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Fuzzy SMC for discrete nonlinear singularly perturbed models with semi-Markovian switching parameters

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***Abstract*—This paper investigates the sliding mode control (SMC) issue for discrete nonlinear semi-Markovian switching singularly perturbed models with incomplete semi-Markovian kernels (SMKs). A T-S fuzzy model is adopted to characterize the underlying model with singularly perturbed parameters, which is more widely used in nonlinear systems. Because the statistical characteristic of a semi-Markovian kernel is laborious to be obtained in practical engineering, the information of semi-Markov kernel is recognized to be incomplete. The main novelty lies in constructing a suitable fuzzy SMC scheme to realize the attainment of the quasi-sliding mode, overcoming the difficulty caused by incomplete SMK. Based on the Lyapunov function related to the fuzzy rule and the elapsed time of current mode, the stability criterion is constructed for the corresponding model. An appropriate SMC approach is then developed to achieve the attainment of a quasi-sliding mode. Finally, the theoretical results are verified by an improved tunnel diode circuit model.**

***Index Terms*—semi-Markovian switching, singularly perturbed models, T-S fuzzy model, semi-Markovian kernel.**

I. Introduction

In many practical physical plants, such as circuit systems and mechanical industry systems, some parasitic parameters lead to multiple time-scales phenomena. Some traditional control design approaches cannot be applied to these complex systems. Fortunately, singularly perturbed systems (SPSs) provide a unified framework for modeling practical systems subject to multiple time-scales phenomena. Consequently, the separation of different time scales in SPSs can be handled by eliminating the singularly perturbed parameter (SPP) or using the upper bound of the SPP, which can reduce the stiffness phenomenon caused by the parasitic parameter [1]. Owing to the modeling advantages, the research on SPSs has attracted much attention [2]–[6]. For SPSs with disturbance rejection [5], a novel composite hierarchical anti-disturbance control has been adopted to compensate the disturbance effectively. The issue of dynamic event-triggered control has been investigated for SPSs in the presence of time delay and sensor saturation [6]. At present, the research on discrete-time slow sampling SPSs is not as extensive as that on discrete-time fast sampling SPSs. However, in actual industrial processes, there are not only fast states such as electromagnetic force, but also slow states such as voltage. Therefore, it is of great importance to design a slow state feedback controller for the SPSs.

It is noted that for SPSs, abrupt changes in parameters and structures are inevitable [7]. For example, some components and faults exist in the circuit systems widely. Markovian switching systems (MSSs) show powerful ability to describe practical systems suffering from sudden alterations [7]–[9]. Although there exist extremely substantial results for MSSs, one of the main drawbacks is that the sojourn time (ST) follows the geometric or exponential distributions with memoryless characteristics. To address this issue, semi-Markovian switching systems (S-MSSs) are adopted to model complex stochastic switching systems, where the ST follows more general probability distribution with memory characteristics. Until now, S-MSSs have received increasing attention, such as stability [10], [11], event-triggered control [12]–[14], output feedback [15], and resilient control [16].

In recent years, the SMK has been introduced to solve the discrete S-MSSs, synthesizing the underlying information about the next and current system modes and the ST [17]–[21]. However, obtaining all the ST-probability density functions (ST-PDFs) and transition probabilities (TPs) is difficult in real practice. For example, the different airspeeds of vertical take-off landing helicopters [22] are described by a semi-Markovian chain, but only some of the TPs and ST-PDFs are convenient to measure. Actually, it is critically important to solving this issue in the light that the designed controller of TPs and ST-PDFs may lead to failure in stabilizing the underlying S-MSSs. The above analysis shows that an incomplete SMK has broader prospects in academia and industry [23]–[28].

In general, many physical systems exhibit nonlinear characteristics. As a result, it is essential to investigate the control problems for nonlinear SPSs. The Takagi-Sugeno (T-S) fuzzy model is recognized as an effective strategy to express complex

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nonlinear systems, which can represent a nonlinear system by describing a set of linear subsystems weighted by the membership functions [29]. In fact, the T-S fuzzy model, formed by a set of IF-THEN rules, establishes the connection between linear and nonlinear systems [30]–[33]. Combining with stochastic switching systems, the corresponding T-S fuzzy S-MSSs have comprehensive utilization, such as stability and stabilization [17], [25], quantization [18], nonfragile control [21], fault-tolerant control [27], and fault detection [34].

Due to sliding mode control (SMC) being a robust control strategy, it is insensitive to external and parametric disturbances with good transient performance and fast response. Very recently, the SMC technology has been widely applied in telerobotic systems [35], permanent magnet synchronous motors [36], unmanned aerial vehicles [37], underactuated surface vehicles [38], and motor driving systems [39]. Further, many significant achievements have already emerged [40], [41]. For example, an adaptive sliding mode controller has been established for MSSs with parameter uncertainties [40].

Furthermore, under the traditional feedback controller for discrete-time fuzzy semi-Markovian switching SPSs [18], [21], the response speed of the system state displays slow characteristics, which may not meet fast-response requirements in practical applications. Next, the Lyapunov function constructed in [17], [19], [28] has no relationship with the elapsed time in the current mode, showing some conservatism in the stability analysis. Under the framework of discrete-time nonlinear semi-Markovian switching SPSs, how to design a suitable SMC law to guarantee the attainability of quasi-sliding mode is an urgent problem to be solved. As such, it is interesting and practical to construct a novel SMC law to realize the fast response of fuzzy semi-Markovian switching slow sampling SPSs. Until now, there has been no relevant theoretical result that specifically promotes this work.

Motivated by these, the aim is to investigate the fuzzy SMC problem for discrete nonlinear semi-Markovian switching SPSs with incomplete SMKs. The main contributions are: (i) Compared with traditional feedback control for discrete fuzzy semi-Markovian switching SPSs [18], [21], it is the first time to apply the SMC strategy to fuzzy semi-Markovian switching SPSs with incomplete SMK. (ii) Different from the traditional Lyapunov function [17], [19], [28], a new Lyapunov function is developed to be associated with the fuzzy rule and the elapsed time of the current mode, reducing some conservatism in the stability analysis of the underlying system. (iii) Based on the fuzzy rule and SPP, an appropriate SMC rule is designed to verify the accessibility of the quasi-sliding mode.

Nomenclature: $\mathbb{R}_{\geq 0}$ represents the set of real matrices. $\mathbb{C}_{\geq T_1}$ and $\mathbb{C}_{[T_1,T_2]}$ represents the set $\{\jmath \in \mathbb{C} \mid \jmath \geq T_1\}$ and $\{\jmath \in \mathbb{C} \mid T_1 \leq \jmath \leq T_2\}$. $\otimes$ signifies the Kronecker product of matrices. $\mathrm{Sym}(U)$ indicates $U + U^\top$. $\mathbf{E}\{\cdot\}$ denotes the expectation.

II. Problem Formulation

Consider the fuzzy semi-Markovian switching SPSs as

Plant Rule p: IF $k_1(l)$ is K_{p1}, $k_2(l)$ is K_{p2}, and ... and $k_\gamma(l)$ is $K_{p\gamma}$, THEN

$$\sigma(l+1) = (A_{p,\mathbf{i}(l)} + \Delta A_{p,\mathbf{i}(l)})E_\varepsilon \sigma(l) + B_{\mathbf{i}(l)}u(l), \qquad (1)$$

where $\sigma(l) = \begin{bmatrix} \sigma_1^{\mathrm{T}}(l) & \sigma_2^{\mathrm{T}}(l) \end{bmatrix}^{\mathrm{T}}$ with slow state $\sigma_1(l) \in \mathbb{R}^{n1}$ and fast state $\sigma_2(l) \in \mathbb{R}^{n2}$. $u(l) \in \mathbb{R}^m$ indicates the input and $E_\varepsilon = \mathrm{diag}\{ I_1 \quad \varepsilon I_2 \}$ and $\varepsilon > 0$ represents the SPP. $\{\mathbf{i}(l)\}_{l \in \mathbb{C}_{>0}}$ denotes a semi-Markovian chain in $\mathbb{L} \triangleq \{1, 2, \ldots, \alpha\}$ governed by the SMK $\Gamma(\wp) \triangleq [\rho_{\mu\tau}(\wp)]$, for μ, $\tau \in \mathbb{L}$ with

$$\begin{aligned}
&\rho_{\mu\tau}(\wp) \triangleq \Pr(\mathfrak{S}_{\beta+1} = \tau, S_{\beta+1} = \wp \mid \mathfrak{S}_\beta = \mu) \\
&= \frac{\Pr(\mathfrak{S}_{\beta+1} = \tau, \mathfrak{S}_\beta = \mu)}{\Pr(\mathfrak{S}_\beta = \mu)} \frac{\Pr(\mathfrak{S}_{\beta+1} = \tau, S_{\beta+1} = \wp, \mathfrak{S}_\beta = \mu)}{\Pr(\mathfrak{S}_{\beta+1} = \tau, \mathfrak{S}_\beta = \mu)} \\
&= \eta_{\mu\tau}\xi_{\mu\tau}(\wp),
\end{aligned}$$

where the mode index at the βth switching is represented as $\mathfrak{S}_\beta$, S_β denotes the ST between the $(\beta-1)$th and βth jumps.

The SMK is considered to be with $\rho_{\mu\mu}(\wp) = 0$, $0 \leq \rho_{\mu\tau}(\wp) \leq 1$, $\forall \mu, \tau \in \mathbb{L}(\mu \neq \tau)$, $\forall \wp \in \mathbb{C}_{\geq 1}$, $\eta_{\mu\tau} \triangleq \frac{\Pr(\mathfrak{S}_{\beta+1}=\tau, \mathfrak{S}_\beta=\mu)}{\Pr(\mathfrak{S}_\beta=\mu)}$ is the TP satisfying $\eta_{\mu\mu} = 0$ with $0 \leq \eta_{\mu\tau} \leq 1$, $\forall \mu, \tau \in \mathbb{L}(\mu \neq \tau)$, and $\xi_{\mu\tau}(\wp) \triangleq \frac{\Pr(\mathfrak{S}_{\beta+1}=\tau, S_{\beta+1}=\wp, \mathfrak{S}_\beta=\mu)}{\Pr(\mathfrak{S}_{\beta+1}=\tau, \mathfrak{S}_\beta=\mu)}$ is the ST-PDF with $\xi_{\mu\mu}(\wp) = 0$, $\forall \mu \in \mathbb{L}$, $\forall \wp \in \mathbb{C}_{\geq 1}$. $\mathbb{I}_\mu \triangleq \{1, 2, \ldots, \kappa_\mu\}$ denotes the set of indices for fuzzy rules, where κ_μ represents the number of IF-THEN rules of μth system mode for $\mathbf{i}(l) = \mu \in \mathbb{L}$. The premise variables are denoted by $k_1(l)$, $k_2(l)$, ..., and $k_\gamma(l)$, and the corresponding fuzzy sets are represented by $K_{p1}, K_{p2}, \ldots, K_{p\gamma}$, where γ means the number of the premise variables. The matrices $A_{p,\mu}$ and B_μ are with appropriate dimension, while $\Delta A_{p,\mu}$ represents the parameter uncertainty. Specifically, $\Delta A_{p,\mu} = F_{p,\mu}D_{p,\mu}H_{p,\mu}$, where $F_{p,\mu}$ and $H_{p,\mu}$ are known matrices and $D_{p,\mu}$ satisfies $D_{p,\mu}^\top D_{p,\mu} \leq I$.

Define the membership function of pth fuzzy rule as

$$h_p(\psi(l)) \triangleq \prod_{\eta=1}^{\gamma} \mathcal{K}_p^{(\eta)}(k_\eta(l)) / \sum_{p=1}^{\kappa_\mu} \prod_{\eta=1}^{\gamma} \mathcal{K}_p^{(\eta)}(k_\eta(l)),$$

where $\mathcal{K}_p^{(\eta)}(k_\eta(l))$ is the membership grade of $k_\eta(l)$ in $\mathcal{K}_{p\eta}$ with $\psi(l) \triangleq [k_1(l)\ k_2(l) \cdots k_\gamma(l)]$, $h_p(\psi(l)) \geq 0$, and $\sum_{p=1}^{\kappa_\mu} h_p(\psi(l)) = 1$.

Based on Refs. [25,29], with the utilization of a center-average defuzzifier, productfuzzy inference, and a singleton fuzzifier, one has following overall system according to (1),

$$\begin{aligned}
\sigma(l+1) = \sum_{p=1}^{\kappa_\mu} h_p(\psi(l))[(A_{p,\mu} + \Delta A_{p,\mu})E_\varepsilon \sigma(l) \\
+ B_\mu u(l)], \qquad (2)
\end{aligned}$$

where $A_{p,\mu} = \begin{bmatrix} A_{11p,\mu} & A_{12p,\mu} \\ A_{21p,\mu} & A_{22p,\mu)} \end{bmatrix}$, $B_\mu = \begin{bmatrix} B_{11\mu} \\ B_{21\mu} \end{bmatrix}$.

Remark 1: Obviously, the TP satisfies $\sum_{\tau \in \mathbb{L}, \tau \neq \mu} \eta_{\mu\tau} = 1$ and $\eta_{\mu\mu} = 0, \forall \mu \in \mathbb{L}$. Moreover, the SMK satisfies $\sum_{\wp=1}^{\infty} \sum_{\tau \in \mathbb{L}} \rho_{\mu\tau}(\wp) = 1$, $\forall \mu$, $\tau \in \mathbb{L}$, $\forall \wp \in \mathbb{C}_{\geq 1}$.

In reality, to obtain comprehensive knowledge about SMK is unattainable. Therefore, this paper considers the incomplete SMK. For all $\mu \in \mathbb{L}, \wp \in \mathbb{C}_{\geq 1}$, define

$$\begin{aligned}
&\mathbb{L}_\mu^{\mathcal{K}} \triangleq \{\mu \in \mathbb{L} \mid \text{if } \rho_{\mu\tau}(\wp) \text{ is known}\}, \\
&\mathbb{L}_\mu^{\mathcal{UK}} \triangleq \{\mu \in \mathbb{L} \mid \text{if } \rho_{\mu\tau}(\wp) \text{ is unknown}\},
\end{aligned}$$

$$\mathbb{L}_{\mu}^{\mathcal{UK}1} \triangleq \{\mu \in \mathbb{L} \mid \text{if } \eta_{\mu\tau} \text{ is unknown, } \xi_{\mu\tau}(\wp) \text{ is known}\},$$
$$\mathbb{L}_{\mu}^{\mathcal{UK}2} \triangleq \{\mu \in \mathbb{L} \mid \text{if } \xi_{\mu\tau}(\wp) \text{ is unknown, } \eta_{\mu\tau} \text{ is known}\},$$
$$\mathbb{L}_{\mu}^{\mathcal{UK}3} \triangleq \{\mu \in \mathbb{L} \mid \text{if } \xi_{\mu\tau}(\wp) \text{ and } \eta_{\mu\tau} \text{ is unknown}\},$$

where $\mathbb{L}_{\mu}^{\mathcal{K}} \triangleq \{\vec{m}_\mu(1), \vec{m}_\mu(2), \ldots, \vec{m}_\mu(\mathcal{M})\}$ with $1 \leq \vec{m}_\mu(1) \leq \vec{m}_\mu(2) \leq \cdots \leq \vec{m}_\mu(\mathcal{M}) \leq \alpha$, and the variable $\mathcal{M}$ represents the overall quantity of elements in $\mathbb{L}_{\mu}^{\mathcal{K}}$. $\vec{m}_\mu(\beta)$ indicates the $\vec{m}_\mu(\beta)$th element of $\mathbb{L}$, which is also recognized as the βth know element of $\mathbb{L}_{\mu}^{\mathcal{K}}$.

Remark 2: From the above definitions with $\rho_{\mu\tau}(\wp) = \eta_{\mu\tau}\xi_{\mu\tau}(\wp)$, it is clear that $\mathbb{L} = \mathbb{L}_{\mu}^{\mathcal{K}} \cup \mathbb{L}_{\mu}^{\mathcal{UK}}, \mathbb{L}_{\mu}^{\mathcal{UK}} = \mathbb{L}_{\mu}^{\mathcal{UK}1} \cup \mathbb{L}_{\mu}^{\mathcal{UK}2} \cup \mathbb{L}_{\mu}^{\mathcal{UK}3}$, $\mathbb{L}_{\mu}^{\mathcal{K}} \cap \mathbb{L}_{\mu}^{\mathcal{UK}} = \emptyset$, $\mathbb{L}_{\mu}^{\mathcal{UK}1} \cap \mathbb{L}_{\mu}^{\mathcal{UK}2} = \emptyset$, $\mathbb{L}_{\mu}^{\mathcal{UK}1} \cap \mathbb{L}_{\mu}^{\mathcal{UK}3} = \emptyset$, $\mathbb{L}_{\mu}^{\mathcal{UK}2} \cap \mathbb{L}_{\mu}^{\mathcal{UK}3} = \emptyset$. When $\mathbb{L}_{\mu}^{\mathcal{UK}} = \emptyset$, $\mathbb{L}_{\mu}^{\mathcal{K}} = \mathbb{L}$. In this paper, we only consider $\mathbb{L}_{\mu}^{\mathcal{K}} \cup \mathbb{L}_{\mu}^{\mathcal{UK}}$, which means that there is some partly knowledge about the information of $\xi_{\mu\tau}(\wp)$ and $\eta_{\mu\tau}$.

Assumption 1: For given constant θ with $\theta \in (0,1)$, the upper bound $T_{max}^{\mu} \in \mathbb{C}_{\geq 1}$, $\mu \in \mathbb{L}$ of ST satisfies $\sum_{\wp=1}^{T_{max}^{\mu}} \xi_{\mu\tau}(\wp) \geq \theta$, $\forall \mu \in \mathbb{L}, \forall \tau \in \mathbb{L}_{\mu}^{\mathcal{UK}1} \cup \mathbb{L}_{\mu}^{\mathcal{UK}2} \cup \mathbb{L}_{\mu}^{\mathcal{UK}3}$.

Definition 1: [23] For the upper bound $T_{max}^{\mu} \in \mathbb{C}_{\geq 1}$, $\mu \in \mathbb{L}$ for μth system of ST, and the initial conditions $\sigma(0)$, $\mathbf{i}(0)$, then system (2) is said to be mean-square stable (MSS), if $\lim_{t\to\infty} \mathbf{E}\{\|\sigma(l)\|^2\}|_{\sigma(0),\mathbf{i}(0)} = 0$.

Lemma 1: [21] For given scalar $\bar{\varepsilon} > 0$ and matrices Ξ_1, Ξ_2, Ξ_3, if $\Xi_1 \geqslant 0$, $\Xi_3 < 0$, $\bar{\varepsilon}^2\Xi_1 + \bar{\varepsilon}\Xi_2 + \Xi_3 < 0$, then for $\forall \varepsilon \in [0, \bar{\varepsilon}]$, $\varepsilon^2\Xi_1 + \varepsilon\Xi_2 + \Xi_3 < 0$.

Lemma 2: [23] For switching systems $\sigma(l+1) = g(\sigma(l), \mathbf{i}(l))$ with bounded ST, the switching points are denoted by $l_0, l_1, \ldots, l_q, \ldots$ with $l_0 = 0$. For given parameter $\iota_{\mathbf{i}(l_q)} > 0$, $\mathbf{i}(l_q) \in \mathbb{L}$, and initial conditions $\sigma(0)$, $\mathbf{i}(0)$, the underlying system is MSS, if we find a Lyapunov function $\mathcal{P}(\sigma(l), \mathbf{i}(l), o(l))$: $\mathbb{R}^{n_x} \to \mathbb{R}$, $l \in \mathbb{C}_{[l_q, l_{q+1})}$, $o(l) \in [1, T_{max}^{\mu}]$ and three class $\mathcal{K}_\infty$ functions $\varsigma_1, \varsigma_2, \varsigma_3$, such that

$$\varsigma_1(\|\sigma(l)\|) \leq \mathcal{P}(\sigma(l), \mathbf{i}(l), o(l)) \leq \varsigma_2(\|\sigma(l)\|), \tag{3}$$
$$\mathcal{P}(\sigma(l), \mathbf{i}(l), o(l)) \leq \iota_{\mathbf{i}(l_q)}\mathcal{P}(\sigma(l_q), \mathbf{i}(l_q), o(l_q)), \quad \forall l \in \mathbb{C}_{[l_q, l_{q+1})}, \tag{4}$$
$$\mathbf{E}\{\mathcal{P}(\sigma(l_{q+1}), \mathbf{i}(l_{q+1}), 1)\}|_{\sigma(l_q),\mathbf{i}(l_q)} - \mathcal{P}(\sigma(l_q), \mathbf{i}(l_q), 1) \leq -\varsigma_3(\|\sigma(l_q)\|). \tag{5}$$

Remark 3: From practical aspect, the energy function serves as a natural alternative for the Lyapunov function. Furthermore, the Lyapunov function is recognized as a potent instrument for theoretical analysis of system stability. It's also important to note that the Lyapunov function is associated with system mode and the elapsed time of current mode, reducing some conservatism.

III. MEAN-SQUARE STABILITY ANALYSIS

Select the sliding function as

$$\Re(l) = G_\mu\sigma(l), \tag{6}$$

where $\Re(l) \in \mathbb{R}^m$, and the matrix $G_\mu \in \mathbb{R}^{n1+n2}$ is designed to ensure that $G_\mu B_\mu$ is nonsingular.

By utilizing the sliding function (6) and system (2), it can be inferred that

$$\Re(l+1) = G_\mu\sigma(l+1) = G_\mu \sum_{p=1}^{\kappa_\mu} h_p(\psi(l))[(A_{p,\mu} + \Delta A_{p,\mu}) E_\varepsilon\sigma(l) + B_\mu u(l)]. \tag{7}$$

This following equation is obtained when the state is driven onto an ideal quasi-sliding mode

$$\Re(l+1) = \Re(l) = 0. \tag{8}$$

As a result, an equivalent fuzzy sliding law is

$$u_{eq}(l) = -(G_\mu B_\mu)^{-1}G_\mu[\sum_{p=1}^{\kappa_\mu} h_p(\psi(l))(A_{p,\mu} + \Delta A_{p,\mu})E_\varepsilon\sigma(l)]. \tag{9}$$

Combing (2) with (9) yields

$$\sigma(l+1) = \bar{\mathcal{A}}_\mu\sigma(l), \tag{10}$$

where $\bar{\mathcal{A}}_\mu \triangleq \sum_{p=1}^{\kappa_\mu} h_{p,l}\mathcal{A}_{p,\mu} \triangleq \sum_{p=1}^{\kappa_\mu} h_{p,l}\hat{A}_{p,\mu}E_\varepsilon \triangleq \sum_{p=1}^{\kappa_\mu} h_{p,l}(I - B_\mu(G_\mu B_\mu)^{-1}G_\mu)(A_{p,\mu} + \Delta A_{p,\mu})E_\varepsilon$. When $\mathbf{i}(l) = \mu$, $h_{p,l}$ is used to represent $h_p(\psi(l))$.

Under incomplete SMK, the subsequent analysis focuses on the mean-square stability of system (10).

Theorem 1: For given scalars $\iota_\mu > 0, T_{max}^{\mu} \in \mathbb{C}_{\geq 1}, \forall \mu \in \mathbb{L}$, $\tau \in \mathbb{L}_{\mu}^{\mathcal{UK}}$, system (10) is MSS, if we find matrices $Q_\mu(g) > 0$, $\mathcal{Q}_\mu(\wp) > 0$, $\forall g \in \mathbb{C}_{[1, T_{max}^{\mu}]}$, such that $\forall p_1, p_2, \cdots, p_g \in \mathbb{I}_\mu$, $\forall g \in \mathbb{C}_{[1, T_{max}^{\mu}-1]}$ and $\forall \wp \in \mathbb{C}_{[1, T_{max}^{\mu}]}$,

$$(\prod_{\vartheta=1}^{g} \mathcal{A}_{p_\vartheta,\mu})^{\mathrm{T}} Q_\mu(g+1) \prod_{\vartheta=1}^{g} \mathcal{A}_{p_\vartheta,\mu} - \iota_\mu Q_\mu(1) < 0, \tag{11}$$

$$\sum_{\wp=1}^{T_{max}^{\mu}} (\prod_{\vartheta=1}^{\wp} \mathcal{A}_{p_\vartheta,\mu})^{\mathrm{T}} \frac{\mathcal{Q}_\mu(\wp)}{d_\mu} \prod_{\vartheta=1}^{\wp} \mathcal{A}_{p_\vartheta,\mu} + (1-\omega_\mu) (\prod_{\vartheta=1}^{\wp} \mathcal{A}_{p_\vartheta,\mu})^{\mathrm{T}} Q_\tau(1) \prod_{\vartheta=1}^{\wp} \mathcal{A}_{p_\vartheta,\mu} - Q_\mu(1) < 0, \tag{12}$$

where $\omega_\mu = \sum_{\wp=1}^{T_{max}^{\mu}} \sum_{\tau\in\mathbb{L}_{\mu}^{\mathcal{K}}} \rho_{\mu\tau}(\wp)$, $\varrho_\mu = \sum_{\wp=1}^{T_{max}^{\mu}} \sum_{\tau\in\mathbb{L}} \rho_{\mu\tau}(\wp)$, $\mathcal{Q}_\mu(\wp) = \sum_{\tau\in\mathbb{L}_{\mu}^{\mathcal{K}}} \rho_{\mu\tau}(\wp)Q_\tau(1)$, $d_\mu = \omega_\mu + (1-\vec{\eta}_\mu)\theta$, and $\vec{\eta}_\mu = \sum_{\tau\in\mathbb{L}} \eta_{\mu\tau}$.

Proof: Select the Lyapunov function

$$\mathcal{P}(\sigma(l), \mathbf{i}(l), o(l) = g) \triangleq \sigma^{\mathrm{T}}(l)\bar{Q}_{\mathbf{i}(l)}(g, \kappa_\mu)\sigma(l), \tag{13}$$

where $\bar{Q}_{\mathbf{i}(l)}(g, \kappa_\mu) \triangleq \sum_{p=1}^{\kappa_\mu} h_{p,l}Q_{\mathbf{i}(l)}(g)$ with $g = l - l_q + 1$.

For $\mathbf{i}(l_q) = \mu$, $\forall l \in \mathbb{C}_{[l_q, l_{q+1})}$, it is inferred that

$$\Im_1\|\sigma(l)\|^2 \leq \mathcal{P}(\sigma(l), \mu, g) \leq \Im_2\|\sigma(l)\|^2, \tag{14}$$

where $\Im_1 \triangleq \inf_{\forall\mu\in\mathbb{L}, g\in\mathbb{C}_{[1,T_{max}^{\mu}]}}\{\lambda_{\min}(Q_\mu(g))\}$ and $\Im_2 \triangleq \sup_{\forall\mu\in\mathbb{L}, g\in\mathbb{C}_{[1,T_{max}^{\mu}]}}\{\lambda_{\max}(Q_\mu(g))\}$.

Define $\varsigma_1(\|\sigma(l)\|) \triangleq \Im_1\|\sigma(l)\|^2$ and $\varsigma_2(\|\sigma(l)\|) \triangleq \Im_2\|\sigma(l)\|^2$, where ς_1 and ς_2 are $\mathcal{K}_\infty$ functions. Then, (3) holds.

For $l \in \mathbb{C}_{[l_q+1, l_{q+1}]}$ and $\mathfrak{S}_{\beta_{\max}(l)} = \mu$, we have $\sigma(l) = \sigma(l_q + g) = \prod_{\vartheta=1}^{g} \bar{\mathcal{A}}_\mu(l_q + \vartheta)\sigma(l_q) = \prod_{\vartheta=1}^{g}(\sum_{p_\vartheta=1}^{\kappa_\mu} h_{p_\vartheta, l_q+\vartheta}\mathcal{A}_{p_\vartheta,\mu})\sigma(l_q)$.
Combining (11), one has

$$\begin{aligned}
&\mathcal{P}(\sigma(l_q+g), \mu, g+1) - \iota_\mu\mathcal{P}(\sigma(l_q), \mu, 1) \\
&= \sigma^T(l_q)(\sum_{p_1'=1}^{\kappa_\mu}\sum_{p_1=1}^{\kappa_\mu}\cdots\sum_{p_g'=1}^{\kappa_\mu}\sum_{p_g=1}^{\kappa_\mu}\sum_{p_{g+1}=1}^{\kappa_\mu} h_{p_{g+1}, l_q+\vartheta}) \\
&(\prod_{\vartheta=1}^{g} h_{p_\vartheta', l_q+\vartheta}h_{p_\vartheta, l_q+\vartheta})[(\prod_{\vartheta=1}^{g} \mathcal{A}_{p_\vartheta,\mu}')^{\mathrm{T}} Q_\mu(g+1)
\end{aligned}$$

$$\prod_{\vartheta=1}^{g}\mathcal{A}_{p_\vartheta,\mu}-\iota_\mu Q_\mu(1)]\sigma(l_q)$$
$$\leq \sigma^T(l_q)(\sum_{p_1=1}^{\kappa_\mu}\cdots\sum_{p_g=1}^{\kappa_\mu})(\prod_{\vartheta=1}^{g}h_{p_\vartheta,l_q+\vartheta})$$
$$[(\prod_{\vartheta=1}^{g}\mathcal{A}_{p_\vartheta,\mu})^{\mathrm{T}}Q_\mu(g+1)\prod_{\vartheta=1}^{g}\mathcal{A}_{p_\vartheta,\mu}-\iota_\mu Q_\mu(1)]\sigma(l_q)$$
$$<0, \tag{15}$$

which further implies (4).

Letting $\mathbf{i}(l_q)=\mu$, $\mathbf{i}(l_{q+1})=\tau$, $\forall\mu,\tau\in\mathbb{L}$, $\mu\neq\tau$ yield

$$\mathbf{G}\{\mathcal{P}(\sigma(l_{q+1}),\tau,1)\}|_{\sigma(l_q)=\mu}-\mathcal{P}(\sigma(l_q),\mu,1)$$
$$=\mathbf{G}\{\sigma^{\mathrm{T}}(l_{q+1})\bar{Q}_\tau(1)\sigma(l_{q+1})\}-\sigma^{\mathrm{T}}(l_q)\bar{Q}_\mu(1)\sigma(l_q)$$
$$=\sigma^{\mathrm{T}}(l_q)\Big\{\sum_{\wp=1}^{T^\mu_{max}}[\sum_{\tau\in\mathbb{L}^{\mathcal{K}}_\mu}\frac{\rho_{\mu\tau}(\wp)}{\varrho_\mu}(\prod_{\vartheta=1}^{\wp}\bar{\mathcal{A}}_\mu(l_q+\vartheta))^{\mathrm{T}}$$
$$\bar{Q}_\tau(1)\prod_{\vartheta=1}^{\wp}\bar{\mathcal{A}}_\mu(l_q+\vartheta)+\sum_{\tau\in\mathbb{L}^{\mathcal{UK}}_\mu}\frac{\rho_{\mu\tau}(\wp)}{\varrho_\mu-\omega_\mu}\frac{\varrho_\mu-\omega_\mu}{\varrho_\mu}$$
$$(\prod_{\vartheta=1}^{\wp}\bar{\mathcal{A}}_\mu(l_q+\vartheta))^{\mathrm{T}}\bar{Q}_\tau(1)\prod_{\vartheta=1}^{\wp}\bar{\mathcal{A}}_\mu(l_q+\vartheta)]\Big\}\sigma(l_q)$$
$$-\sigma^{\mathrm{T}}(l_q)\bar{Q}_\mu(1)\sigma(l_q)$$
$$\leq\sigma^{\mathrm{T}}(l_q)\Big\{\sum_{\wp=1}^{T^\mu_{max}}[\sum_{\tau\in\mathbb{L}^{\mathcal{K}}_\mu}\frac{\rho_{\mu\tau}(\wp)}{\varrho_\mu}(\prod_{\vartheta=1}^{\wp}\bar{\mathcal{A}}_\mu(l_q+\vartheta))^{\mathrm{T}}$$
$$Q_\tau(1)\prod_{\vartheta=1}^{\wp}\bar{\mathcal{A}}_\mu(l_q+\vartheta)+\sum_{\tau\in\mathbb{L}^{\mathcal{UK}}_\mu}\frac{\rho_{\mu\tau}(\wp)}{\varrho_\mu-\omega_\mu}\frac{\varrho_\mu-\omega_\mu}{\varrho_\mu}$$
$$(\prod_{\vartheta=1}^{\wp}\bar{\mathcal{A}}_\mu(l_q+\vartheta))^{\mathrm{T}}Q_\tau(1)\prod_{\vartheta=1}^{\wp}\bar{\mathcal{A}}_\mu(l_q+\vartheta)]\Big\}\sigma(l_q)$$
$$-\sigma^{\mathrm{T}}(l_q)Q_\mu(1)\sigma(l_q), \tag{16}$$

where $\omega_\mu=\sum_{\wp=1}^{T^\mu_{max}}\sum_{\tau\in\mathbb{L}^{\mathcal{K}}_\mu}\rho_{\mu\tau}(\wp)$ and $\varrho_\mu=\sum_{\wp=1}^{T^\mu_{max}}\sum_{\tau\in\mathbb{L}}\rho_{\mu\tau}(\wp)$.

Due to $\sum_{\wp=1}^{T^\mu_{max}}\sum_{\tau\in\mathbb{L}^{\mathcal{UK}}_\mu}\rho_{\mu\tau}(\wp)=\varrho_\mu-\omega_\mu$, $\forall\mu\in\mathbb{L}$, it can be seen that $\sum_{\wp=1}^{T^\mu_{max}}\sum_{\tau\in\mathbb{L}^{\mathcal{UK}}_\mu}\rho_{\mu\tau}(\wp)/(\varrho_\mu-\omega_\mu)=1$ with $0\leq\rho_{\mu\tau}(\wp)/(\varrho_\mu-\omega_\mu)\leq1$ and $0\leq d_\mu\leq\varrho_\mu\leq1$, $\forall\mu\in\mathbb{L},\tau\in\mathbb{L}^{\mathcal{UK}}_\mu$, where $d_\mu=\omega_\mu+(1-\vec{\eta}_\mu)\theta$ and $\vec{\eta}_\mu=\sum_{\tau\in\mathbb{L}^{\mathcal{K}}_\mu}\eta_{\mu\tau}$.

Together with (12) and (16), it yields

$$\mathbf{G}\{\mathcal{P}(\sigma(l_{q+1}),\tau,1)\}|_{\sigma(l_q),\mathfrak{S}_\beta=\mu,S_q\in\mathbb{C}_{[1,T^\mu_{max}]}}-\mathcal{P}(\sigma(l_q),\mu,1)$$
$$\leq\sigma^T(l_q)\sum_{\wp=1}^{T^\mu_{max}}\sum_{\tau\in\mathbb{L}^{\mathcal{UK}}_\mu}\frac{\rho_{\mu\tau}(\wp)}{\varrho_\mu-\omega_\mu}[\sum_{\wp=1}^{T^\mu_{max}}(\prod_{\vartheta=1}^{\wp}$$
$$\bar{\mathcal{A}}_\mu(l_q+\vartheta))^{\mathrm{T}}\frac{\mathcal{Q}_\mu(\wp)}{\varrho_\mu}\prod_{\vartheta=1}^{\wp}\bar{\mathcal{A}}_\mu(l_q+\vartheta)+\frac{\varrho_\mu-\omega_\mu}{\varrho_\mu}(\prod_{\vartheta=1}^{\wp}$$
$$\bar{\mathcal{A}}_\mu(l_q+\vartheta))^{\mathrm{T}}Q_\tau(1)\prod_{\vartheta=1}^{\wp}\bar{\mathcal{A}}_\mu(l_q+\vartheta)-Q_\mu(1)]\sigma(l_q)$$
$$\leq-\sum_{\wp=1}^{T^\mu_{max}}\sum_{\tau\in\mathbb{L}^{\mathcal{UK}}_\mu}\frac{\rho_{\mu\tau}(\wp)}{\omega_\mu-\varrho_\mu}(\sum_{p_1=1}^{\kappa_\mu}\cdots\sum_{p_\wp=1}^{\kappa_\mu})$$
$$(\prod_{\vartheta=1}^{\wp}h_{p_\vartheta,l_q+\vartheta})\lambda_{min}\Big(-\Big[\sum_{\wp=1}^{T^\mu_{max}}(\prod_{\vartheta=1}^{\wp}\mathcal{A}_{p_\vartheta,\mu})^{\mathrm{T}}\frac{\mathcal{Q}_\mu(\wp)}{\varrho_\mu}$$
$$\prod_{\vartheta=1}^{\wp}\mathcal{A}_{p_\vartheta,\mu}+(\omega_\mu-1)(\prod_{\vartheta=1}^{\wp}\mathcal{A}_{p_\vartheta,\mu})^{\mathrm{T}}Q_\tau(1)\prod_{\vartheta=1}^{\wp}\mathcal{A}_{p_\vartheta,\mu}$$
$$-Q_\mu(1)\Big]\Big)\|\sigma(l_q)\|^2$$
$$\leq-\sum_{\wp=1}^{T^\mu_{max}}\sum_{\tau\in\mathbb{L}^{\mathcal{UK}}_\mu}\frac{\rho_{\mu\tau}(\wp)}{\omega_\mu-\varrho_\mu}\Im_3\|\sigma(l_q)\|^2$$
$$=-\Im_3\|\sigma(l_q)\|^2, \tag{17}$$

where

$$\Im_3\triangleq\inf_{\mu\in\mathbb{L},\tau\in\mathbb{L}^{\mathcal{UK}}_\mu,\wp\in\mathbb{C}_{[1,T^\mu_{max}]}}\lambda_{min}\Big(-\Big[\sum_{\wp=1}^{T^\mu_{max}}(\prod_{\vartheta=1}^{\wp}\mathcal{A}_{p_\vartheta,\mu})^{\mathrm{T}}$$
$$\frac{\mathcal{Q}_\mu(\wp)}{\varrho_\mu}\prod_{\vartheta=1}^{\wp}\mathcal{A}_{p_\vartheta,\mu}+(\omega_\mu-1)(\prod_{\vartheta=1}^{\wp}\mathcal{A}_{p_\vartheta,\mu})^{\mathrm{T}}Q_\tau(1)\prod_{\vartheta=1}^{\wp}$$
$$\mathcal{A}_{p_\vartheta,\mu}-Q_\mu(1)\Big]\Big).$$

Let $\varsigma_3\triangleq\Im_3\|\sigma(l_q)\|^2$. The assurance of (5) can be attributed to the characteristic of $\mathcal{K}_\infty$ function ς_3. Therefore, system (10) is MSS. ■

Remark 4: Theorem 1 provides the mean-square stability of semi-Markovian switching SPSs, involving the power of matrix $\mathcal{A}_{p_\vartheta,\mu}$, which is complicated for the controller design. To solve this problem, additional matrix variables are introduced. This strategy includes the SMC that takes the upper bound T^μ_{max} of ST into account to reduce the effects.

Theorem 2: For given scalars $\iota_\mu>0$, $T^\mu_{max}\in\mathbb{C}_{\geq1}$, $\forall\mu\in\mathbb{L}$, $\tau\in\mathbb{L}^{\mathcal{UK}}_\mu$, system (10) is MSS, if we find matrices $L_\mu(g,\varpi)>0$ and $\mathcal{L}_\mu(\wp,j)>0$, $\forall\mu\in\mathbb{L}$, $\tau\in\mathbb{L}^{\mathcal{UK}}_\mu$, $\forall g\in\mathbb{C}_{[1,T^\mu_{max}-1]}$, $\forall\varpi\in\mathbb{C}_{[0,g-1]}$, $\forall\wp\in\mathbb{C}_{[1,T^\mu_{max}]}$, $\forall j\in\mathbb{C}_{[1,\wp-1]}$, such that $\forall\mu\in\mathbb{L}$, $\forall\wp\in\mathbb{C}_{[1,T^\mu_{max}]}$,

$$\sum_{\wp=1}^{T^\mu_{max}}\frac{\mathcal{L}_\mu(\wp,0)}{d_\mu}+(1-\omega_\mu)L_\tau(1,\wp+1)-L_\mu(1,1)<0, \tag{18}$$

$$\sum_{\wp=j+1}^{T^\mu_{max}}(\mathcal{A}^{\mathrm{T}}_{p_{\wp-j},\mu}\frac{\mathcal{L}_\mu(\wp,j+1)}{d_\mu}\mathcal{A}_{p_{\wp-j},\mu}-\frac{\mathcal{L}_\mu(\wp,j)}{d_\mu})<0, \tag{19}$$

$$\mathcal{A}^{\mathrm{T}}_{p_\varpi,\mu}L_\tau(1,\wp-\varpi)\mathcal{A}_{p_\varpi,\mu}-L_\tau(1,\wp-\varpi+1)<0, \tag{20}$$

$$\mathcal{A}^{\mathrm{T}}_{p_{g-\varpi},\mu}L_\mu(g+1,\varpi+2)\mathcal{A}_{p_{g-\varpi},\mu}-L_\mu(g+1,\varpi+1)<0, \tag{21}$$

$$L_\mu(g+1,1)-\iota_\mu L_\mu(1,1)<0, \tag{22}$$

where d_μ and ω_μ are defined in Theorem 1.

Proof: For $\mu\in\mathbb{L}$, $g\in\mathbb{C}$, define $Q_\mu(g)\triangleq L_\mu(g,g)$, $\mathcal{Q}_\mu(g)\triangleq\mathcal{L}_\mu(g,g)$. According to (19), one has

$$\sum_{j=0}^{T^\mu_{max}-1}\{(\prod_{\vartheta=\wp-j+1}^{\wp}\mathcal{A}_{p_\vartheta,\mu})^{\mathrm{T}}[\sum_{\wp=j+1}^{T^\mu_{max}}(\mathcal{A}^{\mathrm{T}}_{p_{\wp-j},\mu}$$
$$\frac{\mathcal{L}_\mu(\wp,j+1)}{d_\mu}\mathcal{A}_{p_{\wp-j},\mu}-\frac{\mathcal{L}_\mu(\wp,j)}{d_\mu})]\prod_{\vartheta=\wp-j+1}^{\wp}\mathcal{A}_{p_\vartheta,\mu}\}<0, \tag{23}$$

which means that

$$\sum_{\wp=1}^{T^\mu_{max}}\sum_{j=0}^{T^\mu_{max}-1}(\prod_{\vartheta=\wp-j+1}^{\wp}\mathcal{A}_{p_\vartheta,\mu})^{\mathrm{T}}[\mathcal{A}^{\mathrm{T}}_{p_{\wp-j},\mu}$$
$$\frac{\mathcal{L}_\mu(\wp,j+1)}{d_\mu}\mathcal{A}_{p_{\wp-j},\mu}-\frac{\mathcal{L}_\mu(\wp,j)}{d_\mu}]\prod_{\vartheta=\wp-j+1}^{\wp}\mathcal{A}_{p_\vartheta,\mu}<0. \tag{24}$$

It can be further obtained that

$$\sum_{\wp=1}^{T^\mu_{max}}[(\prod_{\vartheta=1}^{\wp}\mathcal{A}_{p_\vartheta,\mu})^{\mathrm{T}}\frac{\mathcal{L}_\mu(\wp,\wp)}{d_\mu}\prod_{\vartheta=1}^{\wp}\mathcal{A}_{p_\vartheta,\mu}$$
$$-\frac{\mathcal{L}_\mu(\wp,0)}{d_\mu}]<0. \tag{25}$$

It follows from (20) that

$$\sum_{\varpi=0}^{T^{\mu}_{max}-1}(\prod_{\vartheta=1}^{\varpi}\mathcal{A}_{p_\vartheta,\mu})^{\mathrm{T}}[\mathcal{A}^{\mathrm{T}}_{p_\varpi,\mu}L_\tau(1,\wp-\varpi)\mathcal{A}_{p_\varpi,\mu} - L_\tau(1,\wp-\varpi+1)]\prod_{\vartheta=1}^{\varpi}\mathcal{A}_{p_\vartheta,\mu} = (\prod_{\vartheta=1}^{\wp}\mathcal{A}_{p_\vartheta,\mu})^{\mathrm{T}}L_\tau(1,1)\prod_{\vartheta=1}^{\wp}\mathcal{A}_{p_\vartheta,\mu} - L_\tau(1,\wp+1) < 0. \quad (26)$$

Combining (18), (25), and (26), one can get

$$\sum_{\wp=1}^{T^{\mu}_{max}}[(\prod_{\vartheta=1}^{\wp}\mathcal{A}_{p_\vartheta,\mu})^{\mathrm{T}}\frac{\mathcal{L}_\mu(\wp,\wp)}{d_\mu}\prod_{\vartheta=1}^{\wp}\mathcal{A}_{p_\vartheta,\mu} - \frac{\mathcal{L}_\mu(\wp,0)}{d_\mu}] + (1-\omega_\mu)[(\prod_{\vartheta=1}^{\wp}\mathcal{A}_{p_\vartheta,\mu})^{\mathrm{T}}L_\tau(1,1)\prod_{\vartheta=1}^{\wp}\mathcal{A}_{p_\vartheta,\mu} - L_\tau(1,\wp+1)] + \sum_{\wp=1}^{T^{\mu}_{max}}\frac{\mathcal{L}_\mu(\wp,0)}{d_\mu} + (1-\omega_\mu)L_\tau(1,\wp+1) - L_\mu(1,1)$$
$$= \sum_{\wp=1}^{T^{\mu}_{max}}(\prod_{\vartheta=1}^{\wp}\mathcal{A}_{p_\vartheta,\mu})^{\mathrm{T}}\frac{\mathcal{L}_\mu(\wp,\wp)}{d_\mu}\prod_{\vartheta=1}^{\wp}\mathcal{A}_{p_\vartheta,\mu} + (1-\omega_\mu)(\prod_{\vartheta=1}^{\wp}\mathcal{A}_{p_\vartheta,\mu})^{\mathrm{T}}L_\tau(1,1)\prod_{\vartheta=1}^{\wp}\mathcal{A}_{p_\vartheta,\mu} - L_\mu(1,1) < 0. \quad (27)$$

Replacing $L_\mu(1,1)$, $L_\tau(1,1)$ and $\mathcal{L}_\mu(\wp,\wp)$ with $Q_\mu(1)$, $Q_\tau(1)$ and $\mathcal{Q}_\mu(\wp)$ yields (12).

In addition, from (21), one has

$$\sum_{\varpi=0}^{g-1}(\prod_{\vartheta=g-\varpi+1}^{g}\mathcal{A}_{p_\vartheta,\mu})^{\mathrm{T}}[\mathcal{A}^{\mathrm{T}}_{p_{g-\varpi},\mu}L_\mu(g+1,\varpi+2)\mathcal{A}_{p_{g-\varpi},\mu} - L_\mu(g+1,\varpi+1)]\prod_{\vartheta=g-\varpi+1}^{g}\mathcal{A}_{p_\vartheta,\mu} = (\prod_{\vartheta=1}^{g}\mathcal{A}_{p_\vartheta,\mu})^{\mathrm{T}}L_\mu(g+1,g+1)\prod_{\vartheta=1}^{g}\mathcal{A}_{p_\vartheta,\mu} - L_\mu(g+1,1) < 0. \quad (28)$$

Based on (22) and $Q_\mu(g+1) \triangleq L_\mu(g+1,g+1)$, (11) holds. ■

Remark 5: The introduction of matrix variables $L_\mu(g,\varpi)$ and $\mathcal{L}_\mu(\wp,j)$ in Theorem 2 is a crucial aspect for the controller design. This is necessary because Theorem 1 cannot be solved directly due to the power of $\mathcal{A}_{p_\vartheta,\mu}$.

IV. Parametric solution

To obtain the SMC gain, we will introduce some slack variables J_μ from this step.

Theorem 3: For given scalars $\iota_\mu > 0$, $T^{\mu}_{max} \in \mathbb{C}_{\geq 1}$, $\forall\mu \in \mathbb{L}$, and matrices $\mathcal{F}_{1\mu}$, $\mathcal{F}_{2\mu}$, system (10) is MSS, if we find matrices $J_{11\mu} \in \mathbb{R}$, $J_{22\mu} \in \mathbb{R}$, $M_\mu(g,\varpi)$ with $L_3M_\mu(g,\varpi)L_3^{\mathrm{T}} \geq 0$, $\mathbf{M}_\mu(\wp,0)$ with $L_3\mathbf{M}_\mu(\wp,0)L_3^{\mathrm{T}} \geq 0$, $\forall\mu \in \mathbb{L}$, $\tau \in \mathbb{L}^{\mathcal{UK}}_\mu$, $\forall g \in \mathbb{C}_{[1,T^{\mu}_{max}-1]}$, $\forall\wp \in \mathbb{C}_{[1,T^{\mu}_{max}]}$, $\forall j \in \mathbb{C}_{[0,\wp]}$, $\forall\varpi \in \mathbb{C}_{[0,g]}$, such that $\forall\mu \in \mathbb{L}$, $\tau \in \mathbb{L}^{\mathcal{UK}}_\mu, p_\vartheta \in \mathbb{I}_\mu$,

$$\begin{bmatrix} \Delta_1 & \Delta_2 \\ * & \Delta_3 \end{bmatrix} < 0, \quad (29)$$

$$\mathcal{F}_{2\mu}M_\mu\mathcal{F}^{\mathrm{T}}_{2\mu} - \mathrm{sym}\{\mathcal{F}_{2\mu}\} + \iota_\mu^{-1}M_\mu(g+1,1)^{-1} < 0, \quad (30)$$

$$\begin{bmatrix} \Pi_1 & \Pi_2 \\ * & \Pi_3 \end{bmatrix} < 0, \quad (31)$$

$$\begin{bmatrix} \Omega_1 & \Omega_2 \\ * & \Omega_3 \end{bmatrix} < 0, \quad (32)$$

$$\frac{1}{d_\mu}[\sum_{\wp=1}^{T^{\mu}_{max}}(\mathbf{M}_\mu(\wp,0) - \mathrm{sym}\{J_\mu\})] + (1-\omega_\mu)(M_\tau(1,\wp+1) - \mathrm{sym}\{J_\mu\}) - (M_\mu - \mathrm{sym}\{J_\mu\}) < 0, \quad (33)$$

where

$\Delta_1 \triangleq \begin{bmatrix} \Delta^\flat_{11} & \Delta_{22} \\ * & -M_\mu(g+1,\varpi+2) \end{bmatrix}$,

$\Delta_2 \triangleq \begin{bmatrix} 0 & J^{\mathrm{T}}_\mu H^{\mathrm{T}}_\mu \\ \upsilon N_\mu F_\mu & 0 \end{bmatrix}$, $\Delta_3 \triangleq -\mathrm{diag}\{\upsilon I, \upsilon I\}$,

$\Delta^1_{11} \triangleq \mathcal{F}_{1\mu}L_1M_\mu(g+1,\varpi+1)L_1\mathcal{F}^{\mathrm{T}}_{1\mu} - \mathrm{sym}\{J_\mu\mathcal{F}_{1\mu}\}$,

$\Delta^2_{11} \triangleq \bar{\varepsilon}^2\mathcal{F}_{1\mu}L_2M_\mu(g+1,\varpi+1)L_2\mathcal{F}^{\mathrm{T}}_{1\mu}$,
$+\bar{\varepsilon}\,\mathrm{sym}\{\mathcal{F}_{1\mu}L_1M_\mu(g+1,\varpi+1)L_2\mathcal{F}^{\mathrm{T}}_{1\mu}\} + \Delta^1_{11}$,

$\Delta_{22} \triangleq J^{\mathrm{T}}_\mu\hat{\mathbf{A}}^{\mathrm{T}}_{p_{g-\varpi},\mu}$,

$\Pi_1 \triangleq \begin{bmatrix} \Pi^\flat_{11} & \Pi_{22} \\ * & -\mathrm{diag}\{\hat{M}_\mu, \hat{\mathbf{M}}_\mu(j+1)\} \end{bmatrix}$,

$\Pi_2 \triangleq \begin{bmatrix} 0 & P^{\mathrm{T}}_\mu(j+1)\text{Ł} \\ \epsilon N_\mu F_\mu \otimes I_{\vec{m}(\mathcal{M})+1} & 0 \end{bmatrix}$,

$\text{Ł} \triangleq (J^{\mathrm{T}}_\mu H^{\mathrm{T}}_\mu \otimes I_{\vec{m}(\mathcal{M})+1})$, $\Pi_3 \triangleq \mathrm{diag}\{-\epsilon I, -\epsilon I\}$,

$\Pi^1_{11} \triangleq \sum_{\wp=j+1}^{T^{\mu}_{max}}(\mathcal{F}_{1\mu}L_1\mathbf{M}_\mu(\wp,j)L_1\mathcal{F}^{\mathrm{T}}_{1\mu} - \mathrm{sym}\{J_\mu\mathcal{F}_{1\mu}\})$,

$\Pi^2_{11} \triangleq \sum_{\wp=j+1}^{T^{\mu}_{max}}(\bar{\varepsilon}^2\mathcal{F}_{1\mu}L_2\mathbf{M}_\mu(\wp,j)L_2\mathcal{F}^{\mathrm{T}}_{1\mu}$,
$+\bar{\varepsilon}\,\mathrm{sym}\{\mathcal{F}_{1\mu}L_1\mathbf{M}_\mu(\wp,j)L_2\mathcal{F}^{\mathrm{T}}_{1\mu}\} + \Pi^1_{11})$,

$\Pi_{22} \triangleq P^{\mathrm{T}}_\mu(j+1)(J^{\mathrm{T}}_\mu\hat{\mathbf{A}}^{\mathrm{T}}_{p_{\wp-j},\mu} \otimes I_{\vec{m}(\mathcal{M})+1})$,

$\Omega_1 \triangleq \begin{bmatrix} \Omega^\flat_{11} & \Omega_{22} \\ * & -M_\tau(1,\wp-\varpi) \end{bmatrix}$,

$\Omega_2 \triangleq \begin{bmatrix} 0 & J^{\mathrm{T}}_\mu H^{\mathrm{T}}_\mu \\ \nu N_\mu F_\mu & 0 \end{bmatrix}$, $\Omega_3 = \mathrm{diag}\{-\nu I, -\nu I\}$,

$\Omega^1_{11} \triangleq \mathcal{F}_{1\mu}L_1M_\tau(1,\wp-\varpi+1)L_1\mathcal{F}^{\mathrm{T}}_{1\mu} - \mathrm{sym}\{J_\mu\mathcal{F}_{1\mu}\}$,

$\Omega^2_{11} \triangleq \bar{\varepsilon}^2\mathcal{F}_{1\mu}L_2M_\tau(1,\wp-\varpi+1)L_2\mathcal{F}^{\mathrm{T}}_{1\mu}$,
$+\bar{\varepsilon}\,\mathrm{sym}\{\mathcal{F}_{1\mu}L_1M_\tau(1,\wp-\varpi+1)L_2\mathcal{F}^{\mathrm{T}}_{1\mu}\} + \Omega^1_{11}$,

$\Omega_{22} \triangleq J^{\mathrm{T}}_\mu\hat{\mathbf{A}}^{\mathrm{T}}_{p_\varpi,\mu}$, $\hat{\mathbf{A}}_{p_\vartheta,\mu} \triangleq (I - B_\mu(G_\mu B_\mu)^{-1}G_\mu)A_{p,\mu}$,

$N_\mu \triangleq (I - B_\mu(G_\mu B_\mu)^{-1}G_\mu)$,

$\hat{M}_\mu \triangleq \mathrm{diag}\{M_{\vec{m}(1)}, M_{\vec{m}(2)}, \cdots, M_{\vec{m}(\mathcal{M})}\}$,

$\hat{\mathbf{M}}_\mu(j+1) \triangleq \sum_{\wp=j+1}^{T^{\mu}_{max}}\mathbf{M}_\mu(\wp,j+1)$, $L_1 \triangleq \mathrm{diag}\{I_1, 0\}$,

$L_2 \triangleq \mathrm{diag}\{0, I_2\}$, $L_3 \triangleq [0, I_2]$, $J_\mu \triangleq \mathrm{diag}\{J_{11\mu}, J_{22\mu}\}$,

$P_\mu(j+1) \triangleq [\sqrt{\rho_{\mu\vec{m}(1)}(j+1)}I, \sqrt{\rho_{\mu\vec{m}(2)}(j+1)}I$
$\ldots, \sqrt{\rho_{\mu\vec{m}(\mathcal{M})}(j+1)}I, I]^T$.

Proof: From (29), for $D^{\top}_{p,\mu}D_{p,\mu} \leq I$, $\upsilon > 0$, it can be deduced from Lemma 2 that

$$\begin{bmatrix} \Delta_{11} & J^{\mathrm{T}}_\mu\hat{A}^{\mathrm{T}}_{p_{g-\varpi},\mu} \\ * & -M_\mu(g+1,\varpi+2) \end{bmatrix} < 0, \quad (34)$$

where $\Delta_{11} = \mathcal{F}_{1\mu}E_\varepsilon M_\mu(g+1,\varpi+1)E_\varepsilon\mathcal{F}^{\mathrm{T}}_{1\mu} - \mathrm{sym}\{J_\mu\mathcal{F}_{1\mu}\}$.

Since

$$0 \leq \left(\mathcal{F}_{1\mu}E_\varepsilon M_\mu(g,\varpi)E_\varepsilon - \mathcal{J}^{\mathrm{T}}_\mu\right)\left(E_\varepsilon M_\mu(g,\varpi)E_\varepsilon\right)^{-1}\left(\mathcal{F}_{1\mu}E_\varepsilon M_\mu(g,\varpi)E_\varepsilon - \mathcal{J}^{\mathrm{T}}_\mu\right)^{\mathrm{T}},$$

one can obtain that

$$-\mathcal{J}_\mu^T(E_\varepsilon M_\mu(g,\varpi)E_\varepsilon)^{-1}\mathcal{J}_\mu \leq \mathcal{F}_{1\mu}E_\varepsilon M_\mu(g,\varpi)E_\varepsilon\mathcal{F}_{1\mu}^{\mathrm{T}} - \mathrm{sym}\{J_\mu\mathcal{F}_{1\mu}\}. \quad (35)$$

Combining (34) implies that

$$\begin{bmatrix} -\mathcal{J}_\mu^T(E_\varepsilon M_\mu(g,\varpi)E_\varepsilon)^{-1}\mathcal{J}_\mu & J_\mu^{\mathrm{T}}\hat{A}_{p_g-\varpi,\mu}^{\mathrm{T}} \\ * & -M_\mu(g+1,\varpi+2) \end{bmatrix} < 0. \quad (36)$$

Pre- and post-multiplying (36) by $\mathrm{diag}\{J_\mu^{-\mathrm{T}}, I\}$ and its transpose, then by Schur complement, it can be further got that $\hat{A}_{p_g-\varpi,\mu}^{\mathrm{T}}M_\mu(g+1,\varpi+2)^{-1}\hat{A}_{p_g-\varpi,\mu}-(E_\varepsilon M_\mu(g,\varpi)E_\varepsilon)^{-1} < 0$.

Let $M_\mu(g,\varpi) \triangleq (E_\varepsilon L_\mu(g,\varpi)E_\varepsilon)^{-1}$. Then, (21) can be guaranteed.

Defining $M_\mu \triangleq M_\mu(g,g) \triangleq (E_\varepsilon L_\mu(g,g)E_\varepsilon)^{-1}$, it is inferred from (30) that $E_\varepsilon L_\mu(g+1,1)E_\varepsilon - \iota_\mu E_\varepsilon L_\mu(1,1)E_\varepsilon < 0$.

Furthermore, pre- and post-multiplying E_ε^{-1}, (22) is ensured. Similarly, from (31), it can be obtained that

$$\hat{A}_{p_{\wp-j},\mu}^{\mathrm{T}}E_\varepsilon\mathcal{L}_\mu(j+1,j+1)E_\varepsilon\hat{A}_{p_{\wp-j},\mu} + \sum\nolimits_{\wp=j+2}^{T_{max}^\mu}E_\varepsilon\hat{A}_{p_{\wp-j},\mu}^{\mathrm{T}}\mathcal{L}_\mu(\wp,j+1)E_\varepsilon\hat{A}_{p_{\wp-j},\mu} - \sum\nolimits_{\wp=j+1}^{T_{max}^\mu}\mathcal{L}_\mu(\wp,j) < 0,$$

where $\mathcal{L}_\mu(j+1,j+1) \triangleq \sum_{\tau\in\mathbb{L}_\mu^{\mathcal{K}}}\rho_{\mu\tau}(j+1)L_\mu(j+1,j+1)$. Then, (19) can be guaranteed. Similarly, (18) can can be guaranteed from (33) and (20) can can be guaranteed from (32). This guarantees the stability of system (10) in terms of mean-square sense. ■

Remark 6: The upper bound of SPP is a crucial parameter for SPSs, as it indicates the stable range of the system. Theorem 3 presents a linear matrix inequality for the non-convex condition of the SMC matrix. When the corresponding SMK is fully known, then the condition (32) is unnecessary, and (33) becomes $[\sum_{\wp=1}^{T_{max}^\mu}(\mathbf{M}_\mu(\wp,0)-\mathrm{sym}\{J_\mu\})]-(M_\mu-\mathrm{sym}\{J_\mu\})<0$.

Remark 7: As mentioned in [17], when the ST $T_{max}^\mu \to \infty$, the mean-square stability will approximate to traditional mean-square stability without any approximation error that is generated by truncations on the ST, but obviously the resulting conditions (29)-(33) can not be numerically tested. Nonetheless, the computational complexity of solving linear matrix inequalities increases along with the upper bound of the ST, and it is impractical to choose the parameters that are not too large or too small. Thus, selecting an appropriate value T_{max}^μ is crucial to achieve reasonable calculation costs without compromising the control objective.

V. Reachability analysis

In this section, a predetermined sliding region will be ensured to be reached for the state of system (10) by constructing an appropriate SMC law.

Theorem 4: Consider the matrix G_μ, $\forall\mu\in\mathbb{L}$ solved in Theorem 3. The SMC law can be employed to guarantee the attainability of the sliding region:

$$u(l) = -(G_\mu B_\mu)^{-1}\sum\nolimits_{p=1}^{\kappa_\mu}h_{p,\mu}(\psi(l))[G_\mu A_{p,\mu}E_\varepsilon\sigma(l) + (\|F_{i,\mu}\|\|H_{p,\mu}\|\|G_\mu\|\|E_\varepsilon\|\|\sigma(l)\|+\chi)\,\mathrm{sgn}(\Re(l))], \quad (37)$$

where $\chi > 0$.

Proof: Choose the Lyapunov function

$$\mathcal{P}_\Re(l,\mu) = \frac{1}{2}\Re^T(l)\Re(l). \quad (38)$$

Based on $\triangle\Re(l,\mu) = \Re(l+1)-\Re(l)$ and (7), we have

$$\begin{aligned}\triangle\mathcal{P}_\Re(l,\mu) &= \Re^T(l)\triangle\Re(l,\mu) + \frac{1}{2}\triangle\Re^T(l,\mu)\triangle\Re(l,\mu)\\ &= \Re^T(l)G_\mu\sum\nolimits_{p=1}^{\kappa_\mu}h_{p,\mu}(\psi(l))((A_{p,\mu}+\Delta A_{p,\mu})E_\varepsilon\sigma(l)\\ &+ B_\mu u(l)) - \Re^T(l)\Re(l) + \frac{1}{2}\triangle\Re^T(l,\mu)\triangle\Re(l,\mu)\\ &\leq \Re^T(l)G_\mu\sum\nolimits_{p=1}^{\kappa_\mu}h_{p,\mu}(\psi(l))A_{p,\mu}E_\varepsilon\sigma(l) + \sum\nolimits_{p=1}^{\kappa_\mu}h_{p,\mu}(\psi(l))\\ &\|\Re(l)\|\|G_\mu\|\|\triangle A_{p,\mu}\|\|E_\varepsilon\|\|\sigma(l)\| + \Re^T(l)G_\mu B_\mu u(l)\\ &- \Re^T(l)\Re(l) + \frac{1}{2}\triangle\Re^T(l)\triangle\Re(l).\end{aligned} \quad (39)$$

According to $\|\triangle A_{p,\mu}\| \leq \|F_{p,\mu}\|\|H_{p,\mu}\|$ and the SMC law (37), one has

$$\begin{aligned}\triangle\mathcal{P}_\Re(l,\mu) &\leq -\chi\Re^T(l)\,\mathrm{sgn}(\Re(l)) - \Re^T(l)\Re(l)\\ &+ \frac{1}{2}\Delta\Re^T(l,\mu)\Delta\Re(l,\mu)\\ &= -\chi|\Re(l)| - \Re^T(l)\Re(l) + \frac{1}{2}\Delta\Re^T(l,\mu)\Delta\Re(l,\mu).\end{aligned} \quad (40)$$

Owing to $\|\Re(l)\| \leq |\Re(l)|$, if follows that $\triangle\mathcal{P}_\Re(l,\mu) \leq -\chi\|\Re(l)\| - \Re^T(l)\Re(l) + \frac{1}{2}\Delta\Re^T(l,\mu)\Delta\Re(l,\mu)$.

It is noted that $\Delta\mathcal{P}_\Re(l,\mu) < 0$ can be realized under an appropriate parameter χ, which means that $\Re(l) \neq 0$ is in a finite field. Consequently, $\Delta\Re(l,\mu)$ is constrained, enabling the system to be steered towards a sliding region. ■

Remark 8: In this work, there exists one limitation, that is, the nonlinear system is established by type-1 fuzzy strategy, in which the membership function is exact. Once the complexity of nonlinear plant is studied, the membership function becomes uncertain. Therefore, subsequent research will be conducted on interval type-2 fuzzy to reduce some conservatism.

Remark 9: According to Ref. [42], the dimension of the sliding surface is the same as that of control input. When the control input is m-dimensional, there exist m switching surfaces. First, the system state will enter $(n-1)$ dimensional sliding mode, existing on $\Re_i$, for $i = 1,2,\ldots,m$, where $n = n1+n2$ means the system dimension, and then enter the $(n-2)$ dimensional sliding mode, existing on $\Re_i \cap \Re_j$. And the system state continues until entering the $(n-m)$ dimensional sliding mode, finally arriving at the eventual sliding surface $\Re_E$, where $\Re_E = \Re_1 \cap \Re_2 \cap \ldots \cap \Re_m$.

Remark 10: For the SMC law design, several open parameters need to be selected in advance. First, the system parameters ι_μ, T_{max}^μ, and θ are known a priori based on the practical situation. Second, according to the SMK definition, the underlying parameters $\eta_{\mu\tau}$ and $\xi_{\mu\tau}(\wp)$ are chosen based on the statistical measurement. Furthermore, one has the feasible solution G_μ under the framework of Theorem 3. Finally, in the process of the SMC design, the tuning parameter χ is chosen to realize the fast convergence of sliding motion.

VI. CASE STUDY

Consider a tunnel diode circuit model from [21]. First, based on the dynamic model of the circuit and the switching characteristics, the tunnel diode circuit model is described as fuzzy semi-Markovian switching SPSs. Second, the appropriate sliding surface is established to obtain an equivalent fuzzy sliding control law. Third, by integrating the upper bound of ST, the feasp function based on the MATLAB toolbox is utilized to determine the corresponding controller gains and then the SMC law is derived. Consequently, the SMC law is implemented on the tunnel diode circuit, and its effectiveness is confirmed through MATLAB simulation.

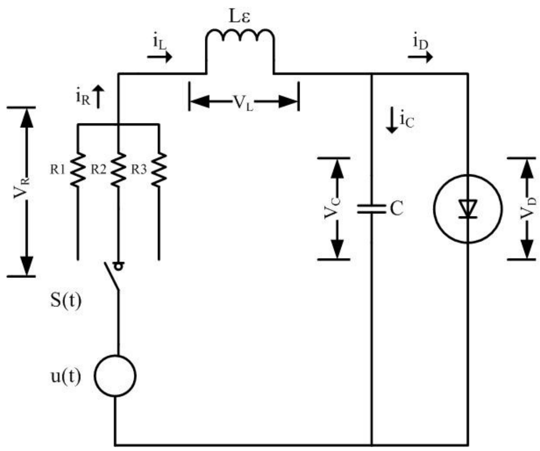

Fig. 1 Tunnel diode circuit

For Fig. 1, the system model is described as

$$i_D(l) = -0.2V_D(l) - 0.05V_D^3(l),$$

where $i_D(l)$ represents the diode current, $V_D(l)$ is the diode voltage, C, L_ε, and R_μ mean the capacitor, the inductance, and the inductance. $V_C(l)$ and $i_C(l)$ represent the capacitor voltage and the capacitor currenty. $V_L(l)$ and $i_L(l)$ are the inductance voltage and the inductance current. $u(l)$ is the input voltage.

The switch $S(l)$ obeys a semi-Markovian chain $\{\mathbf{i}(l)\}_{l\in\mathbb{C}_{>0}}$ with three modes. Let $\sigma_1(l) = V_C(l)$ and $\sigma_2(l) = i_L(l)$. Then, the system model is expressed as

$$\begin{aligned} &C\dot{\sigma}_1(l) = 0.2\sigma_1(l) + 0.05\sigma_1^3(l) + \sigma_2(l), \\ &L_\varepsilon\dot{\sigma}_2(l) = -\sigma_1(l) - R_i\sigma_2(l) + u(l), \mu = 1, 2, 3. \end{aligned} \tag{41}$$

Choosing $C = 0.1F$, $L_\varepsilon = 10^{-3}H$, $R_1 = 90\Omega$, $R_2 = 120\Omega$, and $R_3 = 150\Omega$, system (41) is rewritten as

$$\begin{aligned} \dot{\sigma}_1(l) =& \frac{0.2}{C}\sigma_1(l) + \frac{0.05}{C}\sigma_1^3(l) + \frac{1}{C}\sigma_2(l), \\ \varepsilon_r\dot{\sigma}_2(l) =& -0.1\sigma_1(l) - 0.1R_i\sigma_2(l) + 0.1u(l), \end{aligned} \tag{42}$$

where $\mu = 1, 2, 3$, $\varepsilon_r = 10^{-4}$ is the SPP.

Next, the T-S fuzzy model is adopted to describe the system.

Assuming $\|\sigma_1(l)\| \leq 3$, the membership functions are given as

$$h_1(l) = 1 - \frac{\sigma_1^2(l)}{9}, h_2(l) = 1 - h_1(l). \tag{43}$$

When σ_1 is 0, then $h_1(l) = 1$, $h_2(l) = 0$ and when σ_1 is -3 or 3, then $h_1(l) = 0$, $h_2(l) = 1$.

Together with the parametric uncertainty, one has the following.

Plant Rule 1: IF $\sigma_1(l)$ is 0 volt, THEN

$$\dot{\sigma}(l) = (\bar{A}_{1,\mu}\bar{G}_\epsilon + \Delta A_{1,\mu})\sigma(l) + \bar{B}_\mu u(l);$$

Plant Rule 2: IF $\sigma_1(l)$ is -3 volt or 3 volt, THEN

$$\dot{\sigma}(l) = (\bar{A}_{2,\mu}\bar{G}_\epsilon + \Delta A_{2,\mu})\sigma(l) + \bar{B}_\mu u(l).$$

By the fuzzy blending of a convex combination of linear local systems via the membership function, T-S fuzzy model is obtained as

$$\bar{G}_\epsilon\dot{\sigma}(l) = \sum\nolimits_{p=1}^{2} h_p(l)\left[(A_{p,\mu} + \Delta A_{p,\mu})\sigma(l) + B_\mu u(l)\right],$$

where

$$\begin{aligned} &A_{1,1} = \begin{bmatrix} \frac{0.2}{C} & \frac{1}{C} \\ -0.1 & -0.1R_1 \end{bmatrix}, A_{2,1} = \begin{bmatrix} \frac{0.69}{C} & \frac{1}{C} \\ -0.1 & -0.1R_1 \end{bmatrix}, \\ &A_{1,2} = \begin{bmatrix} \frac{0.2}{C} & \frac{1}{C} \\ -0.1 & -0.1R_2 \end{bmatrix}, A_{2,2} = \begin{bmatrix} \frac{0.69}{C} & \frac{1}{C} \\ -0.1 & -0.1R_2 \end{bmatrix}, \\ &A_{1,3} = \begin{bmatrix} \frac{0.2}{C} & \frac{1}{C} \\ -0.1 & -0.1R_3 \end{bmatrix}, A_{2,3} = \begin{bmatrix} \frac{0.69}{C} & \frac{1}{C} \\ -0.1 & -0.1R_3 \end{bmatrix}, \\ &B_\mu = \begin{bmatrix} 0 & 0.1 \end{bmatrix}^T, F_{p,\mu} = \begin{bmatrix} 0.01 & 0.01 \end{bmatrix}^T, \\ &H_{p,\mu} = \begin{bmatrix} 0.2 & 0.1 \end{bmatrix}, D_{p,\mu} = \sin(l), \bar{G}_\epsilon = \operatorname{diag}\{1, \varepsilon_r\}, \\ &\quad p = 1, 2, \mu = 1, 2, 3. \end{aligned}$$

For sampling time $T_\alpha = 0.14$, the discrete model is obtained as

$$\sigma(l+1) = \sum\nolimits_{p=1}^{2} h_p(l)[(\bar{A}_{p,\mu}\bar{G}_\epsilon + \Delta A_{p,\mu})\sigma(l) + \bar{B}_\mu u(l)],$$

where

$$\begin{aligned} &\bar{A}_{1,1} = \begin{bmatrix} 1.28 & 1.4 \\ -0.014 & -0.26 \end{bmatrix}, \bar{A}_{1,2} = \begin{bmatrix} 1.28 & 1.4 \\ -0.014 & -0.68 \end{bmatrix}, \\ &\bar{A}_{1,3} = \begin{bmatrix} 1.28 & 1.4 \\ -0.014 & -1.1 \end{bmatrix}, \bar{A}_{2,1} = \begin{bmatrix} 1.966 & 1.4 \\ -0.014 & -0.26 \end{bmatrix}, \\ &\bar{A}_{2,2} = \begin{bmatrix} 1.966 & 1.4 \\ -0.014 & -0.68 \end{bmatrix}, \bar{A}_{2,3} = \begin{bmatrix} 1.966 & 1.4 \\ -0.014 & -1.1 \end{bmatrix}, \\ &B_\mu = \begin{bmatrix} 0 & 0.014 \end{bmatrix}^T, \mu = 1, 2, 3. \end{aligned}$$

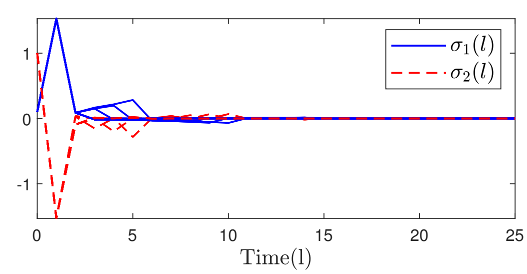

Fig. 2 System state

Consider $T_{max}^1 = 3$, $T_{max}^2 = 3$, $T_{max}^3 = 2$, $\iota_1 = 5$, $\iota_2 = 6$, $\iota_3 = 7$, $\mathcal{F}_{1\mu} = \mathcal{F}_{2\mu} = I_2$, $[\eta_{\mu\tau}] = \begin{bmatrix} 0 & 0.7 & 0.3 \\ 0.4 & 0 & 0.6 \\ \eta_{31} & \eta_{32} & 0 \end{bmatrix}$

and $[\xi_{\mu\tau}(\wp)] = \begin{bmatrix} 0 & \xi_{12}(\wp) & \xi_{13}(\wp) \\ \xi_{21}(\wp) & 0 & \xi_{23}(\wp) \\ \xi_{31}(\wp) & \xi_{32}(\wp) & 0 \end{bmatrix}$. Meanwhile, the known ST-PDF is $\xi_{12}(\wp) = \frac{0.6^{\wp}\cdot 0.4^{10-\wp}\cdot 10!}{(10-\wp)!\cdot \wp!}$, $\xi_{21}(\wp) = 0.9^{(\wp-1)^2} - 0.9^{\wp^2}$, $\xi_{23}(\wp) = \frac{5^{10}\cdot 10!}{(10-\wp)!\cdot \wp!}$, $\xi_{31}(\wp) = 0.4^{(\wp-1)^{1.3}} - 0.4^{\wp^{1.3}}$, and $\xi_{32}(\wp) = 0.3^{(\wp-1)^{0.8}} - 0.3^{\wp^{0.8}}$, while η_{31}, η_{32}, and $\xi_{13}(\wp)$ are supposed to be inaccessible.

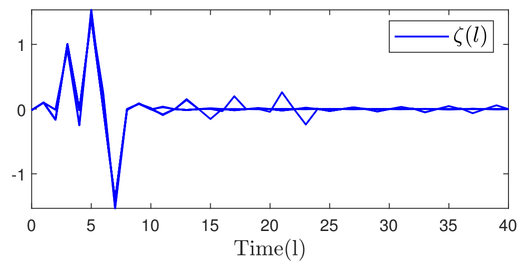

Fig. 3 Sliding function

By solving Theorem 3, through the linear matrix inequality toolbox of MATLAB, after 2.3594s calculation, one has $G_1 = \begin{bmatrix} -0.1636 & 1.0035 \end{bmatrix}$, $G_2 = \begin{bmatrix} -0.0099 & 0.9231 \end{bmatrix}$, $G_3 = \begin{bmatrix} -0.0056 & 0.9118 \end{bmatrix}$.

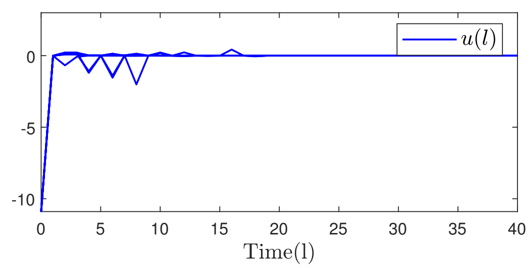

Fig. 4 Control input

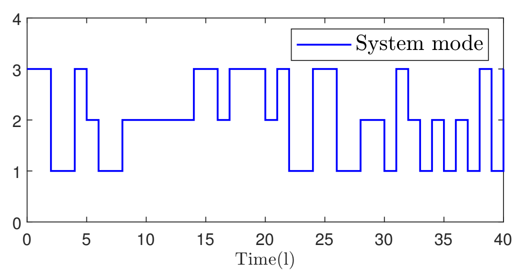

Fig. 5 System mode

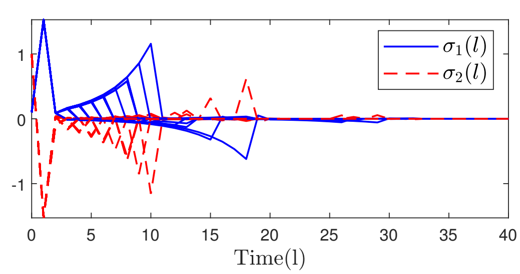

Fig. 6 System state under traditional control [23]

For $\sigma(0) = \begin{bmatrix} 0.1 & 1 \end{bmatrix}^T$, the inductor L_ε serves as the upper bound on SPP with $\varepsilon_r \leq \bar{\varepsilon}_r$. Over 50 realizations, the state depicted in Fig. 2 is achieved, indicating that the voltage and current tend to the origin, ensuring the MSS of the corresponding tunnel diode circuit for $\chi = 0.01$ and $\theta = 0.9$. Fig. 3 describes the sliding function and Fig. 4 plots the controller input. Fig. 5 depicts a randomly generated system mode sequence, which illustrates the impact of time series evolution on the adherence to switching with a semi-Markovian chain. For the same parameters, Fig. 6 shows the system state under traditional control strategy [23]. Compared with traditional control strategy in Fig. 2, the proposed SMC law improves the response speed with an improvement in robustness. Fig. 7 shows the membership functions.

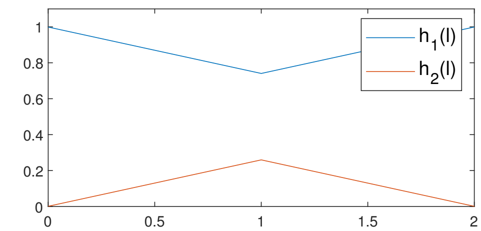

Fig. 7 Membership functions

VII. Conclusion

The issue of fuzzy SMC strategy has been investigated for discrete nonlinear semi-Markovian switching SPSs with incomplete SMK. A fuzzy SMC scheme has been constructed to achieve an effectiveness in the reachability analysis of the sliding region. T-S fuzzy model and singular perturbation parameters have allowed for a more accurate and efficient SMC approach. Establishing the mean-square stability criterion for incomplete SMK information has contributed to a less conservative control strategy. Future research can focus on these findings by exploring the application of this SMC strategy to hidden semi-Markovian systems.

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