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Complex structures on nilpotent Lie algebras

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Abstract. We classify real 6-dimensional nilpotent Lie algebras for which the corresponding Lie group has a left-invariant complex structure, and estimate the dimensions of moduli spaces of such structures.

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Introduction

Let G be a simply-connected real nilpotent Lie group of dimension m. The nilpotency is equivalent to the existence of a basis $\{e^1,\dots,e^m\}$ of left-invariant 1-forms on G such that

$$de^i \in \textstyle\bigwedge^2\langle e^1,\dots,e^{i-1}\rangle, \quad 1\leqslant i\leqslant m, \tag{1}$$

where the right-hand side is interpreted as zero for $i=1$. For any such G with rational structure constants, there is a discrete subgroup Γ such that $M=\Gamma\backslash G$ is a compact manifold [17]. In this case, the differential graded algebra $\mathbb{A}$ of left-invariant forms on G is isomorphic to the de Rham algebra of M, and the Betti numbers b_i of M coincide with the dimensions of the Lie algebra cohomology groups of $\mathfrak{g}$ [21]. Property (1) then implies that $\mathbb{A}$ provides a minimal model for M in the sense of Sullivan, and $\mathbb{A}$ cannot be formal unless G is abelian and M is a torus [9, 3, 13, 5].

Our interest arises from the imposition of extra geometrical structures on M. In particular, suppose that $m=2n$ and J is a complex structure on M associated to a left-invariant tensor on G. We shall work mainly in the language of differential forms, and a starting point for an analysis of low-dimensional examples is the observation that in these circumstances there exists a non-zero closed $(1,0)$-form. More generally, there exists a basis $(\omega^1,\dots,\omega^n)$ of $(1,0)$-forms such that $d\omega^i$ belongs to the ideal generated by $\omega^1,\dots,\omega^{i-1}$. This result is proved in §1 by dualizing the central descending series of $\mathfrak{g}$, and is subsequently used to carry through arguments by induction on dimension.

If the complex structure J makes G a complex Lie group, then the exterior derivative of any left-invariant $(1,0)$-form has type $(2,0)$. On the other hand, the complex structure J is 'abelian' if the derivative of any invariant $(1,0)$-form has type $(1,1)$ [2, 11]. In both cases it follows that

$$d\omega^i \in \textstyle\bigwedge^2\langle \omega^1,\dots,\omega^{i-1},\overline{\omega}^1,\dots,\overline{\omega}^{i-1}\rangle, \quad 1\leqslant i\leqslant n. \tag{2}$$

The properties of complex structures on nilmanifolds with a basis of $(1,0)$-forms satisfying (2) is the subject of papers by Cordero, Fernández, Gray and Ugarte [7, 8]. Our approach had its origin in the realization that not all complex structures have a basis of this form, a fact that makes the general theory all the more richer.

In §3 we embark on a determination of nilpotent Lie algebras of dimension 6 giving rise to complex nilmanifolds, without quoting general classification results. For this reason, our results are likely to serve for a fuller understanding of complex structures on 8-dimensional nilmanifolds. The failure of (2) for $i=2$ is illustrated by the assertion that there exists a unique 6-dimensional nilpotent Lie algebra (NLA) $\mathfrak{g}$ for which G has a left-invariant complex structure J and $b_1=2$. The cases $b_1=3,4$ are more complicated and, in order to streamline the presentation, we precede the classification by a number of preliminary results in §2. Conformal structures are used to construct canonical bases for subspaces of 2-forms in 4 dimensions, and this study leads to a dichotomy in the choice of a canonical basis (Theorem 2.5).

A general classification of nilpotent Lie algebras exists in dimension 7 and less, though 6 is the highest dimension in which there do not exist continuous families. According to published classifications, there are 34 isomorphism classes of NLAs over $\mathbb{R}$, of which 10 are reducible [16, 12]. It is also known which of the corresponding algebras admit symplectic structures, and in §4 we parametrize complex and symplectic structures on 6-dimensional NLAs, and discuss their deformation. This part develops the approach of [1], though here we do not impose compatibility with a metric. Proposition 4.2 concerns the infinitesimal theory, whose non-trivial nature is illustrated by an example of an obstructed cocycle on the Iwasawa manifold. The variety $\mathcal{C}(\mathfrak{g})$ of complex structures on a Lie algebra $\mathfrak{g}$ can be regarded as the fibre of a reduced twistor space over the corresponding Lie group, in the spirit of [4].

Finally, we insert our results into the classification in an appendix that was influenced by [8], and completes the picture given there. Our integrability equations (26) can in fact be solved explicitly for the 6-dimensional NLAs listed, and a more detailed description of the spaces $\mathcal{C}(\mathfrak{g})$ will appear elsewhere.

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1. Invariant differential forms

Let $\mathfrak{g}$ be a Lie algebra of real dimension $2n$. The dual of the Lie bracket gives a linear mapping $\mathfrak{g}^* \to \bigwedge^2\mathfrak{g}^*$ which extends to a finite-dimensional complex

$$0 \to \mathfrak{g}^* \to \textstyle\bigwedge^2\mathfrak{g}^* \to \bigwedge^3\mathfrak{g}^* \to \cdots \to \bigwedge^{2n}\mathfrak{g}^* \to 0. \qquad (3)$$

The vanishing of the composition $\mathfrak{g}^* \to \bigwedge^3\mathfrak{g}^*$ corresponds to the Jacobi identity, and, conversely, any linear mapping $d\colon \mathfrak{g}^* \to \bigwedge^2\mathfrak{g}^*$ whose composition with the natural extension $\bigwedge^2\mathfrak{g}^* \to \bigwedge^3\mathfrak{g}^*$ is zero gives rise to a Lie algebra. Given a Lie algebra $\mathfrak{g}$, we denote by b_k the dimension of the kth cohomology space of (3), which is isomorphic to the Lie algebra cohomology group $H^k(\mathfrak{g})$ [6]. This dimension equals the kth Betti number of $\Gamma\backslash G$ for any discrete cocompact subgroup Γ, by Nomizu's theorem [21].

The descending central series of a Lie algebra $\mathfrak{g}$ is the chain of ideals defined inductively by $\mathfrak{g}^0 = \mathfrak{g}$ and $\mathfrak{g}^i = [\mathfrak{g}^{i-1}, \mathfrak{g}]$ for $i \geqslant 1$. By definition, $\mathfrak{g}$ is s-step nilpotent if $\mathfrak{g}^s = 0$ and $\mathfrak{g}^{s-1} \neq 0$. This condition can easily be interpreted in terms of differential forms as follows. Define a subspaces $\{V_i\}$ of $\mathfrak{g}^*$ inductively by setting $V_0 = \{0\}$, and

$$V_i = \{\sigma \in \mathfrak{g}^* : d\sigma \in \textstyle\bigwedge^2 V_{i-1}\}, \quad i \geqslant 1.$$

Of paramount importance is $V_1 = \ker d$, and the dimension of this equals b_1.

Lemma 1.1 *V_i is the annihilator of $\mathfrak{g}^i$.*

Proof. Suppose inductively that $V_i = (\mathfrak{g}^i)^o$; this is certainly true for $i = 0$. Then $d\sigma \in \bigwedge^2 V_i$ if and only if $d\sigma$ annihilates the subspace $\mathfrak{g} \wedge \mathfrak{g}^i$, i.e. $d\sigma(X, Y) = -\sigma[X, Y]$ vanishes for all $X \in \mathfrak{g}$ and $Y \in \mathfrak{g}^i$. Equivalently, $\sigma \in (\mathfrak{g}^{i+1})^o$. □

A left-invariant almost-complex structure on a Lie group G can be identified with a linear mapping $J\colon \mathfrak{g} \to \mathfrak{g}$ such that $J^2 = -1$. Such a structure determines in the usual way the subspace

$$\Lambda^{1,0} = \{X - iJX : X \in \mathfrak{g}^*\}$$

of the complexification $\mathfrak{g}_c^*$ consisting of left-invariant $(1,0)$-forms, its conjugate $\Lambda^{0,1}$, and more generally subspaces $\Lambda^{p,q}$ of $\bigwedge^{p+q}\mathfrak{g}_c^*$. The almost-complex structure J is said to be *integrable* if

$$[JX, JY] = [X, Y] + J[JX, Y] + J[X, JY], \tag{4}$$

for all $X, Y \in \mathfrak{g}$, and in this case the Newlander-Nirenberg theorem implies that (M, J) is a complex manifold. We shall refer to a pair $(\mathfrak{g}, J)$ consisting of a Lie algebra and an integrable almost-complex structure simply as a 'Lie algebra with a complex structure'. A useful reference for the study of such objects is [23].

The equation (4) holds if and only if $d(\Lambda^{1,0}) \subseteq \Lambda^{2,0} \oplus \Lambda^{1,1}$, and this condition gives rise to a complex

$$0 \to \lambda^{0,0} \to \Lambda^{0,1} \to \Lambda^{0,2} \to \cdots \to \Lambda^{0,n} \to 0, \tag{5}$$

in which each map is the restriction of the ordinary $\overline{\partial}$ operator to a space of left-invariant forms. It follows from (3) that the annihilator

$$\mathfrak{g}^{0,1} = (\Lambda^{1,0})^o \cong (\Lambda^{0,1})^* \tag{6}$$

has the structure of a complex Lie algebra. Only if $J[X, Y] = [JX, Y]$ for all $X, Y \in \mathfrak{g}$, or equivalently if $d(\Lambda^{1,0}) \subseteq \Lambda^{2,0}$, is $\mathfrak{g}$ the the real Lie algebra underlying $\mathfrak{g}^{0,1}$. In this case one can unambiguously declare that $\mathfrak{g}$ is a 'complex Lie algebra'.

Let π_i denote the projection $(V_i)_c \to \Lambda^{0,1}$, and consider the complex subspace

$$V_i^{1,0} = (V_i)_c \cap \Lambda^{1,0} = \ker \pi_i.$$

Observe that $V_i^{1,0} \oplus \overline{V_i^{1,0}}$ is the complexification of the largest J-invariant subspace contained in V_i, namely $V_i \cap JV_i$. It follows that $\dim_{\mathbb{R}}(V_i) \geqslant 2\dim_{\mathbb{C}}(V_i^{1,0})$. Moreover, if $\mathfrak{g}$ is s-step nilpotent then $\dim_{\mathbb{C}}(V_s^{1,0}) = n$.

The following result is an immediate consequence of Lemma 1.1.

Lemma 1.2 *The real subspace underlying $V_i^{1,0}$ is the annihilator of the smallest J-invariant subspace containing $\mathfrak{g}^i$, namely $\mathfrak{g}^i(J) = \mathfrak{g}^i + J\mathfrak{g}^i$.*

The subspace $\mathfrak{g}^i(J)$ is in fact a subalgebra of $\mathfrak{g}$ for every i. This is because $[X, JY]$ belongs to the ideal $\mathfrak{g}^i$ for all $X \in \mathfrak{g}^i$, and $[JX, JY]$ belongs to $\mathfrak{g}^i(J)$ for all $X, Y \in \mathfrak{g}^i$ by (4). It follows that the ideal in the exterior algebra generated by the real and imaginary components of elements of $V_i^{1,0}$ is differential, i.e. closed under d. In fact, more is true. To explain this we denote by $I(\mathcal{E})$ the ideal generated by a set $\{\mathcal{E}\}$ of differential forms in the complexified exterior algebra, and we abbreviate $\overline{\omega^i}$ to $\overline{\omega}^i$.

Theorem 1.3 *An NLA $\mathfrak{g}$ admits a complex structure if and only if $\mathfrak{g}_c^*$ has a basis $\{\omega^1, \ldots, \omega^n, \overline{\omega}^1, \ldots, \overline{\omega}^n\}$ such that*

$$d\omega^{i+1} \in I(\omega^1, \ldots, \omega^i). \tag{7}$$

Proof. Starting from a complex structure J, we construct the ω^i by successively extending a basis of $V_j^{1,0}$ to one of $V_{j+1}^{1,0}$. Given $\omega^1,\ldots,\omega^i$ (or nothing if $i=0$), let j be the least positive integer (dependent on i) such that $V_{j+1}^{1,0}$ contains an element ω^{i+1} for which $\{\omega^1,\cdots,w^{i+1}\}$ is linearly independent. By hypothesis, $\ker\pi_j = V_j^{1,0}$ is spanned by $\{\omega^1,\ldots,\omega^r\}$ for some $r\leqslant i$. It follows that the kernel of the linear mapping

$$\bigwedge^2\pi_j\colon \bigwedge^2(V_j)_c \to \Lambda^{0,2}$$

is a subspace of $\langle\omega^1,\ldots,\omega^i\rangle\wedge\mathfrak{g}_c^*$, and this space therefore contains $d\omega^{i+1}$.

Conversely, a basis $\{\omega^1,\ldots\ldots,\overline{\omega}^n\}$ of $\mathfrak{g}_c^*$ determines an almost-complex structure J on $\mathfrak{g}$ by decreeing $\Lambda^{1,0}$ to be the span of the ω^i. The integrability of J then follows from the condition (7). □

Corollary 1.4 *If $\mathfrak{g}$ is an NLA with a complex structure then $V_1^{1,0}$ is non-zero.*

A complex structure on a Lie algebra $\mathfrak{g}$ is called *abelian* if

$$d(\Lambda^{1,0})\subseteq\Lambda^{1,1}; \tag{8}$$

this is equivalent to asserting that (6) is an abelian Lie algebra [2]. Note that the integrability condition for J is automatically satisfied; it is the special case of (4) for which $[JX,JY]=[X,Y]$ for all $X,Y\in\mathfrak{g}$.

Proposition 1.5 *If $\mathfrak{g}$ is an NLA with an abelian complex structure, then there exists a basis $\{\omega^1,\ldots,\omega^n\}$ of $\Lambda^{1,0}$ satisfying (2).*

Proof. We mimic the proof of Theorem 1.3. Suppose that $\{\omega^1,\ldots,\omega^i\}$ has been found, and let j be the least positive integer such that $V_{j+1}^{1,0}$ contains an element ω^{i+1} with $\{\omega^1,\ldots,\omega^{i+1}\}$ linearly independent. Arguing as above, but with both projections

$$\pi_j\colon(V_j)_c\to\Lambda^{0,1},\quad \pi_j'\colon(V_j)_c\to\Lambda^{1,0},$$

we obtain

$$d\omega^{i+1}\in I(\omega^1,\ldots,\omega^i)\cap I(\overline{\omega}^1,\ldots,\overline{\omega}^i).$$

The result follows. □

2. Choice of bases

This section develops some algebra in dimensions 4 and 6 that will form the basis for the classification of complex structures in §3.

First, we summarize the essential properties of conformal structures on a real oriented 4-dimensional vector space $\mathbb{D}$. Fix a non-zero element $\upsilon\in\bigwedge^4\mathbb{D}$. The bilinear form ϕ on $\bigwedge^2\mathbb{D}$ defined by $\sigma\wedge\tau=\phi(\sigma,\tau)\upsilon$ has signature $+++---$. This is most easily seen by choosing a basis $\{e^1,e^2,e^3,e^4\}$ of $\mathbb{D}$ with $\upsilon=e^{1234}$, and noting that ϕ is diagonalized by the basis consisting of the 6 elements

$$\begin{array}{ll} e^{12}+e^{34}, & e^{12}-e^{34},\\ e^{13}+e^{42}, & e^{13}-e^{42},\\ e^{14}+e^{23}, & e^{14}-e^{23}. \end{array}$$

(Here and in the sequel, we adopt the abbreviation $e^{ijk\cdots}$ for $e^i \wedge e^j \wedge e^k \wedge \cdots$.) This all accords with the fact that the connected component of $O(3,3)$ is double-covered by $SL(4,\mathbb{R})$ [22].

Let g (g for 'metric') be an inner product on $\mathbb{D}$. The underlying conformal structure is the equivalence class $[g] = \{c\,g : c > 0\}$. If $\{e^1, e^2, e^3, e^4\}$ is now an orthonormal basis for g, then

$$\Lambda^2_+ = \langle e^{12} + e^{34},\ e^{13} + e^{42},\ e^{14} + e^{23} \rangle$$

is the $+1$ eigenspace of the so-called $*$ operator, and depends only on $[g]$. Clearly, ϕ is positive definite on Λ_+, and a decomposition

$$\textstyle\bigwedge^2 \mathbb{D} = \Lambda_+ \oplus \Lambda_- \tag{9}$$

is determined by defining Λ_- to be the annihilator of Λ_+ with respect to ϕ. In fact, the correspondence

$$[g] \longleftrightarrow \Lambda_+ \tag{10}$$

is a bijection between the set of conformal classes and the set of 3-dimensional subspaces on which ϕ is positive definite.

Given (9), any 2-form σ equals $\sigma_+ + \sigma_-$ with $\sigma_\pm \in \Lambda^2_\pm$. The 2-form σ is said to be *simple* if it can be expressed as the product $u \wedge v$ of 1-forms. This is the case if and only if $\sigma \wedge \sigma = 0$, or equivalently $|\sigma_+| = |\sigma_-|$, where the norms indicate an inner product induced on $\bigwedge^2 \mathbb{D}$ from one in $[g]$. On the other hand, ignoring the conformal structure, any $\sigma \in \bigwedge^2 \mathbb{D}$ such that $\sigma \wedge \sigma \neq 0$ (a 'generic' element) can be written as $e^{12} + e^{34}$ relative to a suitable basis. The distinction between simple and generic 2-forms in 4 dimensions is crucial, and the next two lemmas illustrate their respective properties. The first is elementary, and its proof is omitted.

Lemma 2.1 *Let σ, τ be linearly-independent simple elements of $\bigwedge^2 \mathbb{D}$. If $\sigma \wedge \tau = 0$ then there exists a basis of $\mathbb{D}$ such that $\sigma = e^{12}$ and $\tau = e^{13}$; otherwise there exists a basis such that $\sigma = e^{12}$ and $\tau = e^{34}$.*

To make (10) more explicit, fix a basis $\{e^1, e^2, e^3, e^4\}$ of $\mathbb{R}^4$, let $a, b, c \in \mathbb{R}$, $a \neq 0$, and set $\Delta = b^2 - 4ac$. Consider the 3-dimensional subspace

$$\Lambda = \langle e^{12} + e^{34},\ e^{13} + e^{42},\ ae^{14} + be^{42} + ce^{23} \rangle \tag{11}$$

of $\bigwedge^2 \mathbb{R}^4$. Relative to the given basis of Λ, the matrix of ϕ is

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & b \\ 0 & b & 2ac \end{pmatrix},$$

so $\phi|\Lambda$ is positive definite if and only if $\Delta < 0$.

Lemma 2.2 *Given (11), there exists a linear transformation θ_-, θ_0 or θ_+ of $\mathbb{R}^4$ (which one depends on Δ) such that*

$$\begin{array}{ll} \Lambda = \theta_-(\langle e^{12} + e^{34},\ e^{13} + e^{42},\ e^{14} + e^{23} \rangle), & \text{if } \Delta < 0, \\ \Lambda = \theta_0(\langle e^{12} + e^{34},\ e^{13},\ e^{14} + e^{23} \rangle), & \text{if } \Delta = 0, \\ \Lambda = \theta_+(\langle e^{12} + e^{34},\ e^{13},\ e^{24} \rangle), & \text{if } \Delta > 0. \end{array}$$

Proof. For convenience, set

$$\sigma = e^{13} + e^{42}, \quad \tau = ae^{14} + be^{42} + ce^{23}.$$

We shall define the required transformations by setting $f^i = \theta_\bullet(e^i)$.

Suppose first that $\Delta < 0$, so that it is possible to solve the equation

$$ax^2 + bx + c = (Ax - B)^2 + (Cx - D)^2$$

over $\mathbb{R}$ with $AD - BC = 1$. Define

$$\begin{array}{ll} f^1 = Ae^1 + Be^2, & f^4 = Ae^4 + Be^3, \\ f^2 = Ce^1 + De^2, & f^3 = Ce^4 + De^3. \end{array}$$

This amounts to applying a transformation in $SL(2,\mathbb{R})$ simultaneously to $\langle e^1, e^2\rangle$ and $\langle e^4, e^3\rangle$, and so $f^{12} = e^{12}$ and $f^{34} = e^{34}$. Moreover, $f^{13} + f^{42} = \sigma$, and

$$\begin{array}{rcl} f^{14} + f^{23} & = & (A^2 + C^2)e^{14} + (AB + CD)(e^{13} - e^{42}) + (B^2 + D^2)e^{23} \\ & = & \tau - \frac{1}{2}b\sigma. \end{array}$$

Thus Λ has the form stated.

Suppose that $\Delta > 0$, $ac \neq 0$, and let s, t be the distinct real solutions of the equation $ax^2 + bx + c = 0$ with $t \neq 0$. Setting

$$\begin{array}{ll} f^1 = e^1 + se^2, & f^2 = e^1 + te^2, \\ f^3 = e^3 + \dfrac{1}{t}e^4, & f^4 = \dfrac{c}{a}e^3 + te^4, \end{array}$$

gives $f^{12} = (t - s)e^{12}$, $f^{34} = (t - s)e^{34}$, and

$$f^{13} = \sigma + \frac{s}{c}\tau, \qquad f^{24} = \frac{1}{a}(c\sigma + t\tau),$$

and $\langle f^{13}, f^{24}\rangle = \langle \sigma, \tau\rangle$, as required. The same conclusion can be verified if $c = 0$.

Suppose that $b^2 = 4ac$, and let $s = -b/(2a)$. This time we define

$$\begin{array}{ll} f^1 = e^1 + se^2, & f^2 = se^1 - e^2, \\ f^3 = se^3 + e^4, & f^4 = e^3 - se^4. \end{array}$$

Then $f^{12} = -(s^2 + 1)e^{12}$, $f^{34} = -(s^2 + 1)e^{34}$, and

$$f^{13} = \frac{1}{2a}(-b\sigma + 2\tau), \qquad f^{14} + f^{23} = (s^2 + 1)\sigma.$$

This completes the proof. $\square$

Let $\mathfrak{g}$ be an NLA of real dimension 6. Let $\{e^1, \ldots, e^6\}$ be a basis of $\mathfrak{g}^*$ satisfying (1), so that there exist constants c^i_{jk} such that

$$de^i = \sum_{j,k<i} c^i_{jk} e^{jk}. \tag{12}$$

These equations imply in particular that $e^1, e^2 \in V_1 = \ker d$. If J is a complex structure on $\mathfrak{g}$, it follows from Corollary 1.4 that we may choose the basis in such a way that

$$e^1 + ie^2 \in V_1^{1,0}. \tag{13}$$

It turns out that in dimensions 6 and less, one may arrange that each structure constant c^i_{jk} be equal to $0, 1$ or -1. Although we do not need to assume this fact, it will make it easy to represent the isomorphism class of a Lie algebra $\mathfrak{g}$ by the equations (12); we write $\mathfrak{g}$ as an m-tuple $(0, 0, de^3, \ldots, de^m)$ abbreviating e^{ij} further to ij. This notation is illustrated by

Proposition 2.3 *A 4-dimensional NLA admitting a complex structure is isomorphic to* $(0,0,0,12)$ *or* $(0,0,0,0)$.

Proof. The symbol $(0,0,0,12)$ means the Lie algebra whose dual has a basis for which $de^i = 0$ for $i = 1,2,3$ and $de^4 = e^{12}$, and $(0,0,0,0)$ is simply an abelian algebra. Let $\mathfrak{g}$ be a 4-dimensional NLA with a complex structure. We may suppose that $\Lambda^{1,0}$ is spanned by a closed 1-form $\omega^1 = e^1 + ie^2$ together with a 1-form $\omega^2 = e^3 + ie^4$ where $de^3 = Ae^{12}$ and $de^4 = Be^{12} + Ce^{13} + De^{23}$ with $A, B, C, D \in \mathbb{R}$. From the proof of Theorem 1.3,

$$0 = \omega^1 \wedge d\omega^2 = (D - iC)e^{123},$$

so that $D = 0 = C$. The result follows by applying a linear transformation of $\langle e^3, e^4 \rangle$. $\square$

It is well known that there is only one other NLA in 4 dimensions, a 2-step one isomorphic to $(0,0,12,13)$, and that this does not admit a complex structure.

Returning to the case of a 6-dimensional Lie algebra with complex structure, choose a basis of $\mathfrak{g}^*$ satisfying (13). The annihilator $\mathfrak{h} = \langle e^1, e^2 \rangle^o$ is then a subalgebra of $\mathfrak{g}$, as in the proof of Lemma 1.2. Its structure can also be described in terms of the quotient $\mathfrak{h}^* = \mathfrak{g}^*/\langle e^1, e^2 \rangle$ with induced operators d and J, and any basis of $\mathfrak{h}^*$ has the form $\{f^i + \langle e^1, e^2 \rangle : 1 \leqslant i \leqslant 4\}$ with $f^i \in \mathfrak{g}^*$. Applying Proposition 2.3, we deduce that there exists a basis of $\mathfrak{g}^*$ satisfying (12) with

$$de^i \in I(e^1, e^2, e^{34}), \quad 1 \leqslant i \leqslant 6,$$

so that de^6 does not involve e^{35} or e^{45}. Stated more explicitly,

Corollary 2.4 *If* $\mathfrak{g}$ *is a 6-dimensional NLA with a complex structure, then* $\mathfrak{g}^*$ *has a basis* $\{e^1, \ldots, e^6\}$ *satisfying (13) and*

$$\begin{aligned} de^3 &\in \langle e^{12} \rangle, \\ de^4 &\in \langle e^{12}, e^{13}, e^{23} \rangle, \\ de^5 &\in \langle e^{12}, e^{13}, e^{14}, e^{23}, e^{24}, e^{34} \rangle, \\ de^6 &\in \langle e^{12}, e^{13}, e^{14}, e^{15}, e^{23}, e^{24}, e^{25}, e^{34} \rangle. \end{aligned} \tag{14}$$

We shall repeatedly choose a basis satisfying both (13) and (14), and then exploit the freedom of choice. In particular, we are at liberty to apply a conformal transformation

$$\begin{aligned} e^1 &\longmapsto A(\ \cos t\, e^1 + \sin t\, e^2), \\ e^2 &\longmapsto A(-\sin t\, e^1 + \cos t\, e^2) \end{aligned} \tag{15}$$

with $A \in \mathbb{R}$, and real 'triangular transformations' of the form

$$e^i \longmapsto \sum_{j=1}^{i} A^i_j e^j, \quad i \geqslant 3. \tag{16}$$

Theorem 2.5 *Let $\mathfrak{g}$ be a 6-dimensional NLA with a complex structure. Then there exists a basis $\{e^1,\dots,e^6\}$ of $\mathfrak{g}^*$ satisfying (14) such that either*

$$\text{(I)}\quad \boxed{\begin{array}{rcl}\omega^1 &=& e^1+ie^2\\ \omega^2 &=& e^3+ie^4\\ \omega^3 &=& e^5+ie^6\end{array}}\qquad \text{or}\qquad \text{(II)}\quad \boxed{\begin{array}{rcl}\omega^1 &=& e^1+ie^2\\ \omega^2 &=& e^4+ie^5\\ \omega^3 &=& e^3+ie^6\end{array}}$$

is a basis of $\Lambda^{1,0}$ satisfying (7), so that

$$\begin{array}{rcl} d\omega^1 &=& 0,\\ \omega^1\wedge dw^2 &=& 0,\\ \omega^1\wedge\omega^2\wedge dw^3 &=& 0.\end{array} \tag{17}$$

Proof. Given a basis $\{e^1,\dots,e^6\}$ of $\mathfrak{g}^*$ satisfying (13) and (14), set $\omega^1=e^1+ie^2$ and define

$$p=\dim\left(\langle e^2,e^3,e^4\rangle_c\cap\Lambda^{1,0}\right),\qquad q=\dim\left(\langle e^2,e^3,e^5\rangle_c\cap\Lambda^{1,0}\right).$$

Since the complexification of a real 3-dimensional subspace cannot contain two linearly independent $(1,0)$-forms, (p,q) must equal one of $(1,0)$, $(0,1)$ or $(0,0)$.

If $p=1$, there exist real 1-forms $f^1,f^2\in\langle e^2,e^3\rangle$ such that

$$\omega^2=f^1+if^2+ie^4\in\Lambda^{1,0}.$$

Modifying the definition of e^3,e^4 using (16), we may assume that $f^1=e^3$ and $f^2=0$. There must also exist $f^3,f^4\in\langle e^2,e^3,e^5\rangle$ such that

$$\omega^3=f^3+if^4+ie^6\in\Lambda^{1,0},$$

and we may modify e^5,e^6 so that $f^3=e^5$ and $f^4=0$. This gives (I). The equations (17) follow from the proof of Theorem 1.3.

In case $(p,q)=(0,1)$, we can modify the definition of e^3,e^5 using (16) so that $e^3+ie^5\in\Lambda^{1,0}$. Then (17) implies that

$$(e^1+ie^2)\wedge de^5=0, \tag{18}$$

so $e^1\wedge de^5=0=e^2\wedge de^5$ and de^5 is a multiple of e^{12}. This being the case, we are at liberty to swap e^4 and e^5 so as to preserve (14), reducing us to Case (I).

Now suppose that $p=0=q$. There must exist $f^1,f^2\in\langle e^2,e^3,e^4\rangle$ such that

$$\omega^2=f^1+if^2+ie^5\in\Lambda^{1,0},$$

and we may modify e^4,e^5 so that $f^1=e^4$ and $f^2=0$. There also exist $f^3,f^4\in\langle e^2,e^3,e^4\rangle$ such that

$$f^3+if^4+ie^6\in\Lambda^{1,0},$$

and we may modify e^6 so that $f^4=0$. Modifying e^3 if necessary, we may also assume that $f^3=e^3+ce^4$ for some $c\in\mathbb{R}$. Then

$$\omega^3=e^3+i(e^6-ce^5)=e^3+ce^4+ie^6-c\omega^2\in\Lambda^{1,0}.$$

Modifying e^6, this gives (II), and (17) follows as before. $\square$

3. A 6-dimensional classification

It is convenient to divide the following analysis into cases according to the value of the first Betti number b_1. Our strategy is based on Theorem 2.5, and we shall always work with a basis $\{e^1,\ldots,e^6\}$ as described there, satisfying (I) or (II). Observe that (I) implies (replacing e^5 by e^4 in (18)) that de^4 is a multiple of e^{12}. But then a non-zero linear combination of e^3,e^4 belongs to V_1, and $b_1\geqslant 3$.

Theorem 3.1 *Any 6-dimensional NLA with* $b_1=2$ *and admitting a complex structure is isomorphic to* $(0,0,12,13,23,14+25)$.

Proof. From the last remark, we can assume that we are in Case (II). Applying a transformation (15) and then rescaling $\omega^2=e^4+ie^5$, we may suppose that $de^3=e^{12}$ and $de^4=Ae^{12}+e^{13}$, where capital letters will always denote real coefficients. Using the equation $d(de^5)=0$, we may write

$$de^5=Be^{12}+Ce^{13}+De^{14}+Ee^{23}.$$

From (17),

$$(C+iE-i)e^{123}+De^{124}=0,$$

so $C=D=0$ and $E=1$. From (14) and the equation $d(de^6)=0$, we obtain

$$de^6=Fe^{12}+Ge^{13}+He^{14}+K(e^{15}+e^{24})+Le^{23}+Me^{25}.$$

The third equation of (17) becomes

$$(G+Li)(e^{1234}+ie^{1235})-(2K+Mi-Hi)e^{1245}=0,$$

so $G=K=L=0$ and $M=H$. To summarize,

$$\begin{aligned} de^5&=Be^{12}+e^{23},\\ de^6&=Fe^{12}+H(e^{14}+e^{25}). \end{aligned}$$

Now $H\neq 0$, for otherwise de^3,de^6 are linearly dependent and $b_1\geqslant 3$. Subtracting multiples of e^3 from e^5 and e^6, and finally rescaling e^6 completes the proof. □

This illustrates the technique we adopt throughout this section. We first apply basis changes that preserve the equations characterizing a hypothetical complex structure, in order to exploit (17). If these equations are verified in a particular case then a complex structure J does exist. Without further reference to J, we then apply linear transformations so as to simplify the description of the real NLA. In the hardest cases ($b_1=3,\ 4$) it is convenient to carry out this simplification *before* using the full force of (17), so it remains to check which of the resulting NLAs do in fact carry a complex structure.

Theorem 3.2 *A 6-dimensional NLA with* $b_1=3$ *admitting a complex structure is isomorphic to one of*

$$\begin{gathered}(0,0,0,12,13,14),\\ (0,0,0,12,13,23),\\ (0,0,0,12,14,24),\\ (0,0,0,12,13,24),\\ (0,0,0,12,13+14,24),\\ (0,0,0,12,13,14+23),\\ (0,0,0,12,14,13+42),\\ (0,0,0,12,13+42,14+23),\\ (0,0,0,12,23,14-35).\end{gathered}$$

Proof. In Case (II), applying (15), we may assume that $de^4 = Ae^{12} + Be^{13}$. Then (17) implies that $de^5 = Ce^{12} + Be^{23}$. Now, $B \neq 0$ for otherwise $b_1 \geqslant 4$, and rescaling $\omega^2 = e^4 + ie^5$ we may assume that $B = 1$. Let

$$de^6 = De^{12} + Ee^{13} + Fe^{14} + Ge^{15} + He^{23} + Ke^{24} + Le^{25} + Me^{34}.$$

The third equation in (17) becomes

$$(E + iH)(e^{1235} - ie^{1234}) + (F - L + iK + iG)e^{1245} + M(-e^{2345} + ie^{1345}) = 0,$$

and so $F = L$, $K = -G$ and $E = H = M = 0$. Next, $d(de^6) = 0$ implies that $G = 0$, whence $de^6 = De^{12} + F(e^{14} + e^{25})$. The term De^{12} may be absorbed into Fe^{14} by modifying e^4. Replacing $-Ce^1 + Ae^2 + e^3$ by a 'new' e^3 gives

$$\mathfrak{g} \cong (0, 0, 0, 13, 23, 14 + 25),$$

which is equivalent to the last one listed in the theorem.

In Case (I), rescaling $\omega^2 = e^3 + ie^4$, we may assume that

$$de^3 = 0, \quad de^4 = e^{12} \tag{19}$$

(cf. (18)). From (17), de^6 has no term in e^{15} or e^{25}, so $de^5, de^6 \in \bigwedge^2 \mathbb{D}$, where

$$\mathbb{D} = \langle e^1, e^2, e^3, e^4 \rangle. \tag{20}$$

Furthermore, $d^2 = 0$ implies that neither de^5 nor de^6 has a term in e^{34}. Consider the following three cases, listed in decreasing generality:

(i) at least one of $de^5 \wedge de^5$, $de^6 \wedge de^6$ is non-zero;

(ii) $de^5 \wedge de^5 = 0 = de^6 \wedge de^6$ and $de^5 \wedge de^6 \neq 0$;

(iii) $de^5 \wedge de^5$, $de^6 \wedge de^6$ and $de^5 \wedge de^6$ are all zero.

In (i), ignoring the complex structure J (and swapping e^5, e^6 if necessary), there exists a transformation of $\mathbb{D}$ of type (16) such that $de^5 = e^{13} + e^{42}$. Thus,

$$de^6 = Ae^{12} + Be^{13} + Ce^{14} + De^{23} + Ee^{24},$$

and if $A \neq 0$ we can eliminate Ae^{12} by replacing $-Ae^1 + De^3$ by a 'new' e^3. Subtracting a multiple of e^5 from e^6 yields

$$de^6 = C'e^{14} - E'e^{42} + D'e^{23}.$$

We may now apply Lemma 2.2 and its proof to find a linear transformation of $\mathbb{D}$ preserving $\langle e^1, e^2 \rangle$ so as to conclude that $\mathfrak{g}$ is isomorphic to one of

$$(0, 0, 0, 12, 13, 14 + 23),$$
$$(0, 0, 0, 12, 14, 13 + 42),$$
$$(0, 0, 0, 12, 13 + 42, 14 + 23).$$

The first of these Lie algebras is characterized by the fact that $de^5 \in \bigwedge^2 V_1$. Each of them admits a complex structure with

$$\Lambda^{1,0} = \langle e^1 + ae^2,\ e^3 + be^4,\ e^5 + ce^6\rangle, \tag{21}$$

where $c = i$ and $a, b \in \mathbb{C}$ solve the equation

$$(e^1 + ae^2) \wedge (e^3 + be^4) \wedge (de^5 + c\, de^6) = 0, \tag{22}$$

by analogy to (17).

We treat (ii) and (iii) by applying Lemma 2.1 to the simple wedge products de^4, de^5, de^6. Since $d^2 = 0$ and (19) holds, e^{34} cannot belong to $d(\mathfrak{g}^*)$. Given (iii), it follows that one may apply a linear transformation of $\mathbb{D}$ preserving the flag $\langle e^1, e^2\rangle \subset V_1$, and a transformation of $\langle e^5, e^6\rangle$, in order that one of the following holds:

$$de^5 = e^{13}, \quad de^6 = e^{14};$$
$$de^5 = e^{13}, \quad de^6 = e^{23};$$
$$de^5 = e^{14}, \quad de^6 = e^{24}.$$

Given (ii), $\mathfrak{g}$ must be isomorphic to one of the algebras

$(0, 0, 0, 12, 13, 24)$,
$(0, 0, 0, 12, 13 + 14, 24)$.

All five of these Lie algebras admit complex structures of the form (21) with $a = i$ and $b, c \in \mathbb{C}$ solving (22). □

Theorem 3.3 *A 6-dimensional NLA with $b_1 = 4$ admitting a complex structure is isomorphic to one of*

$(0, 0, 0, 0, 12, 14 + 25)$,
$(0, 0, 0, 0, 12, 13)$,
$(0, 0, 0, 0, 13 + 42, 14 + 23)$,
$(0, 0, 0, 0, 12, 14 + 23)$,
$(0, 0, 0, 0, 12, 34)$.

If J is a complex structure on any of the last four algebras then V_1 is necessarily J-invariant.

Proof. In Case (II), we may assume that the basis $\{e^1, \ldots, e^6\}$ constructed in the proof of Theorem 2.5 also satisfies $V_1 = \langle e^1, e^2, e^3, e^4\rangle$. Using (18) and rescaling ω^2, we may assume that $de^5 = e^{12}$. Now let

$$de^6 = Qe^{13} + Re^{14} + Se^{15} + Te^{23} + Ue^{24} + Ve^{25} + We^{34}.$$

By applying (15) with $A = 1$ and rescaling e^4, may suppose that $U = 0$. The third equation in (17) becomes

$$-(Q + iT)(e^{1234} + ie^{1235}) + (S - iR + iV)e^{1245} + W(e^{1345} + ie^{2345}) = 0,$$

whence $Q = S = T = W = 0$, $R = V$ and we take $de^6 = e^{14} + e^{25}$, as required.

In Case (I), we may suppose that (19) holds. Applying (15), we may also suppose that de^5 has no e^{23}-component, so that

$$de^5 = Ae^{12} + Be^{13} + Ce^{14} + De^{24} + Ee^{34}.$$

Then (17) implies that $de^6 \in \bigwedge^2\mathbb{D}$ (see (20)), and we argue as in the proof of Theorem 3.2, by applying transformations of $\mathbb{D}$. In case (i), we may take

$$\begin{aligned} de^5 &= e^{13} + e^{42}, \\ de^6 &= C'e^{14} - E'e^{42} + D'e^{23}. \end{aligned}$$

The condition $b_1 = 4$ forces $de^4 = 0$ so that $V_1 = \mathbb{D}$ is J-invariant, and $\mathfrak{g}$ is isomorphic to one of

$$\begin{aligned} &(0, 0, 0, 0, 13, 14 + 23), \\ &(0, 0, 0, 0, 13 + 42, 14 + 23). \end{aligned}$$

Cases (ii) and (iii) are similar, with $\mathfrak{g}$ is isomorphic to one of

$$\begin{aligned} &(0, 0, 0, 0, 12, 34), \\ &(0, 0, 0, 0, 12, 13). \end{aligned}$$

All four of these NLAs admit complex structures of the type described in [1] with $e^5 + ie^6 \in \Lambda^{1,0}$. $\square$

Proposition 3.4 *A 6-dimensional NLA with $b_1 \geqslant 5$ admitting a complex structure is isomorphic to any one of*

$$\begin{aligned} &(0, 0, 0, 0, 0, 0), \\ &(0, 0, 0, 0, 0, 12), \\ &(0, 0, 0, 0, 0, 12 + 34). \end{aligned}$$

Proof. If $\mathfrak{g}$ is non-abelian, then $b_1 = 5$. Choose a basis $\{e^1, \dots, e^6\}$ of $\mathfrak{g}^*$ such that $V_1 = \langle e^1, \dots, e^5\rangle$ and

$$\omega^1 = e^1 + ie^2,\ \omega^2 = e^3 + ie^4 \in \Lambda^{1,0}.$$

Since de^6 is a real 2-form it must have type $(1, 1)$, so that J is abelian and

$$de^6 \in \langle e^{12}, e^{34}, e^{13} - e^{42}, e^{14} - e^{23}\rangle.$$

After a change of basis, we may arrange that $de^6 = e^{12}$ or $de^6 = e^{12} + e^{34}$. $\square$

4. Moduli of complex and symplectic structures

In this section we shall make some observations regarding the spaces of left-invariant complex and (more briefly) symplectic structures structures on a given nilpotent Lie group or algebra.

Fix a Lie algebra $\mathfrak{g}$ of real dimension $2n$. Let $\mathcal{C} = \{J\colon \mathfrak{g} \to \mathfrak{g} : J^2 = -1\}$ denote the set of all almost-complex structures on $\mathfrak{g}$, and let

$$\mathcal{C}(\mathfrak{g}) = \{J \in \mathcal{C} : [JX, JY] = [X, Y] + J[JX, Y] + J[X, JY]\}$$

denote the set of complex structures on $\mathfrak{g}$. The choice of an almost-complex structure on $\mathfrak{g}$ gives an identification

$$\mathcal{C} \cong \frac{GL(2n, \mathbb{R})}{GL(n, \mathbb{C})},$$

and in this sense $\mathcal{C}$ is independent of $\mathfrak{g}$. The space $\mathcal{C}$ has two connected components, corresponding to a choice of orientation, and changing the sign of J flips from one to the other if n is odd. We shall see below (Example (1)) that $\mathcal{C}(\mathfrak{g})$ can have more than two components.

An element J of $\mathcal{C}$ is specified by assigning a complex n-dimensional subspace Λ of $\mathfrak{g}_c^*$ such that $\Lambda\cap\overline{\Lambda}=\{0\}$. This realizes $\mathcal{C}$ as an open set of the Grassmannian $\mathbb{G}r_3(\mathfrak{g}_c^*)$, and furnishes $\mathcal{C}$ with a natural complex structure. The subspace Λ corresponds to the space of $(1,0)$-forms relative to J, so that $\Lambda^{1,0}=\Lambda$ and $\Lambda^{0,1}=\overline{\Lambda}$.

Proposition 4.1 *Let $\mathfrak{g}$ be a non-abelian Lie algebra of dimension $m=2n$. Then $\mathcal{C}(\mathfrak{g})=\mathcal{C}$ if and only $\mathfrak{g}$ has a basis $\{e_0,e_1,\ldots,e_{m-1}\}$ such that $[e_0,e_i]=e_i=-[e_i,e_0]$ for all $i\geqslant 1$ (with all other brackets zero).*

Proof. The existence of a basis as stated is equivalent to the existence of $\alpha\in\mathfrak{g}^*$ such that $d\sigma=\sigma\wedge\alpha$ for all $\sigma\in\mathfrak{g}_c^*$, and this clearly implies that $\mathcal{C}(\mathfrak{g})=\mathcal{C}$. Suppose conversely that $\mathcal{C}(\mathfrak{g})=\mathcal{C}$. Fix $\sigma\in\mathfrak{g}_c^*$ with $\sigma\wedge\overline{\sigma}\neq 0$, and extend it to a basis $\{\sigma,\sigma_2,\ldots,\sigma_n\}$ for a subspace Λ with $\Lambda\cap\overline{\Lambda}=\{0\}$. The integrability condition

$$d\sigma\wedge\sigma\wedge(\sigma_2\wedge\cdots\wedge\sigma_n)=0,$$

valid for all extensions, implies that $d\sigma\wedge\sigma\wedge\eta=0$ for all $\eta\in\bigwedge^{n-1}\mathfrak{g}^*$. It follows that $d\sigma\wedge\sigma=0$ and $d\sigma=\sigma\wedge\alpha$ for some $\alpha\in\mathfrak{g}_c^*$ that depends on σ. This is also valid for a real $\sigma\in\mathfrak{g}^*$, and working with a basis $\{e^0,\ldots,e^{m-1}\}$ of $\mathfrak{g}^*$, it is easy to see that it possible to choose $\alpha\in\mathfrak{g}^*$ such that $de^i=e^i\wedge\alpha$ for all i. $\square$

The real tangent space $T_J\mathcal{C}$ can be identified with the set of endomorphims of $\mathfrak{g}$ that anti-commute with J. Upon complexification, such a mapping reverses the type of 1-forms, and one may identify the holomorphic tangent space $T_J^{1,0}\mathcal{C}$ with $\mathrm{Hom}(\Lambda^{1,0},\Lambda^{0,1})$. To make this more explicit, fix a basis $\{\omega^1,\ldots,\omega^n\}$ of $\Lambda^{1,0}$. If $\tau^i(t)$ is a path in $\Lambda^{1,0}$ with $\tau^i(0)=0$ and $\dot\tau^i(0)=\sigma^i$ for each i, then

$$\Lambda_t=\langle\omega^1+\overline{\tau}^1(t),\ldots,\omega^n+\overline{\tau}^n(t)\rangle \tag{23}$$

is (for sufficiently small t) a path in $\mathbb{G}r_3(\mathfrak{g}_c^*)$ with tangent vector

$$\dot\theta\colon\ \omega^i\ \longmapsto\overline{\sigma}^i \tag{24}$$

at Λ_0.

Given a complex structure J, (5) gives rise to another complex

$$0\to(\Lambda^{0,1})^*\to\mathrm{Hom}(\Lambda^{1,0},\Lambda^{0,1})\stackrel{\overline{\partial}}{\to}\mathrm{Hom}(\Lambda^{1,0},\Lambda^{0,2})\to\cdots\to\mathrm{Hom}(\Lambda^{1,0},\Lambda^{0,n})\to 0. \tag{25}$$

Let K denote the J-invariant subspace of $T_J\mathcal{C}$ underlying $\ker\overline{\partial}$.

Proposition 4.2 *If J is a smooth point of $\mathcal{C}(\mathfrak{g})$, then the tangent space $T_J\mathcal{C}(\mathfrak{g})$ to $\mathcal{C}(\mathfrak{g})$ is a subspace of K.*

Proof. Suppose that

$$\overline{\partial}\omega^i=\sum_{j=1}^n\omega^j\wedge\overline{\alpha}^i_j,\qquad \overline{\alpha}^i_j\in\Lambda^{0,1}.$$

The almost-complex structure defined by (23) is integrable if and only if $d(\omega^i+\overline{\tau}^i(t))$ has no $(0,2)$-component for all i, or equivalently,

$$d(\omega^i + \overline{\tau}^i(t)) \wedge (\omega^1 + \overline{\tau}^1(t)) \wedge \cdots \wedge (\omega^n + \overline{\tau}^n(t)) = 0, \quad i = 1, \ldots, n. \tag{26}$$

If $\dot\theta$ is tangent to $\mathcal{C}(\mathfrak{g})$, this must hold to first order in t, and so

$$(\overline{\partial}\overline{\sigma}^i - \sum_{j=1}^n \overline{\sigma}^j \wedge \overline{\alpha}^i_j) \wedge \omega^{12\cdots n} = 0, \quad i = 1, \ldots, n.$$

This equation can be expressed as

$$\overline{\partial}\overline{\sigma}^i - \dot\theta(\overline{\partial}\omega^i) = 0, \tag{27}$$

which is equivalent to asserting that $\overline{\partial}\dot\theta = 0$ in (25). $\square$

Now let $\mathfrak{g}$ be nilpotent, and suppose that $M = G/\Gamma$ is an associated nilmanifold. The sequence (25) is a subcomplex of the ordinary Dolbeault complex of M tensored with the holomorphic tangent bundle $T = T^{1,0}(M, J)$. The quotient of $\ker\overline{\partial} \subseteq \mathrm{Hom}(\Lambda^{1,0}, \Lambda^{0,1})$ by $\overline{\partial}((\Lambda^{1,0})^*)$ can be identified with the subspace of invariant classes in the sheaf cohomology space $H^1(M, \mathcal{O}(T))$. Theorem 1.3 provides a holomorphic section $\omega^{12\cdots n}$ of the canonical bundle $\bigwedge^n T^*$, and Serre duality implies that

$$H^p(M, \mathcal{O}(T)) \cong H^{n-p}(M, \mathcal{O}(T^*))^* = H^{1,n-p}(M)^*.$$

It follows that the complex dimension of $\mathcal{C}(\mathfrak{g})$ does not exceed

$$n - h^{1,n}(\mathfrak{g}) + h^{1,n-1}(\mathfrak{g}) = n - h^{n-1,0}(\mathfrak{g}) + h^{n-1,1}(\mathfrak{g}) \tag{28}$$

in which the Hodge numbers are computed by tensoring the finite-dimensional complex (5) with $\Lambda^{p,0}$. For the deformation problem it is therefore important to know under what circumstances the terms of (28) coincide with the ordinary Hodge numbers of M, computed with forms that are not necessarily invariant.

The equation (27) can be generalized by examining the component of the left-hand side of (26) of type $(n,2)$. The latter also involves second-order terms and yields the integrability equation

$$\overline{\partial}\theta = \tfrac{1}{2}[\theta, \theta]$$

for the homomorphism $\theta\colon \omega^i \longmapsto \overline{\tau}^i(t)$. Both sides of the equation belong to $\mathrm{Hom}(\Lambda^{1,0}, \Lambda^{0,2})$ and the bracket is defined relative to holomorphic sections of $T = (\Lambda^{1,0})^*$. A necessary condition for $\dot\theta \in \ker\overline{\partial}$ to represent a tangent vector to $\mathcal{C}(\mathfrak{g})$ is therefore that $[\dot\theta, \dot\theta]$ be zero in cohomology (see [15, Theorem 5.1 and (5.86)]). Lemma 4.3 below exhibits a situation in which the vanishing of this primary obstruction is not automatic, so that $\dim_{\mathbb{C}} \mathcal{C}(\mathfrak{g})$ is strictly less than (28).

Let $\mathcal{A}(\mathfrak{g})$ denote the group of automorphisms of the Lie algebra $\mathfrak{g}$, so that an element of $\mathcal{A}(\mathfrak{g})$ is a bijective linear mapping $f\colon \mathfrak{g} \to \mathfrak{g}$ satisfying $f([X,Y]) = [f(X), f(Y)]$. Passing to the dual $\mathfrak{g}^*$, such an f induces a chain mapping of the complex (3) relative to the usual action of the general linear group on forms, and preserves the filtration (V_i). If (4) holds and $f \in \mathcal{A}(\mathfrak{g})$ then $f \circ J \circ f^{-1}$ satisfies (4) in place of J, and in this way we obtain an action of $\mathcal{A}(\mathfrak{g})$ on $\mathcal{C}(\mathfrak{g})$. The tangent space to the orbit is determined by applying the Lie algebra $\mathfrak{d}(\mathfrak{g})$ of $\mathcal{A}(\mathfrak{g})$ to J. An element f of $\mathfrak{d}(\mathfrak{g})$ induces a chain mapping of (3), relative now to its action as a derivation on exterior forms. For example, if $d\alpha = \beta \wedge \gamma$ and $f \in \mathfrak{d}(\mathfrak{g})$, then $d(f\alpha) = f\beta \wedge \gamma + \beta \wedge f\gamma$. In the notation of (24), the element of $T_J\mathcal{C}$ determined by f is given by $\overline{\sigma}^i = (f\omega^i)^{0,1}$.

Now suppose that $n=3$. In accordance with Theorem 1.3, we may write

$$\begin{aligned}\overline{\partial}\omega^1 &= 0,\\ \overline{\partial}\omega^2 &= \omega^1\wedge\overline{\alpha},\\ \overline{\partial}\omega^3 &= \omega^1\wedge\overline{\beta}+\omega^2\wedge\overline{\gamma},\end{aligned}$$

where $\alpha,\beta,\gamma\in\Lambda^{1,0}$, and

$$\partial\alpha=0,\quad \partial\beta=\alpha\wedge\gamma,\quad \partial\gamma=0.$$

Then (27) reduces to

$$\boxed{\begin{aligned}\partial\sigma^1 &= 0\\ \partial\sigma^2 &= \sigma^1\wedge\alpha\\ \partial\sigma^3 &= \sigma^1\wedge\beta+\sigma^2\wedge\gamma\end{aligned}}\qquad(29)$$

The solution space to these equations has dimension (28), and this provides an upper bound for the dimension of $\mathcal{C}(\mathfrak{g})$. An obvious solution is obtained by taking $\sigma^1=\sigma^2=0$ and $\overline{\sigma}^3\in\ker\overline{\partial}$. Indeed, if σ is a *closed* $(1,0)$-form, then the complex structure associated to

$$\Lambda_t=\langle\omega^1,\omega^2,\omega^3+t\overline{\sigma}\rangle,\quad t\in\mathbb{C},$$

belongs to the $\mathcal{A}(\mathfrak{g})$ orbit of J. In any case, $\mathcal{C}(\mathfrak{g})$ never contains isolated points.

Examples (1) Let $\mathfrak{g}$ denote the real Lie algebra $(0,0,0,0,13+42,14+23)$, and define $\omega^1,\omega^2,\omega^3$ by Theorem 2.5(I). Then

$$\begin{cases} d\omega^1=0,\\ d\omega^2=0,\\ d\omega^3=\omega^{12}\end{cases}$$

and both

$$\begin{aligned} J_0\colon\quad \Lambda^{1,0}&=\langle\omega^1,\omega^2,\omega^3\rangle,\\ J_1\colon\quad \Lambda^{1,0}&=\langle\omega^1,\overline{\omega}^2,\overline{\omega}^3\rangle\end{aligned}\qquad(30)$$

are complex structures on $\mathfrak{g}$. Whilst $(\mathfrak{g},J_0)$ is the complex Heisenberg algebra, the integrability of J_1 follows from the fact that $d\omega^3\in\Lambda^{1,1}$. Theorem 3.3 asserts that any complex structure on $\mathfrak{g}$ must preserve $V_1=\mathbb{D}$, and therefore determines an orientation on this 4-dimensional space. Both J_0,J_1 induce the same overall orientation on $\mathfrak{g}$ but different ones on V_1, so $J_0,J_1,-J_0,-J_1$ lie in distinct connected components of $\mathcal{C}(\mathfrak{g})$. It can be shown that there are no other components.

Let $f\in\mathcal{A}(\mathfrak{g})$ and suppose that f preserves the overall orientation of $\mathfrak{g}$. Since $\operatorname{Im}d$ is spanned by the real and imaginary components of ω^{12}, the restriction of f to V_1 commutes with J_0, and we may write

$$f=\begin{pmatrix} B & \beta\\ 0 & \det B\end{pmatrix},$$

where $B\in GL(2,\mathbb{C})$ and $\beta\in\mathbb{C}^2$. The orbit of J_0 under $\mathcal{A}(\mathfrak{g})$ is complex 2-dimensional, whilst it is easy to check from (26) that the connected component of $\mathcal{C}(\mathfrak{g})$ containing J_0 is an open subset of a smooth quadric in $\mathbb{C}^7$. The full deformations of J_0 were considered by Nakamura [19] who showed that the Hodge number $h^{1,2}$ jumps up in value at the point J_0. This will lead to a non-trivial behaviour of the Frölicher spectral sequence over the component of the moduli space $\mathcal{C}(\mathfrak{g})$ containing J_0.

(2) The stabilizer of J_1 in $\mathcal{A}(\mathfrak{g})$ corresponds to the subgroup $S = \mathbb{C}^* \times \mathbb{C}^*$ of diagonal matrices in $GL(2,\mathbb{C})$ that fix the individual subspaces $\langle\omega^1\rangle$, $\langle\omega^2\rangle$. The orbit of J_1 is thus isomorphic to the complex 4-dimensional space $\mathcal{A}(\mathfrak{g})/S$, whilst the component of $\mathcal{C}(\mathfrak{g})$ containing J_1 is again of complex dimension 6. On the other hand, $\ker\overline{\partial}$ has dimension 7, a situation resolved by

Lemma 4.3 *Relative to* J_1, *the element*

$$\dot{\theta}\colon\quad \left\{\begin{array}{l} \omega^1 \longmapsto \overline{\omega}^3\\ \overline{\omega}^2 \longmapsto 0\\ \overline{\omega}^3 \longmapsto 0 \end{array}\right.$$

belongs to $\ker\overline{\partial}$, *but is not tangent to* $\mathcal{C}(\mathfrak{g})$.

Proof. Since J_1 is an abelian complex structure, $\overline{\partial}$ annihilates any $(0,1)$-form, and it follows from (27) that $\overline{\partial}\dot{\theta} = 0$. By above, any complex structure on $\mathfrak{g}$ must preserve V_1, so if $\dot{\theta}$ were the tangent vector to a genuine deformation then $\dot{\theta}(\omega^1) \in \langle\overline{\omega}^1, \omega^2\rangle$, which is false. □

(3) Let $\mathfrak{h}$ be the Lie algebra characterized by Theorem 3.1. In the notation of Theorem 2.5(II), $\mathfrak{h}$ has a complex structure for which $\Lambda^{1,0}$ is spanned by

$$\left\{\begin{array}{l} d\omega^1 = 0,\\ d\omega^2 = \frac{1}{2}(\omega^1\wedge\omega^3 + \omega^1\wedge\overline{\omega}^3),\\ d\omega^3 = \frac{1}{2}i(\omega^1\wedge\overline{\omega}^1 + \omega^1\wedge\overline{\omega}^2 - \omega^2\wedge\overline{\omega}^1). \end{array}\right.$$

In the above notation, $\alpha = \frac{1}{2}\omega^3$, $\beta = -\frac{1}{2}i(\omega^1+\omega^2)$ and $\gamma = \frac{1}{2}i\omega^1$. Now, $\overline{\partial}(\Lambda^{0,1}) = \langle\overline{\omega}^{13}\rangle$ and $\ker\overline{\partial} = \langle\overline{\omega}^1, \overline{\omega}^3\rangle$. The first equation in (29) implies that $\sigma^1 = a\omega^1 + b\omega^3$ with $a, b \in \mathbb{C}$, but then the last equation gives $b = 0$ and $\sigma^2 + a\omega^2 \in \langle\overline{\omega}^1, \overline{\omega}^3\rangle$. The middle equation gives $a = 0$, so that $\sigma^1 = 0$, which is consistent with Corollary 1.4, and the ω^2 component of σ^3 is constrained. Hence, $T^{1,0}_J(\mathcal{C}) \cong \mathbb{C}^4$, a fact confirmed by the calculation of $h^{2,0}(\mathfrak{g}) = 1$ and $h^{2,1}(\mathfrak{g}) = 2$.

The classification of 6-dimensional nilpotent Lie groups admitting left-invariant symplectic forms has been carried out by Goze and Khakimdjanov [12] (the author found an earlier version of this work valuable for the classification given in the appendix). Let G be a Lie group of dimension $2n$. A left-invariant symplectic form can be identified with an element $\sigma \in \bigwedge^2\mathfrak{g}^*$ such that $d\sigma = 0$ and $\sigma^n \neq 0$. The existence question therefore reduces to an examination of

$$L(\mathfrak{g}) = \ker(d\colon \textstyle\bigwedge^2\mathfrak{g}^* \to \bigwedge^3\mathfrak{g}^*).$$

Let $\mathcal{S}$ denote the set of non-degenerate 2-forms on $\mathfrak{g}$. This is an open subset of $\bigwedge^2\mathfrak{g}^*$, though fixing one form gives an identification

$$\mathcal{S} \cong \frac{GL(2n,\mathbb{R})}{Sp(2n,\mathbb{R})}.$$

The set

$$\mathcal{S}(\mathfrak{g}) = \mathcal{S} \cap L(\mathfrak{g})$$

of left-invariant symplectic forms on G is therefore a (possibly empty) open subset of $L(\mathfrak{g})$. If $\sigma \in \mathcal{S}(\mathfrak{g})$ then the tangent space $T_\sigma\mathcal{S}(\mathfrak{g})$ can be identified with $L(\mathfrak{g})$ itself, and

$$\dim \mathcal{S}(\mathfrak{g}) = b_2 + \operatorname{rank}(d\colon \mathfrak{g}^* \to \textstyle\bigwedge^2\mathfrak{g}^*). \tag{31}$$

Now suppose that $\mathfrak{g}$ is nilpotent of real dimension 6. In practice, the following observations help determine whether or not $\mathcal{S}(\mathfrak{g})$ is empty. If $\{e^1, \dots, e^6\}$ is a basis of $\mathfrak{g}^*$ satisfying (12), given $\sigma \in \mathcal{S}(\mathfrak{g})$, we may write

$$\sigma = e^6 \wedge f^1 + e^5 \wedge f^2 + \xi, \quad \xi \in \textstyle\bigwedge^2\mathbb{D}, \tag{32}$$

with $\mathbb{D}$ as in (20), $f^1 \in (\mathbb{D} \oplus \langle e^5\rangle$ and $f^2 \in \mathbb{D}$. Hence,

$$0 = d\sigma = de^6 \wedge f^1 - e^6 \wedge df^1 - e^5 \wedge df^2 + \eta, \quad \eta \in \textstyle\bigwedge^3\mathbb{D},$$

and it follows that $df^1 = 0$. In this way, the choice of basis determines a mapping $\mathcal{S}(\mathfrak{g}) \to V_1$ defined by $\sigma \longmapsto f^1$.

Moreover, if

$$de^6 = e^5 \wedge f^3 + \eta, \quad \nu \in \textstyle\bigwedge^2\mathbb{D},$$

then

$$df^2 = -f^1 \wedge f^3, \tag{33}$$

and

$$\nu \wedge f^1 + de^5 \wedge f^2 \in d(\textstyle\bigwedge^2\mathbb{D}).$$

These equations provide an informal algorithm for determining symplectic forms.

Example (4) The NLA $\mathfrak{g} \cong (0, 0, 12, 13, 14, 34 + 52)$ does not admit a symplectic form σ. For, given σ as in (32) with $f^1 = Ae^1 + Be^2 \in V_1$, (33) yields $df^2 = -Ae^{12}$ and $f^2 = Ce^1 + De^2 - Ae^3$. But then

$$2Ae^{134} + Be^{234} - De^{124} + d\xi = 0,$$

which implies that $A = B = 0$, impossible. Observe that, as a consequence of Theorem 3.1, $\mathfrak{g}$ does not admit a complex structure either.

The moduli spaces $\mathcal{C}(\mathfrak{g})$ and $\mathcal{S}(\mathfrak{g})$ give rise to subsets of the homogeneous space

$$\mathcal{U} \cong \frac{GL(2n, \mathbb{R})}{U(n)},$$

that parametrizes Hermitian structures on $\mathbb{R}^{2n}$. Indeed, there is a double fibration

$$\begin{array}{ccc} & \mathcal{U} & \\ \pi_1 \swarrow & & \searrow \pi_2 \\ \mathcal{C} & & \mathcal{S}, \end{array}$$

and the inverse images $\pi^{-1}(\mathcal{C}(\mathfrak{g}))$, $\pi^{-1}(\mathcal{S}(\mathfrak{g}))$ correspond to the sets of left-invariant Hermitian and almost-Kähler metrics respectively on G. When $n = 3$, the manifolds $\mathcal{C}, \mathcal{U}, \mathcal{S}$ have real dimension $18, 27, 15$ respectively, and the table below shows that in some cases these inverse images each have relatively small codimension in $\mathcal{U}$. On the other hand, the fact that no non-abelian NLA can admit a Kähler metric [3, 5, 7] implies that

$$\pi_1^{-1}(\mathcal{C}(\mathfrak{g})) \cap \pi_2^{-1}(\mathcal{S}(\mathfrak{g})) = \emptyset,$$

unless $\mathfrak{g}$ is abelian.

5. Appendix

Although §2 contains some lemmas relevant to the theory of 6-dimensional NLAs, we have not so far resorted to their full classification. However, in this appendix we tabulate the list given by Magnin [16] (based on an earlier one of Morosov [18]), and insert the results of this paper.

The Lie algebras appear lexicographically with respect to $(b_1, b_2, 6-s)$, where $b_i = \dim H^i(\mathfrak{g})$ and s is the step length (see §1). It is easy to check that each entry satisfies the Jacobi identity $d^2 = 0$, and the choice of sign is often important. For example, with reference to the first, $(0,0,12,13,14+23,34-52)$ does not describe a Lie algebra since (in informal notation) $d(34-52) = 124-142 \neq 0$. The Betti number b_3 may be computed from b_1, b_2 by means of the formula $b_3 = 2(b_2 - b_1 + 1)$, which expresses the vanishing of the Euler characteristic (for example, of an associated nilmanifold G/Γ). Dixmier [10] showed that any NLA has $b_i \geqslant 2$ for all i.

The column headed $\oplus$ indicates the dimensions of irreducible subalgebras in case $\mathfrak{g}$ is not itself irreducible. The numbers in the sixth column refer to the list or irreducible algebras in [16], and those in the seventh to the table of [8]. Referring to the latter, the pairs labelled $13, 15$ and $2, 5$ define the same Lie algebra over $\mathbb{C}$ as do the two 19's and the two 26's.

The number in the eighth column is the upper bound for the complex dimension $\dim_{\mathbb{C}} \mathcal{C}(\mathfrak{g})$ determined by (28). It is computed (naïvely) by picking a particular complex structure J on $\mathfrak{g}$ and determining the solution space to (29). When $\mathfrak{g} = (0,0,0,0,13+42,14+23)$, this solution space has dimension $6, 7$ for the respective structures (30), though both components of $\mathcal{C}(\mathfrak{g})$ have dimension 6. Indeed, the condition that any complex structure preserves V_1 imposes two linear conditions, and the equations describing $\mathcal{C}(\mathfrak{g})$ incorporate an extra quadratic constraint.

An asterisk means that $\mathfrak{g}$ does not admit a complex structure of type (I) in Theorem 2.5, though the one with $b_1 = 4$ does admit a structure satisfying condition (2).

The last column lists the exact real dimension $\dim_{\mathbb{R}} \mathcal{S}(\mathfrak{g})$, which is readily computed using (31). The latter implies that all connected components of $\mathcal{S}(\mathfrak{g})$ have the same dimension, which depends only on the structure of the Lie algebra over $\mathbb{C}$. In this sense, the symplectic case is more straightforward.

As a final application, we quote a result that extends Example 4.

Theorem 5.1 *The 6-dimensional NLAs admitting neither complex nor symplectic structures are*

$$(0,0,12,13,14+23,34+52),$$
$$(0,0,12,13,14,34+52),$$
$$(0,0,0,12,13,14+35),$$
$$(0,0,0,12,23,14+35),$$
$$(0,0,0,0,12,15+34).$$

The first two algebras in this list realize the minimum values $b_i = 2$ for $1 \leqslant i \leqslant 6$.

Six-dimensional real nilpotent Lie algebras

b_1	b_2	$6-s$	Structure	$\oplus$	[16]	[8]	$\dim_{\mathbb{C}} \mathcal{C}(\mathfrak{g})$	$\dim_{\mathbb{R}} \mathcal{S}(\mathfrak{g})$
2	2	1	$(0,0,12,13,14+23,34+52)$		22	32	–	–
2	2	1	$(0,0,12,13,14,34+52)$		21	31	–	–
2	3	1	$(0,0,12,13,14,15)$		2	28	–	7
2	3	1	$(0,0,12,13,14+23,24+15)$		20	30	–	7
2	3	1	$(0,0,12,13,14,23+15)$		19	29	–	7
2	4	2	$(0,0,12,13,23,14)$		11	23	–	8
2	4	2	$(0,0,12,13,23,14-25)$		18	26	–	8
2	4	2	$(0,0,12,13,23,14+25)$		18	26	4^*	8
3	4	2	$(0,0,0,12,14-23,15+34)$		16	27	–	7
3	5	2	$(0,0,0,12,14,15+23)$		17	25	–	8
3	5	2	$(0,0,0,12,14,15+23+24)$		15	24	–	8
3	5	2	$(0,0,0,12,14,15+24)$	$1+5$		22	–	8
3	5	2	$(0,0,0,12,14,15)$	$1+5$		21	–	8
3	5	3	$(0,0,0,12,13,14+35)$		13	18	–	–
3	5	3	$(0,0,0,12,23,14+35)$		14	19	–	–
3	5	3	$(0,0,0,12,23,14-35)$		14	19	4^*	–
3	5	3	$(0,0,0,12,14,24)$	$1+5$		16	6	–
3	5	3	$(0,0,0,12,13+42,14+23)$		10	15	6	8
3	5	3	$(0,0,0,12,14,13+42)$		9	14	5	8
3	5	3	$(0,0,0,12,13+14,24)$		8	13	5	8
3	6	3	$(0,0,0,12,13,14+23)$		6	11	5	9
3	6	3	$(0,0,0,12,13,24)$		7	12	5	9
3	6	3	$(0,0,0,12,13,14)$		1	10	5	9
3	8	4	$(0,0,0,12,13,23)$		3	7	6	9
4	6	3	$(0,0,0,0,12,15+34)$		12	20	–	–
4	7	3	$(0,0,0,0,12,15)$	1+1+4		17	–	9
4	7	3	$(0,0,0,0,12,14+25)$	$1+5$		9	5^*	9
4	8	4	$(0,0,0,0,13+42,14+23)$		5	5	6	10
4	8	4	$(0,0,0,0,12,14+23)$		4	4	6	10
4	8	4	$(0,0,0,0,12,34)$	$3+3$		2	6	10
4	9	4	$(0,0,0,0,12,13)$	$1+5$		6	6	11
5	9	4	$(0,0,0,0,0,12+34)$	$1+5$		3	6	–
5	11	4	$(0,0,0,0,0,12)$	1+1+1+3		8	7	12
6	15	5	$(0,0,0,0,0,0)$	$1+\cdots+1$		1	9	15

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