of the fault-related parameter \( L_i \). By using the control signal (35) and adaptive laws (37)-(38), \( \dot{V} \) is displayed as

\[
\dot{V} \leq \sum_{i=1}^{N} \left\{ \sum_{j=1}^{N} \lambda_{ij} (|\epsilon_3| - \kappa_3)^2 \varphi_i^2 + \frac{\dot{\omega}_i}{\alpha_i} (\alpha_i \rho_i - \dot{\omega}_i) \right. \\
- \sum_{j=1}^{N} \frac{g_{ij}^2}{2} \tilde{L}_i^T \Lambda_{L_i}^{-1} \Lambda_{L_i}^{-1} \tilde{L}_i + (|\epsilon_3| - \kappa_3)^2 \varphi_i^2 \right. \\
- \sum_{j=1}^{N} \frac{g_{ij}^2}{2} \tilde{L}_i^T \Lambda_{L_i}^{-1} \Lambda_{L_i}^{-1} \tilde{L}_i \left. \right\} \quad (41)
\]

where

\[
\Phi_i = - (|\epsilon_3| - \kappa_3)^2 \varphi_i^2 \frac{\partial \alpha_i^2}{\partial \omega_i}^2 \quad (42)
\]

and

\[
\Phi_1 = - (|\epsilon_3| - \kappa_3)^2 \varphi_i^2 \frac{\partial \alpha_i^2}{\partial \omega_i}^2 \\
- \alpha_i^2 (|\epsilon_3| - \kappa_3)^2 \varphi_i^2 \frac{\partial \alpha_i^2}{\partial \omega_i}^2 \quad (43)
\]

By analyzing (42) and (43), we can get \( \Phi_i \leq 0 \). Combining with the adaptive laws designed in (37) and (38), we can get

\[
\dot{V} \leq - \sum_{i=1}^{N} \sum_{j=1}^{N} \lambda_{ij} (|\epsilon_3| - \kappa_3)^2 \varphi_i^2 \quad (44)
\]

Therefore, the Lyapunov function \( V \) is a nonincreasing function within time interval \( [t_r, t_{r+1}) \) for \( \tau = 0, 1, \ldots \). At each fault moment, the part of sudden jump of Lyapunov function is described as

\[
V(t_{r+1}^+) - V(t_r^-) = \sum_{i=1}^{N} \left[ \sum_{j=1}^{N} \frac{g_{ij}^2}{2} \tilde{L}_i^T \Lambda_{L_i}^{-1} \tilde{L}_i \right] \quad (45)
\]

From (45), it can be concluded that the jumping size of \( V \) is determined by error variables \( \tilde{L}_i(t_{r+1}^+) \) and \( \tilde{L}_i(t_r^-) \). Combined with (39), (45) is rewritten as

\[
V(t_{r+1}^+) - V(t_r^-) \leq \sum_{i=1}^{N} \left[ \frac{g_{ij}^2}{2} \tilde{L}_i^T \Lambda_{L_i}^{-1} \tilde{L}_i \right] \quad (46)
\]

where \( \zeta_{\max} = \max(\Lambda_{L_i}^{-1}) \). By analyzing (46), the jumping size of \( \zeta_{\max} \) can decrease when \( \Lambda_{L_i}^{-1} \) continues to increase.

**Remark 7:** Since the coefficient \( \eta_{ij} \) at the next fault instant is unknown, in the worst case (the coefficient \( \eta_{ij} \) is increased randomly at the subsequent fault instant), the piecewise Lyapunov function will increase continuously rather than bounded. To overcome the above difficulty, a predefined accuracy function is introduced and a new fuzzy adaptive fault-tolerant containment control strategy is designed, which fully compensates impacts of the infinite number of actuator faults and also ensures the boundedness of containment errors.

**Remark 8:** According to (46), the jumping error of \( V \) is determined by error variables \( \tilde{L}_i(t_{r+1}^+) \) and \( \tilde{L}_i(t_r^-) \). Traditional methods can only ensure the stability of systems under a finite number of actuator faults. When the number of actuator faults is infinite, the Lyapunov function loses its boundability because the increase cannot be eliminated. Different from existing technologies, we use piecewise Lyapunov function analysis to find the time interval of the jumping. Then, based on properties of the projection operator, i.e., \( \text{Proj}^T(\rho) \Lambda_{\rho}^{-1} \text{Proj}(\rho) \leq \rho^T \Lambda_{\rho}^{-1} \rho \), we design adaptive laws \( \dot{L}_i \) and \( \dot{\omega}_i \) to update \( L_i \) and \( \omega_i \), respectively. Based on the above ideas, we develop a fault-tolerant adaptive control algorithm for the infinite number of actuator faults. This eliminates the growing part of Lyapunov function at each fault interval, and the desired control performance can be recovered in the case of infinite faults.
It is clear from (44) that \( V(t) \) is not increased in any time interval \([\tau, \tau + 1]\). Then, based on (46), we can get
\[
V(t_{m+1}) \leq V(t_m) \leq V(t_{m}) + R_M
\]
where \( M \) represents the total number of actuator failures occurring in the system during \( t \in [\tau, \infty) \). When \( \tau \rightarrow \infty \), the number of parameter jumps \( M \) may be infinite, which may not lead to the conclusion that the closed-loop system is stable. To establish system stability, the following two cases need to be considered in this paper:

(i) When the total number of actuator faults is finite in a time interval \( t \in (0, \infty) \), that is, \( M \) is bounded in time \( t \), the boundedness of the control system is established.

(ii) For an infinite number of actuator faults, the boundedness of \( V \) cannot be proved.

Therefore, how to solve case (ii) is a challenging problem. This can be well handled by the analysis method of piecewise Lyapunov functions used in this paper with the following detailed analysis.

**Definition 2:** The time interval \( T_{\min} \) for any two consecutive actuator faults is defined as
\[
T_{\min} = \min\{t_{\tau+1} - t_{\tau}\}, \quad \tau = 0, 1, \ldots
\]  

**Theorem 1:** Consider MMSs (3) and Assumption 1, the adaptive laws (37)-(38) and designed control signal (35), the designed fault-tolerant adaptive containment control algorithm can guarantee that all signals of MMSs are bounded, and the following results hold.

(i) All output signals of MMSs can converge within the preset range.

(ii) The \( L_2 \) norm transient performance satisfies
\[
\lim_{t \to \infty} ||\hat{x}_1||^2 \leq \frac{2R_M}{1 - e^{-a_0T_{\min}}}
\]
where \( \kappa_0 = 2b_0/a_0, \omega_0 = \min\{2\lambda_1, 1/\alpha, \omega_{\max}\} \), \( b_0 = 0.5\omega_0^2M/\alpha + 0.5N_mg_1M^2L_1^2 \), and \( R_M \) is given in (46).

To facilitate calculation, the auxiliary variables are redefined as
\[
\hat{\varsigma} = [\hat{s}_{11}, \hat{s}_{22}, \hat{s}_{33}]^T, \quad \varsigma_{ij} = (|e_{ij}| - \kappa_{ij})^2\phi_{ij}(i = 1, \ldots, N, j = 1, \ldots, 3).
\]

**Proof:** Based on projection adaptive laws (37) and (38), the error variables \( \hat{\varsigma} \) and \( \hat{L}_i \) are guaranteed to be bounded, i.e., \( ||\hat{\varsigma}|| \leq \omega_{\max} \) and \( ||\hat{L}_i|| \leq \omega_i \). Then, (41) is rewritten as
\[
\dot{V} \leq \sum_{i=1}^{N} \left[ -3 \sum_{j=1}^{N} \lambda_{ij} (|e_{ij}| - \kappa_{ij})^2\phi_{ij} - \frac{1}{2}\omega_i^2 \right]
- \sum_{j=1}^{N} \left[ \frac{g_{ij}}{2} \hat{\varsigma}_{ij} \right] \hat{L}_i \Lambda^{-1}_i \hat{L}_i + \frac{1}{2}N_mg_{ij}M^2L_1^2M
+ \frac{1}{2}N_mg_{ij}M^2L_1^2M
\]
\[
\leq -a_0V + b_0.
\]

By integrating (51) over time \([\tau, \tau + 1]\), we obtain
\[
\int_{\tau}^{\tau+1} e^{a_0t} \dot{V}(t) dt \leq -\int_{\tau}^{\tau+1} a_0e^{a_0t} V(t) dt + a_0e^{a_0t} b_0 dt.
\]

Further, we can get
\[
V(t_{\tau+1}) \leq e^{-a_0(t_{\tau+1} - \tau)} V(t_\tau) + \frac{b_0}{a_0} \left[ 1 - e^{-a_0(t_{\tau+1} - \tau)} \right].
\]

Substituting (46) into (53), \( V(t_{\tau+1}) \) can be further deduced as
\[
V(t_{\tau+1}) \leq e^{-a_0(t_{\tau+1} - \tau)} |V(t_\tau) + R_M| + \frac{b_0}{a_0} \left[ 1 - e^{-a_0(t_{\tau+1} - \tau)} \right].
\]

To facilitate analysis, we define a symbol \( Q_{ij}^2 = e^{-a_0(t_{\tau+1} - \tau)} \).

Therefore, by analyzing \( V(t_{\tau+1}) \) and \( V(t_\tau) \), (54) is further obtained by calculating
\[
V(t_{\tau+1}) \leq Q_{t_{\tau+1}}^{t_\tau} V(t_\tau) + Q_{t_{\tau+1}}^{t_\tau} R_M + \frac{b_0}{a_0} (1 - Q_{t_{\tau+1}}^{t_\tau+1})
\]
\[
\leq Q_{t_{\tau+1}}^{t_\tau} V(t_\tau) + \sum_{j=1}^{\tau} Q_{t_j}^{t_{\tau+1}} R_M + \frac{b_0}{a_0} (1 - Q_{t_{\tau+1}}^{t_\tau+1})
\]
\[
\leq Q_{t_{\tau+1}}^{t_\tau} V(t_\tau) + \sum_{j=1}^{\tau} Q_{t_j}^{t_{\tau+1}} R_M + \frac{b_0}{a_0}.
\]

From (48), it yields
\[
-(t_{\tau+1} - t_\tau) = [-[(t_{\tau+1} - t_\tau) + (t_\tau - t_{\tau-1}) + \cdots + (t_2 - t_1)]
\]
\[
\leq -\tau T_{\min}
\]
and
\[
\sum_{j=1}^{\tau} Q_{t_j}^{t_{\tau+1}} \leq \frac{1 - e^{-a_0\tau T_{\min}}}{1 - e^{-a_0T_{\min}}}.
\]

According to (57), when \( \tau \) is infinite and \( t \to \infty \), (55) can be further represented as
\[
\lim_{t \to \infty} V(t) \leq \frac{R_M}{1 - e^{-a_0T_{\min}}} + \frac{b_0}{a_0}.
\]
of MMSs are guaranteed to be bounded. In addition, due to $\zeta_{i1}/2 \leq V(t)$, we have
\[
\lim_{t \to \infty} \|\zeta_{i1}\|_2 \leq \sqrt{\kappa_{i0} + \frac{2\Re M}{1 - e^{-\alpha_{i0}t_{m}}}}.
\] (59)

Then, the proof has been clearly given.

**Corollary 1:** When a finite number of actuator faults occurs in MMSs, the designed fault-tolerant control algorithm can also ensure the stability of the system.

**Proof:** We have proved that $\dot{V} \leq -\sum_{i=1}^{N} \sum_{j=1}^{3} \lambda_{ij} (|\epsilon_{ij}| - \kappa_{ij})^2 \overline{\varphi}_{ij}$, and by employing the integral operation over $[t, t_{r+1}]$, we can get
\[
V(t_{r+1}) \leq V(t) - \sum_{j=1}^{3} \int_{t}^{t_{r+1}} \lambda_{ij} (|\epsilon_{ij}| - \kappa_{ij})^2 \overline{\varphi}_{ij} dt.
\] (60)

During $[t_{2\Re M}, \infty)$, the total number of actuator faults $\Re$ is finite, that is, there is no fault or the fault mode remains unchanged, so it yields that
\[
V(\infty) \leq V(t_{2\Re M}) - \sum_{j=1}^{3} \int_{t}^{t_{2\Re M}} \lambda_{ij} (|\epsilon_{ij}| - \kappa_{ij})^2 \overline{\varphi}_{ij} dt.
\] (61)

According to the previous definition $\zeta = [\zeta_{i1}, \zeta_{i2}, \zeta_{i3}]^T$, where $\zeta_{ij} = (|\epsilon_{ij}| - \kappa_{ij})^2 \overline{\varphi}_{ij}(i = 1, \ldots, N, j = 1, \ldots, 3)$, we can further get
\[
\int_{0}^{\infty} \zeta^T \zeta dt \leq \frac{V(0)}{\lambda_{i0}} - \frac{V(\infty)}{\lambda_{i0}} + \frac{\Re M}{\lambda_{i0}}.
\] (62)

where $\lambda_{i0} = \min\{\lambda_{ij}\}$. Reviewing the result of (58), we can directly get that $V(\infty)$ is bounded. According to the above analysis, for the case of finite number actuator faults i.e., $\Re$ is finite, we can obtain that $\zeta(t) \in L_2$ is bounded. Due to $\zeta \in L_\infty$, and by applying Barbalas lemma, it can be concluded that $\lim_{t \to \infty} \|\zeta_{i1}\| = 0$. Therefore, the containment error can converge to a user-defined range of intervals $\kappa_{i1}$, that is, $\lim_{t \to \infty} |\epsilon_{i1}| = \kappa_{i1}$.

**Theorem 2:** According to the proposed control scheme, the fault-tolerant control performance of $L_2$ norm transient is derived as:
\[
\|\zeta_{i1}\|_2 \leq \frac{1}{\sqrt{2\lambda_{i1}}} \left[ \frac{\widehat{\omega}_i^2(0)}{\omega_i} + \sum_{j=1}^{m} \left| \frac{g_{ij}}{L_i(0)} \right|^2 \zeta_{max} \hat{L}_i(0) \right]^{\frac{1}{2}} + \Re N \zeta_{max} g_{iM} L_i M.
\] (63)

where $\Re$ is the total number of faults in $t \in [0, \infty)$.

**Proof:** Based on (62), we can get
\[
\|\zeta_{i1}\|_2 \leq \left[ \frac{V(0) + \Re M}{\lambda_{i0}} \right]^{\frac{1}{2}}.
\] (64)

By applying the initialization algorithm in [41], this paper can set $\zeta_{ij}(0) = 0 (i = 1, \ldots, n)$. Then, one has
\[
V(0) = \sum_{i=1}^{N} \left[ \frac{1}{2} \omega_i^2(0) + \sum_{j=1}^{m} \left| \frac{g_{ij}}{L_i(0)} \right|^2 \zeta_{max} \hat{L}_i(0) \right].
\] (65)

By combining (64) and (65), we complete the proof in (63).

**Remark 9:** In view of the situation that the system suffers from an infinite number of actuator faults, both $V \leq 0$ and $\dot{V} \leq -a_0 V + b_0$ must be satisfied to ensure system stability. Between the two adjacent faults, the non-increasing property of Lyapunov function is first ensured, and then the increasing part of Lyapunov function eliminated by adjusting the control coefficient. Therefore, $V \leq 0$ at this time. When the actuator fault occurs, the Lyapunov function will jump, $V \leq 0$ is no longer satisfied, and the system stability cannot be guaranteed. Hence, we need to re-establish the stability of the system. In this paper, the adaptive laws $\hat{\omega}_i$ and $\hat{L}_i$ are designed to ensure that $\hat{\omega}_i$ and $\hat{L}_i$ are bounded, so that $V \leq -a_0 V + b_0$ can be satisfied at each fault time, where $b_0 = 0.5 \omega_i^2(0)/\omega_i + 0.5 N \zeta_{max} g_{iM} L_i^2 M$. In this way, the system stability is re-established at the moment of fault, and hence, both $V \leq 0$ and $V \leq -a_0 V + b_0$ hold simultaneously.

**Remark 10:** By analyzing the design method in this paper, we can improve the steady-state performance of the system by adjusting the parameter $\kappa_{ij}$. If $\kappa_{ij} = 0$, the containment error can be guaranteed to converge to 0. However, this situation will cause the smooth function $sg_{ij}(\cdot)$ to become a discontinuous function, which will lead to the control signal no longer continuous. In addition, $\lambda_{ij}$ and $r_{ij}$ are the gain coefficients of the intermediate control signal controller and control signal, respectively. When $\lambda_{ij}$ and $r_{ij}$ are selected larger, the tracking performance will be better. Hence a larger control input is needed, but this will cause damage to the motor with actual constraints. As a result, there is a trade-off between the control performance and the size of the control input. The parameters need to be selected according to the available changing rate of control resources. The range of state values for MMSs considered in this paper should be within $[0, 2\pi]$. To prevent excessive initial values from affecting the system stability, we gradually increase the initial value of the adaptive law from 0 until the system reaches stability.

V. SIMULATIONS

In this section, it is considered that MMSs are composed of four followers and two leaders as shown in Fig. 2 (a). The structural diagram and schematic diagram for each single-link manipulator system are represented in Fig. 2(b) and (c), respectively.

The trajectory equations of leaders are described as
\[
y_{1d} = 0.3 \sin(t) + 1.05
\]
\[
y_{2d} = 0.3 \sin(t) + 1.2.
\] (66)
We consider the actuator fault models (4), which are given as
\[ u_{i1} = \begin{cases} u_{i1}(t) & t \in [k\Delta T, (k + 1)\Delta T] \\ 0.2 & t \in ((k + 1)\Delta T, (k + 2)\Delta T] \end{cases} \]
\[ u_{i2} = \begin{cases} u_{i2}(t) & t \in [k\Delta T, (k + 1)\Delta T] \\ 0.5u_{i2} & t \in ((k + 1)\Delta T, (k + 2)\Delta T] \end{cases} \]
where \( k \in \mathbb{N} \) and \( \Delta T = 10s \). To reflect the authenticity in simulations, we set the coefficient of the actuator failure part as 0.5.

To approximate uncertain nonlinear functions of the system, the fuzzy membership functions are selected as
\[ \mu_{F_2}(\chi_{i2}) = \exp \left[ -\frac{(\chi_{i2} - \bar{\tau}_{s2})^T(\chi_{i2} - \bar{\tau}_{s2})}{8} \right] \]
with the basis functions
\[ \varphi_s(\chi_{i2}) = \frac{\mu_{F_2}(\chi_{i2})}{\sum_{s=0}^{2} \mu_{F_2}(\chi_{i2})} \]
where
\[ \chi_{i2} = [x_{i1}, x_{j1}, x_{i2}, x_{j2}, y_{5r}, y_{6r}, y_{5r}, y_{6r}, y_{5r}, y_{6r}]^T \]
\[ \bar{\tau}_{s2} = [-4 + 4s, \ldots, -4 + 4s]^T \quad s = 0, 1, 2. \]

The adjacency matrix is given by
\[ A = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \]

In this paper, the initial states of four followers are selected as \( x_{i1}(0) = x_{i2}(0) = x_{i3}(0) = x_{i4}(0) = 0.5 \), and \( \omega_1(0) = \omega_2(0) = \omega_3(0) = \omega_4(0) = 0.5 \). The initial values of the adaptive parameters are chosen as \( \hat{L}_i(0) = [0.5, 0.5, 0.5](i = 1, 2, 3, 4) \). The parameter is selected as \( \Delta L_i = \text{diag}(1.0, 1.0, 1.0, 1.0)(i = 1, 2, 3, 4) \), and external disturbance as \( F_{d2} = 0.001\sin(x_{i1}) \). To achieve control objectives, the parameters related to the control algorithm are shown in TABLE I. The relevant symbols and parameters of MMSs are shown in TABLE II.

Remark 11: Fig. 3 illustrates the control performance of two different algorithms. \( s_{i1} \) is the containment error under the prescribed performance control (PPC) method [24], and \( \epsilon_{i1} \) for designed control method of our work. The existing PPC algorithms can ensure that the containment error converges to a prescribed range, but ignores the chattering phenomenon of containment error in steady state. This paper reduces the chattering size by defining the convergence accuracy of the error. Therefore, the steady-state performance of the system is improved.

The simulation results are shown in Figs. 4-9. Fig. 4 shows that all followers can converge to the convex hull formed by the leaders. Figs. 5 and 6 display adaptive parameters \( \hat{\omega}_i(i = 1, 2, \ldots, 4) \), and the containment error of MMSs converging to a prescribed interval \( \kappa_{ij}(i = 1, 2, \ldots, 4, j = 1, 2, 3) \). In Fig. 7, \( s_{i1} \) represents the containment error in the algorithm without compensating for actuator faults, and \( \epsilon_{i1} \) represents the containment error in our designed algorithm. Obviously, under the same parameters and initial states, the designed control

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**Fig. 2.** The containment control principle framework of the MMSs.

**Fig. 3.** The synchronization errors of \( s_{i1} \) and \( \epsilon_{i1} \) (\( i = 1, 2, 3, 4 \)).

**Fig. 4.** Trajectories of the followers \( x_{i1} \) and leaders \( y_{6d}, y_{6d} \).
algorithm of our work can reduce the chattering size of the system error and improve the steady-state performance. Fig. 8 shows the adaptive laws $\hat{L}_{ij}(i = 1, 2, 3, 4, j = 1, 2, 3)$ for compensating actuator faults, and Fig. 9 shows the control signal that undergoes actuator faults.

Fig. 5. Parameter estimate $\hat{\omega}_i$ ($i = 1, 2, 3, 4$).

Fig. 6. Trajectories of the containment error $\epsilon_i$.

Fig. 7. Trajectories of the containment error $s_{i1}$ and $\epsilon_{i1}$.

Fig. 8. Parameter estimate $\hat{L}_{ij}$ ($i = 1, 2, 3, 4, j = 1, 2, 3$).

Fig. 9. Performance of controllers with the infinite number of actuator faults with $T = 50s$.

The simulation results of MMSs show that the designed algorithm can guarantee the expected control performance even if there are infinite faults in the actuators. As shown in Corollary 1, it is proved that the system control performance can also be guaranteed when the actuator suffers a finite number of faults.

<table>
<thead>
<tr>
<th>TABLE I. Parameters of the fault-tolerant adaptive control method.</th>
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<tbody>
<tr>
<td>$\lambda_{11} = 5.5$</td>
</tr>
<tr>
<td>$\lambda_{12} = 5$</td>
</tr>
<tr>
<td>$\lambda_{13} = 5$</td>
</tr>
<tr>
<td>$\kappa_{11} = 0.60$</td>
</tr>
<tr>
<td>$\kappa_{12} = 0.60$</td>
</tr>
<tr>
<td>$\kappa_{13} = 0.60$</td>
</tr>
<tr>
<td>$o_1 = 0.05$</td>
</tr>
<tr>
<td>$r_{11} = 6.5$</td>
</tr>
<tr>
<td>$r_{12} = 6.5$</td>
</tr>
<tr>
<td>$r_{13} = 6.5$</td>
</tr>
<tr>
<td>$\ell_{11} = 0.01$</td>
</tr>
<tr>
<td>$\ell_{12} = 0.01$</td>
</tr>
<tr>
<td>$\ell_{13} = 0.01$</td>
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<tr>
<th>TABLE II. Notations in the manipulator system.</th>
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<tbody>
<tr>
<td>Notation</td>
</tr>
<tr>
<td>rotor inertia</td>
</tr>
<tr>
<td>mass</td>
</tr>
<tr>
<td>load mass</td>
</tr>
<tr>
<td>link length</td>
</tr>
<tr>
<td>radius of the load</td>
</tr>
<tr>
<td>gravity coefficient</td>
</tr>
<tr>
<td>coefficient of viscous friction at the joint</td>
</tr>
<tr>
<td>armature inductance</td>
</tr>
<tr>
<td>protective resistance</td>
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<tr>
<td>torque coefficient</td>
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<td>back-emf coefficient</td>
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VI. CONCLUSIONS

For uncertain nonlinear multi-manipulator systems with actuator faults, the fault-tolerant adaptive fuzzy containment control scheme has been designed, where the nonlinear function of the system has been handled by using the characteristic of fuzzy logic systems approximating the nonlinear term. Then through employing a class of turning functions, a fault-tolerant controller has been constructed compensating for actuator faults. In addition, the designed control method by the predefined accuracy function and new Lyapunov function avoids the influence of some positive terms generated by the Young’s inequality and FLSs on the control algorithm design. This ensures the boundedness of all signals of the system. In future work for practical applications, we will consider that boundary of the actuator fault coefficient is an unknown constant, and design corresponding control algorithms based on fixed-time stability theory for $n$-link flexible-joint robots.

REFERENCES


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