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On isomorphism of the space of α-Hölder continuous functions with finite p-th variation.

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Abstract

We study the concept of (generalized) p-th variation of a real-valued continuous function along a general class of refining sequence of partitions. We show that the finiteness of the p-th variation of a given function is closely related to the finiteness of ℓ^p-norm of the coefficients along a Schauder basis, similar to the fact that Hölder coefficient of the function is connected to ℓ^∞-norm of the Schauder coefficients. This result provides an isomorphism between the space of α-Hölder continuous functions with finite (generalized) p-th variation along a given partition sequence and a subclass of infinite-dimensional matrices equipped with an appropriate norm, in the spirit of Ciesielski.

Keywords— p-th variation, Hölder regularity, Ciesielski's isomorphism, Schauder basis, Variation index, Refining partition sequences

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1 Introduction

In the seminal paper [14], Föllmer derived the pathwise Itô's formula for a class of real functions with a finite quadratic variation. In particular, for a twice differentiable function F and a one-dimensional continuous function x with finite quadratic variation along a partition sequence $\pi = (\pi^n)_{n\in\mathbb{N}}$, the pathwise Itô formula is given as

$$F\big(x(t)\big) = F\big(x(0)\big) + \int_0^t F'\big(x(s)\big)d^{\pi}x(s) + \frac{1}{2}\int_0^t F''\big(x(s)\big)d[x]_{\pi}(s). \tag{1.1}$$

Here, the first integral is defined as a left Riemann sum

$$\int_0^t F'\big(x(s)\big)d^{\pi}x(s) := \lim_{n\to\infty}\sum_{\pi^n\ni t_j^n\le t} F'\big(x(t_j^n)\big)\big(x(t_{j+1}^n)-x(t_j^n)\big),$$

and the integrator $[x]_{\pi}(\cdot)$ of the second integral is the quadratic variation of x along the partition sequence π, defined as the following uniform limit in t:

$$[x]_{\pi^n}(t) := \sum_{\pi^n\ni t_j^n\le t}\big|x(t_{j+1}^n)-x(t_j^n)\big|^2 \xrightarrow{n\to\infty} [x]_{\pi}(t). \tag{1.2}$$

This pathwise Itô's formula has been generalized in several aspects [1, 4, 9, 11, 12, 17, 22]. Among these, Cont and Perkowski [11] defined the notion of p-th variation of continuous functions along π by raising the exponent in (1.2) to any even integers $p \in 2\mathbb{N}$, and derived high-order pathwise change-of-variable formula; more recently, Cont and Jin [10] developed fractional pathwise Itô formula for functions with p-th variation for any $p \ge 1$, with a fractional Itô remainder term. These pathwise calculus formulae, including Föllmer's original one (1.1), require the continuous function x to have finite p-th variation along π. In other words, the existence of the limit

$$[x]^{(p)}_{\pi^n}(t) := \sum_{\pi^n\ni t_j^n\le t}\big|x(t_{j+1}^n)-x(t_j^n)\big|^p \xrightarrow{n\to\infty} [x]^{(p)}_{\pi}(t) \tag{1.3}$$

is the crucial assumption when applying these formulae. It is then natural to study a class V^p_{π} of functions x such that the limit (1.3) exists for a fixed partition sequence π and $p \ge 1$.

In this regard, Schied [21] showed that the space V^p_{π} is not a vector space by constructing an example of two continuous functions x and y on $[0,1]$ such that $[x]^{(2)}_{\mathbb{T}}$ and $[y]^{(2)}_{\mathbb{T}}$ exist, but $[x+y]^{(2)}_{\mathbb{T}}$ does not exist, along the dyadic partition sequence $\mathbb{T} = (\mathbb{T}^n)_{n\in\mathbb{N}}$ with $\mathbb{T}^n := \{k2^{-n} : k = 0, 1, \cdots, 2^n\}$. These two functions x and y belong to a class of so-called generalized Takagi functions, constructed via the Schauder representation of continuous functions. From the Schauder representation of x and y along $\mathbb{T}$, one can obtain explicit expressions of both terms in the following strict inequality to show that $[x+y]^{(2)}_{\mathbb{T}}$ does not exist:

$$\liminf_{n\to\infty}[x+y]^{(2)}_{\mathbb{T}^n}(t) < \limsup_{n\to\infty}[x+y]^{(2)}_{\mathbb{T}^n}(t).$$

Since Schied's example implies that requiring the existence of the limit (1.3) restricts the function space V^p_{π} too much, in this paper we study a larger space $\mathcal{X}^p_{\pi} \supset V^p_{\pi}$ of functions x that satisfy

$$\limsup_{n\to\infty}[x]^{(p)}_{\pi^n}(t) = \limsup_{n\to\infty}\sum_{\pi^n\ni t_j^n\le t}\big|x(t_{j+1}^n)-x(t_j^n)\big|^p < \infty, \tag{1.4}$$

but does not require the limit to exist. With an appropriate norm, we prove that the space $\mathcal{X}^p_{\pi}$ is a Banach space (see definition (2.7) and Proposition 2.5 below).

Even though we may not apply the aforementioned pathwise change-of-variable formulae to every function in $\mathcal{X}^p_{\pi}$, we shall study the Banach space $\mathcal{X}^p_{\pi}$, instead of V^p_{π}, because the notion of variation index, i.e., the infimum number $p \ge 1$ such that the condition (1.4) holds (see Definition 2.3 below), can be used for measuring 'roughness' of a given function (or a path of a stochastic process) [2, 6]. It is well known that (almost every path of) a fractional Brownian motion (fBM) B^H with Hurst index $H \in (0,1)$, has Hölder exponent equal to $H-$, whereas its variation index along 'reasonable' partition sequences (e.g., dyadic partition sequence $\mathbb{T}$) is equal to $1/H$. These facts are closely related to the self-similarity property of fBMs, but it is generally not true for general continuous functions that the reciprocal of the variation index is equal to (the supremum of) Hölder exponent. In a recent work [2], a specific example of (1/4)-Hölder continuous function with variation index along the dyadic partition sequence equal to 2 is constructed, thus, the variation index should be considered as an alternative way of measuring function's roughness.

With the help of Schauder representation along a general class of partition sequences, our main result provides a necessary and sufficient condition for elements of the Banach space $\mathcal{X}^p_\pi$, in terms of their Schauder coefficients (see Theorem 4.3). More specifically, the condition (1.4) is equivalent to the ℓ^∞-finiteness of the sequence composed of ℓ^p-norm of Schauder coefficients of functions along each partition π^n, scaled by a $(p/2)$-power of the mesh size of π^n.

When the Schauder coefficients of functions are arranged in an infinite dimensional matrix, this result gives rise to an isomorphism between the space of α-Hölder continuous functions with finite (generalized) p-th variation along a partition sequence π and a subspace of infinite-dimensional matrices with an appropriate matrix norm (see Theorem 5.3). Our isomorphism result reminds that of Ciesielski's in 1960 [5], between the space of α-Hölder continuous functions and the space of bounded real sequences, using Schauder representation along the dyadic partition sequence $\mathbb{T}$, which has been generalized recently by [2] along a wider class of partition sequences.

Preview: This paper is organized as follows. Section 2 introduces the notion of variation index and defines the Banach space $\mathcal{X}^p_\pi$. Section 3 provides some notations and reviews preliminary results regarding Schauder representation of continuous functions. Section 4 states and proves our main result, the characterization of generalized p-th variation in terms of a function's Schauder coefficients. Section 5 includes the isomorphism, as an important consequence of the result. Finally, Appendix A provides an explicit expression of the p-th variation in terms of Schauder coefficients, for a limited case of even integers p along the dyadic partition sequence, which is of independent interest.

2 Variation index and the Banach space $\mathcal{X}^p_\pi$

2.1 p-th variation and variation index

First, we introduce some relevant notations and definitions for partition sequences. For a fixed $T > 0$, we shall consider a (deterministic) sequence of partitions $\pi = (\pi^n)_{n\geq 0}$ of $[0,T]$

$$\pi^n = \left(0 = t^n_0 < t^n_1 < t^n_2 < \cdots < t^n_{N(\pi^n)} = T\right),$$

where we denote $N(\pi^n)$ the number of intervals in the partition π^n. By convention, $\pi^0 = \{0, T\}$. For example, the dyadic partition sequence, denoted by $\mathbb{T} \equiv \pi$, contains partition points $t^n_k = kT/2^n$ for $n \in \mathbb{N}$, $k = 0, \cdots, 2^n$.

Definition 2.1 (Refining sequence of partitions)**.** A sequence of partitions $\pi = (\pi^n)_{n\geq 0}$ is said to be *refining (or nested)*, if $t \in \pi^m$ implies $t \in \cap_{n\geq m}\pi^n$ for every $m \in \mathbb{N}$. In particular, we have $\pi^1 \subseteq \pi^2 \subseteq \cdots$.

For a partition sequence $\pi = (\pi^n)_{n\geq 0}$, we write

$$\underline{\pi^n} := \inf_{i=0,\cdots,N(\pi^n)-1} |t^n_{i+1} - t^n_i|, \qquad |\pi^n| := \sup_{i=0,\cdots,N(\pi^n)-1} |t^n_{i+1} - t^n_i|, \tag{2.1}$$

the size of the smallest and the largest interval of π^n, respectively. In the following, we denote $\Pi([0,T])$ the collection of all refining partition sequences π of $[0,T]$ with vanishing mesh, i.e., $|\pi^n| \to 0$ as $n \to \infty$.

Let us denote $C^0([0,T])$ the space of real-valued continuous functions defined on $[0,T]$. In this subsection, we fix a partition sequence $\pi = (\pi^n)_{n\geq 0} \in \Pi([0,T])$ and $x \in C^0([0,T])$. For $p \geq 1$, we denote

$$[x]^{(p)}_{\pi^n}(t) := \sum_{\pi^n \ni t^n_j \leq t} \left| x(t^n_{j+1}) - x(t^n_j) \right|^p \tag{2.2}$$

the p-th variation of x along a partition π^n for each level $n \in \mathbb{N}$.

Remark 2.2. If there exists a continuous, non-decreasing function $[x]^{(p)}_{\pi}$ such that

$$\lim_{n\to\infty} [x]^{(p)}_{\pi^n}(t) = [x]^{(p)}_{\pi}(t), \qquad \forall\, t \in [0,T], \tag{2.3}$$

then we say x admits finite p-th variation along π, and the above convergence is uniform in t ([11, Definition 1.1 and Lemma 1.3]). We write V^p_π the space of such functions x admitting finite p-th variation along π. In the particular case of $p = 2$ (then V^2_π is often denoted as Q_π) and π given as the dyadic partition sequence $\mathbb{T}$, it is shown in [21, Proposition 2.7] that $V^2_{\mathbb{T}}$ is not a vector space.

Even though the p-th variation of x along a given sequence π defined in Remark 2.2 may not exist, one can always define its variation index along π as the following.

Definition 2.3 (Variation index along a partition sequence, Definition 2.3 of [6])**.** The *variation index* of $x \in C^0([0,T])$ along $\pi \in \Pi([0,T])$ is defined as

$$p^\pi(x) := \inf\big\{p \geq 1 : \limsup_{n\to\infty} [x]^{(p)}_{\pi^n}(T) < \infty\big\}. \tag{2.4}$$

Thanks to the continuity of x, it is straightforward to show

$$\limsup_{n\to\infty} [x]^{(q)}_{\pi^n}(T) = \begin{cases} 0, & q > p^\pi(x), \\ \infty, & q < p^\pi(x), \end{cases} \tag{2.5}$$

Therefore, the definition (2.4) can be formulated as

$$p^\pi(x) = \inf\big\{p \geq 1 \,:\, \limsup_{n\to\infty} [x]^{(p)}_{\pi^n}(T) = 0\big\}.$$

Moreover, since $\limsup_{n\to\infty} [x]^{(p)}_{\pi^n}(T) < \infty$ if and only if $\sup_{n\in\mathbb{N}} [x]^{(p)}_{\pi^n}(T) < \infty$, we also have

$$p^\pi(x) = \inf\big\{p \geq 1 : \sup_{n\in\mathbb{N}} [x]^{(p)}_{\pi^n}(T) < \infty\big\}. \tag{2.6}$$

Now that the quantity $[x]^{(p)}_{\pi^n}(t)$ in (2.2) can be recognized as the p-th power of ℓ^p-norm of the real sequence $\{x(t^n_{j+1}) - x(t^n_j)\}_{t^n_j \in \pi^n,\, t^n_j \leq t}$, we provide the following definition.

Definition 2.4. For $x \in C^0([0,T])$, $p \geq 1$, and $\pi \in \Pi([0,T])$, we denote

$$\|x\|^{(p)}_\pi := |x(0)| + \sup_{n\in\mathbb{N}} \Big([x]^{(p)}_{\pi^n}(T)\Big)^{\frac{1}{p}}$$

and consider the subspace of $C^0([0,T])$:

$$\mathcal{X}^p_\pi := \{x \in C^0([0,T]) \,:\, \|x\|^{(p)}_\pi < \infty\}. \tag{2.7}$$

We say $\mathcal{X}^p_\pi$ is the class of continuous functions with finite *(generalized) p-th variation* along π.

The space $\mathcal{X}^p_\pi$ turns out to be a Banach space, in contrast to the space V^p_π.

Proposition 2.5. *The mapping $\mathcal{X}^p_\pi \ni x \mapsto \|x\|^{(p)}_\pi$ is a norm, and the space $(\mathcal{X}^p_\pi, \|\cdot\|^{(p)}_\pi)$ is a Banach space.*

Proof. We first prove that the mapping is a norm. For any scalar r, the identity $\|rx\|_\pi^{(p)} = |r|\|x\|_\pi^{(p)}$ is straightforward. Thanks to Minkowski's inequality, it is also easy to prove the subadditive property (triangle inequality). These imply, in particular, that $\mathcal{X}_\pi^p$ is a vector space. Finally, if $\|x\|_\pi^{(p)} = 0$, then x has zero value on every partition point t_j^n of π for all j, n. Since $|\pi^n| \to 0$ as $n \to \infty$, the set $P := \bigcup_{n\in\mathbb{N}} \pi^n$ of all partition points of π is dense in $[0, T]$, and the continuity of x with $x(0) = 0$ concludes $x \equiv 0$. This shows that $\|x\|_\pi^{(p)}$ is a norm.

To prove the space $\mathcal{X}_\pi^p$ is a Banach space, we fix a Cauchy sequence $(x_\ell)_{\ell\in\mathbb{N}}$ of $\mathcal{X}_\pi^p$, i.e., for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $\|x_k - x_m\|_\pi^{(p)} < \epsilon$ for all $k, m \geq N$. In particular, for every $k, m \geq N$, we have $|x_k(0) - x_m(0)| < \epsilon$ and

$$[x_k - x_m]_{\pi^n}^{(p)}(T) = \sum_{t_j^n \in \pi^n} \Big| \big(x_k(t_{j+1}^n) - x_m(t_{j+1}^n)\big) - \big(x_k(t_j^n) - x_m(t_j^n)\big) \Big|^p < \epsilon^p \tag{2.8}$$

holds for each $n \in \mathbb{N}$. Since $\{x_\ell(0)\}_{\ell\in\mathbb{N}}$ is a real Cauchy sequence, its limit $\lim_{\ell\to\infty} x_\ell(0) = \tilde{x}(0)$ exists. Moreover, we fix an arbitrary $n \in \mathbb{N}$, then for all indices j such that t_j^n belongs to π^n, we have

$$\Big| \big(x_k(t_{j+1}^n) - x_k(t_j^n)\big) - \big(x_m(t_{j+1}^n) - x_m(t_j^n)\big) \Big|^p = \Big| \big(x_k(t_{j+1}^n) - x_m(t_{j+1}^n)\big) - \big(x_k(t_j^n) - x_m(t_j^n)\big) \Big|^p < \epsilon^p$$

for every $k, m \geq N$, in other words, $\big(x_k(t_{j+1}^n) - x_k(t_j^n)\big)_{k\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$ for each j. Again by the completeness of $\mathbb{R}$, the limit $d(t_j^n) := \lim_{k\to\infty} \big(x_k(t_{j+1}^n) - x_k(t_j^n)\big) \in \mathbb{R}$ exists for each index j and $n \in \mathbb{N}$.

Let us recall the set $P = \bigcup_{n\in\mathbb{N}} \pi^n$ of all partition points of π, and define a function $\tilde{x}$ on P

$$\tilde{x}(t_j^n) = \tilde{x}(0) + \sum_{i=1}^{j-1} d(t_i^n), \qquad \text{for every } t_j^n \in \pi^n \text{ and } n \in \mathbb{N}.$$

Since P is a dense subset of $[0, T]$ and a function defined on a dense set can be extended to a continuous function, there exists $x \in C^0([0, T])$ such that $x(t_j^n) = \tilde{x}(t_j^n)$ holds for all points t_j^n of P. Furthermore, we have $x(0) = \tilde{x}(0) = \lim_{k\to\infty} x_k(0)$ as well as

$$x(t_{j+1}^n) - x(t_j^n) = \tilde{x}(t_{j+1}^n) - \tilde{x}(t_j^n) = d(t_j^n) = \lim_{k\to\infty} \big(x_k(t_{j+1}^n) - x_k(t_j^n)\big),$$

thus $x(t_j^n) = \lim_{k\to\infty} x_k(t_j^n)$ for each $t_j^n \in P$.

Sending $m \to \infty$ in (2.8), we have for each $n \in \mathbb{N}$

$$\sum_{t_j^n \in \pi^n} \Big| \big(x_k(t_{j+1}^n) - x(t_{j+1}^n)\big) - \big(x_k(t_j^n) - x(t_j^n)\big) \Big|^p < \epsilon^p, \qquad \text{for } k \geq N. \tag{2.9}$$

Minkowski's inequality now yields for each $n \in \mathbb{N}$

$$\begin{aligned}\bigg(\sum_{t_j^n \in \pi^n} \Big| x(t_{j+1}^n) - x(t_j^n) \Big|^p \bigg)^{\frac{1}{p}} &\leq \bigg(\sum_{t_j^n \in \pi^n} \Big| \big(x_k(t_{j+1}^n) - x(t_{j+1}^n)\big) - \big(x_k(t_j^n) - x(t_j^n)\big) \Big|^p \bigg)^{\frac{1}{p}} \\ &\quad + \bigg(\sum_{t_j^n \in \pi^n} \Big| x_k(t_{j+1}^n) - x_k(t_j^n) \Big|^p \bigg)^{\frac{1}{p}} \\ &\leq \epsilon + \|x_k\|_\pi^{(p)} < \infty, \qquad \text{for } k \geq N,\end{aligned}$$

and this proves $x \in \mathcal{X}_\pi^p$. Furthermore, the inequality (2.9) implies $\|x_k - x\|_\pi^{(p)} < \epsilon$ for all large enough numbers k. This concludes that the Cauchy sequence $(x_\ell)_{\ell\in\mathbb{N}}$ converges to x in $\|\cdot\|_\pi^{(p)}$ norm. ∎

In line with Proposition 2.5, it is well-known that the space $(C^{0,\alpha}([0, T]), \|\cdot\|_{C^{0,\alpha}})$ of α-Hölder continuous functions, is also a Banach space. We next note the inclusion

$$\mathcal{X}_\pi^p \subset \mathcal{X}_\pi^q, \qquad \text{for } 1 \leq p \leq q < \infty, \tag{2.10}$$

due to the straightforward inequality $([x]^{(q)}_{\pi^n}(T))^{\frac{1}{q}} \le ([x]^{(p)}_{\pi^n}(T))^{\frac{1}{p}}$ for every $n \ge 0$. We conclude this subsection with the following property that adding a function with vanishing p-th variation does not affect the variation index.

Lemma 2.6. *For $x, y \in C^0([0,T])$, $p \ge 1$, $t \in [0,T]$, and $\pi \in \Pi([0,T])$, suppose that*

$$\limsup_{n\to\infty} [y]^{(p)}_{\pi^n}(t) = 0$$

holds. Then, we have

$$\limsup_{n\to\infty} [x]^{(p)}_{\pi^n}(t) < \infty \qquad \text{if and only if} \qquad \limsup_{n\to\infty} [x+y]^{(p)}_{\pi^n}(t) < \infty,$$

therefore $p^\pi(x) = p^\pi(x+y)$. In particular, the identity $[x]^{(p)}_\pi(t) = [x+y]^{(p)}_\pi(t)$ holds, provided that the limit $[x]^{(p)}_\pi(t)$ exists in the sense of Remark 2.2.

Proof. Applying Minkowski's inequality twice yields

$$\big([x]^{(p)}_{\pi^n}(t)\big)^{\frac{1}{p}} - \big([y]^{(p)}_{\pi^n}(t)\big)^{\frac{1}{p}} \le \big([x+y]^{(p)}_{\pi^n}(t)\big)^{\frac{1}{p}} \le \big([x]^{(p)}_{\pi^n}(t)\big)^{\frac{1}{p}} + \big([y]^{(p)}_{\pi^n}(t)\big)^{\frac{1}{p}}.$$

Taking lim sup or lim respectively gives the result. ■

2.2 Variation index along different partition sequences

A continuous function x can have different p-th variations, $[x]^{(p)}_\pi$ and $[x]^{(p)}_\rho$, along two different refining partition sequences π and ρ. In this subsection, we study the variation index of x along different partition sequences. We first introduce Proposition 2.8, inspired by Freedman [15], whose proof needs a preliminary result.

Lemma 2.7. *For any given numbers $q > 1$, $\epsilon > 0$, and $x \in C^0([0,T])$, there exists a finite set $\pi = \{0 = t_0, t_1, \cdots, t_m = T\}$ in $[0,T]$ such that the q-th variation of x along π is less than ϵ, i.e.,*

$$[x]^{(q)}_\pi(T) = \sum_{j=0}^{m-1} \Big| x(t_{j+1}) - x(t_j) \Big|^q < \epsilon.$$

Proof. If $x(0) = x(T)$, then we just take $\pi = \{0, T\}$. Thus, we suppose that $x(T) > x(0)$; the other case $x(T) < x(0)$ can be handled by applying the same argument to $y(t) = x(T-t)$.

We assume without loss of generality that $x(0) = 0$, $T = 1$, and $x(T) = 1$. For given $q > 1$ and $\epsilon > 0$, we choose $N \in \mathbb{N}$ large enough so that $N^{1-q} < \epsilon$, and define $t^N_j := \min\{t \ge 0 : x(t) = j/N\}$ for $j = 0, \cdots, N$. Let $\pi = \{t^N_0, \cdots, t^N_N\}$ if $t^N_N = 1$, or $\pi = \{t^N_0, \cdots, t^N_N, 1\}$ otherwise. Now it is simple to check $[x]^{(q)}_\pi(1) = N^{1-q} < \epsilon$. ■

Proposition 2.8. *For any $x \in C^0([0,T])$, we have*

$$\inf\big\{p^\pi(x) : \pi \in \Pi([0,T])\big\} = 1.$$

Proof. Let us fix $x \in C^0([0,T])$. For any $q > 1$, we shall show that there exists a sequence $\pi = (\pi^n)_{n\ge0} \in \Pi([0,T])$ satisfying

$$[x]^{(q)}_\pi(T) = \limsup_{n\to\infty} [x]^{(q)}_{\pi^n}(T) = 0. \tag{2.11}$$

Then, the identity (2.11), together with (2.5), implies that for any $q > 1$ there exists $\pi \in \Pi([0,T])$ satisfying $p^\pi(x) \le q$, which in turn proves the result.

We choose a decreasing real sequence $\epsilon_n \downarrow 0$, and set $\pi^0 = \{0, T\}$. We shall inductively define π^n for each $n \ge 0$. Suppose π^n is defined, and let ρ^{n+1} be a partition of $[0,T]$ satisfying $\pi^n \subset \rho^{n+1}$ and

$|\rho^{n+1}| \leq \epsilon_{n+1}$. Suppose that ρ^{n+1} has $m+1$ points, dividing $[0,T]$ into m subintervals. From Lemma 2.7, we construct a partition π^{n+1} of $[0,T]$ with $\rho^{n+1} \subset \pi^{n+1}$, such that for each pair $t_j^{\rho^{n+1}}, t_{j+1}^{\rho^{n+1}}$ of consecutive points of ρ^{n+1} we have

$$[x]^{(q)}_{\nu_j^{n+1}} \leq \frac{\epsilon_{n+1}}{m},$$

where $\nu_j^{n+1} := \pi^{n+1} \cap [t_j^{\rho^{n+1}}, t_{j+1}^{\rho^{n+1}}]$ and $[x]^{(q)}_{\nu_j^{n+1}}$ is the q-th variation along ν_j^{n+1} on the interval $[t_j^{\rho^{n+1}}, t_{j+1}^{\rho^{n+1}}]$. Then, we obtain $[x]^{(q)}_{\pi^{n+1}}(T) \leq \epsilon_{n+1}$ and $|\pi^{n+1}| \leq |\rho^{n+1}| \leq \epsilon_{n+1}$, therefore, $\pi = (\pi^n)$ satisfies condition (2.11). ■

On the other hand, the rough path theory asserts that an α-Hölder continuous function $x \in C^{0,\alpha}([0,T])$ has finite $(\frac{1}{\alpha})$-variation, i.e., $\|x\|_{\frac{1}{\alpha}-var} < \infty$, with

$$\|x\|_{p-var} := \left(\sup_{\rho} \sum_{t_j, t_{j+1} \in \rho} \left| x(t_{j+1}) - x(t_j) \right|^p \right)^{\frac{1}{p}},$$

where the supremum is taken over all partitions ρ of $[0,T]$. This implies that for a given refining partition sequence $\pi \in \Pi([0,T])$ with vanishing mesh, the variation index $p^\pi(x)$ of $x \in C^{0,\alpha}([0,T])$ should be bounded above by the reciprocal of its Hölder exponent α (see Lemma 4.3 of [2] for the proof), namely

$$p^\pi(x) \leq \frac{1}{\alpha}.$$

We formalize the above arguments into the following theorem.

Theorem 2.9. *For any $x \in C^0([0,T])$, we have*

$$\inf \left\{ p^\pi(x) : \pi \in \Pi([0,T]) \right\} = 1.$$

Moreover, for any $x \in C^{0,\alpha}([0,T])$, we have

$$\sup \left\{ p^\pi(x) : \pi \in \Pi([0,T]) \right\} \leq \frac{1}{\alpha}. \tag{2.12}$$

This result implies that an α-Hölder continuous function x can have any variation index $p^\pi(x)$ between 1 and $1/\alpha$, along a given partition sequence $\pi \in \Pi([0,T])$. Moreover, the inclusion (2.10) shows that $x \in \mathcal{X}^q_\pi$ for any $q > p^\pi(x)$.

Example 1. The inequality (2.12) can be strict. Consider the increasing function $y(t) = \sqrt{t}$ defined on $[0,1]$, which is $\frac{1}{2}$-Hölder continuous. The function y has finite 1-variation along any partition sequence π, thus $p^\pi(y) = 1$, as it is an increasing function. □

Example 2. A uniformly continuous function z defined on $[0, \frac{1}{2}]$

$$z(t) = \begin{cases} \frac{1}{\log t}, & t \in (0, \frac{1}{2}], \\ 0, & t = 0, \end{cases}$$

is not α-Hölder continuous for any $\alpha > 0$. However, it is a decreasing function on the compact support, thus of bounded variation. As in the previous example, $p^\pi(z) = 1$ for every $\pi \in \Pi([0, \frac{1}{2}])$, which implies the left-hand side of (2.12) for z is 1. □

In what follows, we shall characterize conditions for x to belong to the Banach space $\mathcal{X}^p_\pi$, in terms of the Schauder coefficients of x along π.

3 Schauder representation along a general class of partition sequences

In this section, we provide several definitions and preliminary results, mostly taken from [7, 8], regarding Schauder representation of continuous functions along a general class of partition sequences. This type of representation was originally introduced by Schauder [20]. After that, we shall provide our results in the next sections.

3.1 Properties of partition sequence

Let us recall Definition 2.1 and the notations (2.1). We introduce a subclass of refining sequence of partitions with a 'finite branching' property at every level $n \in \mathbb{N}$.

Definition 3.1 (Finitely refining sequence of partitions)**.** A sequence of partitions $\pi = (\pi^n)_{n\geq 0}$ in $\Pi([0,T])$ is said to be *finitely refining*, if there exists a positive integer M such that the number of partition points of π^{n+1} within any two consecutive partition points of π^n is always bounded above by M, irrespective of $n \geq 0$. In particular, we have $\sup_{n\geq 0} \frac{N(\pi^n)}{M^n} \leq 1$.

The following definition provides a condition that the ratio of the biggest step size to the smallest step size at each level is bounded.

Definition 3.2 (Balanced sequence of partitions)**.** A sequence of partitions $\pi = (\pi^n)_{n\geq 0}$ is said to be *balanced*, if there exists a constant $c > 1$ such that

$$\frac{|\pi^n|}{\underline{\pi^n}} \leq c \tag{3.1}$$

holds for every $n \in \mathbb{N}$.

We now give two conditions of refining partition sequences involving the biggest step sizes of two consecutive levels.

Definition 3.3 (Complete refining sequence of partitions)**.** A finitely refining sequence of partitions $\pi = (\pi^n)_{n\geq 0}$ is said to be *complete refining*, if there exist positive constants a and b such that

$$1 + a \leq \frac{|\pi^n|}{|\pi^{n+1}|} \leq b \tag{3.2}$$

holds for every $n \in \mathbb{N}$.

Definition 3.4 (Convergent refining sequence of partitions)**.** A complete refining sequence of partitions is said to be *convergent refining*, if the following limit exists:

$$\lim_{n\to\infty} \frac{|\pi^n|}{|\pi^{n+1}|} = r \in (1, \infty). \tag{3.3}$$

Remark 3.5 (Notation)**.** Throughout this paper, we shall use the same symbols $M, c, a, b,$ and r to refer to the constants that appeared in Definitions 3.1 - 3.4.

3.2 Generalized Haar basis and Schauder representation

This subsection recalls some relevant definitions of generalized Haar and Schauder functions, which were introduced in [7].

Let us fix $\pi \in \Pi([0,T])$ and denote $p(n,k) := \inf\{j \geq 0 : t_j^{n+1} \geq t_k^n\}$. Since π is refining, we have the following inequality for every $k = 0, \cdots, N(\pi^n) - 1$

$$0 \leq t_k^n = t_{p(n,k)}^{n+1} < t_{p(n,k)+1}^{n+1} < \cdots < t_{p(n,k+1)}^{n+1} = t_{k+1}^n \leq T. \tag{3.4}$$

With the notation $\Delta_{i,j}^n := t_j^n - t_i^n$, we now define the generalized Haar basis associated with π.

Definition 3.6 (Generalized Haar basis)**.** The *generalized Haar basis* associated with a finitely refining sequence $\pi = (\pi^n)_{n\geq 0}$ of partitions is a collection of piecewise constant functions $\{\psi^\pi_{m,k,i} : m = 0, 1, \cdots, \ k = 0, \cdots, N(\pi^m) - 1, \ i = 1, \cdots, p(m, k+1) - p(m, k)\}$ defined as follows:

$$\psi^\pi_{m,k,i}(t) = \begin{cases} 0, & \text{if } t \notin \left[t^{m+1}_{p(m,k)}, t^{m+1}_{p(m,k)+i}\right) \\ \left(\frac{\Delta^{m+1}_{p(m,k)+i-1,p(m,k)+i}}{\Delta^{m+1}_{p(m,k),p(m,k)+i-1}} \times \frac{1}{\Delta^{m+1}_{p(m,k),p(m,k)+i}}\right)^{\frac{1}{2}}, & \text{if } t \in \left[t^{m+1}_{p(m,k)}, t^{m+1}_{p(m,k)+i-1}\right) \\ -\left(\frac{\Delta^{m+1}_{p(m,k),p(m,k)+i-1}}{\Delta^{m+1}_{p(m,k)+i-1,p(m,k)+i}} \times \frac{1}{\Delta^{m+1}_{p(m,k),p(m,k)+i}}\right)^{\frac{1}{2}}, & \text{if } t \in \left[t^{m+1}_{p(m,k)+i-1}, t^{m+1}_{p(m,k)+i}\right) \end{cases}. \tag{3.5}$$

We note that the function values of $\psi^\pi_{m,k,i}$ are chosen to satisfy $\int \psi^\pi_{m,k,i}(t)dt = 0$ and $\int (\psi^\pi_{m,k,i}(t))^2 dt = 1$ so that the collection $\{\psi^\pi_{m,k,i}\}$ is an orthonormal basis in $L^2([0, T])$. The Schauder functions $e^\pi_{m,k,i} : [0, T] \to \mathbb{R}$ are obtained by integrating the generalized Haar basis:

$$e^\pi_{m,k,i}(t) := \int_0^t \psi^\pi_{m,k,i}(s)ds = \left(\int_{t^{m+1}_{p(m,k)}}^{t\wedge t^{m+1}_{p(m,k)+i}} \psi^\pi_{m,k,i}(s)ds\right) \mathbb{1}_{[t^m_k, t^{m+1}_{p(m,k)+i}]}(t).$$

To further simplify the notations in what follows, we introduce

$$t^{m,k,i}_1 := t^{m+1}_{p(m,k)}, \qquad t^{m,k,i}_2 := t^{m+1}_{p(m,k)+i-1}, \qquad t^{m,k,i}_3 := t^{m+1}_{p(m,k)+i},$$
$$\Delta^{m,k,i}_1 := \Delta^{m+1}_{p(m,k),p(m,k)+i-1} = t^{m,k,i}_2 - t^{m,k,i}_1, \qquad \Delta^{m,k,i}_2 := \Delta^{m+1}_{p(m,k)+i-1,p(m,k)+i} = t^{m,k,i}_3 - t^{m,k,i}_2.$$

Definition 3.7 (Generalized Schauder function)**.** For every index m, k, i of Definition 3.6, the following function $e^\pi_{m,k,i}$ is called *generalized Schauder function* associated with $\pi = (\pi^n)_{n\geq 0}$:

$$e^\pi_{m,k,i}(t) = \begin{cases} 0, & \text{if } t \notin [t^{m,k,i}_1, t^{m,k,i}_3) \\ \left(\frac{\Delta^{m,k,i}_2}{\Delta^{m,k,i}_1} \times \frac{1}{\Delta^{m,k,i}_1 + \Delta^{m,k,i}_2}\right)^{\frac{1}{2}} \times (t - t^{m,k,i}_1), & \text{if } t \in [t^{m,k,i}_1, t^{m,k,i}_2) \\ \left(\frac{\Delta^{m,k,i}_1}{\Delta^{m,k,i}_2} \times \frac{1}{\Delta^{m,k,i}_1 + \Delta^{m,k,i}_2}\right)^{\frac{1}{2}} \times (t^{m,k,i}_3 - t), & \text{if } t \in [t^{m,k,i}_2, t^{m,k,i}_3) \end{cases}. \tag{3.6}$$

Note that generalized Schauder functions are continuous, triangle-shaped (and not differentiable) functions. The following result shows that any continuous function defined on $[0, T]$ admits a unique Schauder representation along a given partition sequence π.

Proposition 3.8 (Theorem 3.8 of [7])**.** *Let π be a finitely refining partition sequence of $[0, T]$. Then, every continuous function $x : [0, T] \to \mathbb{R}$ has a unique Schauder representation along π:*

$$x(t) = x(0) + \big(x(T) - x(0)\big)t + \sum_{m=0}^{\infty} \sum_{k=0}^{N(\pi^m)-1} \sum_{i=1}^{p(m,k+1)-p(m,k)} \theta^{x,\pi}_{m,k,i} e^\pi_{m,k,i}(t), \qquad \forall\, t \in [0, T], \tag{3.7}$$

with a closed-form representation of the Schauder coefficient

$$\theta^{x,\pi}_{m,k,i} = \frac{\big(x(t^{m,k,i}_2) - x(t^{m,k,i}_1)\big)(t^{m,k,i}_3 - t^{m,k,i}_2) - \big(x(t^{m,k,i}_3) - x(t^{m,k,i}_2)\big)(t^{m,k,i}_2 - t^{m,k,i}_1)}{\sqrt{(t^{m,k,i}_2 - t^{m,k,i}_1)(t^{m,k,i}_3 - t^{m,k,i}_2)(t^{m,k,i}_3 - t^{m,k,i}_1)}}. \tag{3.8}$$

Remark 3.9. A family of Schauder functions $\{e^\pi_{m,k,i}\}_{m,k,i}$ in Definition 3.7 can be reordered as $\{e^\pi_{m,k}\}_{m,k}$, such that for each $m \geq 0$ the values of k run from 0 to $N(\pi^{m+1}) - N(\pi^m) - 1$ after reordering. We shall frequently use this reordering to simplify the notation and denote the index set

$$I_m := \{0, 1, \cdots, N(\pi^{m+1}) - N(\pi^m) - 1\} \tag{3.9}$$

for each m. The corresponding Schauder coefficients $\{\theta^{x,\pi}_{m,k,i}\}_{m,k,i}$ in Proposition (3.8) can be reordered as $\{\theta^{x,\pi}_{m,k}\}_{m,k}$ for $k \in I_m$ and $m \geq 0$ in the same manner.

4 Characterization of variation index

In this section, we characterize the variation index $p^\pi(x)$ of $x \in C^0([0,T])$ along $\pi \in \Pi([0,T])$, in terms of the Schauder coefficients $\{\theta^{x,\pi}_{m,k}\}_{m,k}$ introduced in Section 3.2. We recall the definition (2.2) of the p-th variation, as well as Definitions 3.1-3.4.

Remark 4.1. Any $x \in C^0([0,T])$ can be translated to $\bar{x} \in C^0([0,T])$ with $\bar{x}(0) = \bar{x}(T) = 0$, by adding a linear function. For any $p > 1$, the p-th variation of a linear function y along any element $\pi = (\pi^n)_{n\geq 0}$ of $\Pi([0,T])$ is zero, i.e., $\limsup_{n\to\infty}[y]^{(p)}_{\pi^n} = 0$. Moreover, the subadditive property of the norm $\|\cdot\|^{(p)}_\pi$ in Definition 2.4 implies $\|\bar{x}\|^{(p)}_\pi < \infty$ if and only if $\|x\|^{(p)}_\pi < \infty$. Since we are only interested in the conditions regarding the finiteness of $\|x\|^{(p)}_\pi$-norm (or $\limsup_{n\to\infty}[x]^{(p)}_{\pi^n}$), we shall assume without loss of generality $x(0) = x(T) = 0$ in what follows. Then, the Schauder representation (3.7) of any $x \in C^0([0,T])$ becomes simpler:

$$x(t) = \sum_{m=0}^{\infty} \sum_{k=0}^{N(\pi^m)-1} \sum_{i=1}^{p(m,k+1)-p(m,k)} \theta^{x,\pi}_{m,k,i} e^{\pi}_{m,k,i}(t), \qquad \forall\, t \in [0,T]. \tag{4.1}$$

The above triple sum can be expressed as a double sum after re-indexing as in Remark 3.9.

4.1 Results

We provide Proposition 4.2 and Theorem 4.3 below, and their proofs are given in the next subsection.

Proposition 4.2. *For any $p > 1$, $x \in C^0([0,T])$, and a balanced, complete refining partition sequence $\pi = (\pi^n)_{n\geq 0}$ of $[0,T]$, we denote*

$$\eta^{\pi,(p)}_n := |\pi^n|^{p-1} \left(\sum_{m=0}^{n-1} |\pi^m|^{\frac{1}{p}-\frac{1}{2}} \left(\sum_{k\in I_m} |\theta^{x,\pi}_{m,k}|^p \right)^{\frac{1}{p}} \right)^p. \tag{4.2}$$

Then, we have

$$\limsup_{n\to\infty} [x]^{(p)}_{\pi^n}(T) < \infty \quad \textit{if and only if} \quad \limsup_{n\to\infty} \eta^{\pi,(p)}_n < \infty. \tag{4.3}$$

For any balanced, complete refining partition sequence π, Proposition 4.2 immediately provides the sufficient and necessary condition for $x \in C^0([0,T])$ to belong to the Banach space $\mathcal{X}^p_\pi$ in (2.7), in terms of its Schauder coefficients through the sequence $(\eta^{\pi,(p)}_n)_{n\geq 0}$:

$$x \in \mathcal{X}^p_\pi \quad \Longleftrightarrow \quad \limsup_{n\to\infty} \eta^{\pi,(p)}_n < \infty.$$

Moreover, it also yields the equivalent formulation of the variation index in (2.4):

$$p^\pi(x) = \inf\big\{p > 1 : \limsup_{n\to\infty} \eta^{\pi,(p)}_n < \infty\big\}. \tag{4.4}$$

Thus, the (lim sup)-finiteness of the sequence $(\eta^{\pi,(p)}_n)_{n\geq 0}$ can provide useful path property of x along any balanced, complete refining partition sequences, and each term $\eta^{\pi,(p)}_n$ contains the Schauder coefficients of x up to level $n-1$, namely $\{\theta^{x,\pi}_{m,k}\}_{m=0,\cdots,n-1,\, k\in I_m}$. However, with nominal additional conditions on the partition sequence, we have a much simpler condition involving Schauder coefficients.

Theorem 4.3. *For any $p > 1$, $x \in C^0([0,T])$, and a balanced, convergent refining partition sequence $\pi = (\pi^n)_{n\geq 0}$ of $[0,T]$, we denote*

$$\xi^{\pi,(p)}_n = |\pi^n|^{\frac{p}{2}} \left(\sum_{k\in I_n} |\theta^{x,\pi}_{n,k}|^p \right), \qquad \forall\, n \geq 0. \tag{4.5}$$

Then, we have

$$\limsup_{n\to\infty} [x]^{(p)}_{\pi^n}(T) < \infty \quad \textit{if and only if} \quad \limsup_{n\to\infty} \xi^{\pi,(p)}_n < \infty. \tag{4.6}$$

Thus, we also have

$$x \in \mathcal{X}^p_\pi \quad \textit{if and only if} \quad \limsup_{n\to\infty} \xi^{\pi,(p)}_n < \infty.$$

In the definition (4.5), the quantity $\xi^{\pi,(p)}_n$ only contains the Schauder coefficients $\{\theta^{x,\pi}_{n,k}\}_{k\in I_n}$ of x that belong to the n-th level, for each $n \in \mathbb{N}$. Theorem 4.3 also provides a similar equivalent formulation of the variation index in (2.4).

Corollary 4.4. *Let π be a balanced, convergent refining partition sequence. Then, we have*

$$p^\pi(x) = \inf\big\{p > 1 : \limsup_{n\to\infty} \xi^{\pi,(p)}_n < \infty\big\}. \tag{4.7}$$

Remark 4.5. In all of the previous results, we considered the (generalized) p-th variation up to the terminal time T. However, we can derive similar results for any partition points $t \in \cup_{n\in\mathbb{N}}\pi^n$. For $x \in C^0([0,T])$, let us recall the definition (1.3) of $[x]^{(p)}_{\pi^n}(t)$ such that the mapping $t \mapsto \limsup_{n\to\infty} [x]^{(p)}_{\pi^n}(t)$ is nondecreasing. We also introduce the notations

$$\eta^{\pi,(p)}_n(t) := |\pi^n|^{p-1}\left(\sum_{m=0}^{n-1} |\pi^m|^{\frac{1}{p}-\frac{1}{2}}\Bigg(\sum_{\substack{k\in I_m \\ supp(e^\pi_{m,k})\subset[0,t]}} |\theta^{x,\pi}_{m,k}|^p\Bigg)^{\frac{1}{p}}\right)^p, \tag{4.8}$$

$$\xi^{\pi,(p)}_n(t) := |\pi^n|^{\frac{p}{2}}\Bigg(\sum_{\substack{k\in I_n \\ supp(e^\pi_{n,k})\subset[0,t]}} |\theta^{x,\pi}_{n,k}|^p\Bigg). \tag{4.9}$$

Then, the results (4.3) and (4.6) can be replaced by

$$\limsup_{n\to\infty} [x]^{(p)}_{\pi^n}(t) < \infty \quad \text{if and only if} \quad \limsup_{n\to\infty} \eta^{\pi,(p)}_n(t) < \infty, \quad \text{and} \tag{4.10}$$

$$\limsup_{n\to\infty} [x]^{(p)}_{\pi^n}(t) < \infty \quad \text{if and only if} \quad \limsup_{n\to\infty} \xi^{\pi,(p)}_n(t) < \infty, \quad \text{for every } t \in \cup_{n\in\mathbb{N}}\pi^n. \tag{4.11}$$

To show (4.10) and (4.11), we first define a 'stopped function' $x_t(s) := x(t\wedge s)$ for $s \in [0,T]$. Furthermore, we define

$$\widetilde{\theta}^{x,\pi}_{m,k} := \begin{cases} \theta^{x,\pi}_{m,k}, & \text{if } \mathrm{supp}(e^\pi_{m,k}) \subset [0,t], \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\widetilde{x}(t) := \sum_{m=0}^{\infty}\sum_{k\in I_m} \widetilde{\theta}^{x,\pi}_{m,k} e^\pi_{m,k}(t).$$

For $t \in \cup_{n\in\mathbb{N}}\pi^n =: P$, the two functions x_t and $\widetilde{x}$ differ only by a finite sum of piecewise linear functions, say y, which hence satisfies $[y]^{(p)}_\pi \equiv 0$ for every $p > 1$. Lemma 2.6 therefore yields that $\limsup_{n\to\infty} [\widetilde{x}]^{(p)}_{\pi^n}(T) = \limsup_{n\to\infty} [x_t]^{(p)}_{\pi^n}(T) = \limsup_{n\to\infty} [x]^{(p)}_{\pi^n}(t)$. Now applying Proposition 4.2 and Theorem 4.3 to $\widetilde{x}$ with the quantities (4.8) and (4.9), proves (4.10) and (4.11).

For $t \notin P$, we can choose a point $s \in P$ which is sufficiently close and bigger than t, and check the finiteness of $\limsup_{n\to\infty} \eta^{\pi,(p)}_n(s)$, or $\limsup_{n\to\infty} \xi^{\pi,(p)}_n(s)$, to conclude the finiteness $\limsup_{n\to\infty}[x]^{(p)}_{\pi^n}(t) \le \limsup_{n\to\infty}[x]^{(p)}_{\pi^n}(s) < \infty$.

4.2 Proofs

Before proving Proposition 4.2 and Theorem 4.3, we first introduce some preliminary lemmata.

Lemma 4.6. *Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be real sequences such that $b_n > 0$, $\frac{b_{n+1}}{b_n} =: \beta_n > 1$ for every $n \in \mathbb{N}$, and the limit $\lim_{n\to\infty} \beta_n = \beta > 1$ exists. Then, we have the inequality*

$$\limsup_{n\to\infty} \left(\frac{a_{n+1} - a_n}{b_{n+1} - b_n} \right) \leq \frac{\beta}{\beta - 1} \limsup_{n\to\infty} \left(\frac{a_{n+1}}{b_{n+1}} \right) - \frac{1}{\beta - 1} \liminf_{n\to\infty} \left(\frac{a_n}{b_n} \right). \tag{4.12}$$

Proof of Lemma 4.6. Taking lim sup to the both sides of the following identity

$$\frac{a_{n+1} - a_n}{b_{n+1} - b_n} = \frac{1}{\frac{b_{n+1}}{b_n} - 1} \left(\frac{a_{n+1}}{b_{n+1}} \times \frac{b_{n+1}}{b_n} - \frac{a_n}{b_n} \right) = \frac{1}{\beta_n - 1} \left(\beta_n \frac{a_{n+1}}{b_{n+1}} - \frac{a_n}{b_n} \right) \tag{4.13}$$

with the following properties for any real sequences $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}$ proves the result:

$$\limsup_{n\to\infty} (x_n + y_n) \leq \limsup_{n\to\infty} x_n + \limsup_{n\to\infty} y_n, \qquad \limsup_{n\to\infty} (-x_n) = -\liminf_{n\to\infty} x_n, \tag{4.14}$$
$$\limsup_{n\to\infty} (x_n y_n) = \big(\lim_{n\to\infty} x_n \big) \big(\limsup_{n\to\infty} y_n \big), \quad \text{provided that } \lim_{n\to\infty} x_n \text{ exists and is positive.}$$

■

Lemma 4.7. *Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be real sequences such that $(b_n)_{n\in\mathbb{N}}$ is strictly increasing and $\lim_{n\to\infty} b_n = \infty$. Then, we have the following inequalities*

$$\liminf_{n\to\infty} \left(\frac{a_{n+1} - a_n}{b_{n+1} - b_n} \right) \leq \liminf_{n\to\infty} \left(\frac{a_n}{b_n} \right) \leq \limsup_{n\to\infty} \left(\frac{a_n}{b_n} \right) \leq \limsup_{n\to\infty} \left(\frac{a_{n+1} - a_n}{b_{n+1} - b_n} \right). \tag{4.15}$$

Proof of Lemma 4.7. The middle inequality is obvious. We shall show the last inequality; the first inequality then follows from (4.14). If the right-most term of (4.15) diverges to infinity, there is nothing to show. Thus, we assume

$$\limsup_{n\to\infty} \left(\frac{a_{n+1} - a_n}{b_{n+1} - b_n} \right) = L < \infty.$$

For any $r > L$, there exists $N \in \mathbb{N}$ such that

$$\frac{a_{n+1} - a_n}{b_{n+1} - b_n} < r, \qquad \text{or} \qquad a_{n+1} - a_n < r(b_{n+1} - b_n),$$

holds for every $n > N$. Fix an arbitrary integer m greater than N, and sum up the last inequalities for $n = N, \cdots, m - 1$ to obtain

$$a_m - a_N = \sum_{n=N}^{m-1} (a_{n+1} - a_n) < r \sum_{n=N}^{m-1} (b_{n+1} - b_n) = r(b_m - b_N), \quad \text{thus} \quad \frac{a_m - a_N}{b_m} < r - r\frac{b_N}{b_m}.$$

Sending m to infinity and using the fact $\lim_{m\to\infty} b_m = \infty$ yields the inequality

$$\limsup_{m\to\infty} \left(\frac{a_m}{b_m} \right) < r.$$

Since this should hold for any $r > L$, we conclude that the last inequality of (4.15) holds. ■

Lemma 4.8. *Let $A = (a_{n,m})_{n\geq 0, m\geq 0}$ be an infinite-dimensional matrix satisfying the following properties:*

(i) $\lim_{n\to\infty} a_{n,m} = 0$ *for every* $m \geq 0$*;*

(ii) $\lim_{n\to\infty} \sum_{m=0}^{\infty} a_{n,m} = 1$*;*

(iii) $\sup_{n\geq 0} \sum_{m=0}^{\infty} |a_{n,m}| < \infty$.

Then, for any real sequence $(s_n)_{n\geq 0}$ with nonnegative terms, i.e., $s_n \geq 0$ for all $n \geq 0$, we have

$$\limsup_{n\to\infty} \sum_{m=0}^{\infty} a_{n,m}s_m \leq \limsup_{n\to\infty} s_n. \tag{4.16}$$

Remark 4.9. We note that Lemma 4.8 was inspired by the Silverman-Toeplitz Theorem (see, e.g., [3]), which states that the real sequence $(s_n)_{n\geq 0}$ converges to s, if and only if

$$\lim_{n\to\infty} \Big(\sum_{m=0}^{n} a_{n,m}s_m \Big) = s, \tag{4.17}$$

for $A = (a_{n,m})_{n\geq 0, m\geq 0}$ satisfying the conditions of Lemma 4.8.

Proof of Lemma 4.8. If $\limsup_{n\to\infty} s_n = \infty$, then there is nothing to prove; thus, we assume $\limsup_{n\to\infty} s_n =: s < \infty$. This implies that there exists $K < \infty$ such that $s_n \leq K$ for all $n \geq 0$. We denote $L := \sup_{n\geq 0} \sum_{m=0}^{\infty} |a_{n,m}| < \infty$ in condition (iii), and fix an arbitrary $\epsilon > 0$. Then, there exists $M_1 \in \mathbb{N}$ such that

$$s_m \leq s + \frac{\epsilon}{4L}, \qquad \text{for every } m > M_1. \tag{4.18}$$

Condition (i) implies that there exist constants $N_0, N_1, \cdots, N_{M_1}$ such that

$$|a_{n,m}| \leq \frac{\epsilon}{4(M_1+1)(K+1)}, \qquad \text{for every } 0 \leq m \leq M_1 \text{ and } n > N_m.$$

Set $\tilde{N} := \max\{N_0, N_1, \cdots, N_{M_1}\}$, then

$$\sum_{m=0}^{M_1} a_{n,m}s_m \leq \sum_{m=0}^{M_1} |a_{n,m}s_m| \leq \sum_{m=0}^{M_1} \frac{s_m \epsilon}{4(M_1+1)(K+1)} < \frac{\epsilon}{4}, \qquad \text{for every } n > \tilde{N}.$$

On the other hand, we have from (4.18)

$$\sum_{m=M_1+1}^{\infty} a_{n,m}s_m \leq s \sum_{m=M_1+1}^{\infty} |a_{n,m}| + \frac{\epsilon}{4L} \sum_{m=M_1+1}^{\infty} |a_{n,m}| \leq s \sum_{m=M_1+1}^{\infty} |a_{n,m}| + \frac{\epsilon}{4}.$$

Combining the last two inequalities,

$$\sum_{m=0}^{\infty} a_{n,m}s_m = \sum_{m=0}^{M_1} a_{n,m}s_m + \sum_{m=M_1+1}^{\infty} a_{n,m}s_m \leq s \sum_{m=M_1+1}^{\infty} |a_{n,m}| + \frac{\epsilon}{2} \quad \text{for every } n > \tilde{N}. \tag{4.19}$$

We now claim that $(\sum_{m=0}^{\infty} a_{n,m}s_n)_{n\geq 0}$ is an absolutely convergence sequence

$$\sum_{m=0}^{\infty} |a_{n,m}s_m| \leq K \sum_{m=0}^{\infty} |a_{n,m}| \leq KL < \infty,$$

thanks to condition (iii). Therefore, taking the limit as $n \to \infty$ in (4.19), together with condition (ii), we conclude

$$\lim_{n\to\infty} \sum_{m=0}^{\infty} a_{n,m}s_m \leq s + \frac{\epsilon}{2}.$$

Since ϵ is chosen arbitrarily, this proves the result. ∎

We are now ready to prove Proposition 4.2 and Theorem 4.3.

Proof of Proposition 4.2. Using the Schauder representation (4.1), we expand the p-th variation of x along π^n for each $n \in \mathbb{N}$

$$
\begin{aligned}
[x]^{(p)}_{\pi^n}(T) &= \sum_{\ell=0}^{N(\pi^n)-1} \left| x(t^n_{\ell+1}) - x(t^n_\ell) \right|^p \qquad (4.20)\\
&= \sum_{\ell=0}^{N(\pi^n)-1} \left| \sum_{m=0}^{n-1} \sum_{k=0}^{N(\pi^m)-1} \sum_{i=1}^{p(m,k+1)-p(m,k)} \theta^{x,\pi}_{m,k,i} \Big(e_{m,k,i}(t^n_{\ell+1}) - e_{m,k,i}(t^n_\ell) \Big) \right|^p .
\end{aligned}
$$

Since π is finitely refining, for each fixed pair (m, ℓ) with $m < n$ and $\ell < N(\pi^n)$, the cardinality of the set $I(m,\ell) := \{(k,i) : e_{m,k,i}(t^n_{\ell+1}) - e_{m,k,i}(t^n_\ell) \neq 0\}$ has an upper bound M. Also, in Definition 3.7, we note that

$$
\underline{\pi^{m+1}} \le \Delta^{m,k,i}_1 \le M|\pi^{m+1}|, \qquad \underline{\pi^{m+1}} \le \Delta^{m,k,i}_2 \le |\pi^{m+1}|,
$$

as $\Delta^{m,k,i}_1$ is a length of an interval containing at most M many consecutive intervals of π^{m+1}, whereas $\Delta^{m,k,i}_2$ is a length of a single interval of π^{m+1}. From the balanced and complete refining property, we have

$$
\begin{aligned}
\left| e_{m,k,i}(t^n_{\ell+1}) - e_{m,k,i}(t^n_\ell) \right| &\le \frac{1}{\sqrt{\Delta^{m,k,i}_1 + \Delta^{m,k,i}_2}} \left(\max\left(\sqrt{\frac{\Delta^{m,k,i}_2}{\Delta^{m,k,i}_1}}, \sqrt{\frac{\Delta^{m,k,i}_1}{\Delta^{m,k,i}_2}} \right) \right) |\pi^n| \\
&\le \frac{1}{\sqrt{\underline{\pi^{m+1}}}} \sqrt{\frac{M|\pi^{m+1}|}{\underline{\pi^{m+1}}}} |\pi^n| \le \frac{\sqrt{cM}}{\sqrt{\underline{\pi^{m+1}}}} |\pi^n| \le \frac{c\sqrt{M}}{\sqrt{|\pi^{m+1}|}} |\pi^n| = \frac{c\sqrt{bM}|\pi^n|}{\sqrt{|\pi^m|}}.
\end{aligned}
$$

Thus, we have from (4.20)

$$
\begin{aligned}
[x]^{(p)}_{\pi^n}(T) &\le \sum_{\ell=0}^{N(\pi^n)-1} \left| \sum_{m=0}^{n-1} M \Big(\max_{(k,i)\in I(m,\ell)} |\theta^{x,\pi}_{m,k,i}| \Big) \frac{c\sqrt{bM}|\pi^n|}{\sqrt{|\pi^m|}} \right|^p \\
&= \Big(Mc\sqrt{bM}|\pi^n| \Big)^p \sum_{\ell=0}^{N(\pi^n)-1} \left| \sum_{m=0}^{n-1} \Big(\max_{(k,i)\in I(m,\ell)} |\theta^{x,\pi}_{m,k,i}| \Big) |\pi^m|^{-\frac{1}{2}} \right|^p =: Q_n.
\end{aligned}
$$

We now set $\epsilon := p - \lfloor p \rfloor$ and expand the $\lfloor p \rfloor$-th power to obtain

$$
\begin{aligned}
\frac{Q_n}{\big(Mc\sqrt{bM}|\pi^n|\big)^p} &= \sum_{\ell=0}^{N(\pi^n)-1} \left| \sum_{m=0}^{n-1} \Big(\max_{(k,i)\in I(m,\ell)} |\theta^{x,\pi}_{m,k,i}| \Big) |\pi^m|^{-\frac{1}{2}} \right|^{\lfloor p \rfloor} \left| \sum_{m=0}^{n-1} \Big(\max_{(k,i)\in I(m,\ell)} |\theta^{x,\pi}_{m,k,i}| \Big) |\pi^m|^{-\frac{1}{2}} \right|^{\epsilon} \\
&= \sum_{\ell=0}^{N(\pi^n)-1} \sum_{0 \le m_1, \cdots, m_{\lfloor p \rfloor} \le n-1} \left(\prod_{j=1}^{\lfloor p \rfloor} \Big(\max_{(k,i)\in I(m_j,\ell)} |\theta^{x,\pi}_{m_j,k,i}| \Big) |\pi^{m_j}|^{-\frac{1}{2}} \right) \left| \sum_{m=0}^{n-1} \Big(\max_{(k,i)\in I(m,\ell)} |\theta^{x,\pi}_{m,k,i}| \Big) |\pi^m|^{-\frac{1}{2}} \right|^{\epsilon} \\
&= \sum_{0 \le m_1, \cdots, m_{\lfloor p \rfloor} \le n-1} \left(\prod_{j=1}^{\lfloor p \rfloor} |\pi^{m_j}|^{-\frac{1}{2}} \right) \sum_{\ell=0}^{N(\pi^n)-1} \left(\prod_{j=1}^{\lfloor p \rfloor} \max_{(k,i)\in I(m_j,\ell)} |\theta^{x,\pi}_{m_j,k,i}| \right) \left| \sum_{m=0}^{n-1} \Big(\max_{(k,i)\in I(m,\ell)} |\theta^{x,\pi}_{m,k,i}| \Big) |\pi^m|^{-\frac{1}{2}} \right|^{\epsilon} \\
&\le \sum_{0 \le m_1, \cdots, m_{\lfloor p \rfloor} \le n-1} \left(\prod_{j=1}^{\lfloor p \rfloor} |\pi^{m_j}|^{-\frac{1}{2}} \right) \\
&\qquad \times \prod_{j=1}^{\lfloor p \rfloor} \left(\sum_{\ell=0}^{N(\pi^n)-1} \max_{(k,i)\in I(m_j,\ell)} |\theta^{x,\pi}_{m_j,k,i}|^p \right)^{\frac{1}{p}} \left(\sum_{\ell=0}^{N(\pi^n)-1} \left| \sum_{m=0}^{n-1} \Big(\max_{(k,i)\in I(m,\ell)} |\theta^{x,\pi}_{m,k,i}| \Big) |\pi^m|^{-\frac{1}{2}} \right|^{\epsilon \cdot \frac{p}{\epsilon}} \right)^{\frac{\epsilon}{p}} \\
&= \sum_{0 \le m_1, \cdots, m_{\lfloor p \rfloor} \le n-1} \left(\prod_{j=1}^{\lfloor p \rfloor} |\pi^{m_j}|^{-\frac{1}{2}} \right) \prod_{j=1}^{\lfloor p \rfloor} \left(\sum_{\ell=0}^{N(\pi^n)-1} \max_{(k,i)\in I(m_j,\ell)} |\theta^{x,\pi}_{m_j,k,i}|^p \right)^{\frac{1}{p}} \left(\frac{Q_n}{\big(Mc\sqrt{bM}|\pi^n|\big)^p} \right)^{\frac{\epsilon}{p}} .
\end{aligned}
$$

Here, the inequality follows from generalized Hölder inequality with $\frac{1}{p} \times \lfloor p \rfloor + \frac{\epsilon}{p} = 1$. We further derive

$$
\begin{aligned}
(Q_n)^{1-\frac{\epsilon}{p}} &\le \Big(Mc\sqrt{bM}|\pi^n|\Big)^{\lfloor p \rfloor} \sum_{0 \le m_1 \cdots m_{\lfloor p \rfloor} \le n-1} \Big(\prod_{j=1}^{\lfloor p \rfloor} |\pi^{m_j}|^{-\frac{1}{2}}\Big) \prod_{j=1}^{\lfloor p \rfloor} \Big(\sum_{\ell=0}^{N(\pi^n)-1} \max_{(k,i) \in I(m_j, \ell)} |\theta^{x,\pi}_{m_j,k,i}|^p \Big)^{\frac{1}{p}} \\
&\le \Big(Mc\sqrt{bM}|\pi^n|\Big)^{\lfloor p \rfloor} \sum_{0 \le m_1 \cdots m_{\lfloor p \rfloor} \le n-1} \Big(\prod_{j=1}^{\lfloor p \rfloor} |\pi^{m_j}|^{-\frac{1}{2}}\Big) \prod_{j=1}^{\lfloor p \rfloor} \Big(\frac{c|\pi^{m_j}|}{|\pi^n|} \sum_{k,i} |\theta^{x,\pi}_{m_j,k,i}|^p \Big)^{\frac{1}{p}} \\
&= \Big(Mc\sqrt{bM}|\pi^n|\Big)^{\lfloor p \rfloor} \Bigg(\sum_{m=0}^{n-1} |\pi^m|^{-\frac{1}{2}} \Big(\frac{c|\pi^m|}{|\pi^n|}\Big)^{\frac{1}{p}} \Big(\sum_{k,i} |\theta^{x,\pi}_{m,k,i}|^p\Big)^{\frac{1}{p}} \Bigg)^{\lfloor p \rfloor}.
\end{aligned}
$$

Here, the second inequality uses the fact that for a fixed m_j there are at most $\frac{|\pi^{m_j}|}{\pi^n}$ many partition points of π^n sharing the same $\theta^{x,\pi}_{m_j,k,i}$, and this number is bounded by $\frac{c|\pi^{m_j}|}{|\pi^n|}$ due to the balanced condition Therefore, we obtain

$$
\begin{aligned}
[x]^{(p)}_{\pi^n}(T) &\le Q_n = \big(Q_n^{1-\frac{\epsilon}{p}}\big)^{\frac{p}{\lfloor p \rfloor}} \qquad (4.21) \\
&\le \Big(Mc\sqrt{bM}|\pi^n|\Big)^p \Bigg(\sum_{m=0}^{n-1} |\pi^m|^{-\frac{1}{2}} \Big(\frac{c|\pi^m|}{|\pi^n|}\Big)^{\frac{1}{p}} \Big(\sum_{k,i} |\theta^{x,\pi}_{m,k,i}|^p\Big)^{\frac{1}{p}} \Bigg)^p = c\Big(Mc\sqrt{bM}\Big)^p \eta^{\pi,(p)}_n,
\end{aligned}
$$

from the definition (4.2) (after re-indexing k, i into k as in Remark 3.9).

On the other hand, using the expression (3.8) of the Schauder coefficients, we obtain the following bound on the p-th power of $\theta^{x,\pi}_{m,k,i}$, thanks to the balanced condition

$$
\begin{aligned}
|\theta^{x,\pi}_{m,k,i}|^p \le \Big(\frac{c}{|\pi^{m+1}|}\Big)^{\frac{3p}{2}} \Big| &\big(x(t_2^{m,k,i}) - x(t_1^{m,k,i})\big)(t_3^{m,k,i} - t_2^{m,k,i}) \qquad (4.22) \\
&- \big(x(t_3^{m,k,i}) - x(t_2^{m,k,i})\big)(t_2^{m,k,i} - t_1^{m,k,i}) \Big|^p.
\end{aligned}
$$

Here, note that $t_2^{m,k,i}$ and $t_3^{m,k,i}$ are consecutive partition points of π^{m+1}, but $t_1^{m,k,i}$ and $t_2^{m,k,i}$ may not be. Recalling the notations in (3.4), we use the telescoping sum

$$
x(t_2^{m,k,i}) - x(t_1^{m,k,i}) = \sum_{j=1}^{i-1} \Big(x(t^{m+1}_{p(m,k)+j}) - x(t^{m+1}_{p(m,k)+j-1}) \Big)
$$

with the bound $\max\{|t_2^{m,k,i} - t_1^{m,k,i}|, |t_3^{m,k,i} - t_2^{m,k,i}|\} \le M|\pi^{m+1}|$, and apply Jensen's inequality to the right-hand side of (4.22) to obtain

$$
\begin{aligned}
|\theta^{x,\pi}_{m,k,i}|^p &\le \Big(\frac{c}{|\pi^{m+1}|}\Big)^{\frac{3p}{2}} (i+1)^{p-1} \Bigg(\sum_{j=1}^{i-1} \Big| \big(x(t^{m+1}_{p(m,k)+j}) - x(t^{m+1}_{p(m,k)+j-1})\big)(t_3^{m,k,i} - t_2^{m,k,i}) \Big|^p \\
&\qquad + \Big| \big(x(t_3^{m,k,i}) - x(t_2^{m,k,i})\big)(t_2^{m,k,i} - t_1^{m,k,i}) \Big|^p \Bigg) \\
&\le \frac{M^p c^{\frac{3p}{2}} (i+1)^{p-1}}{|\pi^{m+1}|^{\frac{3p}{2}-p}} \Bigg(\sum_{j=1}^{i-1} \big| x(t^{m+1}_{p(m,k)+j}) - x(t^{m+1}_{p(m,k)+j-1}) \big|^p + \big| x(t_3^{m,k,i}) - x(t_2^{m,k,i}) \big|^p \Bigg).
\end{aligned}
$$

We note that the quantities inside the last big parenthesis is the p-th variation of x along the partition points of π^{m+1} that belong to the interval $[t^n_k, t^n_{k+1}]$, and these intervals are disjoint for different values of k. We now derive the following inequality

$$
\sum_{k=0}^{N(\pi^m)-1} \sum_{i=1}^{p(m,k+1)-p(m,k)} |\theta^{x,\pi}_{m,k,i}|^p \le \frac{M^p c^{\frac{3p}{2}} (M+1)^{p-1}}{|\pi^{m+1}|^{\frac{p}{2}}} M [x]^{(p)}_{\pi^{m+1}}(T) < \frac{c^{\frac{3p}{2}} (M+1)^{2p}}{|\pi^{m+1}|^{\frac{p}{2}}} [x]^{(p)}_{\pi^{m+1}}(T),
$$

since the largest value i can take is $p(m,k+1) - p(m,k) \le M$ and the first p-th power increment $|x(t^{m+1}_{p(m,k)+1}) - x(t^{m+1}_{p(m,k)})|^p$ (which has been most repeatedly added) has been added at most M many times.

Plugging the last expression into (4.2) with the complete refining property, we obtain

$$\begin{aligned}
\eta^{\pi,(p)}_n &\le (M+1)^{2p} c^{\frac{3p}{2}} |\pi^n|^{p-1} \left(\sum_{m=0}^{n-1} |\pi^m|^{\frac{1}{p}-\frac{1}{2}} |\pi^{m+1}|^{-\frac{1}{2}} \Big([x]^{(p)}_{\pi^{m+1}}(T) \Big)^{\frac{1}{p}} \right)^p \\
&\le (M+1)^{2p} c^{\frac{3p}{2}} |\pi^n|^{p-1} \left(\sum_{m=0}^{n-1} b^{\frac{1}{2}} |\pi^m|^{\frac{1}{p}-1} \Big([x]^{(p)}_{\pi^{m+1}}(T) \Big)^{\frac{1}{p}} \right)^p \\
&= (M+1)^{2p} c^{\frac{3p}{2}} b^{\frac{p}{2}} \left(\sum_{m=0}^{n-1} \left(\frac{|\pi^n|}{|\pi^m|} \right)^{1-\frac{1}{p}} \Big([x]^{(p)}_{\pi^{m+1}}(T) \Big)^{\frac{1}{p}} \right)^p \\
&\le (M+1)^{2p} c^{\frac{3p}{2}} b^{\frac{p}{2}} \left(\sum_{m=0}^{n-1} (1+a)^{(m-n)(1-\frac{1}{p})} \Big([x]^{(p)}_{\pi^{m+1}}(T) \Big)^{\frac{1}{p}} \right)^p. \qquad (4.23)
\end{aligned}$$

We now define an infinite-dimensional matrix $A = (a_{n,m})_{n\ge 0, m\ge 0}$ with entries

$$a_{n,m} := \begin{cases} \Big(1-(1+a)^{\frac{1}{p}-1}\Big) \times (1+a)^{(m-n)(1-\frac{1}{p})}, & \text{for } m \le n, \\ 0, & \text{for } m > n, \end{cases}$$

and we shall show that the matrix A satisfies properties (i) - (iii) of Lemma 4.8. First, condition (i) is obvious. In order to show (ii), we use the geometric series to derive

$$\begin{aligned}
\lim_{n\to\infty} \sum_{m=0}^{\infty} a_{n,m} &= \lim_{n\to\infty} \Big(1-(1+a)^{\frac{1}{p}-1}\Big) \left(\sum_{m=0}^{n} (1+a)^{(m-n)(1-\frac{1}{p})} \right) \\
&= \lim_{n\to\infty} \Big(1-(1+a)^{\frac{1}{p}-1}\Big) \left(\frac{1-(1+a)^{(\frac{1}{p}-1)(n+1)}}{1-(1+a)^{\frac{1}{p}-1}} \right) \\
&= \lim_{n\to\infty} 1-(1+a)^{(\frac{1}{p}-1)(n+1)} = 1.
\end{aligned}$$

Condition (iii) is also obvious from (ii); $\sup_{n\ge 0} \sum_{m=0}^{\infty} |a_{n,m}| = 1 < \infty$.

Therefore, we apply Lemma 4.8 to the inequality (4.23) to obtain

$$\begin{aligned}
\limsup_{n\to\infty} \eta^{\pi,(p)}_n &\le \frac{(M+1)^{2p} c^{\frac{3p}{2}} b^{\frac{p}{2}}}{\Big(1-(1+a)^{\frac{1}{p}-1}\Big)^p} \limsup_{n\to\infty} \left(\sum_{m=0}^{\infty} a_{n,m} \Big([x]^{(p)}_{\pi^{m+1}}(T) \Big)^{\frac{1}{p}} \right)^p \\
&\le \frac{(M+1)^{2p} c^{\frac{3p}{2}} b^{\frac{p}{2}}}{\Big(1-(1+a)^{\frac{1}{p}-1}\Big)^p} \left(\limsup_{n\to\infty} \Big([x]^{(p)}_{\pi^{n}}(T) \Big)^{\frac{1}{p}} \right)^p \\
&= \frac{(M+1)^{2p} c^{\frac{3p}{2}} b^{\frac{p}{2}}}{\Big(1-(1+a)^{\frac{1}{p}-1}\Big)^p} \limsup_{n\to\infty} [x]^{(p)}_{\pi^n}(T). \qquad (4.24)
\end{aligned}$$

Combining (4.24) with the inequality after taking lim sup to (4.21), yields the result (4.3). ■

Proof of Theorem 4.3. For fixed p, x, and π satisfying the conditions of Theorem 4.3, let us define

$$a_n := \sum_{m=0}^{n-1} |\pi^m|^{\frac{1}{p}-\frac{1}{2}} \left(\sum_{k\in I_m} |\theta^{x,\pi}_{m,k}|^p \right)^{\frac{1}{p}}, \qquad b_n := |\pi^n|^{\frac{1}{p}-1}, \qquad \forall\, n \in \mathbb{N}$$

such that

$$a_{n+1} - a_n = |\pi^n|^{\frac{1}{p}-\frac{1}{2}} \left(\sum_{k\in I_n} |\theta^{x,\pi}_{n,k}|^p \right)^{\frac{1}{p}}, \qquad b_{n+1} - b_n = |\pi^{n+1}|^{\frac{1}{p}-1} - |\pi^n|^{\frac{1}{p}-1}.$$

Moreover, from the notation (4.2), we have

$$\frac{a_n}{b_n} = \left(\eta_n^{\pi,(p)}\right)^{\frac{1}{p}}, \qquad \frac{a_{n+1}-a_n}{b_{n+1}-b_n} = \frac{|\pi^n|^{\frac{1}{p}-\frac{1}{2}}\left(\sum_{k\in I_n}|\theta_{n,k}^{x,\pi}|^p\right)^{\frac{1}{p}}}{|\pi^{n+1}|^{\frac{1}{p}-1}-|\pi^n|^{\frac{1}{p}-1}} = \frac{\left(\xi_n^{\pi,(p)}\right)^{\frac{1}{p}}}{\left(\frac{|\pi^{n+1}|}{|\pi^n|}\right)^{\frac{1}{p}-1}-1}, \tag{4.25}$$

and the complete refining property provides the bounds

$$\frac{\left(\xi_n^{\pi,(p)}\right)^{\frac{1}{p}}}{b^{1-\frac{1}{p}}-1} \le \frac{a_{n+1}-a_n}{b_{n+1}-b_n} \le \frac{\left(\xi_n^{\pi,(p)}\right)^{\frac{1}{p}}}{(1+a)^{1-\frac{1}{p}}-1}. \tag{4.26}$$

We further define

$$\beta_n := \frac{b_{n+1}}{b_n} = \left(\frac{|\pi^{n+1}|}{|\pi^n|}\right)^{\frac{1}{p}-1} > 1, \qquad \forall\, n \in \mathbb{N}, \tag{4.27}$$

then, the limit $\beta := \lim_{n\to\infty}\beta_n = r^{\frac{1}{p}-1} > 1$ exists, thanks to the convergent refining property of π. Applying (4.12) of Lemma 4.6 with the bounds (4.26), (4.24) yields

$$\begin{aligned}\limsup_{n\to\infty}\frac{\left(\xi_n^{\pi,(p)}\right)^{\frac{1}{p}}}{b^{1-\frac{1}{p}}-1} &\le \frac{\beta}{\beta-1}\limsup_{n\to\infty}\left(\eta_n^{\pi,(p)}\right)^{\frac{1}{p}} - \frac{1}{\beta-1}\liminf_{n\to\infty}\left(\eta_n^{\pi,(p)}\right)^{\frac{1}{p}} \le \frac{\beta}{\beta-1}\limsup_{n\to\infty}\left(\eta_n^{\pi,(p)}\right)^{\frac{1}{p}}\\ &\le \left(\frac{\beta}{\beta-1}\right)\left(\frac{(M+1)^2c^{\frac{3}{2}}b^{\frac{1}{2}}}{1-(1+a)^{\frac{1}{p}-1}}\right)\limsup_{n\to\infty}\left([x]_{\pi^n}^{(p)}6(T)\right)^{\frac{1}{p}}.\end{aligned}$$

This implies $\limsup_{n\to\infty}[x]_{\pi^n}^{(p)}(T) < \infty \Longrightarrow \limsup_{n\to\infty}\xi_n^{\pi,(p)} < \infty$.

For the opposite direction, we take lim sup to (4.21), and use Lemma 4.7 with (4.26) to obtain

$$\begin{aligned}\frac{1}{c\left(Mc\sqrt{bM}\right)^p}\limsup_{n\to\infty}[x]_{\pi^n}^{(p)}(T) &\le \limsup_{n\to\infty}\eta_n^{\pi,(p)} = \limsup_{n\to\infty}\left(\frac{a_n}{b_n}\right)^p\\ &\le \limsup_{n\to\infty}\left(\frac{a_{n+1}-a_n}{b_{n+1}-b_n}\right)^p = \frac{1}{\left((1+a)^{1-\frac{1}{p}}-1\right)^p}\limsup_{n\to\infty}\xi_n^{\pi,(p)}.\end{aligned}$$

This proves the result (4.6). ■

5 Isomorphism on $\mathcal{X}_\pi^p$

In this section, we shall use several function norms and matrix norms, thus we note that Table 1 at the end of this section lists all the norms with their definitions for the convenience of readers.

Recall the space $C^{0,\alpha}([0,T])$ of α-Hölder continuous functions with the norm

$$\|x\|_{C^{0,\alpha}} := \|x\|_\infty + |x|_{C^{0,\alpha}} \quad \text{with} \quad \|x\|_\infty = \sup_{t\in[0,T]}|x(t)| \quad \text{and} \quad |x|_{C^{0,\alpha}} := \sup_{\substack{s,t\in[0,T]\\ s\neq t}}\frac{|x(s)-x(t)|}{|s-t|^\alpha}. \tag{5.1}$$

Ciesielski [5] proved that the following mapping $T_\alpha^{\mathbb{T}}$ is an isomorphism between $C^{0,\alpha}([0,T])$ and the space $\ell^\infty(\mathbb{R})$ of all bounded real sequences, equipped with the supremum norm $\|\cdot\|_\infty$:

$$\begin{aligned}T_\alpha^{\mathbb{T}} : C^{0,\alpha}([0,T]) &\longrightarrow \quad \ell^\infty(\mathbb{R})\\ x \quad &\longmapsto \left\{2^{(m+1)(\alpha-\frac{1}{2})}|\theta_{m,k}^{x,\mathbb{T}}|\right\}_{m,k}.\end{aligned}$$

Here, $\theta^{x,\mathbb{T}}_{m,k}$'s are the Schauder coefficients of x along the dyadic partition sequence $\mathbb{T}$, and the double-indexed set $\{2^{(m+1)(\alpha-\frac{1}{2})}|\theta^{x,\mathbb{T}}_{m,k}|\}_{m,k}$ can be identified as a real sequence by flattening it. A recent work [2] extends this isomorphism to any balanced, complete refining partition sequence π:

$$\begin{aligned} T^\pi_\alpha : C^{0,\alpha}([0,T]) &\longrightarrow \quad \ell^\infty(\mathbb{R}) \\ x \quad &\longmapsto \Big\{|\pi^{m+1}|^{\frac{1}{2}-\alpha}|\theta^{x,\pi}_{m,k}|\Big\}_{m,k}. \end{aligned} \tag{5.2}$$

We may arrange each element of the sequence $\big\{|\pi^{m+1}|^{\frac{1}{2}-\alpha}|\theta^{x,\pi}_{m,k}|\big\}_{m,k}$ in a matrix without flattening it. Let us denote $\mathcal{M}$ the space of infinite-dimensional matrices and fix a partition sequence $\pi = (\pi^n)_{n\geq 0}$ of $[0,T]$. For each $m \geq 0$, recall the index set I_m of (3.9) corresponding to π, and consider the subspace

$$\mathcal{M}_\pi := \{A \in \mathcal{M} : A_{m,k} = 0 \quad \text{if } k > |I_m|\} \subset \mathcal{M}, \tag{5.3}$$

composed of infinite-dimensional matrices whose m-th row vector can take nonzero values only for the first $|I_m|$ components. We now construct a 'Schauder coefficient matrix' $\Theta^{x,\pi}$ in $\mathcal{M}_\pi$ to arrange the Schauder coefficients:

$$(\Theta^{x,\pi})_{m,k} = \begin{cases} \theta^{x,\pi}_{m,k}, & \text{if } k \in I_m, \\ 0, & \text{otherwise,} \end{cases} \qquad m \geq 0, \quad k \geq 0.$$

We also define a diagonal matrix $D^\pi_\alpha \in \mathcal{M}$ with each (m,m)-th entry equal to $|\pi^{m+1}|^{\frac{1}{2}-\alpha}$:

$$(D^\pi_\alpha)_{m,k} = \begin{cases} |\pi^{m+1}|^{\frac{1}{2}-\alpha}, & \text{if } m = k, \\ 0, & \text{otherwise.} \end{cases} \tag{5.4}$$

From this construction, we have the identity

$$\sup_{m,k}\Big(|\pi^{m+1}|^{\frac{1}{2}-\alpha}|\theta^{x,\pi}_{m,k}|\Big) = \|D^\pi_\alpha \Theta^{x,\pi}\|_{sup}, \tag{5.5}$$

where $\|A\|_{sup} := \sup_{m,k\geq 0}|A_{m,k}|$ is the supremum norm for matrices; in the mapping T^π_α of (5.2), the condition $\big\{|\pi^{m+1}|^{\frac{1}{2}-\alpha}|\theta^{x,\pi}_{m,k}|\big\}_{m,k} \in \ell^\infty(\mathbb{R})$ is then equivalent to $\|D^\pi_\alpha\Theta^{x,\pi}\|_{sup} < \infty$.

We now restate the isomorphism in (5.2) along any balanced and complete refining partition sequence.

Proposition 5.1. *For any balanced, complete refining partition sequence π and $\alpha \in (0,1)$, the mapping*

$$\begin{aligned} T^\pi_\alpha : \Big(C^{0,\alpha}([0,T]), \|\cdot\|_{C^{0,\alpha}}\Big) &\longrightarrow \Big(\mathcal{M}^\alpha_\pi, \|\cdot\|^\alpha_{sup}\Big) \\ x \qquad &\longmapsto \qquad \Theta^{x,\pi} \end{aligned} \tag{5.6}$$

is an isomorphism, where

$$\mathcal{M}^\alpha_\pi := \{A \in \mathcal{M}_\pi : \|A\|^\alpha_{sup} < \infty\}, \qquad \|A\|^\alpha_{sup} := \|D^\pi_\alpha A\|_{sup}.$$

Moreover, we have the following bounds for the operator norms:

$$\|T^\pi_\alpha\|_{op} \leq 2(\sqrt{c})^3, \qquad \|(T^\pi_\alpha)^{-1}\|_{op} \leq \max\Big(2M\sqrt{c}K^\alpha_1 + 2MK^\alpha_2,\, MK^\alpha_2|\pi^1|^\alpha\Big), \tag{5.7}$$

where $K^\alpha_1 := \frac{1}{1-(1+a)^{\alpha-1}}$ and $K^\alpha_2 := \frac{1}{1-(1+a)^{-\alpha}}$ with the constants a, c, M in Remark 3.5.

Proof of Proposition 5.1. From [2, Theorem 3.4] and the identity (5.5), it is easy to show that the mapping T^π_α is bijective. We note that the notation $\|\cdot\|_{C^\alpha([0,T])}$ in the bounds [2, Equation (3.2)] represents the Hölder semi-norm ($|\cdot|_{C^{0,\alpha}}$ in (5.1) of this paper).

The bound for operator norm $\|T^{\pi}_{\alpha}\|_{op}$ is also straightforward from [2, Theorem 3.4] and (5.5):

$$\|\Theta^{x,\pi}\|^{\alpha}_{sup} = \sup_{m,k}\Big(|\pi^{m+1}|^{\frac{1}{2}-\alpha}|\theta^{x,\pi}_{m,k}|\Big) \le 2(\sqrt{c})^3|x|_{C^{0,\alpha}} \le 2(\sqrt{c})^3\|x\|_{C^{0,\alpha}}.$$

The same theorem also yields the inequality

$$|x|_{C^{0,\alpha}} \le (2M\sqrt{c}K_1^{\alpha} + 2MK_2^{\alpha})\|\Theta^{x,\pi}\|^{\alpha}_{sup}. \tag{5.8}$$

Furthermore, we can derive that

$$\begin{aligned}\|x\|_{\infty} &\le \sup_{t\in[0,T]}\Big(\sum_{m=0}^{\infty}\sum_{k\in I_m}|\theta^{x,\pi}_{m,k}||e^{\pi}_{m,k}(t)|\Big) \le M\sum_{m=0}^{\infty}\Big(\sup_{k\in I_m}|\theta^{x,\pi}_{m,k}|\Big)|\pi^{m+1}|^{\frac{1}{2}} \\ &\le M\Big(\sum_{m=0}^{\infty}|\pi^{m+1}|^{\alpha}\Big)\Big(\sup_{m,k}\Big(|\theta^{x,\pi}_{m,k}||\pi^{m+1}|^{\frac{1}{2}-\alpha}\Big)\Big) \le MK_2^{\alpha}|\pi^1|^{\alpha}\|\Theta^{x,\pi}\|^{\alpha}_{sup}.\end{aligned}$$

Here, the second inequality and the last inequality follow from [2, bound (2.4) and Lemma 3.2], respectively. Combining this with (5.8) yields the bound for $\|(T^{\pi}_{\alpha})^{-1}\|_{op}$. ■

Let us fix $x \in C^{0,\alpha}([0,T])$ and $\pi \in \Pi([0,T])$, and recall from Theorem 2.9 that x belongs to $\mathcal{X}^q_{\pi}$ for some $q \in [1, \frac{1}{\alpha}]$. In what follows, we shall characterize such functions $x \in C^{0,\alpha}([0,T]) \cap \mathcal{X}^q_{\pi}$ in terms of its Schauder coefficients.

We now fix $p > 1$ and define a diagonal matrix E^{π} in $\mathcal{M}$ such that every (m,m)-th entry is equal to $|\pi^m|^{\frac{1}{2}}$:

$$(E^{\pi})_{m,k} := \begin{cases} |\pi^m|^{\frac{1}{2}}, & \text{if } m = k, \\ 0, & \text{otherwise.}\end{cases} \tag{5.9}$$

With the matrix norm

$$\|A\|_{p,\infty} := \sup_{k\ge 0}\Big(\sum_{m\ge 0}|A_{m,k}|^p\Big)^{\frac{1}{p}}, \qquad \text{for any } p > 1, \tag{5.10}$$

we define

$$\mathcal{M}^{(p)}_{\pi} := \{A \in \mathcal{M}_{\pi} : \|A\|_{(p)} < \infty\}, \qquad \text{where} \qquad \|A\|_{(p)} := \|(E^{\pi}A)^{\top}\|_{p,\infty}. \tag{5.11}$$

Recalling the definition (4.5), we obtain the identity from (5.11)

$$\|\Theta^{x,\pi}\|_{(p)} = \|(E^{\pi}\Theta^{x,\pi})^{\top}\|_{p,\infty} = \sup_{n\ge 0}\big(\xi^{\pi,(p)}_n\big)^{\frac{1}{p}}. \tag{5.12}$$

Therefore, the condition (4.6) of Theorem 4.3 is also equivalent to $\|\Theta^{x,\pi}\|_{(p)} < \infty$. We are now ready to provide the following results regarding the intersection space $C^{0,\alpha}([0,T]) \cap \mathcal{X}^p_{\pi}$.

Proposition 5.2. *For any $\alpha \in (0,1)$, $p \in (1, \frac{1}{\alpha}]$, and $\pi \in \Pi([0,T])$, the space $\big(C^{0,\alpha}([0,T]) \cap \mathcal{X}^p_{\pi}, \|\cdot\|_{C^{0,\alpha}} + \|\cdot\|^{(p)}_{\pi}\big)$ is a Banach space.*

Proof of Proposition 5.2. Since $\big(C^{0,\alpha}([0,T]), \|\cdot\|_{C^{0,\alpha}}\big)$ and $\big(\mathcal{X}^p_{\pi}, \|\cdot\|^{(p)}_{\pi}\big)$ are Banach spaces (Proposition 2.5), it is obvious that $\|\cdot\|_{C^{0,\alpha}} + \|\cdot\|^{(p)}_{\pi}$ is a norm in the intersection space, and it is enough to show the completeness of $C^{0,\alpha}([0,T]) \cap \mathcal{X}^p_{\pi}$. Fix any Cauchy sequence $(x_\ell)_{\ell\in\mathbb{N}} \in C^{0,\alpha}([0,T]) \cap \mathcal{X}^p_{\pi}$ in $\|\cdot\|_{C^{0,\alpha}} + \|\cdot\|^{(p)}_{\pi}$-norm. Then, $(x_\ell)_{\ell\in\mathbb{N}}$ is also Cauchy in $\|\cdot\|_{C^{0,\alpha}}$-norm, thus it has a limit $x \in C^{0,\alpha}([0,T])$ such that $\|x_\ell - x\|_{C^{0,\alpha}} \to 0$ as $\ell \to \infty$; in particular, $\{x_\ell(t)\}_{\ell\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$, and $x_\ell(t) \to x(t)$ as $\ell \to \infty$ for each $t \in [0,T]$. Moreover, since $\{x_\ell\}_{\ell\in\mathbb{N}}$ is also a Cauchy sequence in $\|\cdot\|^{(p)}_{\pi}$-norm, there exists a limit $\tilde{x} \in \mathcal{X}^p_{\pi}$ such that $\|x_\ell - \tilde{x}\|^{(p)}_{\pi} \to 0$ as $\ell \to \infty$. As in the proof of Proposition 2.5, we have $\lim_{\ell\to\infty} x_\ell(t^n_j) = \tilde{x}(t^n_j) = x(t^n_j)$ for every partition point t^n_j of $P := \bigcup_{n\ge 0}\pi^n$. In other words, x and $\tilde{x}$ coincide on the dense set P, thus the unique continuous extension of $\tilde{x}$ must be x, thus $(x_\ell)_{\ell\in\mathbb{N}}$ converges to $x \in C^{0,\alpha}([0,T]) \cap \mathcal{X}^p_{\pi}$ in $\|\cdot\|_{C^{0,\alpha}} + \|\cdot\|^{(p)}_{\pi}$-norm. ■

In addition to Ciesielski's isomorphism, we have the following isomorphism from the intersection space.

Theorem 5.3 (Isomorphism on the Banach space $\mathcal{X}^p_\pi$)**.** *For any $\alpha \in (0,1)$, $p \in (1, \frac{1}{\alpha}]$, and a balanced, convergent refining partition sequence π, the mapping*

$$T^\pi_{\alpha,(p)} : \Big(C^{0,\alpha}([0,T]) \cap \mathcal{X}^p_\pi, \, \|\cdot\|_{C^{0,\alpha}} + \|\cdot\|^{(p)}_\pi\Big) \longrightarrow \Big(\mathcal{M}^\alpha_\pi \cap \mathcal{M}^{(p)}_\pi, \, \|\cdot\|^\alpha_{sup} + \|\cdot\|_{(p)}\Big)$$
$$x \longmapsto \Theta^{x,\pi} \tag{5.13}$$

is an isomorphism. Furthermore, we have the following bounds for the operator norms:

$$\|T^\pi_{\alpha,(p)}\|_{op} \le \max\left(2(\sqrt{c})^3, \, \frac{(M+1)^2 c^{\frac{3}{2}} b^{\frac{3}{2}-p}}{\Big((1+a)^{1-\frac{1}{p}} - 1\Big)^{\frac{1}{p}}}\right), \tag{5.14}$$

$$\|(T^\pi_{\alpha,(p)})^{-1}\|_{op} \le 1 + \max\Big(2M\sqrt{c}K^\alpha_1 + 2MK^\alpha_2, \, MK^\alpha_2|\pi^1|^\alpha\Big) + \frac{c^{\frac{1}{p}}(Mc\sqrt{bM})}{(1+a)^{1-\frac{1}{p}} - 1}. \tag{5.15}$$

Proof of Theorem 5.3. We shall prove the result in the following parts.

Part 1: For any $x \in C^{0,\alpha}([0,T]) \cap \mathcal{X}^p_\pi$, we shall prove $T^\pi_{\alpha,(p)}(x) \in \mathcal{M}^\alpha_\pi \cap \mathcal{M}^{(p)}_\pi$.

We fix $x \in C^{0,\alpha}([0,T]) \cap \mathcal{X}^p_\pi$. Proposition 5.1 proves $\Theta^{x,\pi} \in \mathcal{M}^\alpha_\pi$, thus we need to show $\Theta^{x,\pi} \in \mathcal{M}^{(p)}_\pi$, which is equivalent to $\sup_{n\ge 0}\big(\xi^{(p)}_n\big) < \infty$ from (5.12).

Recalling the inequality (4.23) and computing the geometric series, we have for each $n \ge 0$

$$\begin{aligned}
\eta^{\pi,(p)}_{\pi^n} &\le (M+1)^{2p} c^{\frac{3p}{2}} b^{\frac{p}{2}} \Big(\|x\|^{(p)}_\pi\Big)^p \left(\sum_{m=0}^{n-1} (1+a)^{(m-n)(1-\frac{1}{p})}\right)^p \\
&= (M+1)^{2p} c^{\frac{3p}{2}} b^{\frac{p}{2}} \Big(\|x\|^{(p)}_\pi\Big)^p \left(\frac{1 - (1+a)^{-n(1-\frac{1}{p})}}{(1+a)^{1-\frac{1}{p}} - 1}\right) \le \frac{(M+1)^{2p} c^{\frac{3p}{2}} b^{\frac{p}{2}}}{(1+a)^{1-\frac{1}{p}} - 1} \Big(\|x\|^{(p)}_\pi\Big)^p.
\end{aligned}$$

Furthermore, recalling the notations (4.25) and (4.27) with the identity (4.13), we derive

$$\big(\xi^{\pi,(p)}_n\big)^{\frac{1}{p}} = (\beta_n - 1)\frac{a_{n+1} - a_n}{b_{n+1} - b_n} = \beta_n \frac{a_{n+1}}{b_{n+1}} - \frac{a_n}{b_n} \le \beta_n \frac{a_{n+1}}{b_{n+1}} \le b^{1-\frac{1}{p}} \Big(\eta^{\pi,(p)}_{n+1}\Big)^{\frac{1}{p}}.$$

Here, the last inequality uses the fact that β_n has an upper bound $b^{1-\frac{1}{p}}$ from the complete refining property.

Combining the last two inequalities, we obtain for each $n \ge 0$

$$\big(\xi^{\pi,(p)}_n\big)^{\frac{1}{p}} \le \frac{(M+1)^2 c^{\frac{3}{2}} b^{\frac{3}{2}-p}}{\Big((1+a)^{1-\frac{1}{p}} - 1\Big)^{\frac{1}{p}}} \|x\|^{(p)}_\pi. \tag{5.16}$$

Since $x \in \mathcal{X}^p_\pi$, we have $\sup_{n\ge 0}\big(\xi^{(p)}_n\big) < \infty$, which shows $\Theta^{x,\pi} \in \mathcal{M}^{(p)}_\pi$.

Part 2: For any $\Theta \in \mathcal{M}^\alpha_\pi \cap \mathcal{M}^{(p)}_\pi$, we shall prove $(T^\pi_{\alpha,(p)})^{-1}\Theta \in C^{0,\alpha}([0,T]) \cap \mathcal{X}^p_\pi$.

We fix $\Theta \in \mathcal{M}^\alpha_\pi \cap \mathcal{M}^{(p)}_\pi$. Using the entries $\Theta_{m,k}$ of Θ as Schauder coefficients along π, we can construct an α-Hölder continuous function x from Proposition 5.1. The identity (5.12) with Corollary 4.4 and (2.6) imply $x \in \mathcal{X}^p_\pi$.

Part 3: We shall prove that the mapping $T^\pi_{\alpha,(p)}$ is bounded.

For any $x \in C^{0,\alpha}([0,T]) \cap \mathcal{X}^p_\pi$, consider $\Theta^{x,\pi} = T^\pi_{\alpha,(p)}x$. From (5.12) and (5.16), we have

$$\|\Theta^{x,\pi}\|_{(p)} \le \frac{(M+1)^2 c^{\frac{3}{2}} b^{\frac{3}{2}-p}}{\Big((1+a)^{1-\frac{1}{p}} - 1\Big)^{\frac{1}{p}}} \|x\|^{(p)}_\pi.$$

Moreover, from Proposition 5.1, we have $\|\Theta^{x,\pi}\|^{\alpha}_{sup} \leq 2(\sqrt{c})^3\|x\|_{C^{0,\alpha}}$. Combining the two bounds concludes (5.14).

Part 4: We shall prove that the inverse mapping $(T^{\pi}_{\alpha,(p)})^{-1}$ is bounded.

For any $\Theta \in \mathcal{M}^{\alpha}_{\pi} \cap \mathcal{M}^{(p)}_{\pi}$, we write $x = (T^{\pi}_{\alpha,(p)})^{-1}\Theta$ and consider its Schauder coefficients $\{\theta^{x,\pi}_{m,k} = \Theta_{m,k}\}_{m,k}$. Recalling the inequality (4.21) and the notation (4.5), we obtain for any $n \geq 0$

$$\begin{aligned}
[x]^{(p)}_{\pi^n}(T) &\leq \Big(Mc\sqrt{bM}|\pi^n|\Big)^p \left(\sum_{m=0}^{n-1} |\pi^m|^{-\frac{1}{2}} \left(\frac{c|\pi^m|}{|\pi^n|}\right)^{\frac{1}{p}} \left(\sum_{k,i} |\theta^{x,\pi}_{m,k,i}|^p\right)^{\frac{1}{p}} \right)^p \\
&\leq \Big(Mc\sqrt{bM}|\pi^n|\Big)^p \left(\sum_{m=0}^{n-1} |\pi^m|^{-\frac{1}{2}} \left(\frac{c|\pi^m|}{|\pi^n|}\right)^{\frac{1}{p}} |\pi^m|^{-\frac{1}{2}} (\xi^{\pi,(p)}_m)^{\frac{1}{p}} \right)^p \\
&= c\Big(Mc\sqrt{bM}\Big)^p |\pi^n|^{p-1} \left(\sum_{m=0}^{n-1} |\pi^m|^{\frac{1}{p}-1} \right)^p \Big(\sup_{m\geq 0} \xi^{\pi,(p)}_m \Big).
\end{aligned}$$

From the complete refining property and computing the geometric series, we have for each $n \geq 0$

$$\begin{aligned}
\sum_{m=0}^{n-1} |\pi^m|^{\frac{1}{p}-1} &\leq |\pi^n|^{\frac{1}{p}-1} \sum_{m=0}^{n-1} (1+a)^{(\frac{1}{p}-1)(n-m)} \\
&= |\pi^n|^{\frac{1}{p}-1}(1+a)^{\frac{1}{p}-1} \frac{1-(1+a)^{(\frac{1}{p}-1)n}}{1-(1+a)^{\frac{1}{p}-1}} \leq |\pi^n|^{\frac{1}{p}-1} \frac{(1+a)^{\frac{1}{p}-1}}{1-(1+a)^{\frac{1}{p}-1}} = \frac{|\pi^n|^{\frac{1}{p}-1}}{(1+a)^{1-\frac{1}{p}}-1}.
\end{aligned}$$

Combining the last two inequalities,

$$[x]^{(p)}_{\pi^n}(T) \leq c\Big(Mc\sqrt{bM}\Big)^p |\pi^n|^{p-1} \left(\frac{|\pi^n|^{\frac{1}{p}-1}}{(1+a)^{1-\frac{1}{p}}-1} \right)^p \Big(\sup_{m\geq 0} \xi^{\pi,(p)}_m \Big) = \frac{c\Big(Mc\sqrt{bM}\Big)^p}{\Big((1+a)^{1-\frac{1}{p}}-1\Big)^p} \Big(\sup_{m\geq 0} \xi^{\pi,(p)}_m \Big).$$

Moreover, thanks to (5.12), we have

$$\|x\|^{(p)}_{\pi} \leq |x(0)| + \frac{c^{\frac{1}{p}}(Mc\sqrt{bM})}{(1+a)^{1-\frac{1}{p}}-1} \Big(\sup_{m\geq 0} \xi^{\pi,(p)}_m \Big)^{\frac{1}{p}} = |x(0)| + \frac{c^{\frac{1}{p}}(Mc\sqrt{bM})}{(1+a)^{1-\frac{1}{p}}-1} \|\Theta^{x,\pi}\|_{(p)}.$$

Also, Proposition 5.1 yields a bound $\|x\|_{C^{0,\alpha}} \leq \max\Big(2M\sqrt{c}K^{\alpha}_1 + 2MK^{\alpha}_2,\, MK^{\alpha}_2|\pi^1|^{\alpha}\Big)\|\Theta\|^{\alpha}_{sup}$. Combining these bounds proves (5.15). ■

Remark 5.4. From Proposition 5.1 and Theorem 5.3, one may expect that the following mapping would also be an isomorphism:

$$\begin{aligned}
T^{\pi}_{(p)} : \Big(\mathcal{X}^p_{\pi},\, \|\cdot\|^{(p)}_{\pi}\Big) &\longrightarrow \Big(\mathcal{M}^{(p)}_{\pi},\, \|\cdot\|_{(p)}\Big) \\
x &\longmapsto \Theta^{x,\pi}.
\end{aligned}$$

However, this is not an isomorphism, since $x \in \mathcal{X}^{(p)}_{\pi}$ is a subclass of continuous functions, and the continuity is not guaranteed without additional conditions if one constructs a function from Schauder coefficients. In the following, we provide an example of function x constructed from a given Schauder matrix $\Theta \in \mathcal{M}^{(2)}_{\pi}$, satisfying the condition $\|x\|^{(2)}_{\pi} < \infty$, but $x \notin C^0([0,T],\mathbb{R})$.

Let us consider the dyadic partition sequence $\mathbb{T}$ on a unit interval $[0,1]$ and a matrix $\Theta \in \mathcal{M}$ such that for each $m \geq 0$ the components of m-th row are given by $\Theta_{m,0} = 2^{\frac{m}{2}}$ and $\Theta_{m,k} = 0$ for all $k \geq 1$. Then, it is easy to verify that $\|\Theta\|_{(2)} = \|(E^{\mathbb{T}}\Theta)^{\top}\|_{2,\infty} < \infty$. We now construct a function

$x(\cdot) := \sum_{m=0}^{\infty} \sum_{k \in I_m} \Theta_{m,k} e^{\mathbb{T}}_{m,k}(\cdot)$ on $[0,1]$. It turns out that x is not continuous at 0; we take $t_n = 2^{-n}$ for each $n \in \mathbb{N}$, then we have

$$x(t_n) = \sum_{m=0}^{n-1} \Theta_{m,0} e^{\mathbb{T}}_{m,0}(t_n) = \sum_{m=0}^{n-1} 2^{\frac{m}{2}} 2^{\frac{m}{2}} t_n = 2^{-n} \sum_{m=0}^{n-1} 2^m = 1 - 2^{-n},$$

thus $0 = x(0) = x(\lim_{n\to\infty} t_n) \neq \lim_{n\to\infty} x(t_n) = 1$, so $x \notin C^0([0,1], \mathbb{R})$.

Function norm	Definition
$\\|x\\|^{(p)}_{\pi}$	$\lvert x(0)\rvert + \sup_{n\in\mathbb{N}} \left([x]^{(p)}_{\pi^n}(T) \right)^{\frac{1}{p}}$ in Definition (2.4)
$\\|x\\|_{\infty}$	$\sup_{t\in[0,T]} \lvert x(t)\rvert$
$\lvert x\rvert_{C^{0,\alpha}}$	$\sup_{s,t\in[0,T],\ s\neq t} \frac{\lvert x(s)-x(t)\rvert}{\lvert s-t\rvert^{\alpha}}$
$\\|x\\|_{C^{0,\alpha}}$	$\\|x\\|_{\infty} + \lvert x\rvert_{C^{0,\alpha}}$ in (5.1)
Matrix norm	**Definition**
$\\|A\\|_{sup}$	$\sup_{m,k\geq 0} \lvert A_{m,k}\rvert$
$\\|A\\|^{\alpha}_{sup}$	$\\|D^{\pi}_{\alpha} A\\|_{sup}$ where D^{π}_{α} is the matrix defined in (5.4)
$\\|A\\|_{p,\infty}$	$\sup_{k\geq 0} \left(\sum_{m\geq 0} \lvert A_{m,k}\rvert^p \right)^{\frac{1}{p}}$ in (5.10)
$\\|A\\|_{(p)}$	$\\|(E^{\pi} A)^{\top}\\|_{p,\infty}$ where E^{π} is the matrix defined in (5.9)

Table 1: List of norms used in this section

In the following table, x represents a (continuous) function defined on $[0,T]$, and A represents an infinite dimensional matrix.

A The case of even integers, $p \in 2\mathbb{N}$, along the dyadic sequence

The concept of pathwise quadratic variation, that is, the limit $[x]^{(2)}_{\pi}$ in (1.3), was introduced in [14], and was extended in [11] to even integers p. However, as mentioned earlier, the existence of the limit $[x]^{(p)}_{\pi}$ is a strong assumption, indicated by the fact that the class V^p_{π} is not a vector space in general. Moreover, a closed-form formula of the p-th variation $[x]^{(p)}_{\pi}$ is known only for the quadratic case $p = 2$ (along the dyadic partition sequence [18] and along general finitely refining partition sequences [7]). In this appendix, we provide a generalized closed-form expression of the p-th variation for even integers p along the dyadic partition sequence, which can be of independent interest.

We first write the dyadic partition sequence $\mathbb{T} = (\mathbb{T}^n)_{n\geq 0}$ as in the beginning of Section 2.1. From Propositions 4.1 and 4.4 of [7], the quadratic variation $[x]^{(2)}_{\mathbb{T}}$ of $x \in C^0([0,T])$ along the n-th dyadic partition $\mathbb{T}^n$ has a simple expression in terms of its Faber-Schauder coefficients:

$$[x]^{(2)}_{\mathbb{T}^n}(T) = 2^{-n} \sum_{m=0}^{n-1} \sum_{k=0}^{2^m-1} (\theta^{x,\mathbb{T}}_{m,k})^2, \qquad \forall\, n \in \mathbb{N}. \tag{A.1}$$

Here, the Schauder coefficients $\theta^{x,\mathbb{T}}$ along the dyadic sequence $\mathbb{T}$ are often called 'Faber-Schauder' coefficients, as Faber [13] earlier constructed a basis by integrating the orthonormal basis along the dyadic partitions introduced by Haar [16] in 1910.

This expression (A.1) can be generalized to any even integers $p \in 2\mathbb{N}$ along the dyadic partitions $\mathbb{T}^n$ in the following.

Proposition A.1. *For a fixed $p \in 2\mathbb{N}$, the p-th variation $[x]^{(p)}_{\mathbb{T}^n}$ of $x \in C^0([0,T])$ along the n-th dyadic partition $\mathbb{T}^n$ can be expressed as:*

$$[x]^{(p)}_{\mathbb{T}^n}(T) = \sum_{m=0}^{n-1} \sum_{k=0}^{2^m-1} 2^{n-m} \times \left(2^{\frac{m}{2}} \times 2^{-n}\right)^p (\theta^{x,\mathbb{T}}_{m,k})^p, \tag{A.2}$$

Proof of Proposition A.1. We recall the identity (4.20) with the fact that for any dyadic partition $\mathbb{T}^n$ there is a unique $k = k(m, \ell, n)$ such that $e^{\mathbb{T}}_{m,k}(\ell/2^n) \neq 0$, to derive

$$\begin{aligned} [x]^{(p)}_{\mathbb{T}^n}(T) &= \sum_{\ell=0}^{2^n-1} \left| \sum_{m=0}^{n-1} \sum_{k=0}^{2^m-1} \theta^{x,\mathbb{T}}_{m,k} \left(e^{\mathbb{T}}_{m,k}(\frac{\ell+1}{2^n}) - e^{\mathbb{T}}_{m,k}(\frac{\ell}{2^n}) \right) \right|^p \\ &= \sum_{\ell=0}^{2^n-1} \left(\sum_{m=0}^{n-1} \sum_{\{k:\psi^{\mathbb{T}}_{m,k}(\ell/2^n)\neq 0\}} \theta^{x,\mathbb{T}}_{m,k} \psi^{\mathbb{T}}_{m,k}(\frac{\ell}{2^n}) 2^{-n} \right)^p, \end{aligned} \tag{A.3}$$

where $\psi^{\mathbb{T}}_{m,k}$ is the Haar basis associated with the Faber-Schauder function $e^{\mathbb{T}}_{m,k}$ (Definition 3.6).

The coefficient of the p-th power term $(\theta^{x,\mathbb{T}}_{m,k})^p$ for each pair (m, k) is

$$\sum_{\{\ell:\psi^{\mathbb{T}}_{m,k}(\ell/2^n)\neq 0\}} \left(\psi^{\mathbb{T}}_{m,k}(\frac{\ell}{2^n}) 2^{-n} \right)^p = 2^{n-m} \times \left(2^{\frac{m}{2}} \times 2^{-n}\right)^p$$

Here, the number of indices ℓ of the set $|\{\ell : \psi^{\mathbb{T}}_{m,k}(\ell/2^n) \neq 0\}|$ is equal to 2^{n-m}, and the absolute values $|\psi^{\mathbb{T}}_{m,k}(\ell/2^n)|$ for such ℓ's are all equal to $2^{\frac{m}{2}}$.

In order to handle the coefficients of the cross-terms like $\prod_{i=1}^p \theta^{x,\mathbb{T}}_{m_i,k_i}$ in (A.3), we fix p pairs (m_1, k_1), $\cdots$, (m_p, k_p) such that at least one pair among the p pairs is different, and consider the following two cases.

Case 1. Suppose that there exist two pairs with disjoint support, i.e., $\exists\ 1 \leq i < j \leq n$ such that $\text{supp}(\psi^{\mathbb{T}}_{m_i,k_i}) \cap \text{supp}(\psi^{\mathbb{T}}_{m_j,k_j}) = \emptyset$. Then, $\psi^{\mathbb{T}}_{m_i,k_i}(t)\psi^{\mathbb{T}}_{m_j,k_j}(t) = 0$ for any t, thus the coefficient of the cross-term in this case is zero.

Case 2. The only remaining case is $\text{supp}(\psi^{\mathbb{T}}_{m_1,k_1}) \subset \text{supp}(\psi^{\mathbb{T}}_{m_2,k_2}) \subset \cdots \subset \text{supp}(\psi^{\mathbb{T}}_{m_p,k_p})$, after some re-numbering of the indices. This is because if we have two pairs $(m_i, k_i), (m_j, k_j)$ such that $m_i = m_j$ but $k_i \neq k_j$, then the supports of $\psi^{\mathbb{T}}_{m_i,k_i}$ and $\psi^{\mathbb{T}}_{m_j,k_j}$ should be disjoint, which is of Case 1. Thus, the values of m_i should be all different. The coefficient of the cross-term $\prod_{i=1}^p \theta^{x,\mathbb{T}}_{m_i,k_i}$ in (A.3) is given by

$$\begin{aligned} &\sum_{\substack{(m_1,k_1),\cdots,(m_p,k_p) \\ m_1<\cdots<m_p}} \sum_{\{\ell:\psi^{\mathbb{T}}_{m_1,k_1}(\ell/2^n)\neq 0\}} \left(\prod_{i=1}^p \psi^{\mathbb{T}}_{m_i,k_i}(\frac{\ell}{2^n}) \right) 2^{-np} \\ =& \sum_{\substack{(m_1,k_1),\cdots,(m_p,k_p) \\ m_1<\cdots<m_p}} \sum_{\{\ell:\psi^{\mathbb{T}}_{m_1,k_1}(\ell/2^n)\neq 0\}} \left(\psi^{\mathbb{T}}_{m_1,k_1}(\frac{\ell}{2^n}) \times \prod_{i=2}^p \psi^{\mathbb{T}}_{m_i,k_i}(t_1^{m_1,k_1}) \right) 2^{-np} \\ =& 2^{-np} \sum_{\substack{(m_1,k_1),\cdots,(m_p,k_p) \\ m_1<\cdots<m_p}} \prod_{i=2}^p \psi^{\mathbb{T}}_{m_i,k_i}(t_1^{m_1,k_1}) \left(\sum_{\{\ell:\psi^{\mathbb{T}}_{m_1,k_1}(\ell/2^n)\neq 0\}} \psi^{\mathbb{T}}_{m_1,k_1}(\frac{\ell}{2^n}) \right) \end{aligned}$$

where $t_1^{m_1,k_1}$ is the left-end point of the support of $\psi^{\mathbb{T}}_{m_1,k_1}$. Now, the values of $\psi^{\mathbb{T}}_{m_1,k_1}(\frac{\ell}{2^n})$ take positive values for exactly half of the indices ℓ in the set $\{\ell : \psi^{\mathbb{T}}_{m_1,k_1}(\ell/2^n) \neq 0\}$; for the remaining half of the indices ℓ of the set, $\psi^{\mathbb{T}}_{m_1,k_1}(\frac{\ell}{2^n})$ take the same absolute, but negative values. Therefore, the last summation is zero.

This concludes that there are no cross-terms in (A.3) and the result (A.2) follows. ■

Remark A.2. For an odd integer p, the argument in the proof of Proposition A.1 does not work in general, so we don't expect such a simple expression of the p-th variation in terms of Faber-Schauder coefficients. For an odd integer p, the identity (A.3) becomes

$$[x]^{(p)}_{\mathbb{T}^n}(T) = \sum_{\ell=0}^{2^n-1} \left| \Big(\sum_{m=0}^{n-1} \sum_{\{k:\psi^{\mathbb{T}}_{m,k}(\ell/2^n)\neq 0\}} \theta^{x,\mathbb{T}}_{m,k} \psi^{\mathbb{T}}_{m,k}(\frac{\ell}{2^n}) 2^{-n} \Big)^p \right|.$$

After expanding the p-th power inside the parenthesis, we can argue as before to conclude that the coefficients of the cross-terms of **Case 1** still vanish. However, the p-th power terms and **Case 2** cross-terms don't vanish, because the outermost summation and the absolute value symbol cannot be exchanged in the following equation.

$$[x]^{(p)}_{\mathbb{T}^n}(T) = 2^{-np} \sum_{\ell=0}^{2^n-1} \left| \sum_{m=0}^{n-1} \sum_{k=0}^{2^m-1} (\theta^{x,\mathbb{T}}_{m,k})^p \big(\psi^{\mathbb{T}}_{m,k}(\frac{\ell}{2^n})\big)^p + \sum_{\substack{(m_1,k_1),\cdots,(m_p,k_p) \\ m_1<\cdots<m_p}} \prod_{i=1}^{p} \Big[\theta^{x,\mathbb{T}}_{m_i,k_i} \psi^{\mathbb{T}}_{m_i,k_i}(\frac{\ell}{2^n}) \Big] \right|.$$

Thanks to Proposition A.1, in the case of $p \in 2\mathbb{N}$, we have the following strengthening of Theorem 4.3.

Theorem A.3. *For $p \in 2\mathbb{N}$ in Theorem 4.3, x has finite p-th variation along $\mathbb{T}$, i.e., the limit $[x]^{(p)}_{\mathbb{T}^n}(T)$ exists, if and only if the limit $\xi^{\mathbb{T},(p)}_n$ exists as $n \to \infty$. In particular, we have the identity*

$$\lim_{n\to\infty} [x]^{(p)}_{\mathbb{T}^n}(T) = \frac{1}{2^{p-1}-1} \lim_{n\to\infty} \xi^{\mathbb{T},(p)}_n. \tag{A.4}$$

Proof. We recall from (4.5) and (A.2)

$$2^{\frac{np}{2}} \times \xi^{\mathbb{T},(p)}_n = \sum_{k=0}^{2^n-1} (\theta^{x,\mathbb{T}}_{n,k})^p,$$

$$[x]^{(p)}_{\mathbb{T}^n}(T) = 2^{-n(p-1)} \sum_{m=0}^{n-1} \sum_{k=0}^{2^m-1} 2^{m(\frac{p}{2}-1)} (\theta^{x,\mathbb{T}}_{m,k})^p = 2^{-n(p-1)} \sum_{m=0}^{n-1} 2^{m(p-1)} \xi^{\mathbb{T},(p)}_m.$$

Let us define

$$c_n := \sum_{m=0}^{n-1} 2^{m(p-1)} \xi^{\mathbb{T},(p)}_m, \qquad \text{and} \qquad d_n := 2^{n(p-1)},$$

then we have $c_{n+1} - c_n = 2^{n(p-1)} \xi^{\mathbb{T},(p)}_n$, $d_{n+1} - d_n = 2^{n(p-1)}(2^{p-1}-1)$, and

$$\frac{c_{n+1}-c_n}{d_{n+1}-d_n} = \frac{\xi^{\mathbb{T},(p)}_n}{2^{p-1}-1}, \qquad \frac{c_n}{d_n} = [x]^{(p)}_{\mathbb{T}^n}(T).$$

From Lemma A.4 below, the limit of $\xi^{\mathbb{T},(p)}_n$ exists if and only if the limit of $[x]^{(p)}_{\mathbb{T}^n}(T)$ exists, and the result (A.4) follows. ■

Lemma A.4 (Theorems 1.22, 1.23 of [19])**.** *Let (a_n) and (b_n) be real sequences such that (b_n) is strictly monotone, divergent, and satisfies $\lim_{n\to\infty} \frac{b_{n+1}}{b_n} = \beta \neq 1$. Then, we have the following equivalence*

$$\lim_{n\to\infty} \left(\frac{a_{n+1}-a_n}{b_{n+1}-b_n} \right) = \ell \in [-\infty,\infty] \quad \Longleftrightarrow \quad \lim_{n\to\infty} \left(\frac{a_n}{b_n} \right) = \ell \in [-\infty,\infty]. \tag{A.5}$$

The proof of Lemma A.4 can be found in [19]. We note that the implication '$\Longrightarrow$' of Lemma A.4 is known as the Stolz-Cesaro theorem.

By applying Lemma A.4 again to (4.25), we can further enhance the identity (A.4):

$$\lim_{n\to\infty} [x]^{(p)}_{\mathbb{T}^n} = \frac{1}{2^{p-1}-1} \lim_{n\to\infty} \xi^{(p)}_n = \frac{\big(2^{1-\frac{1}{p}}-1\big)^p}{2^{p-1}-1} \lim_{n\to\infty} \eta^{(p)}_n, \tag{A.6}$$

and the three limits exist if any one of them exists. This is a higher-order generalization to Proposition 2.1 of [18].

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