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Non-standard Skorokhod convergence of Lévy-driven convolution integrals in Hilbert spaces

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Abstract

We study the convergence in probability in the non-standard M_1 Skorokhod topology of the Hilbert valued stochastic convolution integrals of the type $\int_0^t F_\gamma(t-s)\,dL(s)$ to a process $\int_0^t F(t-s)\,dL(s)$ driven by a Lévy process L. In Banach spaces we introduce strong, weak and product modes of M_1-convergence, prove a criterion for the M_1-convergence in probability of stochastically continuous càdlàg processes in terms of the convergence in probability of the finite dimensional marginals and a good behaviour of the corresponding oscillation functions, and establish criteria for the convergence in probability of Lévy driven stochastic convolutions. The theory is applied to the infinitely dimensional integrated Ornstein–Uhlenbeck processes with diagonalisable generators.

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1 Introduction

In many problems of engineering, physics or finance, the evolution of a random system can be described by stochastic convolution integrals of some kernel with respect to a noise process, see e.g. Barndorff–Nielsen and Shephard [2], Elishakoff [7], Pavlyukevich and Sokolov [16].

The present work is originally motivated by the paper by Chechkin et al. [5], where the authors consider a simple model for the motion of a charged particle in a constant external magnetic field subject to α-stable Lévy perturbation. The particle's position $x \in \mathbb{R}^3$ is described by the second-order Newtonian equation

$$\ddot{x} = \dot{x} \times B - \nu\dot{x} + \varepsilon\dot{\ell},$$

where $B \in \mathbb{R}^3$ is the direction of the magnetic field, ν, $\varepsilon > 0$ and ℓ is an isometric three-dimensional α-stable Lévy process with the characteristic function $Ee^{i\langle u,\ell(t)\rangle} = e^{-t|u|^\alpha}$

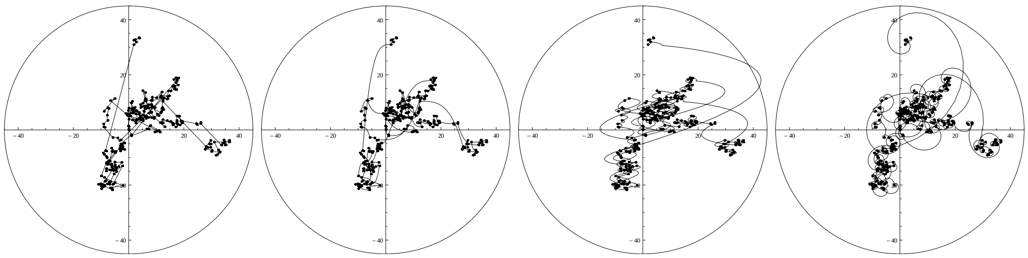

Figure 1: Sample paths of the convolution integrals $A_j X_\gamma^{(j)}(t) = \int_0^t (1 - e^{-\gamma A_j (t-s)})\, dL(s)$, $j = 1, \dots, 4$, driven by a 1.5-stable Lévy process L for large $\gamma > 0$ (from left to right).

for all $u \in \mathbb{R}^3$. Denoting the velocity $\dot{x} = v$ and the linear operator $Av := -v \times B + \nu v$, we obtain that the velocity process v satisfies the linear Ornstein–Uhlenbeck equation

$$\dot{v} = -Av + \varepsilon \dot{\ell}, \tag{1.1}$$

whereas the coordinate is obtained by integration of the velocity v. Assuming that $v_0 = x_0 = 0$ we solve equation (1.1) explicitly to obtain $v(t) = \varepsilon \int_0^t e^{-A(t-s)}\, d\ell(s)$, and Fubini's theorem yields $x(t) = \varepsilon A^{-1} \int_0^t (1 - e^{-A(t-s)})\, d\ell(s)$.

It is possible to study the dynamics of x in the regime of the small noise perturbation by letting $\varepsilon \to 0$. Indeed, performing a convenient time-change $t \mapsto \varepsilon^{-\alpha} t$, using the self-similarity of α-stable processes, i.e.

$$\big(\varepsilon \ell(t/\varepsilon^\alpha) \colon t \geqslant 0\big) \stackrel{\mathcal{D}}{=} \big(\ell(t) \colon t \geqslant 0\big),$$

and taking for convenience another copy $L = \ell$ of the driving process ℓ, we transfer the small noise amplitude into the large friction coefficient; that is the stochastic processes X and V, defined by $X(t) := x(t/\varepsilon^\alpha)$ and $V(t) := v(t/\varepsilon^\alpha)$ for all $t \geqslant 0$, satisfy the equations

$$\dot{V} = -\frac{1}{\varepsilon^\alpha} AV + \dot{L}, \qquad \dot{X} = \frac{1}{\varepsilon^\alpha} V.$$

By denoting the large parameter $\gamma := \varepsilon^{-\alpha}$ we obtain the solutions

$$V_\gamma(t) = \int_0^t e^{-\gamma A(t-s)}\, dL(s), \qquad AX_\gamma(t) = \int_0^t (1 - e^{-\gamma A(t-s)})\, dL(s).$$

It can be shown (see Lemma 4.1) that if the eigenvalues of A have strictly positive real parts, then $AX_\gamma \to L$ in probability with respect to an appropriate metric (M_1) in the sample path space.

As an example, consider a two dimensional integrated Ornstein–Uhlenbeck process driven by an α-stable Lévy process L as well as the corresponding sample paths $t \mapsto A_j X_\gamma^{(j)}(t)$ of the integrated Ornstein–Uhlenbeck processes for the following matrices A_j (see Figure 1):

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad A_4 = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

Obviously, the sample paths differ significantly. This example determines the scope of this paper: we will establish convergence of general stochastic convolution integrals driven by Lévy processes in the Skorokhod M_1 topology in infinite dimensional spaces. The particular example (1.1) of this introduction in one dimension is considered by Hintze and Pavlyukevich in [8].

As one of four topologies, the M_1 topology in the path space $D([0,1],\mathbb{R})$, the space of càdlàg functions $f\colon [0,T] \to \mathbb{R}$, was introduced in the seminal paper by Skorokhod [22]. An excellent account on convergence in the M_1 topology in a multi-dimensional setting can be found in Whitt [28]. To the best of our knowledge, the M_1 topology has not yet been considered in an infinite dimensional setting. Note, that in the M_1 topology it is possible that a continuous function, as the sample paths of AX, converges to a discontinuous function, as the sample paths of the Lévy process L.

In the present paper we study the following aspects of the M_1 topology. First, we notice that in a typical setting, such as considered in Whitt [28], one often obtains convergence in the M_1 topology not only in the weak sense but also in probability. Second, we generalise the finite-dimensional setting of Skorokhod and Whitt to stochastic processes with values in separable Banach spaces. Here it turns out, that in addition to the two kinds of M_1 topologies in multi-dimensional spaces, a third kind of M_1 topology arises in infinite dimensional spaces.

The second part of our work is concerned with convergence of stochastic convolution integrals in Hilbert spaces in the M_1 topology. By considering stochastic convolution integrals we may abandon the semimartingale setting. It is known, see Basse and Pedersen [3] and Basse–O'Connor and Rosiński [4], that even one-dimensional convolution integrals $\int_0^t F(t-s)\, dL(s)$ define a semimartingale if and only if F is absolutely continuous with sufficiently regular density.

As a specific example, the case of integrated Ornstein–Uhlenbeck processes in a Hilbert space is considered in the last section of this paper. It turns out that only in the case of a diagonalisable operator, convergence can be established, and then, only in the weakest sense. This result corresponds to the two-dimensional example above, where only in cases $j = 1$ and $j = 2$ the stochastic convolution integrals converge in the M_1 topology.

Notation: For two values $a, b \in \mathbb{R}$ we denote $a \wedge b := \min\{a, b\}$ and $a \vee b := \max\{a, b\}$. The Euclidean norm in $\mathbb{R}^d$, $d \geq 1$, is denoted by $|\cdot|$. A partition $(t_i)_{i=1}^m$ of an interval $[0,T]$ is a finite sequence of numbers $t_i \in [0,T]$ satisfying $t_1 < \cdots < t_m$. For functions $f\colon [0,T] \to S$, where S is a linear space with a norm $\|\cdot\|_S$, we define the supremum norm $\|f\|_\infty := \sup_{t\in[0,T]} \|f(t)\|_S$. The 2-variation of a function $f\colon [0,T] \to S$ is defined by

$$\|f\|_{TV_2}^2 := \sup \sum_{k=0}^{m} \|f(t_k) - f(t_{k-1})\|^2 ,$$

where the supremum is taken over all partitions of $[0,T]$.

Let U be a separable Banach space with norm $\|\cdot\|$. The dual space is denoted by U^* with dual pairing $\langle u, u^*\rangle$. The Borel σ-algebra in U is denoted by $\mathcal{B}(U)$. For another separable Banach space V the space of bounded, linear operators from U to V is denoted by $\mathcal{L}(U,V)$ equipped with the norm topology $\|\cdot\|_{U\to V}$.

Let $(\Omega, \mathcal{A}, P)$ be a probability space. The space of equivalence classes of measurable functions $f\colon \Omega \to U$ is denoted by $L_P^0(\Omega; U)$ and it is equipped with the topology of convergence in probability. The space of equivalence classes of measurable functions whose p-th power has finite integral is denoted by $L_P^p(\Omega; U)$ for $p \geqslant 1$.

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2 The Skorokhod space

In this section, we introduce the Skorokhod space and some of its topologies. Let V denote a separable Banach space. For a fixed time $T > 0$, the space of V-valued càdlàg functions is denoted by $D([0,T];V)$. For each $f \in D([0,T];V)$ we define the set of discontinuities by

$$J(f) := \{t \in (0,T] \colon f(t-) \neq f(t)\}.$$

The set $J(f)$ is countably finite. The jump size at t is defined by $(\Delta f)(t) = f(t) - f(t-)$. For two elements $v_1, v_2 \in V$ we define the *segment* as the straight line between v_1 and v_2:

$$[\![v_1, v_2]\!] := \{v \in V \colon v = \alpha v_1 + (1-\alpha)v_2 \text{ for } \alpha \in [0,1]\}.$$

In order to define a metric on $D([0,T];V)$, the so-called *(strong) M_1 metric*, we define for each $f \in D([0,T];V)$ the *extended graph of f* by

$$\Gamma(f) := \{(t,v) \in [0,T] \times V \colon v \in [\![f(t-), f(t)]\!]\},$$

where $f(0-) := f(0)$. The projection of $\Gamma(f)$ to its spatial component in V is given by

$$\pi(\Gamma(f)) := \{v \in V \colon (t,v) \in \Gamma(f) \quad \text{for some } t \in [0,T]\}.$$

A total order relation on $\Gamma(f)$ is given by

$$(t_1, v_1) \leqslant (t_2, v_2) \;\Leftrightarrow\; \begin{cases} t_1 < t_2 \quad \text{or} \\ t_1 = t_2 \text{ and } \|f_1(t_1-) - v_1\| \leqslant \|f_1(t_1-) - v_2\|. \end{cases}.$$

A *parametric representation* of the extended graph of f is a continuous, non-decreasing, surjective function

$$(r,u) : [0,1] \to \Gamma(f), \qquad (r,u)(0) = (0, f(0)),\ (r,u)(1) = (T, f(T)).$$

Let $\Pi(f)$ denote the set of all parametric representations of f.

2.1 Strong M_1 topology

For $f_1, f_2 \in D([0,T];V)$ we define

$$d_M(f_1, f_2) := \inf \Big\{ |r_1 - r_2|_\infty \vee \|u_1 - u_2\|_\infty : (r_i, u_i) \in \Pi(f_i),\ i = 1,2 \Big\}.$$

As in the finite dimensional situation, cf. [28, Theorem 12.3.1], it follows that d_M is a metric on $D([0,T];V)$, and we call it the *strong M_1 metric*. The metric space $\big(D([0,T];V), d_M\big)$ is separable but not complete.

Convergence of a sequence of functions in the metric d_M can be described by quantifying the oscillation of the functions. For $v, v_1, v_2 \in V$ the *distance from v to the segment between v_1 and v_2* is defined by

$$M(v_1, v, v_2) := \inf_{\alpha \in [0,1]} \|v - (\alpha v_1 + (1-\alpha)v_2)\|.$$

The distance M obeys for every $v, v_1, v_2, v', v_1', v_2' \in V$ the inequality

$$M(v_1, v, v_2) \leqslant M(v_1', v', v_2') + \|v - v'\| + \|v_1 - v_1'\| + \|v_2 - v_2'\|\,, \tag{2.1}$$

and, instead of a triangular inequality, it satisfies

$$M(v_1 + v_1', v + v', v_2 + v_2') \leqslant M(v_1, v, v_2) + \|v'\| + \big(\|v_1'\| \vee \|v_2'\|\big). \tag{2.2}$$

For functions $f, g \in D([0,T]; V)$ and $0 \leqslant t_1 \leqslant t \leqslant t_2 \leqslant T$ it follows from (2.2) that

$$M\big(f(t_1) + g(t_1), f(t) + g(t), f(t_2) + g(t_2)\big) \leqslant M\big(f(t_1), f(t), f(t_2)\big) + 2\,\|g\|_\infty\,, \tag{2.3}$$

and if $t_2 - t_1 \leqslant \delta$ then

$$\begin{aligned} &M\big(f(t_1) + g(t_1), f(t) + g(t), f(t_2) + g(t_2)\big) \\ &\qquad \leqslant M\big(f(t_1), f(t), f(t_2)\big) + \sup_{\substack{s_1, s_2 \in [0,T] \\ |s_2 - s_1| \leqslant \delta}} \|g(s_1) - g(s_2)\|\,. \end{aligned} \tag{2.4}$$

Define for $f \in D([0,T]; V)$ and $\delta > 0$ the oscillation function by

$$M(f; \delta) := \sup\Big\{M\Big(f(t_1), f(t), f(t_2)\Big) \colon 0 \leqslant t_1 < t < t_2 \leqslant T \text{ and } t_2 - t_1 \leqslant \delta\Big\}.$$

Lemma 2.1. *Let f be in $D([0,T]; V)$ and let $0 \leqslant t_1 \leqslant t_2 \leqslant t_3 \leqslant T$ with $(t_i, v_i) \in \Gamma(f)$ for some $v_i \in V$ and $i = 1, 2, 3$. If $t_3 - t_1 \leqslant \delta$ for some $\delta > 0$ then*

$$M(v_1, v_2, v_3) \leqslant M(f; \delta).$$

Proof. Follows as Lemma 12.5.2 in Whitt [28]. □

Lemma 2.2. *If $f \in D([0,T]; V)$ then $\lim_{\delta \searrow 0} M(f; \delta) = 0$.*

Proof. Follows as Lemma 12.5.3 in [28]. □

Lemma 2.3. *Let $f_n^\gamma, f^\gamma, f_n, f$, $n \in \mathbb{N}$, $\gamma > 0$, be functions in $D([0,T]; \mathbb{R})$ satisfying*

(i) $\displaystyle\lim_{n\to\infty} \left(\|f_n - f\|_\infty + \limsup_{\gamma\to\infty} \|f_n^\gamma - f^\gamma\|_\infty \right) = 0$;

(ii) $\displaystyle\lim_{\gamma\to\infty} f_n^\gamma = f_n$ *in* $(D([0,T]; \mathbb{R}), d_M)$ *for all* $n \in \mathbb{N}$.

Then it follows that $\displaystyle\lim_{\gamma\to\infty} f^\gamma = f$ *in* $\big(D([0,T]; \mathbb{R}), d_M\big)$.

Proof. Fix $\varepsilon > 0$ and choose $n_0 \in \mathbb{N}$ such that

$$\|f_n - f\|_\infty \leqslant \varepsilon \quad \text{and} \quad \limsup_{\gamma\to\infty} \|f_n^\gamma - f^\gamma\|_\infty \leqslant \varepsilon \qquad \text{for all } n \geqslant n_0.$$

Thus, there exists $\gamma_0 = \gamma_0(n_0)$ such that

$$\big\|f_{n_0}^\gamma - f^\gamma\big\|_\infty \leqslant 2\varepsilon \quad \text{for all } \gamma \geqslant \gamma_0.$$

Condition (2) implies by part (iv) in [28, Theorem 12.5.1] that there exists a dense subset $D \subseteq [0,T]$ including 0 and T such that for each $t \in D$ there exists a $\gamma_1 = \gamma_1(t, n_0) > 0$ with

$$\left|f_{n_0}^{\gamma}(t) - f_{n_0}(t)\right| \leqslant \varepsilon \qquad \text{for all } \gamma \geqslant \gamma_1, \tag{2.5}$$

and that there exists a $\delta_0 = \delta_0(n_0) > 0$ such that

$$\limsup_{\gamma\to\infty} M(f_{n_0}^{\gamma}, \delta) \leqslant \varepsilon \qquad \text{for all } \delta \leqslant \delta_0. \tag{2.6}$$

Consequently, we can conclude from (2.5) for each $t \in D$ and $\gamma \geqslant \max\{\gamma_0, \gamma_1\}$ that

$$|f^{\gamma}(t) - f(t)| \leqslant \left|f^{\gamma}(t) - f_{n_0}^{\gamma}(t)\right| + \left|f_{n_0}^{\gamma}(t) - f_{n_0}(t)\right| + |f_{n_0}(t) - f(t)| \leqslant 4\varepsilon.$$

Thus we have shown that

$$\lim_{\gamma\to\infty} f^{\gamma}(t) = f(t) \qquad \text{for all } t \in D. \tag{2.7}$$

It follows from (2.6) for each $\delta \leqslant \delta_0$ by inequality (2.3) that

$$\limsup_{\gamma\to\infty} M(f^{\gamma}, \delta) \leqslant \limsup_{\gamma\to\infty} M(f_{n_0}^{\gamma}, \delta) + 2\limsup_{\gamma\to\infty} \left\|f_{n_0}^{\gamma} - f^{\gamma}\right\| \leqslant 3\varepsilon. \tag{2.8}$$

By (2.7) and (2.8) a final application of Theorem 12.5.1 in [28] completes the proof. □

The metric space $(D([0,T];V), d_M)$ is not complete. However, one can define another metric $\hat{d}_M$ on $D([0,T];V)$ such that $(D([0,T];V), \hat{d}_M)$ is complete and the two topological spaces $(D([0,T];V),\, d_M)$ and $(D([0,T];V),\, \hat{d}_M)$ are homeomorphic, that is there exists a bijective function $i\colon (D([0,T];V),\, d_M) \to (D([0,T];V),\, \hat{d}_M)$ such that both i and its inverse are continuous, see [28, Theorem 12.8.1]. The last property, i.e. the existence of a homeomorphic mapping between the metric spaces, is called the *topological equivalence* of $(D([0,T];V),\, d_M)$ and $(D([0,T];V),\, \hat{d}_M)$. In particular, this means that open, closed and compact sets are the same in both spaces but also the spaces of real-valued, continuous functions on $D([0,T];V)$ coincide for both metrics. Moreover, since $(D([0,T];V),\, d_M)$ is separable, the space $(D([0,T];V),\, \hat{d}_M)$ is also separable, and thus it is Polish, i.e. a topological space which is metrisable as a complete separable space.

2.2 Product M_1 topology

For the product M_1 topology we assume that the Banach space V has a Schauder basis $e = (e_k)_{k\in\mathbb{N}}$ and that $(e_k^*)_{k\in\mathbb{N}}$ denotes the bi-orthogonal functionals. Instead of equipping $D([0,T];V)$ with the strong M_1 topology we can consider the space as the Cartesian product space $\prod_{k=1}^{\infty} D([0,T];\mathbb{R})$ and equip it with the product metric

$$d_M^e(f,g) := \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{d_M(\langle f, e_k^*\rangle, \langle g, e_k^*\rangle)}{1 + d_M(\langle f, e_k^*\rangle, \langle g, e_k^*\rangle)} \qquad \text{for all } f, g \in D([0,T];V).$$

The metric on the right hand side refers to the metric on the space $D([0,T];\mathbb{R})$ introduced in the previous Section 2.1. Clearly, convergence in d_M^e depends on the chosen

Schauder basis e of V. Alternatively, we can use the topological equivalent metric $\hat{d}_M$ on $D([0,T];\mathbb{R})$ to define the product metric

$$\hat{d}^e_M(f,g) := \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{\hat{d}_M(\langle f, e_k^* \rangle, \langle g, e_k^* \rangle)}{1 + \hat{d}_M(\langle f, e_k^* \rangle, \langle g, e_k^* \rangle)} \qquad \text{for all } f, g \in D([0,T];V).$$

Since $(D([0,T];\mathbb{R}), \hat{d}_M)$ is a Polish space it follows that $(D([0,T];V), \hat{d}^e_M)$ is Polish, too. Analogously, we obtain that $(D([0,T];V), d^e_M)$ is topological equivalent to $(D([0,T];V), \hat{d}^e_M)$. Recall, that the product topology is the topology of *point-wise convergence*, i.e. a sequence $(f_n)_{n\in\mathbb{N}}$ converges to f in $(D([0,T];V), d^e_M)$ if and only if for any $k \in \mathbb{N}$

$$\lim_{n\to\infty} d_M\big(\langle f_n, e_k^* \rangle, \langle f, e_k^* \rangle\big) = 0.$$

2.3 Weak M_1 topology

In an infinite dimensional Banach space V there is a third mode of convergence in the M_1 sense. A sequence $(f_n)_{n\in\mathbb{N}} \subseteq D([0,T];V)$ is said to *converge weakly* to $f \in D([0,T];V)$ if for all $v^* \in V^*$ we have

$$\lim_{n\to\infty} \langle f_n, v^* \rangle = \langle f, v^* \rangle \qquad \text{in } \big(D([0,T];\mathbb{R}), d_M\big).$$

Note, that if V is infinite dimensional the induced topology is not metrisable. The three different modes of convergence are related as shown by the following diagram:

$$\text{strong } M_1 \;\Rightarrow \text{weak } M_1 \;\Rightarrow \text{product } M_1.$$

The first implication follows from the fact that $f_1, f_2 \in D([0,T];V)$ obey the inequality

$$d_M(\langle f_1, v^* \rangle, \langle f_2, v^* \rangle) \leqslant (\|v^*\| \vee 1)\, d_M(f_1, f_2) \qquad \text{for all } v^* \in V^*.$$

Since the product topology is the point-wise convergence it is immediate that it is implied by weak convergence.

If V is finite dimensional it is known that the weak and strong topology coincide, see Theorem 12.7.2 in [28]. In the infinite dimensional situation the situation differs as illustrated by the following example.

Example 2.4. Let V be an arbitrary Hilbert space with orthonormal basis $(e_k)_{k\in\mathbb{N}}$. The functions $f_n \colon [0,T] \to V$ can be chosen as $f_n(t) := e_n$ for all $t \in [0,T]$ and $n \in \mathbb{N}$. It follows that

$$\sup_{t\in[0,T]} |\langle f_n(t), v \rangle| = |\langle e_n, v \rangle| \to 0 \qquad \text{as } n \to \infty \text{ for all } v \in V,$$

and thus, the sequence $(f_n)_{n\in\mathbb{N}}$ converges weakly to 0 in $D([0,T];V)$. However, since $\|f_n(t)\| = 1$ for all $n \in \mathbb{N}$ and $t \in [0,T]$ it does not converge strongly.

Example 2.5. It is well known that in a finite dimensional Hilbert space V convergence in the product topology does not imply strong convergence in the M_1 metric. Since in the finite dimensional situation, the strong M_1 topology coincides with the topology of weak convergence in $D([0,T];V)$, this is also an example that convergence in the product topology does not imply weak convergence.

Finally let us remark that our notion of the modes of convergence in the M_1 sense does not coincide with the one in the literature such as [28]. There the product topology is called weak topology, whereas our weak topology does not have a name as it coincides with the strong topology in finite dimensional spaces. Since addition is not continuous in $D([0,T];V)$ our notion of weak convergence can not be confused with the usual weak topology in a linear topological vector space.

2.4 Random variables in the Skorokhod space

Let $\mathcal{B}(D)$ denote the Borel-σ-algebra generated by open sets in $(D([0,T];V), d_M)$. As in the finite dimensional situation, see [28, Theorem 11.5.2], it can be shown that $\mathcal{B}(D)$ coincides with the σ-algebra, generated by the coordinate mappings

$$\pi_{t_1,\dots,t_n}\colon D([0,T];V) \to V^n, \qquad \pi_{t_1,\dots,t_n}(f) = (f(t_1),\dots,f(t_n))$$

for each $t_1,\dots,t_n \in [0,T]$ and $n \in \mathbb{N}$. The analogous result for the Skorokhod J_1 topology in separable metric spaces can be found in [9]. On the other hand, the Borel-σ-algebra generated by open sets in $(D([0,T];V), d_M^e)$ equals the product of Borel-σ-algebras in $(D([0,T];\mathbb{R}), d_M)$. Consequently, both Borel-σ-algebras, generated by open sets with respect to d_M and d_M^e coincide.

Let $(\Omega, \mathcal{A}, P)$ be a probability space and let $X := (X(t)\colon t \in [0,T])$ be a V-valued stochastic process with càdlàg paths. Since the Borel-σ-algebra $\mathcal{B}(D)$ is generated by the coordinate mappings, it follows that X is a $D([0,T];V)$-valued random variables.

3 Convergence in probability

3.1 Strong topology

In this section we consider the convergence in probability in the strong metric d_M of stochastic processes $(X_n)_{n\in\mathbb{N}}$ to a stochastic process X in the space $D([0,T];V)$. If the stochastic processes X and X_n have càdlàg paths then X and X_n are $D([0,T);V)$-valued random variables. Since $\big(D([0,T);V), d_M\big)$ is separable, convergence in probability is well defined in the sense that $(X_n)_{n\in\mathbb{N}}$ converges to X in probability in $\big(D([0,T);V), d_M\big)$ if

$$\lim_{n\to\infty} P\Big(d_M(X_n, X) \geqslant \varepsilon\Big) = 0 \qquad \text{for all } \varepsilon > 0.$$

Lemma 3.1. *A V-valued stochastically continuous stochastic process $(X(t)\colon t \in [0,T])$ with càdlàg trajectories obeys*

$$\lim_{\delta\searrow 0} \sup_{t\in[0,T]} P\left(\sup_{\substack{s\in[0,T]\\ |s-t|\leqslant\delta}} \|X(t) - X(s)\| \geqslant \varepsilon \right) = 0 \qquad \textit{for all } \varepsilon > 0.$$

Proof. Define for each $\delta > 0$ and $t \in [0,T]$ the random variable

$$Z(t,\delta) := \sup_{\substack{s\in[0,T]\\ |s-t|\leqslant\delta}} \|X(t) - X(s)\|$$

and assume for a contradiction that there exist $\varepsilon_1, \varepsilon_2 > 0$, a sequence $(\delta_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}_+$ converging to 0, and a sequence $(t_n)_{n\in\mathbb{N}} \subseteq [0,T]$ such that

$$\lim_{n\to\infty} P\big(Z(t_n,\delta_n) \geqslant \varepsilon_1\big) \geqslant \varepsilon_2.$$

By passing to a subsequence if necessary we can assume that $t_n \to t_0$ for some $t_0 \in [0,T]$. Then, for each $\delta > 0$ there exists $n(\delta) \in \mathbb{N}$ such that

$$[t_n - \delta_n, t_n + \delta_n] \subseteq [t_0 - \delta, t_0 + \delta] \qquad \text{for all } n \geqslant n(\delta),$$

which implies by the definition of Z that

$$Z(t_n, \delta_n) \leqslant Z(t_0, \delta) \qquad \text{for all } n \geqslant n(\delta).$$

Consequently, we obtain for every $\delta > 0$ that there exists $n(\delta) \in \mathbb{N}$ such that

$$\varepsilon_2 \leqslant E\big[\, \mathbb{1}_{\{Z(t_n,\delta_n) \geqslant \varepsilon_1\}} \big] \leqslant E\big[\, \mathbb{1}_{\{Z(t_0,\delta) \geqslant \varepsilon_1\}} \big] \qquad \text{for all } n \geqslant n(\delta). \tag{3.1}$$

On the other hand, since $Z(t_0, \delta) \to |\Delta X(t_0)|$ P-a.s. as $\delta \searrow 0$, Lebesgue's theorem of dominated convergence implies that

$$0 = P\big(|\Delta X(t_0)| \geqslant \varepsilon_1 \big) = E\left[\lim_{\delta \searrow 0} \mathbb{1}_{\{Z(t_0,\delta) \geqslant \varepsilon_1\}} \right] = \lim_{\delta \searrow 0} E\left[\mathbb{1}_{\{Z(t_0,\delta) \geqslant \varepsilon_1\}} \right],$$

which contradicts (3.1). □

Theorem 3.2. *For V-valued, stochastically continuous stochastic processes $(X(t) \colon t \in [0,T])$ and $(X_n(t) \colon t \in [0,T])$, $n \in \mathbb{N}$, with càdlàg trajectories the following are equivalent:*

(a) *$X_n \to X$ in probability in $\big(D([0,T];V),\, d_M\big)$ as $n \to \infty$.*

(b) *the following two conditions are satisfied:*

  (i) *for every $t \in [0,T]$ we have $\lim_{n\to\infty} X_n(t) = X(t)$ in probability*

  (ii) *for every $\varepsilon > 0$ the oscillation function obeys*

$$\lim_{\delta \searrow 0} \limsup_{n \to \infty} P\big(M(X_n, \delta) \geqslant \varepsilon \big) = 0. \tag{3.2}$$

Proof. (a) ⇒ (b) To establish property (i), let $t \in [0,T]$ and $\varepsilon_1, \varepsilon_2 > 0$ be given. Lemma 3.1 guarantees that there exists $\delta > 0$ such that the set

$$E(\varepsilon_1, \delta) := \left\{ \omega \in \Omega \colon \sup_{\substack{s \in [0,T] \\ |s-t| \leqslant \delta}} \|X(t)(\omega) - X(s)(\omega)\| \leqslant \varepsilon_1 \right\},$$

satisfy $P\big(E(\varepsilon_1, \delta)\big) \geqslant 1 - \frac{\varepsilon_2}{2}$. By the assumed condition (a) there is $n_0 \in \mathbb{N}$ such that

$$P\big(d_M(X_n, X) < (\varepsilon_1 \wedge \delta)\big) \geqslant 1 - \frac{\varepsilon_2}{2} \qquad \text{for all } n \geqslant n_0.$$

Consequently, the set

$$F(\varepsilon_1, \delta, n) := E(\varepsilon_1, \delta) \cap \{d_M(X_n, X) < (\varepsilon_1 \wedge \delta)\}$$

satisfies $P\big(F(\varepsilon_1, \delta, n)\big) \geqslant 1 - \varepsilon_2$ for every $n \geqslant n_0$. Define for $\omega \in F(\varepsilon_1, \delta, n)$ the functions $f_n := X_n(\cdot)(\omega)$ and $f := X(\cdot)(\omega)$. It follows that there are parametric representations $(r, u) \in \Pi(f)$ and $(r_n, u_n) \in \Pi(f_n)$ satisfying

$$|r - r_n|_\infty \vee \|u - u_n\|_\infty \leqslant (\varepsilon_1 \wedge \delta) \qquad \text{for all } n \geqslant n_0. \tag{3.3}$$

For every $t \in [0,T]$ denote $\tau, \tau_n \in [0,1]$ for $n \geqslant n_0$ such that

$$(t, f(t)) = (r(\tau), u(\tau)) \quad \text{and} \quad (t, f_n(t)) = (r_n(\tau_n), u_n(\tau_n)).$$

Since $u(\tau_n) \in [\![f(r(\tau_n)-), f(r(\tau_n))]\!]$ for every $n \geqslant n_0$, there is $\alpha_n \in [0,1]$ such that

$$u(\tau_n) = \alpha_n f(r(\tau_n)-) + (1-\alpha_n) f(r(\tau_n)).$$

Since $t = r_n(\tau_n)$ and $|r(\tau_n) - r_n(\tau_n)| \leqslant \delta$ for all $n \geqslant n_0$ by (3.3), we have $|r(\tau_n) - t| \leqslant \delta$. Another application of inequality (3.3) implies

$$\begin{aligned}
\|u(\tau_n) - u(\tau)\| &= \|\alpha_n f(r(\tau_n)-) + (1-\alpha_n) f(r(\tau_n)) - f(t)\| \\
&\leqslant \sup_{\substack{s\in[0,T] \\ |s-t|\leqslant\delta}} \sup_{\alpha\in[0,1]} \|\alpha f(s-) + (1-\alpha) f(s) - f(t)\| \\
&= \sup_{\substack{s\in[0,T] \\ |s-t|\leqslant\delta}} \sup_{\alpha\in[0,1]} \left\|\alpha\big(f(s-) - f(t)\big) + (1-\alpha)\big(f(s) - f(t)\big)\right\| \\
&\leqslant \sup_{\substack{s\in[0,T] \\ |s-t|\leqslant\delta}} \|f(s-) - f(t)\| + \sup_{\substack{s\in[0,T] \\ |s-t|\leqslant\delta}} \|f(s) - f(t)\| \\
&\leqslant 2\varepsilon_1.
\end{aligned} \tag{3.4}$$

Inequalities (3.3) and (3.4) imply for every $n \geqslant n_0$ that

$$\begin{aligned}
\|f(t) - f_n(t)\| &= \|u(\tau) - u_n(\tau_n)\| \\
&\leqslant \|u(\tau) - u(\tau_n)\| + \|u(\tau_n) - u_n(\tau_n)\| \leqslant 3\varepsilon_1,
\end{aligned}$$

which establishes Condition (i) in (b).

In order to show Condition (ii) fix some $\varepsilon_1, \varepsilon_2 > 0$. Lemma 2.2 guarantees that there exists $\delta_0 > 0$ such that the set

$$G(\varepsilon_1, \delta) := \{\omega \in \Omega \colon M(X(\omega), \delta) \leqslant \varepsilon_1\}$$

satisfies $P(G(\varepsilon_1, \delta)) \geqslant 1 - \frac{\varepsilon_2}{2}$ for all $\delta \in [0, \delta_0]$. For each $\delta > 0$ there is $n_0 \in \mathbb{N}$ by the assumed condition (a) such that

$$P\big(d_M(X_n, X) < (\varepsilon_1 \wedge \delta)\big) \geqslant 1 - \frac{\varepsilon_2}{2} \qquad \text{for all } n \geqslant n_0.$$

Together we obtain for every $\delta \in [0, \delta_0]$

$$\liminf_{n\to\infty} P\big(G(\varepsilon_1, \delta) \cap \{d_M(X_n, \delta) < (\varepsilon_1 \wedge \delta)\}\big) \geqslant 1 - \varepsilon_2.$$

Fix $\omega \in G(\varepsilon_1, \delta) \cap \{d_M(X_n, \delta) < (\varepsilon_1 \wedge \delta)\}$ and define $f := X(\cdot)(\omega)$ and $f_n := X_n(\cdot)(\omega)$. It follows that there are parametric representations $(r, u) \in \Pi(f)$ and $(r_n, u_n) \in \Pi(f_n)$ satisfying

$$|r - r_n|_\infty \vee \|u - u_n\|_\infty \leqslant (\varepsilon_1 \wedge \delta) \qquad \text{for all } n \geqslant n_0.$$

For every $0 \leqslant t_1 \leqslant t_2 \leqslant t_3$ denote $\tau_i, \tau_{i,n} \in [0,1]$ such that $(t_i, f(t_i)) = (r(\tau_i), u(\tau_i))$ and $(t_i, f_n(t_{i,n})) = (r_n(\tau_{i,n}), u_n(\tau_{i,n}))$ for $i = 1,2,3$. Inequality (2.1) and Lemma 2.1 imply for every $n \geqslant n_0$ that

$$\begin{aligned}
M\big(f_n(t_1), f_n(t_2), f_n(t_3)\big) &= M\big(u_n(\tau_{1,n}), u_n(\tau_{2,n}), u_n(\tau_{3,n})\big) \\
&\leqslant M\big(u(\tau_{1,n}), u(\tau_{2,n}), u(\tau_{3,n})\big) + 3\|u - u_n\|_\infty \\
&\leqslant M(f, \delta) + 3\|u - u_n\|_\infty \\
&\leqslant \varepsilon_1 + 3\varepsilon_1 = 4\varepsilon_1,
\end{aligned}$$

which completes the proof of the implication (a) ⇒ (b).
(b) ⇒ (a). Let $\varepsilon_1, \varepsilon_2 > 0$ be fixed. Define for each $n \in \mathbb{N}$ and $\delta > 0$ the sets

$$G(\varepsilon_1, \delta) := \Big\{\omega \in \Omega\colon\, M(X(\omega), \delta) < \tfrac{\varepsilon_1}{512}\Big\},$$
$$G_n(\varepsilon_1, \delta) := \Big\{\omega \in \Omega\colon\, M(X_n(\omega), \delta) < \tfrac{\varepsilon_1}{512}\Big\}.$$

Condition (ii) guarantees that there exist $\delta_1 > 0$ and $n_1 \in \mathbb{N}$ such that

$$\limsup_{n\to\infty} P\big(G_n^c(\varepsilon_1, \delta)\big) \leqslant \frac{\varepsilon_2}{8} \qquad \text{for all } \delta \in [0, \delta_1].$$

Consequently, for each $\delta \in [0, \delta_1]$ there exists $n_1 = n_1(\delta)$ such that

$$\sup_{n\geqslant n_1} P\big(G_n^c(\varepsilon_1, \delta)\big) \leqslant \frac{\varepsilon_2}{4}, \tag{3.5}$$

whereas Lemma 2.2 implies that there exist $\delta_2 > 0$ such that

$$P\big(G(\varepsilon_1, \delta)\big) \geqslant 1 - \frac{\varepsilon_2}{4} \qquad \text{for all } \delta \in [0, \delta_2]. \tag{3.6}$$

Define for $c > 0$ the set

$$B(c) := \Big\{\omega \in \Omega\colon\, \|X(\omega)\|_\infty \leqslant c - 1\Big\}.$$

Since X is a random variable with values in $D([0, T]; V)$ there exists $c > 0$ such that

$$P\big(B(c)\big) \geqslant 1 - \frac{\varepsilon_2}{4}. \tag{3.7}$$

Choose a partition $\pi = (t_i)_{i=0}^m$ of the interval $[0, T]$ such that

$$0 = t_0 < t_1 < \cdots < t_m = T \quad \text{and} \quad \max_{i\in\{1,\dots,m\}} |t_i - t_{i-1}| \leqslant \min\{\delta_1, \delta_2, \tfrac{\varepsilon_1}{16}\},$$

and define the set

$$F_n(\varepsilon_1, \pi) := \Big\{\omega \in \Omega\colon\, \max_{i=1,\dots,m} \|X_n(t_i)(\omega) - X(t_i)(\omega)\| < \tfrac{\varepsilon_1}{512}\Big\}.$$

Condition (i) guarantees that there exists $n_2 \in \mathbb{N}$ such that

$$\sup_{n\geqslant n_2} P\Big(F_n^c(\varepsilon_1, \pi)\Big) \leqslant \frac{\varepsilon_2}{4}. \tag{3.8}$$

It follows from (3.5) to (3.8) that for $\delta := \delta_1 \wedge \delta_2$ the set $E_n(\varepsilon_1, \delta, c, \pi) := G_n(\varepsilon_1, \delta) \cap G(\varepsilon_1, \delta) \cap B(c) \cap F_n(\varepsilon_1, \pi)$ obeys

$$P\Big(E_n(\varepsilon_1, \delta, c, \pi)\Big) \geqslant 1 - \varepsilon_2 \qquad \text{for all } n \geqslant n_1 \vee n_2.$$

For $\omega \in E_n(\varepsilon_1, \delta, c, \pi)$ define $f_0(\cdot) := X(\cdot)(\omega)$ and $f_n(\cdot) := X_n(\cdot)(\omega)$. Let N denote the integers $\{n_1 \vee n_2, \dots\}$ and N_0 the union $N \cup \{0\}$. For $n \in N_0$ and $i \in \{1, \dots, m\}$ let Γ_i^n be the graph of f_n between $(t_{i-1}, f_n(t_{i-1}))$ and $(t_i, f_n(t_i))$. By defining d_i to be the smallest integer larger than $\|f_0(t_{i-1}) - f_0(t_i)\| \frac{16}{\varepsilon_1}$ we can divide the segment $[\![f_0(t_{i-1}), f_0(t_i)]\!]$ in equidistant points

$$\xi_{i,j} := f_0(t_{i-1}) + \alpha_{i,j}(f_0(t_i) - f_0(t_{i-1})) \quad \text{for } \alpha_{i,j} := \frac{j}{d_i}, \quad j = 0, \dots, d_i.$$

We claim that for each $i \in \{1, \dots, m\}$ the balls $B_{i,j} := \big\{h \in V \colon \|h - \xi_{i,j}\| < \frac{\varepsilon_1}{25}\big\}$ covers each of the graphs Γ_n^i for $n \in N_0$, i.e.

$$\pi(\Gamma_n^i) \subseteq \bigcup_{j=0}^{d_i} B_{i,j} \qquad \text{for all } n \in N_0. \tag{3.9}$$

Indeed, let $(t,h) \in \Gamma_i^n$ be of the form $h = \alpha f_n(t-) + (1-\alpha) f_n(t)$ for some $\alpha \in [0,1]$. Since $t \in [t_{i-1}, t_i]$ and $t_i - t_{i-1} \leqslant \delta_2$ it follows from the definition of $M(f_n, \delta_2)$ that there exists $\ell_n, r_n \in [\![f_n(t_{i-1}), f_n(t_i)]\!]$ such that

$$\|f_n(t-) - \ell_n\| \leqslant \tfrac{\varepsilon_1}{512} \quad \text{and} \quad \|f_n(t) - r_n\| \leqslant \tfrac{\varepsilon_1}{512} \qquad \text{for all } n \in N_0.$$

Since $u_n := \alpha \ell_n + (1-\alpha) r_n \in [\![f_n(t_{i-1}), f_n(t_i)]\!]$ we have $M(f_n(t_{i-1}), u_n, f_n(t_i)) = 0$. Inequality (2.1) implies that

$$\begin{aligned} M(f_0(t_{i-1}), u_n, f_0(t_i)) &\leqslant M(f_n(t_{i-1}), u_n, f_n(t_i)) + \|f_0(t_{i-1}) - f_n(t_{i-1})\| + \|f_0(t_i) - f_n(t_i)\| \\ &\leqslant 0 + 2 \max_{i \in \{0,\dots,m\}} \|f_n(t_i) - f_0(t_i)\| \\ &< 2\frac{\varepsilon_1}{512}. \end{aligned}$$

Consequently, there exists $u_0 \in [\![f_0(t_{i-1}), f_0(t_i)]\!]$ such that $\|u_n - u_0\| \leqslant \frac{2\varepsilon_1}{512}$. (If $n = 0$ we can choose $u_0 = u_n$.) Since $u_0 \in [\![f_0(t_{i-1}), f_0(t_i)]\!]$ we can choose the closest node $\xi_{i,j}$ for some $j = 0, \dots, d_i$ such that $\|u_0 - \xi_{i,j}\| \leqslant \frac{\varepsilon_1}{32}$. It follows

$$\begin{aligned} \|h - \xi_{i,j}\| &= \big\|\big(\alpha f_n(t-) + (1-\alpha) f_n(t)\big) - \xi_{i,j}\big\| \\ &\leqslant \alpha \|f_n(t-) - \ell_n\| + (1-\alpha)\|f_n(t) - r_n\| + \|u_n - u_0\| + \|u_0 - \xi_{i,j}\| \\ &\leqslant \frac{\varepsilon_1}{512} + \frac{2\varepsilon_1}{512} + \frac{\varepsilon_1}{32} \leqslant \frac{\varepsilon_1}{25}, \end{aligned}$$

which shows (3.9).

In the following, we define for each $i \in \{1, \dots, m\}$ and $n \in N_0$ an ordered sequence of points

$$\big((r_{i,0}^n, z_{i,0}^n), \dots, (r_{i,m_i}^n, z_{i,m_i}^n)\big) \in \big(\Gamma_i^n \times \dots \times \Gamma_i^n\big),$$

for some $m_i \in \mathbb{N}$, independent of n, such that they satisfy for every $j = 1, \dots, m_i$:

$$\sup_{(z,r) \in \Gamma_{i,j}^n} \max\big\{\big\|z - z_{i,j-1}^n\big\|, \big\|z - z_{i,j}^n\big\|, \big|r - r_{i,j-1}^n\big|, \big|r - r_{i,j}^n\big|\big\} \leqslant \frac{\varepsilon_1}{4}, \tag{3.10}$$

where $\Gamma_{i,j}^n := \{(r,z) \in \Gamma_i^n \colon (r_{i,j-1}^n, z_{i,j-1}^n,) \leqslant (r,z) \leqslant (r_{i,j}^n, z_{i,j}^n)\}$.

If $d_i = 1$ we define $m_i = 1$ and for every $n \in N_0$ the points:

$$(r_{i,0}^n, z_{i,0}^n) := (t_{i-1}, f_n(t_{i-1})), \quad (r_{i,1}^n, z_{i,1}^n) := (t_i, f_n(t_i)).$$

It follows from (3.9) that for each $(r,z) \in \Gamma_i^n$ there is $k \in \{0,1\}$ such that $z \in B_{i,k}$. For $k=0$ this results in

$$\begin{aligned}\left\|z - z_{i,0}^n\right\| \vee \left\|z - z_{i,1}^n\right\| &\leqslant \|z-\xi_{i,0}\| + \|\xi_{i,0}-\xi_{i,1}\| + \|\xi_{i,1} - f_n(t_i)\| \\ &\leqslant \frac{\varepsilon_1}{25} + \frac{\varepsilon_1}{16} + \frac{\varepsilon_1}{512}\,\mathbb{1}_N(n) \leqslant \frac{\varepsilon_1}{4},\end{aligned} \tag{3.11}$$

and analogously for $k=1$. Since each $r \in [r_{i,0}^n, r_{i,1}^n]$ satisfies $\left|r - r_{i,j}^n\right| \leqslant \left|r_{i,0}^n - r_{i,1}^n\right| \leqslant \frac{\varepsilon_1}{16}$ for $j \in \{0,1\}$ we obtain the inequality (3.10).

If $d_i = 2$ we define $m_i = 3$ but we distinguish two cases. Firstly, assume that $\pi(\Gamma_i^n) \subseteq B_{i,0} \cup B_{i,2}$. Then we define for each $n \in N_0$ the points

$$(r_{i,0}^n, z_{i,0}^n,) := (t_{i-1}, f_n(t_{i-1})), \quad (r_{i,3}^n, z_{i,3}^n) := (t_i, f_n(t_i))$$

and we choose $z_{i,1}^n, z_{i,2}^n \in \pi(\Gamma_i^n) \cap B_{i,0} \cap B_{i,2}$ and $r_{i,1}^n, r_{i,2}^n \in [0,1]$ such that

$$(r_{i,0}^n, z_{i,0}^n) < (r_{i,1}^n, z_{i,1}^n) < (r_{i,2}^n, z_{i,2}^n) < (r_{i,3}^n, z_{i,3}^n).$$

In the case $\pi(\Gamma_n^i) \not\subseteq B_{i,0} \cup B_{i,2}$ we define for every $n \in N_0$ the points

$$\begin{aligned}&(r_{i,0}^n, z_{i,0}^n) := (t_{i-1}, f_n(t_{i-1})),\\ &(r_{i,1}^n, z_{i,1}^n) := \inf\{(r,z) \in \Gamma_n^i \colon (r,z) > (r_{i,0}^n, z_{i,0}^n) \text{ and } z \in \partial B_{i,1}\},\\ &(r_{i,2}^n, z_{i,2}^n) := \inf\{(r,z) \in \Gamma_n^i \colon (r,z) > (r_{i,1}^n, z_{i,1}^n) \text{ and } z \in \partial B_{i,2}\},\\ &(r_{i,3}^n, z_{i,3}^n) := (t_i, f_n(t_i)).\end{aligned}$$

If $d_i \geqslant 3$ we define $m_i = d_i + 1$. Since $\|\xi_{i,j} - \xi_{i,j-1}\| \geqslant \frac{d_i - 1}{d_i}\frac{\varepsilon_1}{16} > \frac{\varepsilon_1}{25}$ we have $B_{i,j} \cap B_{i,j+2} = \emptyset$ for all $j = 1, \dots, d_i$. Thus, we can define the following increasing sequence:

$$\begin{aligned}(r_{i,0}^n, z_{i,0}^n) &:= (t_{i-1}, f_n(t_{i-1})),\\ (r_{i,j}^n, z_{i,j}^n) &:= \inf\{(r,z) \in \Gamma_i^n \colon (r,z) > (r_{i,j-1}^n, z_{i,j-1}^n) \text{ and } z \in \partial B_{i,j}\}, \quad j = 1, \dots, d_i,\\ (r_{i,m_i}^n, z_{i,m_i}^n) &:= (t_i, f_n(t_i)).\end{aligned}$$

In both cases for $d_i = 2$ and in the case $d_i \geqslant 3$ it follows for each $n \in N$ that

$$\begin{aligned}\left\|z_{i,0}^n - \xi_{i,0}\right\| &= \|f_n(t_{i-1}) - f_0(t_{i-1})\| < \frac{\varepsilon_1}{512},\\ \left\|z_{i,m_i}^n - \xi_{i,d_i}\right\| &= \|f_n(t_i) - f_0(t_i)\| < \frac{\varepsilon_1}{512}.\end{aligned} \tag{3.12}$$

Consequently, $z_{i,0}^n \in B_{i,0}$ and $z_{i,m_i}^n \in B_{i,d_i}$ and thus $z_{i,0}^n, z_{i,1}^n \in \bar{B}_{i,0}$ and $z_{i,m_i-1}^n, z_{i,m_i}^n \in \bar{B}_{i,d_i}$ for every $n \in N_0$. Since $z_{i,j-1}^n, z_{i,j}^n \in \bar{B}_{i,j-1}$ for all $j = 2, \dots, m_i - 1$ by construction, we obtain

$$\left\|z_{i,j-1}^n - z_{i,j}^n\right\| \leqslant 2\frac{\varepsilon_1}{25} \qquad \text{for all } j \in \{1, \dots, m_i\},\ n \in N_0. \tag{3.13}$$

If $(r,z) \in \Gamma_{i,j}^n$ for some $j \in \{1, \dots, m_i\}$ and $n \in N_0$ then $M(z_{i,j-1}^n, z, z_{i,j}^n) < \frac{\varepsilon_1}{512}$ since $\left|r_{i,j-1}^n - r_{i,j}^n\right| \leqslant |t_{i-1} - t_i| \leqslant \delta_1$. Thus, there exists $z_0 \in [\![z_{i,j-1}^n, z_{i,j}^n,]\!]$ such that $\|z - z_0\| \leqslant \frac{\varepsilon_1}{512}$. Together with (3.13) it follows for each $(r,z) \in \Gamma_{i,j}^n$ and $k \in \{j-1, j\}$ for $j = 1, \dots, m_i$ that

$$\left\|z - z_{i,k}^n\right\| \leqslant \|z - z_0\| + \left\|z_0 - z_{i,k}^n\right\| \leqslant \|z - z_0\| + \left\|z_{i,j-1}^n - z_{i,j}^n\right\| \leqslant \frac{\varepsilon_1}{512} + 2\frac{\varepsilon_1}{25} \leqslant \frac{\varepsilon_1}{4}.$$

Since we also have that

$$\left|r - r_{i,k}^n\right| \leqslant |t_i - t_{i-1}| \leqslant \frac{\varepsilon_1}{16},$$

we obtain (3.10).

The constructed sequence exhibits a further property: since for every $i \in \{1, \ldots, m\}$ and $n \in N$ the points $z_{i,j}^0$ and $z_{i,j}^n$ are in the same closed ball $\bar{B}_{i,j}$ for $j \in \{0, m_i\}$ by (3.12) and for $j \in \{1, \ldots, m_i - 1\}$ by construction , it follows that

$$\sup_{\substack{i \in \{1,\ldots,m\} \\ j \in \{0,\ldots,m_i\}}} \left\{ \left\| z_{i,j}^0 - z_{i,j}^n \right\| \right\} \leqslant 2\frac{\varepsilon_1}{25}.$$

Since $\left|r_{i,j}^0 - r_{i,j}^n\right| \leqslant |t_i - t_{i-1}| \leqslant \frac{\varepsilon_1}{16}$ we obtain that

$$\sup_{\substack{i \in \{1,\ldots,m\} \\ j \in \{0,\ldots,m_i\}}} \max\left\{ \left\| z_{i,j}^0 - z_{i,j}^n \right\|, \left|r_{i,j}^0 - r_{i,j}^n\right| \right\} \leqslant 2\frac{\varepsilon_1}{25}. \qquad \text{for all } n \in N. \tag{3.14}$$

By gluing together we obtain for each $n \in N_0$ an ordered sequence

$$\big((r_{1,0}^n, z_{1,0}^n), \ldots, (r_{1,m_1}^n, z_{1,m_1}^n), (r_{2,0}^n, z_{2,0}^n), \ldots, (r_{m,m_m}^n, z_{m,m_m}^n)\big) \in (\Gamma_n \times \cdots \times \Gamma_n),$$

satisfying the inequalities (3.10) and (3.14). It follows as in the proof of the implication $(vi) \Rightarrow (i)$ of Theorem 12.5.1 in [28], that one can define for every parametric representation $(r, u) \in \Pi(f_0)$ a parametric representation $(r_n, u_n) \in \Pi(f_n)$ such that

$$|r - r_n|_\infty \vee \|u - u_n\|_\infty \leqslant 2\frac{\varepsilon_1}{4} + \frac{2\varepsilon_1}{25} \qquad \text{for all } n \in N.$$

Thus, we have shown that for each $\omega \in E_n(\varepsilon_1, \delta, c, \pi)$ we have

$$d_M(X(\omega), X_n(\omega)) \leqslant 2\frac{\varepsilon_1}{4} + \frac{2\varepsilon_1}{25} \qquad \text{for all } n \in N,$$

which completes the proof. □

3.2 Product topology

In this part we equip the space $D([0,T];V)$ with the product topology d_M^e for a fixed Schauder basis $e := (e_k)_{k\in\mathbb{N}}$ of V with bi-orthogonal sequence $(e_k^*)_{k\in\mathbb{N}}$, and we consider the convergence in probability of stochastic processes $(X_n)_{n\in\mathbb{N}}$ to a stochastic process X. For stochastic process X and $(X_n)_{n\in\mathbb{N}}$ with càdlàg trajectories we say that $(X_n)_{n\in\mathbb{N}}$ converges to X in probability in $\big(D([0,T];V), d_M^e\big)$ if

$$\lim_{n\to\infty} P\Big(d_M^e(X_n, X) \geqslant \varepsilon\Big) = 0 \qquad \text{for all } \varepsilon > 0.$$

Since the product topology corresponds to point-wise convergence, the stochastic processes $(X_n)_{n\in\mathbb{N}}$ converges to X in probability in $\big(D([0,T];V), d_M^e\big)$ if and only if for every $k \in \mathbb{N}$

$$\lim_{n\to\infty} P\Big(d_M\big(\langle X_n, e_k^*\rangle, \langle X, e_k^*\rangle\big) \geqslant \varepsilon\Big) = 0 \qquad \text{for all } \varepsilon > 0, \tag{3.15}$$

see [10, Lemma 4.4.4]. Consequently, we obtain as an analogue of Theorem 3.2:

Corollary 3.3. *Let $(e_k)_{k\in\mathbb{N}}$ be a Schauder basis of V with bi-orthogonal sequence $(e_k^*)_{k\in\mathbb{N}}$. For V-valued, stochastically continuous stochastic processes $(X(t)\colon t\in[0,T])$ and $(X_n(t)\colon t\in[0,T])$, $n\in\mathbb{N}$, with càdlàg trajectories the following are equivalent:*

(a) *$X_n \to X$ in probability in $\big(D([0,T];V),\, d_M^e\big)$ as $n\to\infty$;*

(b) *the following two conditions are satisfied for every $k\in\mathbb{N}$:*

  (i) *for every $t\in[0,T]$ we have $\lim_{n\to\infty}\langle X_n(t),e_k^*\rangle = \langle X(t),e_k^*\rangle$ in probability;*

  (ii) *for every $\varepsilon>0$ the oscillation function obeys*

$$\lim_{\delta\searrow 0}\limsup_{n\to\infty} P\big(M(\langle X_n,e_k^*\rangle,\delta)\geqslant\varepsilon\big)=0.$$

Proof. Follows immediately from Theorem 3.2 and (3.15). □

3.3 Weak topology

Recall that the weak M_1 topology in an infinite dimensional Hilbert space is not metrisable. A sequence $(X_n)_{n\in\mathbb{N}}$ of stochastic processes $(X_n)_{n\in\mathbb{N}}$ with trajectories in $D([0,T];V)$ is said to *converge weakly in M_1 in probability* to a process X with trajectories in $D([0,T];V)$ if for all $v^*\in V^*$ we have

$$\lim_{n\to\infty}\langle X_n,v^*\rangle=\langle X,v^*\rangle \qquad \text{in probability in } \big(D([0,T];\mathbb{R}),d_M\big).$$

Equivalently, by using the metric d_M in $D([0,T];\mathbb{R})$, this convergence takes place if and only if for each $v^*\in V^*$ we have

$$\lim_{n\to\infty} P\Big(d_M\big(\langle X_n,v^*\rangle,\langle X,v^*\rangle\big)\geqslant\varepsilon\Big)=0 \qquad \text{for all } \varepsilon>0. \tag{3.16}$$

By comparing (3.16) with (3.15) one can colloquially describe the difference between convergence in the weak sense and in the product topology by testing the one-dimensional projections either with all elements, i.e. $\langle X_n,v^*\rangle$ for all $v^*\in V^*$, or only with the bi-orthogonal elements of V^*, i.e. $\langle X_n,e_k^*\rangle$ for all $k\in\mathbb{N}$. Clearly, the first one is independent of the chosen basis.

Corollary 3.4. *For V-valued, stochastically continuous stochastic processes $(X(t)\colon t\in[0,T])$ and $(X_n(t)\colon t\in[0,T])$, $n\in\mathbb{N}$, with càdlàg trajectories the following are equivalent:*

(a) *$X_n\to X$ weakly in M_1 in probability in $D([0,T];V)$ as $n\to\infty$;*

(b) *the following two conditions are satisfied for every $v^*\in V^*$:*

  (i) *for every $t\in[0,T]$ we have $\lim_{n\to\infty}\langle X_n(t),v^*\rangle=\langle X(t),v^*\rangle$ in probability;*

  (ii) *for every $\varepsilon>0$ the oscillation function obeys*

$$\lim_{\delta\searrow 0}\limsup_{n\to\infty} P\big(M(\langle X_n,v^*\rangle,\delta)\geqslant\varepsilon\big)=0.$$

Proof. Follows immediately from Theorem 3.2 and (3.16). □

Remark 3.5. In this part we always require that the considered stochastic processes have càdlàg paths in the Hilbert space V. If V is infinite dimensional this might be a too restrictive assumption. In fact the definition of weak convergence only requires that the stochastic processes have *cylindrical càdlàg trajectories*, that is

$$\big(\langle X(t), v^*\rangle \colon t \in [0,T]\big), \qquad \big(\langle X_n(t), v^*\rangle \colon t \in [0,T]\big), \quad n \in \mathbb{N},$$

have càdlàg trajectories for all $v^* \in V^*$. Since Corollary 3.4 is just proved by the application of Theorem 3.2 to these real-valued stochastic processes we could easily soften our assumption on the path regularities of the considered stochastic processes accordingly.

The same comment applies to convergence in the product topology $(D([0,T];V), d_M^e)$ for a Schauder basis $e = (e_k)_{k\in\mathbb{N}}$ of V with bi-orthogonal sequence $e = (e_k^*)_{k\in\mathbb{N}}$. Here it is sufficient to require that the considered stochastic processes have *D-cylindrical càdlàg trajectories* for $D = \{e_1^*, e_2^*, \dots\}$, that is

$$\big(\langle X(t), e_k^*\rangle \colon t \in [0,T]\big), \qquad \big(\langle X_n(t), e_k^*\rangle \colon t \in [0,T]\big), \quad n \in \mathbb{N},$$

have càdlàg trajectories for all $k \in \mathbb{N}$. The notions of cylindrical càdlàg and D-cylindrical càdlàg paths can be found in [19].

In order to have a clearer presentation of our paper, and not at least since our focus is rather on the different modes of convergence instead of the subtle issue of temporal regularity, we require stochastic processes to have càdlàg trajectories in the underlying Banach space. However, if necessary, it should be obvious how to extend our results to stochastic processes with càdlàg trajectories only in the cylindrical sense.

4 Convergence of stochastic convolution integrals

In this section we apply our results of Section 3 to the convergence of stochastic convolution integrals with respect to Lévy process. Although it would be possible to continue with the general setting of Banach spaces with a Schauder basis, we restrict ourselves here to Hilbert spaces in order to make use of standard integration theory as in [6]. In this case, we identify the dual spaces U^* and V^* with the separable Hilbert spaces U and V.

Let ξ be an infinitely divisible Radon measure on $\mathcal{B}(U)$. Then the characteristic function of ξ is given by

$$\varphi_\xi \colon U \to \mathbb{C}, \qquad \varphi_\xi(u) = \exp\big(\Psi(u)\big),$$

where the *Lévy symbol* $\psi \colon U \to \mathbb{C}$ is defined by

$$\Psi(u) = i\langle a, u\rangle - \tfrac{1}{2}\langle Qu, u\rangle + \int_U \Big(e^{i\langle u, r\rangle} - 1 - i\langle u, r\rangle\, \mathbb{1}_{B_U}(r)\Big)\, \nu(dr),$$

where $a \in U$, $Q \colon U \to U$ is the covariance operator of a Gaussian Radon measure on $\mathcal{B}(U)$ and ν is a σ-finite measure on $\mathcal{B}(U)$ with $\nu(\{0\}) = 0$ and

$$\int_U \big(\,\|r\|^2 \wedge 1\big)\, \nu(dr) < \infty.$$

Consequently, the triplet (a, Q, ν) characterises the distribution of the Radon measure ξ and thus, it is called it the *characteristics of* ξ. If X is an U-valued random variable which is infinitely divisible then we call the characteristics of its probability distribution the characteristics of X. The Lévy symbol $\Psi\colon U^* \to \mathbb{C}$ is sequentially weakly continuous and satisfies

$$|\Psi(u)| \leqslant c(1 + \|u\|^2) \qquad \text{for all } u \in U, \tag{4.1}$$

for a constant $c > 0$ depending on the underlying infinitely divisible distribution.

Let $\{\mathcal{F}_t\}_{t\geqslant 0}$ be a filtration for the probability space $(\Omega, \mathcal{A}, P)$. An adapted stochastic process $L := (L(t)\colon t \geqslant 0)$ with values in U is called a *Lévy process* if $L(0) = 0$ P-a.s., L has independent and stationary increments and L is continuous in probability. It follows that there exists a version of L with paths which are continuous from the right and have limits from the left (càdlàg paths). In the sequel we always assume that a Lévy process has càdlàg paths. Clearly, the random variable $L(1)$ is infinitely divisible and we call its characteristics the characteristics of L.

In the work [6], Chojnowska–Michalik introduces a theory of stochastic integration for deterministic, operator-valued integrands with respect to a U-valued Lévy process. Another approach in a more general setting can be found in [21] but we follow here [6]. Let V be another separable Hilbert space and define

$$\mathcal{H}^2(U,V) := \left\{ F\colon [0,T] \to \mathcal{L}(U,V)\colon F \text{ is measurable, } \int_0^T \|F(s)\|^2_{U\to V}\, ds < \infty \right\}.$$

For $F \in \mathcal{H}^2(U,V)$ we denote by $F^*(t)$ the adjoint operator $(F(t))^*\colon V \to U$ for each $t \in [0,T]$. In [6], the author starts with step functions in $\mathcal{H}^2(U,V)$ to define a stochastic integral and finally shows, that for each element in $\mathcal{H}^2(U,V)$ this stochastic integral exists as the limit of the stochastic integrals for step functions in $\mathcal{H}^2(U,V)$ in the topology of convergence in probability. We denote this stochastic integral for $F \in \mathcal{H}^2(U,V)$ with respect to the Lévy process L by

$$I(F) := \int_0^T F(s)\, dL(s).$$

If Ψ is the Lévy symbol of L then the stochastic integral $I(F)$ is infinitely divisible and has the characteristic function

$$\varphi_{I(F)}\colon V \to \mathbb{C}, \qquad \varphi_{I(F)}(v) = \exp\left(\int_0^T \Psi(F^*(s)v)\, ds \right). \tag{4.2}$$

By firstly considering step functions and then passing to the limit, one can show that for each $F \in \mathcal{H}^2(U,V)$ the stochastic integral $I(F)$ obeys

$$\langle \int_0^T F(s)\, dL(s), v \rangle = \int_0^T F^*(s)v\, dL(s) \qquad P\text{-a.s. for all } v \in V. \tag{4.3}$$

Here, the right hand side is understood as the same stochastic integral but for the integrand $F^*(\cdot)v \in \mathcal{H}^2(U,\mathbb{R})$. If $F \in \mathcal{H}^2(U,V)$ is for some $v \in V$ of the special form

$$F^*(t)v = \varphi(t)Gv \qquad \text{for all } t \in [0,T],$$

for a function $\varphi\colon\ \mathbb{R} \to \mathbb{R}$ and $G \in \mathcal{L}(V,U)$ then one obtains

$$\int_0^T F^*(s)v\, dL(s) = \int_0^T \varphi(s)\, d\ell(s), \tag{4.4}$$

where ℓ denotes the real-valued Lévy process defined by $\ell(t) := \langle L(t), Gv\rangle$. If $F \in \mathcal{H}^2(U,V)$ is of the special form $F(\cdot) = S(\cdot)G$ for some $G \in \mathcal{L}(U,V)$ and $S \in \mathcal{H}^2(V,V)$, we obtain

$$\int_0^T G^* S^*(s)v\, dL(s) = \int_0^T S^*(s)v\, dK(s), \tag{4.5}$$

where K is the Lévy process in V defined by $K(t) := GL(t)$ for all $t \geqslant 0$.

For a function $F \in \mathcal{H}^2(U,V)$ we define the *stochastic convolution integral process* $F * L := (F * L(t) \colon t \in [0,T])$ by

$$F * L(t) := \int_0^t F(t-s)\, dL(s) \qquad \text{for all } t \in [0,T].$$

In this section we apply our results of Section 3 to the convergence of stochastic convolution integral processes in the weak and product topology M_1, that is for functions F, $F_\gamma \in \mathcal{H}^2(U,V)$, depending on a parameter $\gamma > 0$, we establish the convergence

$$\lim_{\gamma\to\infty} F_\gamma * L = F * L$$

in probability in the weak and product topology.

The study of the limiting behaviour requires that the stochastic processes have càdlàg paths in V, or at least in the appropriate cylindrical sense as pointed out in Remark 3.5. There is no condition for regularities of trajectories available covering our rather general setting but for numerous specific situations one knows sufficient conditions guaranteeing either continuous or càdlàg trajectories of stochastic convolution integrals. For example, classical results on continuity of Gaussian processes can be found in [14] and [23], and on regularity of infinitely divisible processes in [24]; temporal path regularity of stochastic convolution integrals are considered in [11] and [13], the infinite-dimensional Ornstein-Uhlenbeck process is treated in [15] and [17]. As our work is focused on the convergence rather than regularities of trajectories we will assume the following in this section:

Assumption A: For all considered functions F,$F_\gamma \in \mathcal{H}^2(U,V)$, $\gamma > 0$, the stochastic processes $F * L$ and $F_\gamma * L$, $\gamma > 0$, have càdlàg trajectories.

Furthermore, if W denotes the Gaussian part of L then the stochastic process $F * W$ has continuous trajectories.

We do not assume that $F_\gamma * W$ has continuous paths but only the prospective limit $F * W$. This is a quite natural assumption for the M_1 topology that only the limit is continuous.

4.1 Convergence of the marginals

Lemma 4.1. *Let F, F_γ, $\gamma > 0$, be functions in $\mathcal{H}^2(U,V)$ satisfying for a subset $D \subseteq V$ and all $u \in U$*

(i) $F^*(\cdot)v,\ F_\gamma^*(\cdot)v \in D([0,T];U)$ *for all $v \in D$ and $\gamma > 0$;* (4.6)

(ii) $\sup_{\gamma>0} \|\langle F_\gamma(\cdot)u, v\rangle\|_\infty < \infty$ *for all $v \in D$;* (4.7)

(iii) *for each $v \in D$ there exists a Lebesgue null set $B \in \mathcal{B}([0,T])$ such that*

$$\lim_{\gamma\to\infty} \langle \big(F_\gamma(s) - F(s)\big)u, v\rangle = 0 \quad \text{for all } s \in B^c,\ u \in U. \tag{4.8}$$

Then for each $t \in [0,T]$ and $v \in D$ we have

$$\lim_{\gamma\to\infty} \langle F_\gamma * L(t), v\rangle = \langle F * L(t), v\rangle$$

in probability.

Proof. Define for each $t \in [0,T]$ and $\gamma > 0$ the random variable

$$X_\gamma(t) := \int_0^t \big(F_\gamma(t-s) - F(t-s)\big)\, dL(s).$$

By linearity of the stochastic integral and since the Euclidean topology in $\mathbb{R}^n$ coincides with the product topology it is sufficient to prove that for each $v \in D$ and $t \in [0,T]$ we have

$$\langle X_\gamma(t), v\rangle \to 0 \qquad \text{weakly in } \mathbb{R} \text{ as } \gamma \to \infty.$$

Let Ψ denote the Lévy symbol of L. Due to equality (4.3) we obtain for the characteristic function of X_γ for $\beta \in \mathbb{R}$ that

$$\begin{aligned} E\big[\exp\big(i\beta\langle X_\gamma(t), v\rangle\big)\big] &= E\left[\exp\left(i\beta \int_0^t \big(F_\gamma^*(t-s) - F^*(t-s)\big) v\, dL(s)\right)\right] \\ &= \exp\left(\int_0^t \Psi\left(\big(F_\gamma^*(t-s) - F^*(t-s)\big)(\beta v)\right)\, ds\right) \\ &= \exp\left(\int_0^t \Psi\left(\big(F_\gamma^*(s) - F^*(s)\big)(\beta v)\right)\, ds\right). \end{aligned} \tag{4.9}$$

Fix $v \in D$ and define for each $\gamma > 0$ the linear and continuous mapping

$$T_\gamma \colon U \to D([0,T], \|\cdot\|_\infty), \qquad T_\gamma u := \langle u, F_\gamma^*(\cdot)v\rangle.$$

Condition (4.7) guarantees for each $u \in U$ that

$$\sup_{\gamma>0} \|T_\gamma u\|_\infty = \sup_{\gamma>0} \big\|\langle u, F_\gamma^*(\cdot)v\rangle\big\|_\infty < \infty.$$

Thus, the uniform boundedness principle implies

$$M := \sup_{\gamma>0} \sup_{s\in[0,T]} \big\|F_\gamma^*(s)v\big\|_U = \sup_{\gamma>0} \sup_{s\in[0,T]} \sup_{\|u\|\leqslant 1} \big|\langle u, F_\gamma^*(s)v\rangle\big| = \sup_{\gamma>0} \|T_\gamma\|_{U\to D} < \infty.$$

The estimate (4.1) for the Lévy symbol Ψ implies that there exists a constant $c > 0$ such that for each $\gamma > 0$ and $s \in [0,T]$

$$\begin{aligned} \big|\Psi\big((F_\gamma^*(s) - F^*(s))(\beta v)\big)\big| &\leqslant c\left(1 + \big\|\big(F_\gamma^*(s) - F^*(s)\big)(\beta v)\big\|^2\right) \\ &\leqslant c\left(1 + 2\,|\beta|^2 \left(\big\|F_\gamma^*(s)v\big\|^2 + \|F^*(s)v\|^2\right)\right) \\ &\leqslant c\left(1 + 2\,|\beta|^2 \left(M^2 + \|v\|^2 \|F(s)\|_{U\to V}^2\right)\right). \end{aligned}$$

Since Condition (4.8) implies by the sequentially weak continuity of the Lévy symbol $\Psi \colon U^* \to \mathbb{C}$ that

$$\lim_{\gamma\to\infty} \Psi\left(\big(F_\gamma^*(s) - F^*(s)\big)(\beta v)\right) = 0 \qquad \text{for Lebesgue almost all } s \in [0,T],$$

Lebesgue's theorem of dominated convergence enables us to conclude

$$\lim_{\gamma\to\infty} \int_0^t \Psi\left(\big(F_\gamma^*(s) - F^*(s)\big)(\beta v)\right)\, ds = 0,$$

which completes the proof by (4.9). □

4.2 The reproducing kernel Hilbert space

In this subsection we fix a Lévy process L in U with characteristics (a, Q, ν) and let α be a positive constant. The Lévy process L can be decomposed into

$$L(t) = W(t) + X_\alpha(t) + Y_\alpha(t) \qquad \text{for all } t \geqslant 0, \tag{4.10}$$

where W is a Wiener process with covariance operator $Q\colon U \to U$, and X_α and Y_α are U-valued Lévy processes with characteristic functions

$$\varphi_{X_\alpha(t)}(u) = \exp\left(-t\int_{\|r\|\leqslant\alpha}\left(e^{i\langle r,u\rangle} - 1 - i\langle r,u\rangle\right)\,\nu(dr)\right),$$

$$\varphi_{Y_\alpha(t)}(u) = \exp\left(it\langle b_\alpha, u\rangle - t\int_{\alpha<\|r\|}\left(e^{i\langle r,u\rangle} - 1\right)\,\nu(dr)\right),$$

for $u \in U$ and $t \geqslant 0$. The element $b_\alpha \in U$ is determined by the characteristics of L and by the constant α. Since $X_\alpha(1)$ has finite moments we can define the covariance operator $R_\alpha\colon U \to U$ of $X_\alpha(1)$ by

$$\langle R_\alpha u_1, u_2\rangle = E\big[\langle X_\alpha(1), u_1\rangle\langle X_\alpha(1), u_2\rangle\big] \qquad \text{for all } u_1, u_2 \in U.$$

Since R_α is positive and symmetric there exists a separable Hilbert space H_α with norm $\|\cdot\|_{H_\alpha}$ and an embedding $i_\alpha\colon H \to U$ satisfying $R_\alpha = i_\alpha i_\alpha^*$. In particular, we have

$$\lim_{\alpha\to 0}\|i_\alpha\|_{H_\alpha\to U} = 0, \tag{4.11}$$

which follows from the estimate

$$\|i_\alpha^* u\|_{H_\alpha}^2 = \langle R_\alpha u, u\rangle = \int_{\|r\|\leqslant\alpha}\langle u, r\rangle^2\,\nu(dr) \leqslant \|u\|^2\int_{\|r\|\leqslant\alpha}\|r\|^2\,\nu(dr)$$

for each $u \in U$. One obtains for $\alpha \leqslant \beta$

$$\langle R_\alpha u, u\rangle = \int_{\|r\|\leqslant\alpha}\langle u, r\rangle^2\,\nu(dr) \leqslant \int_{\|r\|\leqslant\beta}\langle u, r\rangle^2\,\nu(dr) = \langle R_\beta u, u\rangle \qquad \text{for all } u \in U.$$

One can deduce from Riesz representation theorem, see [26, Proposition 1.1] or [27, Section 1.1], that $H_\alpha \subseteq H_\beta$ and the embedding $H_\alpha \to H_\beta$ is contractive.

Since the range of i_α^* is dense in H_α and H_α is separable there exits a basis $(h_k^\alpha)_{k\in\mathbb{N}} \subseteq i_\alpha^*(U)$. We choose $u_k^\alpha \in U$ such that $i_\alpha^* u_k^\alpha = h_k^\alpha$ for all $k \in \mathbb{N}$ and we define real-valued Lévy processes ℓ_k^α by $\ell_k^\alpha(t) := \langle X_\alpha(t), u_k^\alpha\rangle$ for all $t \in [0, T]$. The Lévy process X_α can be represented by

$$X_\alpha(t) = \sum_{k=1}^\infty i_\alpha h_k^\alpha \ell_k^\alpha(t) \qquad \text{for all } t \in [0, T], \tag{4.12}$$

where the sum converges weakly in $L_P^2(\Omega; U)$, i.e.

$$\langle X_\alpha(t), u\rangle = \sum_{k=1}^\infty \langle i_\alpha h_k^\alpha, u\rangle \ell_k(t) \qquad \text{in } L_P^2(\Omega;\mathbb{R}) \text{ for all } t \in [0,T] \text{ and } u \in U.$$

The representation (4.12) is called *Karhunen–Loève expansion*, and it can be derived in the same way as it is done in Riedle [20] for Wiener processes. Note, that it follows easily from their definition that the Lévy processes $(\ell_k)_{k\in\mathbb{N}}$ are uncorrelated.

4.3 Estimating the small jumps

We begin with a generalisation to the Hilbert space setting of a result of Marcus and Rosiński in [12] on a maximal inequality of convolution integrals. We apply here the decomposition (4.10) of the Lévy process L for some $\alpha > 0$.

Lemma 4.2. *A function $f \in \mathcal{H}^2(U,\mathbb{R})$ satisfies for each $\alpha > 0$ the estimate*

$$E\left[\sup_{t\in[0,T]}\left|\int_0^t f(t-s)\,dX_\alpha(s)\right|\right] \leqslant \varkappa\sqrt{2T}\,\sum_{k=1}^\infty \|\langle f(\cdot), i_\alpha h_k^\alpha\rangle\|_{TV_2}\,,$$

where $\varkappa := 32\sqrt{2}\,\int_0^1 \sqrt{\ln(1/s)}\;ds$.

Proof. Since $\alpha > 0$ is fixed we neglect its notation in the following. According to (4.12) the Lévy process X can be represented by

$$X(t) = \sum_{k=1}^\infty ih_k\ell_k(t) \qquad \text{for all } t\in[0,T],$$

where the sum converges weakly in $L^2_P(\Omega;U)$. Since the real-valued Lévy processes $(\ell_k)_{k\in\mathbb{N}}$ are uncorrelated we obtain

$$\begin{aligned} E\left[\sup_{t\in[0,T]}\left|\int_0^t f(t-s)\,dX(s)\right|\right] &= E\left[\sup_{t\in[0,T]}\left|\sum_{k=1}^\infty\int_0^t \langle f(t-s), ih_k\rangle\,d\ell_k(s)\right|\right]\\ &\leqslant \sum_{k=1}^\infty E\left[\sup_{t\in[0,T]}\left|\int_0^t \langle f(t-s), ih_k\rangle\,d\ell_k(s)\right|\right]. \end{aligned}$$

In order to estimate the expectation of the real-valued stochastic integrals on the right hand side we follow some arguments by Marcus and Rosiński in [13]. Let ℓ_k' be an independent copy of ℓ_k and define the symmetrisation $\hat{\ell}_k := \ell_k - \ell_k'$ for each $k\in\mathbb{N}$. Since $E[\ell_k(t)] = 0$ for all $t \geqslant 0$ and $k\in\mathbb{N}$ one obtains

$$\begin{aligned} &E\left[\sup_{t\in[0,T]}\left|\int_0^t \langle f(t-s), ih_k\rangle\,d\ell_k(s)\right|\right]\\ &\quad= E\left[\sup_{t\in[0,T]}\left|E\left[\int_0^t \langle f(t-s), ih_k\rangle\,d\ell_k(s) - \int_0^t \langle f(t-s), ih_k\rangle\,d\ell'(s)\,\middle|\,\ell_k(T)\right]\right|\right]\\ &\quad\leqslant E\left[\sup_{t\in[0,T]} E\left[\left|\int_0^t \langle f(t-s), ih_k\rangle\,d\hat{\ell}_k(s)\right|\,\middle|\,\ell_k(T)\right]\right]\\ &\quad\leqslant E\left[E\left[\sup_{t\in[0,T]}\left|\int_0^t \langle f(t-s), ih_k\rangle\,d\hat{\ell}_k(s)\right|\,\middle|\,\ell_k(T)\right]\right]\\ &\quad= E\left[\sup_{t\in[0,T]}\left|\int_0^t \langle f(t-s), ih_k\rangle\,d\hat{\ell}_k(s)\right|\right]. \end{aligned}$$

Theorem 1.1 in [12] guarantees for all $k\in\mathbb{N}$ that

$$E\left[\sup_{t\in[0,T]}\left|\int_0^t \langle f(t-s), ih_k\rangle\,d\hat{\ell}_k(s)\right|\right] \leqslant \varkappa E\left[\left|\int_0^T \left\|\langle f(\cdot), ih_k\rangle\,\mathbb{1}_{[0,T-s]}\right\|_{TV_2}\,d\hat{\ell}_k(s)\right|\right].$$

Since the Lévy measure λ_k of ℓ_k is given by $\lambda_k := \nu_\alpha \circ \langle \cdot, u_k \rangle^{-1}$ where ν_α denotes the Lévy measure of X we conclude

$$\begin{aligned}
&\left(E \left| \int_0^T \left\| \langle f(\cdot), ih_k \rangle \, \mathbb{1}_{[0,T-s]} \right\|_{TV_2} \, d\hat{\ell}_k(s) \right| \right)^2 \\
&\qquad \leqslant E \left| \int_0^T \left\| \langle f(\cdot), ih_k \rangle \, \mathbb{1}_{[0,T-s]} \right\|_{TV_2} \, d\hat{\ell}_k(s) \right|^2 \\
&\qquad = 2 \int_0^T \int_{\mathbb{R}} \left\| \langle f(\cdot), ih_k \rangle \, \mathbb{1}_{[0,T-s]} \right\|_{TV_2}^2 \beta^2 \, \lambda_k(d\beta) \, ds \\
&\qquad = 2 \int_0^T \left\| \langle f(\cdot), ih_k \rangle \, \mathbb{1}_{[0,T-s]} \right\|_{TV_2}^2 \, ds \int_U \langle u, u_k \rangle^2 \, \nu_\alpha(du) \\
&\qquad = 2 \int_0^T \left\| \langle f(\cdot), ih_k \rangle \, \mathbb{1}_{[0,T-s]} \right\|_{TV_2}^2 \, ds \, \langle Ru_k, u_k \rangle \\
&\qquad = 2 \int_0^T \left\| \langle f(\cdot), ih_k \rangle \, \mathbb{1}_{[0,T-s]} \right\|_{TV_2}^2 \, ds \\
&\qquad \leqslant 2T \left\| \langle f(\cdot), ih_k \rangle \right\|_{TV_2}^2 ,
\end{aligned} \tag{4.13}$$

which completes the proof. □

The bound on the right hand side in Lemma 4.2 depends on the regularity of the function f and of the covariance structure of the underlying Lévy process. It is a natural generalisation to the infinite dimensional setting of the result in [12].

We will later consider the special case, that the integrands of the stochastic convolution integrals can be diagonalised with respect to an orthonormal basis. In this case, we can improve the analogue estimate of Lemma 4.2.

Lemma 4.3. *Let V be a Hilbert space with an orthonormal basis $(e_k)_{k\in\mathbb{N}}$ and let $F \in \mathcal{H}^2(U,V)$ be a function of the form*

$$F^*(\cdot)e_k = \varphi_k(\cdot)\, Ge_k \qquad \text{for all } k \in \mathbb{N},$$

for càdlàg functions $\varphi_k\colon [0,T] \to \mathbb{R}$ and $G \in \mathcal{L}(V,U)$. Then it follows for each $\alpha > 0$ and $v \in V$ that

$$\begin{aligned}
&E \left[\sup_{t\in[0,T]} \left| \int_0^t F^*(t-s)v \, dX_\alpha(s) \right| \right] \\
&\qquad \leqslant \varkappa \sqrt{2T} \sum_{k=1}^\infty |\langle v, e_k \rangle| \, \|\varphi_k(\cdot)\|_{TV_2} \left(\int_{\|u\|\leqslant\alpha} |\langle u, Ge_k \rangle|^2 \, \nu(du) \right)^{1/2} ,
\end{aligned}$$

where $\varkappa := 32\sqrt{2} \int_0^1 \sqrt{\ln(1/s)} \, ds$.

Proof. For each $k \in \mathbb{N}$ define the real-valued Lévy process x_k^α by defining $x_k^\alpha(t) =$

$\langle X_\alpha(t), Ge_k\rangle$ for all $t \geqslant 0$. It follows from (4.4) for each $v \in V$ that

$$\begin{aligned} E\left[\sup_{t\in[0,T]}\left|\int_0^t F^*(t-s)v\,dX_\alpha(s)\right|\right] &= E\left[\sup_{t\in[0,T]}\left|\sum_{k=1}^\infty \langle v, e_k\rangle \int_0^t \varphi_k(t-s)Ge_k\,dX_\alpha(s)\right|\right] \\ &= E\left[\sup_{t\in[0,T]}\left|\sum_{k=1}^\infty \langle v, e_k\rangle \int_0^t \varphi_k(t-s)\,dx_k^\alpha(s)\right|\right] \\ &\leqslant \sum_{k=1}^\infty |\langle v, e_k\rangle|\, E\left[\sup_{t\in[0,T]}\left|\int_0^t \varphi_k(t-s)\,dx_k^\alpha(s)\right|\right]. \end{aligned}$$

As in the proof of Lemma 4.2 we obtain

$$E\left[\sup_{t\in[0,T]}\left|\int_0^t \varphi_k(t-s)\,dx_k^\alpha(s)\right|\right] \leqslant \varkappa E\left[\left|\int_0^T \left\|\varphi_k(\cdot)\,\mathbb{1}_{[0,T-s]}\right\|_{TV_2}\, d\hat{x}_k^\alpha(s)\right|\right],$$

where $\hat{x}_k^\alpha := x_k^\alpha - x_k^{\alpha\prime}$ denotes the symmetrisation of x_k^α for each $k \in \mathbb{N}$ by an independent copy $x_k^{\alpha\prime}$ of x_k^α. Since the Lévy measure λ_k^α of x_k^α is given by $\lambda_k^\alpha = \nu_\alpha \circ \langle \cdot, Ge_k\rangle^{-1}$ where ν_α denotes the Lévy measure of X_α we conclude by a similar calculation as in (4.13) that

$$\left(E\left[\left|\int_0^T \left\|\varphi_k(\cdot)\,\mathbb{1}_{[0,T-s]}\right\|_{TV_2}\, d\hat{x}_k^\alpha(s)\right|\right]\right)^2 \leqslant 2T\,\|\varphi_k(\cdot)\|_{TV_2}^2 \int_{\|u\|\leqslant\alpha} \langle u, Ge_k\rangle^2\,\nu(du).$$

Summarising the estimates above completes the proof. □

4.4 The general case

In this section we consider the convergence in the weak topology and the product topology. For the latter we assume that $e = (e_k)_{k\in\mathbb{N}}$ is an orthonormal basis of V.

Theorem 4.4. *Let $F, F_\gamma \in \mathcal{H}^2(U,V)$, $\gamma > 0$, be functions satisfying for a subset $D \subseteq V$*

(i) $F^*(\cdot)v,\ F_\gamma^*(\cdot)v \in D([0,T];U)$ *for all $v \in D$ and $\gamma > 0$;* (4.14)

(ii) $\sup_{\gamma>0} \|\langle F_\gamma(\cdot)u, v\rangle\|_\infty < \infty$ *for all $u \in U$, $v \in D$;* (4.15)

(iii) $\lim_{\alpha\to 0}\sup_{\gamma>0} \sup \sum_{k=1}^\infty \left\|\langle F_\gamma^*(\cdot)v, i_\alpha h_k^\alpha\rangle\right\|_{TV_2} = 0$ *for all $v \in D$;* (4.16)

(iv) $\lim_{\gamma\to\infty} d_M\big(\langle F_\gamma(\cdot)u, v\rangle,\ \langle F(\cdot)u, v\rangle\big) = 0$ *for all $u \in U$, $v \in D$.* (4.17)

(1) *If $\{e_1, e_2, \dots\} \subseteq D$ then it follows*

$$\lim_{\gamma\to\infty} \big(F_\gamma * L(t)\colon t \in [0,T]\big) = \big(F * L(t)\colon t \in [0,T]\big)$$

in probability in the product topology $\big(D([0,T];V), d_M^e\big)$.

(2) *If $V = D$ then it follows*

$$\lim_{\gamma\to\infty} \big(F_\gamma * L(t)\colon t \in [0,T]\big) = \big(F * L(t)\colon t \in [0,T]\big)$$

weakly in probability in $D([0,T];V)$.

Proof. We show that each $v \in D$ the stochastic processes X and X_γ, $\gamma > 0$, defined by

$$X(t) := \langle \int_0^t F(t-s)\, dL(s), v\rangle, \quad X_\gamma(t) := \langle \int_0^t F_\gamma(t-s)\, dL(s), v\rangle, \qquad t \in [0,T],$$

satisfy the conditions in Theorem 3.2. Note, that for $0 \leqslant t_1 \leqslant t_2 \leqslant T$ and $\beta \in \mathbb{R}$ we have

$$\begin{aligned} &E\left[\exp\left(i\beta(X(t_1) - X(t_2))\right)\right] \\ &\quad = \exp\left(\int_0^{t_1} \Psi\big((F^*(t_1 - s) - F^*(t_2 - s))(\beta v)\big)\, ds\right) \exp\left(\int_{t_1}^{t_2} \Psi\big((F^*(t_2 - s)(\beta v)\big)\, ds\right). \end{aligned}$$

Since $F^*(\cdot)(\beta v)$ is Lebesgue almost everywhere continuous due to (4.14) it follows that the stochastic process X and analogously X_γ are stochastically continuous.

It follows from Condition (4.17) by Theorem 12.5.1 in [28] for each $u \in U$ and $v \in D$ that

$$\lim_{\gamma\to\infty} \langle F_\gamma(t)u, v\rangle = \langle F(t)u, v\rangle \qquad \text{for all } t \in \big(J(\langle F(\cdot)u, v\rangle)\big)^c. \tag{4.18}$$

Since the set $J(F^*(\cdot)v)$ of discontinuities of $F^*(\cdot)v$ is a Lebesgue null set by Condition (4.14) and satisfies $\big(J(\langle F(\cdot)u, v\rangle)\big) \subseteq \big(J(F^*(\cdot)v)\big)$ for every $u \in U$, Lemma 4.1 guarantees that Condition (i) in Theorem 3.2 is satisfied.

In order to show Condition (ii) we have to establish for every $\varepsilon > 0$ and $v \in D$ that

$$\lim_{\delta\searrow 0} \limsup_{\gamma\to\infty} P\left(M\left(\langle \int_0^\cdot F_\gamma(\cdot - s)\, dL(s), v\rangle,\, \delta\right) \geqslant \varepsilon\right) = 0. \tag{4.19}$$

For this purpose, fix some constants $\varepsilon_1, \varepsilon_2 > 0$ and $v \in D$. Condition (4.16) enables us to choose a constant $\alpha > 0$ such that

$$\sup_{\gamma>0} \sum_{k=1}^\infty \big\| \langle F_\gamma^*(\cdot)v, i_\alpha h_k^\alpha\rangle \big\|_{TV_2} \leqslant \frac{\varepsilon_1\varepsilon_2}{2\varkappa\sqrt{2T}} \tag{4.20}$$

for $\varkappa := 32\sqrt{2}\, \int_0^1 \sqrt{\ln(1/s)}\;\, ds$. As in (4.10) we decompose the Lévy process L into $L(t) = W(t) + X_\alpha(t) + Y_\alpha(t)$ for all $t \geqslant 0$, but we suppress the notion of α in the sequel. Here, W is a Wiener process with covariance operator $Q\colon U \to U$, and X and Y are U-valued Lévy processes with characteristic functions

$$\begin{aligned} \varphi_{X(t)}(u) &= \exp\left(-t \int_{\|r\|\leqslant\alpha} \left(e^{i\langle r,u\rangle} - 1 - i\langle r, u\rangle\right)\, \nu(dr)\right), \\ \varphi_{Y(t)}(u) &= \exp\left(it\langle b, u\rangle - t\int_{\alpha<\|r\|} \left(e^{i\langle r,u\rangle} - 1\right)\, \nu(dr)\right), \end{aligned}$$

for all $t \geqslant 0$, $u \in U$. It follows from (4.3) for every $t \in [0,T]$ and $\gamma > 0$ that

$$\langle \int_0^t F_\gamma(t-s)\, dL(s), v\rangle = \int_0^t F_\gamma^*(t-s)v\, dL(s).$$

By the decomposition of L we obtain the representation

$$I_\gamma(t) := \int_0^t F_\gamma^*(t-s)v\, dL(s) = C_\gamma(t) + A_\gamma(t) + B_\gamma(t),$$

where we define the real-valued random variables

$$C_\gamma(t) := \int_0^t F_\gamma^*(t-s)v\, dW(s),$$
$$A_\gamma(t) := \int_0^t F_\gamma^*(t-s)v\, dY(s), \quad B_\gamma(t) := \int_0^t F_\gamma^*(t-s)v\, dX(s).$$

In the sequel, we will consider the three stochastic integrals separately.
1) We show that

$$\lim_{\delta\searrow 0}\limsup_{\gamma\to\infty} P\left(\sup_{\substack{t_1,t_2\in[0,T]\\ |t_2-t_1|\leqslant\delta}} |C_\gamma(t_2)-C_\gamma(t_1)| \geqslant \varepsilon_1\right) = 0. \tag{4.21}$$

Let the covariance operator Q of W be decomposed into $Q = i_Q i_Q^*$ for some $i_Q \in \mathcal{L}(H_Q, U)$ and for a Hilbert space H_Q. Since the characteristic function $\varphi_{W(1)}$ of $W(1)$ is sequentially weakly continuous and is given by

$$\varphi_{W(1)}(u) = \exp\left(-\tfrac{1}{2}\|i^*u\|_{H_Q}^2\right) \qquad \text{for all } u \in U,$$

also the function $i^*\colon U \to H_Q$ is sequentially weakly continuous. Consequently, we can conclude from (4.18) that

$$\lim_{\gamma\to\infty}\left\|i_Q^*\big(F_\gamma^*(s)v - F^*(s)v\big)\right\|_{H_Q}^2 = 0 \qquad \text{for Lebesgue-a.a. } s\in[0,T].$$

Due to (4.15) Lebesgue's theorem of dominated convergence implies

$$\lim_{\gamma\to\infty}\int_0^T \left\|i_Q^*\big(F_\gamma^*(s)v - F^*(s)v\big)\right\|_{H_Q}^2\, ds = 0. \tag{4.22}$$

Let $I := [0,T]\cap\mathbb{Q}$ and define for each $\gamma > 0$ the stochastic processes $G_\gamma := (G_\gamma(t)\colon t\in I)$ where $G_\gamma(t) := C_\gamma(t) - \langle F * W(t), v\rangle$. Since G_γ has independent increments it is a real-valued Gaussian process with a.s. bounded sample paths due to Assumption A. By denoting $M_\gamma := \sup_{t\in I} G_\gamma(t)$ it follows for each $\delta > 0$ that

$$\begin{aligned}&\sup_{\substack{t_1,t_2\in I\\ |t_2-t_1|\leqslant\delta}} |C_\gamma(t_2)-C_\gamma(t_1)|\\ &\qquad\leqslant 2\sup_{t\in I}|G_\gamma(t)-E[M_\gamma]| + \sup_{\substack{t_1,t_2\in I\\ |t_2-t_1|\leqslant\delta}} |\langle F*W(t_2)-F*W(t_1), v\rangle|.\end{aligned} \tag{4.23}$$

Borell's inequality, see [1, Theorem 2.1], (or Borell–Tsirelson–Ibragimov–Sudakov inequality) implies $E[M_\gamma] < \infty$ and that for every $\varepsilon > 0$

$$P\left(\sup_{t\in I}|G_\gamma(t)-E[M_\gamma]|\geqslant\varepsilon\right) \leqslant 2P\left(\sup_{t\in I}G_\gamma(t)-E[M_\gamma]\geqslant\varepsilon\right) \leqslant \exp\left(-\frac{\varepsilon^2}{2\sigma_\gamma^2}\right),$$

where $\sigma_\gamma^2 := \sup_{t\in I} E[G_\gamma(t)^2]$. Since equation (4.22) guarantees

$$\sigma_\gamma^2 = \int_0^T \left\|i_Q^*\left(F_\gamma^*(s)v - F^*(s)v\right)\right\|_{H_Q}^2\, ds \to 0 \quad \text{as } \gamma\to\infty,$$

we can conclude

$$\lim_{\gamma\to\infty} P\left(\sup_{t\in I} |G_\gamma(t) - E[M_\gamma]| \geqslant \frac{\varepsilon_1}{4}\right) = 0. \tag{4.24}$$

As the stochastic process $(\langle F * W(t), v\rangle \colon t \in [0,T])$ is continuous according to Assumption A it follows

$$\lim_{\delta\searrow 0} \sup_{\substack{t_1,t_2\in I\\ |t_2-t_1|\leqslant\delta}} |\langle F * W(t_2) - F * W(t_1), v\rangle| = 0 \quad P\text{-a.s.} \tag{4.25}$$

Equations (4.24) and (4.25) establish (4.21).
2) Lemma 4.2 implies by Markov inequality for each $\gamma > 0$

$$P\left(\sup_{t\in[0,T]} |B_\gamma(t)| \geqslant \varepsilon_1\right) \leqslant \frac{\varkappa\sqrt{2T}}{\varepsilon_1} \sum_{k=1}^{\infty} \left\|\langle F_\gamma^*(\cdot)v, i_\alpha h_k^\alpha\rangle\right\|_{TV_2} \leqslant \frac{\varepsilon_2}{2}. \tag{4.26}$$

3) Let $(N(t) \colon t \in [0,T])$ denote the counting process for Y, i.e.

$$N(t) := \sum_{s\in[0,t]} \mathbb{1}_{\{\|\Delta Y(s)\| > \alpha\}} \qquad \text{for all } t \in [0,T],$$

and let τ_j, $j \in \mathbb{N} \cup \{0\}$, denote the jump times of Y, recursively defined by $\tau_0 = 0$ and

$$\tau_j := \inf\left\{t > \tau_{j-1} \colon \ \|\Delta Y(t)\| > \alpha\right\} \qquad \text{for all } j \in \mathbb{N}$$

with $\inf \emptyset = \infty$. Note, that the jump times of Y are countable in increasing order since the jump size of Y is bounded from below by α. Since Y is a pure jump process with drift b, we obtain

$$A_\gamma(t) = \int_0^t F_\gamma^*(t-s)v \, dY(s) = R_\gamma(t) + S_\gamma(t),$$

where we define for every $t \in [0,T]$ and $\gamma > 0$

$$R_\gamma(t) := \sum_{j=0}^{N(t)} R_\gamma^j(t), \qquad R_\gamma^j(t) := \langle F_\gamma^*(t-\tau_j)v, \Delta Y(\tau_j)\rangle \, \mathbb{1}_{[\tau_j,\infty)}(t),$$

$$S_\gamma(t) := \int_0^t \langle F_\gamma^*(t-s)v, b\rangle \, ds.$$

Analogously, we have

$$A(t) = \int_0^t F^*(t-s)v \, dY(s) = R(t) + S(t),$$

where we define for every $t \in [0,T]$

$$R(t) := \sum_{j=0}^{N(t)} R^j(t), \qquad R^j(t) := \langle F^*(t-\tau_j)v, \Delta Y(\tau_j)\rangle \, \mathbb{1}_{[\tau_j,\infty)}(t),$$

$$S(t) := \int_0^t \langle F^*(t-s)v, b\rangle \, ds.$$

Due to Condition (4.14) the stochastic process $R^j := (R^j(t)\colon t \in [0,T])$ has càdlàg paths for each $j \in \mathbb{N} \cup \{0\}$ and the random set $J(R^j)$ of its discontinuities satisfies

$$J(R^j) \subseteq \Big\{ t \in [0,T]\colon t - \tau_j \in J\big(F^*(\cdot)v\big),\, t \in (\tau_j, T] \Big\} \cup \{\tau_j\}.$$

Consequently, the set of joint discontinuities of R^i and R^j for $i < j$ satisfies

$$J(R^i) \cap J(R^j) \subseteq \Big\{ t \in [0,T]\colon t = \tau_j(\omega) - \tau_i(\omega) \in J\big(F^*(\cdot)v\big) - J\big(F^*(\cdot)v\big) \text{ for some } \omega \in \Omega \Big\},$$

where we apply the convention $\infty - \infty = \infty$. Since the deterministic set

$$J\big(F^*(\cdot)v\big) - J\big(F^*(\cdot)v\big) = \{ t \in [0,T]\colon t = s_1 - s_2 \text{ for some } s_1, s_2 \in J\big(F^*(\cdot)v\big)\}$$

is at most countable and the random vector (τ_i, τ_j) is absolutely continuous for every $i < j$ it follows that

$$\begin{aligned} P\big(R^i \text{ and } R^j \text{ have joint discontinuities}\big) &= P\big(J(R^i) \cap J(R^j)\big) \\ &\leqslant P\big(\tau_j - \tau_i \in J\big(F^*(\cdot)v\big) - J\big(F^*(\cdot)v\big)\big) = 0. \end{aligned}$$

Consequently, for the stochastic processes $(R^j)_{j\in\mathbb{N}}$ restricted to the complement S^c of the null set

$$S := \bigcup_{i=2}^{\infty} \bigcup_{j=1}^{i-1} \big\{ \omega \in \Omega\colon \tau_i(\omega) - \tau_j(\omega) \in J\big(F^*(\cdot)v\big) \cap J\big(F^*(\cdot)v\big) \big\},$$

there does not exist any $i \neq j$ such that R^i and R^j have joint discontinuities. Since Condition (4.17) guarantees that for each $\omega \in \Omega$

$$\langle F^*_\gamma(\cdot - \tau_j(\omega))v, \Delta Y(\tau_j)(\omega)\rangle \to \langle F^*(\cdot - \tau_j(\omega))v, \Delta Y(\tau_j)(\omega)\rangle$$

in $(D([0,T];\mathbb{R}), d_M)$ it follows that $R^j_\gamma(\omega) \to R^j(\omega)$ in $(D([0,T];\mathbb{R}), d_M)$ for all $\omega \in \Omega$ and $j \in \mathbb{N}$. Due to the disjoint sets of discontinuities on S^c, Corollary 12.7.1 in [28] guarantees that $R_\gamma(\omega)$ converges to $R(\omega)$ in $(D([0,T];\mathbb{R}), d_M)$ for all $\omega \in S^c$. The deterministic analogue of Theorem 3.2, i.e. [28, Theorem 12.5.1], implies that

$$\lim_{\delta \searrow 0} \limsup_{\gamma\to\infty} M(R_\gamma(\omega);\delta) = 0 \qquad \text{for all } \omega \in S^c.$$

By combining with Fatou's Lemma it follows that there exists $\delta_0 > 0$ such that for all $\delta \leqslant \delta_0$ we have

$$\begin{aligned} \limsup_{\gamma\to\infty} P\Big(M(R_\gamma;\delta) \geqslant \tfrac{1}{2}\varepsilon_1\Big) &\leqslant P\Big(\limsup_{\gamma\to\infty} \{M(R_\gamma;\delta) \geqslant \tfrac{1}{2}\varepsilon_1\}\Big) \\ &= P\Big(\limsup_{\gamma\to\infty} M(R_\gamma;\delta) \geqslant \tfrac{1}{2}\varepsilon_1\Big) \leqslant \varepsilon_2. \end{aligned}$$

Since inequality (2.4) implies

$$\begin{aligned} M(A_\gamma;\delta) = M(R_\gamma + S_\gamma;\delta) &\leqslant M(R_\gamma;\delta) + \sup_{\substack{s_1,s_2\in[0,T] \\ |s_2-s_1|\leqslant\delta}} |S_\gamma(s_1) - S_\gamma(s_2)| \\ &\leqslant M(R_\gamma;\delta) + \delta \sup_{\gamma>0} \left\|\langle v, F_\gamma(\cdot)b\rangle\right\|_\infty, \end{aligned}$$

it follows that for all $\delta \leqslant \delta_1$ where $\delta_1 := \min\{\delta_0, \frac{\varepsilon_1}{3}(\sup_{\gamma>0} \|\langle v, F_\gamma(\cdot)b\rangle\|_\infty)^{-1}\}$ we have

$$\limsup_{\gamma\to\infty} P\Big(M(A_\gamma;\delta) \geqslant \varepsilon_1\Big) \leqslant \limsup_{\gamma\to\infty} P\Big(M(R_\gamma;\delta) \geqslant \tfrac{1}{2}\varepsilon_1\Big) \leqslant \varepsilon_2. \tag{4.27}$$

4) It follows from (4.21) and(4.26) that there exists $\delta_2 > 0$ such that for each $\gamma > 0$ the set

$$E_\gamma := \left\{ \sup_{|t_2-t_1|\leqslant\delta_2} |C_\gamma(t_2) - C_\gamma(t_1)| \leqslant \varepsilon_1 \right\} \cap \left\{ \sup_{t\in[0,T]} |B_\gamma(t)| \leqslant \varepsilon_1 \right\}$$

satisfies $P(E_\gamma) \geqslant 1 - \varepsilon_2$. It follows from (4.27) by the inequalities (2.3) and (2.4) for all $\delta \leqslant \min\{\delta_1, \delta_2\}$ that

$$\limsup_{\gamma\to\infty} P\Big(M(I_\gamma;\delta)) \geqslant 4\varepsilon_1\Big) \leqslant \limsup_{\gamma\to\infty} P\big(M(A_\gamma;\delta) \geqslant \varepsilon_1\big) + \limsup_{\gamma\to\infty} P(E_\gamma^c) \leqslant 2\varepsilon_2,$$

which shows (4.19) and thus, Condition (ii) in Theorem 3.2. □

4.5 The diagonal case

In this section we consider the special case that the integrands can be diagonalised. For that purpose, assume that $(e_k)_{k\in\mathbb{N}}$ is an orthonormal basis of V.

Corollary 4.5. *Let $F, F_\gamma \in \mathcal{H}^2(U,V)$, $\gamma > 0$, be functions of the form*

$$F^*(s)e_k = \varphi^k(s)\, Ge_k, \qquad F_\gamma^*(s)e_k = \varphi_\gamma^k(s)\, Ge_k \qquad \text{for all } s \in [0,T],\ k \in \mathbb{N},$$

for càdlàg functions φ^k, $\varphi_\gamma^k \colon [0,T] \to \mathbb{R}$ and $G \in \mathcal{L}(V,U)$. If

$$\text{(i) } \sup_{\gamma>0} \big\|\varphi_\gamma^k\big\|_\infty < \infty \qquad \text{for all } k \in \mathbb{N}; \tag{4.28}$$

$$\text{(ii) } \sup_{\gamma>0} \big\|\varphi_\gamma^k\big\|_{TV_2} < \infty \qquad \text{for all } k \in \mathbb{N}; \tag{4.29}$$

$$\text{(iii) } \lim_{\gamma\to\infty} \varphi_\gamma^k = \varphi^k \, in \ \big(D([0,T];\mathbb{R}), d_M\big) \qquad \text{for all } k \in \mathbb{N}\,. \tag{4.30}$$

then it follows for each $\varepsilon > 0$ and $k \in \mathbb{N}$ that

$$\lim_{\gamma\to\infty} \big(F_\gamma * L(t) \colon t \in [0,T]\big) = \big(F * L(t) \colon t \in [0,T]\big)$$

in probability in the product topology $\big(D([0,T];V), d_M^e\big)$.

Proof. The proof is analogously to the proof of Theorem 4.4 for $D = \{e_1, e_2, \dots\}$ but only the estimate (4.26) is derived in the following way: for each $k \in \mathbb{N}$ and $\gamma > 0$ Lemma 4.3 implies

$$E\left[\sup_{t\in[0,T]} \left|\int_0^t F_\gamma^*(t-s)e_k\, dX(s)\right|\right] \leqslant \varkappa\sqrt{2T}\ \|\varphi_k(\cdot)\|_{TV_2} \left(\int_{\|u\|\leqslant\alpha} |\langle u, Ge_k\rangle|^2\, \nu(du)\right)^{1/2},$$

where $\varkappa := 32\sqrt{2}\ \int_0^1 \sqrt{\ln(1/s)}\ ds$ and $\alpha > 0$ denotes the bound of the truncation function. Condition (4.29) enables us to choose for every ε_1, $\varepsilon_2 > 0$ and $k \in \mathbb{N}$ the constant $\alpha > 0$ small enough such that

$$P\left(\sup_{t\in[0,T]} \left|\int_0^t F_\gamma^*(t-s)e_k\, dX(s)\right| \geqslant \varepsilon_1\right) \leqslant \frac{\varepsilon_2}{2} \qquad \text{for all } \gamma > 0,$$

which corresponds to inequality (4.26). Now we can follow the proof of Theorem 4.4 once the Conditions (4.14), (4.15) and (4.17) are established. The first two conditions follow directly from the càdlàg property of φ^k, φ^k_γ and (4.28). Condition (4.17) is satisfied since Condition (4.30) implies by Theorem 12.7.2 in [28] for each $k \in \mathbb{N}$ and $u \in U$ that

$$\lim_{\gamma\to\infty} \langle F_\gamma(\cdot)u, e_k\rangle = \lim_{\gamma\to\infty} \varphi^k_\gamma(\cdot)\langle u, Ge_k\rangle = \varphi^k(\cdot)\langle u, Ge_k\rangle = \langle F(\cdot)u, e_k\rangle,$$

in $\big(D([0,T];\mathbb{R}), d_M\big)$. ☐

Corollary 4.6. *Let $F, F_\gamma \in \mathcal{H}^2(U,V)$, $\gamma > 0$, be functions of the form*

$$F^*(s)e_k = \varphi^k(s)\, Ge_k, \qquad F^*_\gamma(s)e_k = \varphi^k_\gamma(s)\, Ge_k \qquad \text{for all } s \in [0,T],\ k \in \mathbb{N},$$

for càdlàg functions φ^k, $\varphi^k_\gamma\colon [0,T] \to \mathbb{R}$ and $G \in \mathcal{L}(V,U)$. If

(i) $F^*(\cdot)v,\ F^*_\gamma(\cdot)v \in D([0,T];U)$ *for all $v \in V$ and $\gamma > 0$;* (4.31)

(ii) $\sup_{\gamma>0}\sup_{k\in\mathbb{N}} \|\varphi^k_\gamma\|_\infty < \infty;$ (4.32)

(iii) $\sup_{\gamma>0}\sup_{k\in\mathbb{N}} \|\varphi^k_\gamma\|_{TV_2} < \infty;$ (4.33)

(iv) *for each $n \in \mathbb{N}$ we have*

$$\lim_{\gamma\to\infty} \big(\varphi^1_\gamma, \dots, \varphi^n_\gamma\big) = \big(\varphi^1, \dots, \varphi^n\big) \quad \text{in } \big(D([0,T];\mathbb{R}^n), d_M\big), \tag{4.34}$$

then it follows that

$$\lim_{\gamma\to\infty} \big(F_\gamma * L(t)\colon t\in[0,T]\big) = \big(F * L(t)\colon t \in [0,T]\big)$$

weakly in probability in $D([0,T];V)$.

Proof. The proof is analogous to the proof of Theorem 4.4 for $D = V$ but only the estimate (4.26) is derived in the following way: for each $v \in V$ and $\gamma > 0$ Lemma 4.3 and Cauchy–Schwarz inequality imply

$$\begin{aligned}
&E\left[\sup_{t\in[0,T]}\left|\int_0^t F^*_\gamma(t-s)v\, dX(s)\right|\right]\\
&\qquad\leqslant \varkappa\sqrt{2T}\ \sum_{k=1}^\infty |\langle v, e_k\rangle|\, \|\varphi^k_\gamma(\cdot)\|_{TV_2}\left(\int_{\|u\|\leqslant\alpha} |\langle u, Ge_k\rangle|^2\ \nu(du)\right)^{1/2}\\
&\qquad\leqslant \varkappa\sqrt{2T}\ \sup_{k\in\mathbb{N}} \|\varphi^k_\gamma(\cdot)\|_{TV_2}\left(\sum_{k=1}^\infty \langle v, e_k\rangle^2\right)^{1/2}\left(\sum_{k=1}^\infty\int_{\|u\|\leqslant\alpha}\langle u, Ge_k\rangle^2\,\nu(du)\right)^{1/2}\\
&\qquad= \varkappa\sqrt{2T}\ \sup_{k\in\mathbb{N}} \|\varphi^k_\gamma(\cdot)\|_{TV_2}\,\|v\|\left(\int_{\|u\|\leqslant\alpha}\|G^*u\|^2\ \nu(du)\right)^{1/2},
\end{aligned}$$

where $\varkappa := 32\sqrt{2}\ \int_0^1 \sqrt{\ln(1/s)}\ ds$ and $\alpha > 0$ denotes the bound of the truncation function. Condition (4.33) enables us to choose for every ε_1, $\varepsilon_2 > 0$ the constant $\alpha > 0$ small enough such that

$$P\left(\sup_{t\in[0,T]}\left|\int_0^t F^*_\gamma(t-s)v\, dX(s)\right| \geqslant \varepsilon_1\right) \leqslant \frac{\varepsilon_2}{2} \qquad \text{for all } \gamma > 0,$$

which corresponds to inequality (4.26). Now we can follow the proof of Theorem 4.4 once the Conditions (4.15) and (4.17) are established. Cauchy–Schwarz inequality implies for each $u \in U$ and $v \in V$

$$\sup_{\gamma>0} \|\langle F_\gamma(\cdot)u, v\rangle\|_\infty^2 \leqslant \|G^*u\|^2 \|v\|^2 \sup_{\gamma>0} \sup_{k\in\mathbb{N}} \left\|\varphi_\gamma^k\right\|_\infty^2 < \infty,$$

which establishes Condition (4.15). For the last part, fix $u \in U$ and $v \in V$ and define for each $\gamma > 0$ and $n \in \mathbb{N}$ the functions

$$\begin{aligned} f &:= \langle F(\cdot)u, v\rangle, & f^\gamma &:= \langle F_\gamma(\cdot)u, v\rangle, \\ f_n &:= \sum_{k=1}^n \langle u, Ge_k\rangle\langle v, e_k\rangle \varphi^k, & f_n^\gamma &:= \sum_{k=1}^n \langle u, Ge_k\rangle\langle v, e_k\rangle \varphi_\gamma^k. \end{aligned}$$

Cauchy–Schwarz inequality implies

$$\begin{aligned} \sup_{\gamma>0} \|f_n^\gamma - f^\gamma\|_\infty^2 &= \sup_{\gamma>0} \left\| \sum_{k=n+1}^\infty \varphi_\gamma^k(\cdot)\langle u, Ge_k\rangle\langle v, e_k\rangle \right\|_\infty^2 \\ &\leqslant \sup_{\gamma>0} \sup_{k\in\mathbb{N}} \left\|\varphi_\gamma^k\right\|_\infty^2 \left(\sum_{k=n+1}^\infty \langle G^*u, e_k\rangle^2 \right) \left(\sum_{k=n+1}^\infty \langle v, e_k\rangle^2 \right) \to 0, \quad n \to \infty, \end{aligned}$$

and analogously we obtain $\|f_n - f\|_\infty \to 0$ as $n \to \infty$. Since Condition (4.34) guarantees by Theorem 12.7.2 in [28] that $f_n^\gamma \to f_n$ as $\gamma \to \infty$ in $\big(D([0,T];\mathbb{R}), d_M\big)$ for all $n \in \mathbb{N}$, Lemma 2.3 implies

$$\lim_{\gamma\to\infty} \langle F_\gamma(\cdot)u, v\rangle = \langle F(\cdot)u, v\rangle \qquad \text{in } \big(D([0,T];\mathbb{R}), d_M\big),$$

which shows Condition (4.17) and completes the proof. □

5 Integrated Ornstein-Uhlenbeck process

Let V be a separable Hilbert space and let A be the generator of a strongly continuous semigroup $(S(t))_{t\geqslant 0}$ in V. For $\gamma > 0$ consider the equation

$$\begin{aligned} dY_\gamma(t) &= \gamma A Y_\gamma(t)\, dt + G\, dL(t) \qquad \text{for } t \in [0,T], \\ Y_\gamma(0) &= 0, \end{aligned} \tag{5.1}$$

where L denotes a Lévy process in U and $G \in \mathcal{L}(U,V)$. A progressively measurable stochastic process $(Y_\gamma(t)\colon t \in [0,T])$ is called a *weak solution of* (5.1) if it satisfies for every $v \in \mathcal{D}(A^*)$ and $t \in [0,T]$ P-a.s. the equation

$$\langle Y_\gamma(t), v\rangle = \int_0^t \langle Y_\gamma(s), A^*v\rangle\, ds + \langle L(t), G^*v\rangle.$$

Since $A_\gamma := \gamma A\colon \mathcal{D}(A) \subseteq V \to V$ is the generator of the C_0-semigroup $(S_\gamma(t))_{t\geqslant 0}$ where $S_\gamma(t) := S(\gamma t)$ for all $t \geqslant 0$, Theorem 2.3 in [6] implies that there exists a unique weak solution $Y_\gamma := (Y_\gamma(t)\colon t \in [0,T])$ of (5.1), and which can be represented by

$$Y_\gamma(t) = \int_0^t S_\gamma(t-s) G\, dL(s) \qquad \text{for all } t \in [0,T].$$

We require that Y_γ has càdlàg trajectories, which is satisfied for example if the semigroup $(S(t))_{t\geqslant 0}$ is analytic or contractive; see [18]. Then the integrated Ornstein-Uhlenbeck process $(X_\gamma(t)\colon t\in[0,T])$ is defined by

$$X_\gamma(t) := \gamma\int_0^t Y_\gamma(s)\,ds \qquad \text{for all } t\in[0,T].$$

Corollary 5.1. *Assume that the semigroup $(S(t))_{t\geqslant 0}$ is diagonalisable, i.e. there exists an orthonormal basis $e:=(e_k)_{k\in\mathbb{N}}$ of V such that*

$$S(t)e_k = e^{-\lambda_k t}e_k \qquad \text{for all } t\geqslant 0,\ k\in\mathbb{N},$$

for some $\lambda_k>0$ with $\lambda_k\to\infty$ for $k\to\infty$. Then the integrated Ornstein–Uhlenbeck process X_γ satisfies

$$\lim_{\gamma\to\infty}\big(AX_\gamma(t)\colon t\in[0,T]\big) = \big(-GL(t)\colon t\in[0,T]\big)$$

in probability in the product topology $\big(D([0,T];V),d_M^e\big)$.

Proof. For every $t\in[0,T]$ and $v\in\mathcal{D}(A^*)$ Fubini's theorem implies by (4.3)

$$\begin{aligned}\langle AX_\gamma(t),v\rangle &= \gamma\int_0^t\langle Y_\gamma(s),A^*v\rangle\,ds\\ &= \gamma\int_0^t\left(\int_0^s G^*S_\gamma^*(s-r)A^*v\,dL(r)\right)ds\\ &= \gamma\int_0^t\left(\int_r^t G^*S_\gamma^*(s-r)A^*v\,ds\right)dL(r).\end{aligned}\tag{5.2}$$

It follows from properties of the adjoint semigroup (see [25, Proposition 1.2.2]) that for every $r\in[0,t]$ we have

$$\int_r^t G^*S_\gamma^*(s-r)A^*v\,ds = \frac{1}{\gamma}G^*A_\gamma^*\int_0^{t-r}S_\gamma^*(s)v\,ds = \frac{1}{\gamma}\left(G^*S_\gamma^*(t-r)v - G^*v\right).\tag{5.3}$$

Define a Lévy process K in V by $K(t):=GL(t)$ for all $t\geqslant 0$. By combining (5.3) with (5.2) we obtain from (4.3) and (4.5) that

$$\langle AX_\gamma(t),v\rangle = \int_0^t\left(G^*S_\gamma^*(t-r)v-G^*v\right)dL(r) = \langle\int_0^t\left(S_\gamma(t-r)-\mathrm{Id}\right)dK(r),v\rangle.$$

Consider the functions

$$F\colon[-1,T]\to\mathcal{L}(V,V),\qquad F(t)=\begin{cases}-\mathrm{Id}, & \text{if } t\in[0,T],\\ 0, & \text{if } t\in[-1,0),\end{cases}$$

$$F_\gamma\colon[-1,T]\to\mathcal{L}(V,V),\qquad F_\gamma(t)=\begin{cases}S(\gamma t)-\mathrm{Id}, & \text{if } t\in[0,T],\\ 0, & \text{if } t\in[-1,0).\end{cases}$$

Defining the functions above on $[-1,T]$ and not only on $[0,T]$ with a jump at 0 enables us to consider càdlàg functions. By means of Corollary 4.5 (with an obvious adaption for considering the interval $[-1,T]$) we show that

$$\lim_{\gamma\to\infty}\left(\int_0^t\left(S_\gamma(t-r)-\mathrm{Id}\right)dK(r)\colon t\in[-1,T]\right) = \left(\int_0^t F(t-s)\,dK(s)\colon t\in[-1,T]\right)\tag{5.4}$$

in probability in the product topology $\big(D([-1,T];V),d_M^e\big)$. The functions F_γ and F are of the form

$$F^*(t)e_k = \varphi^k(t)\, e_k, \qquad F_\gamma^*(t)e_k = \varphi_\gamma^k(t)e_k \qquad \text{for all } t\in[-1,T],\ k\in\mathbb{N},$$

where the real-valued functions φ^k, $\varphi_\gamma^k\colon [-1,T]\to\mathbb{R}$ are defined by

$$\varphi^k(t) = -\,\mathbb{1}_{[0,T]}(t), \qquad \varphi_\gamma^k(t) = \mathbb{1}_{[0,T]}(t)(e^{-\lambda_k\gamma t}-1).$$

For every $k\in\mathbb{N}$ the sequence $(\varphi_\gamma^k)_{\gamma>0}$ meets Condition (4.28), as

$$\sup_{\gamma>0}\big\|\varphi_\gamma^k\big\|_\infty = \sup_{\gamma>0}\sup_{s\in[0,T]}\big|e^{-\lambda_k\gamma s}-1\big| \leqslant 1.$$

Since φ_γ^k is a decreasing function it has finite variation and thus $\big\|\varphi_\gamma^k\big\|_{TV_2}=0$ which verifies Condition (4.29). Since φ_γ^k is monotone for each $\gamma>0$, $k\in\mathbb{N}$ and satisfies for each $k\in\mathbb{N}$

$$\lim_{\gamma\to\infty}\varphi_\gamma^k(s)=\varphi^k(s) \qquad \text{for all } s\in[-1,T]\backslash\{0\},$$

it follows from Corollary 12.5.1 in [28], that $\varphi_\gamma^k\to\varphi^k$ as $\gamma\to\infty$ in $\big(D([0,T];\mathbb{R}),d_M\big)$, which is Condition (4.30). Thus, we can apply Corollary 4.5 to conclude (5.4). □

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