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The Tate Form on Steroids: Resolution and Higher Codimension Fibers

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F-theory on singular elliptically fibered Calabi-Yau four-folds provides a setting to geometrically study four-dimensional $\mathcal{N} = 1$ supersymmetric gauge theories, including matter and Yukawa couplings. The gauge degrees of freedom arise from the codimension 1 singular loci, the matter and Yukawa couplings are generated at enhanced singularities in higher codimension. We construct the resolution of the singular Tate form for an elliptic Calabi-Yau four-fold with an ADE type singularity in codimension 1 and study the structure of the fibers in codimension 2 and 3. We determine the fibers in higher codimension which in general are of Kodaira type along minimal singular loci, and are thus consistent with the low energy gauge-theoretic intuition. Furthermore, we provide a complementary description of the fibers in higher codimension, which will also be applicable to non-minimal singularities. The irreducible components in the fiber in codimension 2 correspond to weights of representations of the ADE gauge group. These can split further in codimension 3 in a way that is consistent with the generation of Yukawa couplings. Applying this reasoning, we then venture out to study non-minimal singularities, which occur for A type along codimension 3, and for D and E also in codimension 2. The fibers in this case are non-Kodaira, however some insight into these singularities can be gained by considering the splitting of fiber components along higher codimension, which are shown to be consistent with matter and Yukawa couplings for the corresponding gauge groups.

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1 Introduction

It has long been known that F-theory is an ideal framework for geometric engineering of gauge theories [1–3]. F-theory compactified on singular elliptically fibered Calabi-Yau n -manifolds Y_n , realizes supersymmetric gauge theories at low energies, which can be understood to have their origin in 7-branes wrapping cycles in the compactification manifold. Let B_{n-1} denote the base of the elliptic fibration, which we shall assume to be described in terms of a Weierstrass model

$$y^2 = x^3 + fx + g, \tag{1.1}$$

where f and g are sections of $K_{B_{n-1}}^{-4}$ and $K_{B_{n-1}}^{-6}$. The singular elliptic fibers in codimension 1 in the base were classified by Kodaira [4], and the Kodaira type of the fiber determines the gauge group of the effective theory on the 7-branes that wrap the codimension 1 locus in the base as well as the transverse flat space. From an M-theory perspective, the abelian part of the gauge bosons arise by dimensional reduction of the C_3 -form along the \mathbb{P}^1 s in the resolved fiber, and the remaining components arise from wrapped M2-branes.

Whenever $n \geq 3$, in addition, matter transforming in highest weight representations of the gauge group is known to come from codimension 2 loci in the base [5–10], which confirms the physical intuition that 7-branes generate matter along their intersection loci. Naturally, one can extend this for $n \geq 4$ and the study of codimension 3 fibers. Indeed, more recently, F-theory on Calabi-Yau fourfolds has played a key role in building four dimensional supersymmetric grand unified theories (GUTs) [11–13]. For a complete description of a GUT model,

we need in addition to gauge and matter degrees of freedom, a way to generate the Yukawa couplings of the matter fields. 7-brane intuition tells us that this should happen at the intersection of multiple branes. Geometrically, these are realized in codimension 3 in the base B_3 .

The generation of matter and Yukawa couplings from higher codimension loci, where the singularity type enhances, has a counterpart in the effective field on the 7-branes, there matter and Yukawa can be thought to arise from Higgsing of a higher rank gauge group. For gauge groups embedded into E_8 , this has a precise description in terms of the Higgs bundle spectral cover [14–17], and more generally including additional gluing data [18–21]. For instance for $SU(5)$ GUTs the matter and Yukawa couplings can be thought to arise by Higgsing $SU(6)$ or $SO(10)$ for matter, and $SO(12)$, or E_6 for Yukawa couplings. This gauge theoretic expectation was confirmed by studying the resolved Calabi-Yau fourfold and the fiber structure in [8, 9], except for the absence of a Kodaira E_6 fiber in codimension 3, which however was shown to not be an obstruction to generating the top Yukawa coupling [9]: along the putative E_6 codimension 3 locus the fiber component corresponding to a $\mathbf{10}$ matter representation splits further into a $\mathbf{10}$ and a $\mathbf{5}$.

In this context there are two natural questions that arise, which we will address in this paper: from a mathematical point of view, one can ask whether a codimension 1 singular Kodaira fiber degenerates in higher codimension to Kodaira fibers or not. The physical counterpart to this question is whether in general the geometry corroborates the gauge theoretic description of matter and Yukawas in terms of Higgsing of a higher rank gauge group.

Key to understanding the fibers in higher codimension is the resolution of the singularities. One of the main objectives of this paper is therefore to resolve elliptically fibered Calabi-Yau fourfolds with an ADE type singularity in codimension 1. Our starting point is the Tate form [22, 23] of the Kodaira singularity in codimension 1¹. In each case we construct the fully resolved geometry, and study the fibers in higher codimension, giving two, slightly distinct, points of view: we can either consider the intersections of the fibers in higher codimension. Or, alternatively, we can study the splitting of the fibers along the enhanced singularities. The latter will in particular be of use for the non-minimal singularities, which we will discuss in a moment.

Concretely, for codimension 2, we show that generically the intersections of the fibers along minimal singularities are Kodaira A or D type. Likewise, following the fibers in codimension

¹Note that we will not assume any further structure of the sections appearing in the Tate form. Relaxing this conditions, one can of course obtain less generic fibers, e.g. models with extra sections will not be considered here. Starting with the Tate form comes with a few minor caveats, which will be discussed in the next section.

1 to the codimension 2 locus, we show that additional components in the fiber correspond to weights² of highest weight representations of the gauge group. In codimension 3 the fibers along minimal singular loci are again generically Kodaira A or D type, or alternatively put, the fiber components become reducible, and in each case we identify these with a consistent Yukawa coupling between matter in codimension 2.

What we said so far applies to singularities which are so-called minimal, characterized by a vanishing order of the sections f and g in the Weierstrass form as well as the discriminant Δ not all exceeding 3, 5 and 11, respectively. In codimension 1 we will always assume that this condition of minimality is satisfied. However, when passing to higher codimension, we will also consider non-minimal loci. These occur for A_n type singularities in codimension 3, generalizing the E -type Dynkin diagrams that one would obtain as fiber intersection graphs for low values of n . The non-minimality is clear from the additional one-dimension higher (i.e. surface) components in the fiber, and the fibers are most definitely not Kodaira, signaling additional degrees of freedom in the low energy effective theory. Nevertheless we can study how the matter surfaces in codimension 2 split along the non-minimal loci, which is shown to be completely consistent with the generation of Yukawa couplings (generalizing the E_6 Yukawa for $SU(5)$, for instance). To depict the fibers we either consider only the components that descend from the matter surfaces, yielding generalizations of E -type Dynkin diagrams or we include the additional one-dimension higher components.

In a similar way we study the non-minimal matter locus in codimension 2 for D_n , and show that the Cartans split into spin representations and the fundamental representation V . The fiber along these non-minimal loci will depend on the particular small resolution that one applies to the singular geometry. To exemplify this, we provide an alternative resolution for the non-minimal matter locus for D_n in appendix D, where in addition the splitting includes matter in the representations $\Lambda^i V$. We consider these results as a starting point for studying these non-minimal singularities and their low energy effective description in more detail.

Related work for compactifications of F-theory to six dimensions has appeared [7], where the matter representations and fibers were studied by resolving the local singularities in codimension 2. An alternative point of view was presented in [10, 24], where a generalized version of the Tate algorithm was developed for matter. The present result seems to be in agreement with the matter found in [10] for minimal singularities. Finally, in [25] an example of a codimension 3 non-minimal singularity for an E_7 -type singularity was considered, and studied from the point of view of singular spectral covers. This opens up the possibility of a

²As we will make precise in the main text, this means, that their intersections with the codimension 1 fibers, i.e. roots, are given by Dynkin labels of highest weight representations.

connection with the constructions in [21]. It would be interesting to apply similar methods to resolve the non-minimal loci that we encounter for A_n and D_n .

Beyond the clarification of the codimension 2 and 3 structure of the fibers, the resolved Tate forms that we determine will be useful for the study of additional data that are necessary in F-theory compactification. In particular G -flux $G_4 \in H^{2,2}(Y, \mathbb{Z})$ can be constructed in terms of surfaces in the resolved geometry, and proper quantization can be checked using the c_2 of the resolved fourfold [9, 17, 21, 26–29]. Likewise for general gauge groups, the resolved geometries in this paper can be applied to construct such G -flux.

Resolutions of singularities for some low gauge group relevant for GUT models have been studied in the literature: for $SU(5)$ in [8, 9], for E_6 and $SO(10)$ codimension 1 singularities, the resolution of CY fourfolds has appeared in [30, 31], and for other models related to the $SU(5)$ case, e.g. with additional $U(1)$ factors and connection to orientifold limits, the resolved geometries were studied in [27, 29, 32–37].

The outline of the paper is as follows. In section 2 we summarize the theory used to resolve the singular geometry and the intersection theory necessary to calculate the properties of the fibers in higher codimensions, in particular their splitting and charges. For each singularity in codimension 1, we summarized the higher codimension loci of enhanced symmetry in table 2.13. In the subsequent sections we resolve the A_n , D_n and E_n singularities in turn. Each section can be read independently, although the most detailed analysis will be presented for $SU(2k + 1)$. For each type we first fully resolve the Tate form for general n , then compute the intersections of exceptional divisors in codimension 1. In particular, we will consider the divisors that are obtained by fibering the irreducible components of the fibers over the codimension 1 locus, which we will denote by *Cartan divisors*, and their intersections reproduce the affine Dynkin diagrams of ADE type. For each of the codimension 2 loci in table 2.13 we determine the splitting of the codimension 1 fibers, and compute the intersections of the irreducible fiber components in codimension 2 with the Cartan divisors (labeled by the simple roots), which gives rise to weights of representation of the ADE gauge group. The third subsection in each ADE case contains the discussion of fibers in codimension 3, and in each locus listed in table 2.13 we show what Yukawa coupling they generate. For A_n and D_n we furthermore discuss the fibers along the non-minimal loci in a separate subsection. Appendix A lists the Tate forms, appendix B contains low rank examples. Finally, we include extensive appendices containing the details of the intersection calculations and the resolved geometries.

2 Resolution of Singularities

2.1 The Tate Form

Consider a singular elliptically fibered Calabi Yau n -fold. The elliptic fibration has singularities in codimension 1, and for sufficiently large n higher codimension singularities can arise, whose structure is less understood. The present case of interest, $n = 4$, allows in addition codimension 2 and 3 singularities. We will study these in turn for various Tate forms, by first resolving the geometries in all codimensions and then studying the intersections along the loci of singularity enhancement.

The starting point for our analysis is the Weierstrass form for a singular elliptically fibered Calabi-Yau fourfold Y_4 over a base B_3

$$y^2 = x^3 + fx + g. \quad (2.1)$$

Let ζ_0 be a local coordinate on the base threefold. For a singularity with Kodaira fibers of a specific type along a divisor $\zeta_0 = 0$, one can globally put the Weierstrass form into the Tate form (or a suitable generalization thereof) [22, 23]

$$T = y^2w - x^3 + b_1xyw\zeta_0^{i_1} - b_2wx^2\zeta_0^{i_2} + b_3yw^2\zeta_0^{i_3} - b_4xw^2\zeta_0^{i_4} - b_6\zeta_0^{i_6}w^3 = 0, \quad (2.2)$$

which is realized inside a \mathbb{P}^2 bundle

$$X_5 = \mathbb{P}(\mathcal{O} \oplus K_{B_3}^{-2} \oplus K_{B_3}^{-3}). \quad (2.3)$$

There is a subtlety associated with this starting point. The Tate form guarantees that a singularity of a specific type is realized. However, there are some caveats related to bringing a given Weierstrass form globally into the Tate form [23]. In particular for the cases

$$\begin{aligned} SU(m), & \quad m = 6, 7, 8, 9, \\ Sp(m), & \quad m = 3, 4 \\ SO(m), & \quad m = 13, 14, \end{aligned} \quad (2.4)$$

the Tate form cannot be achieved. For the I_{2k+1} singularities with monodromy, which in the table in appendix A, are denoted by I^{ns} or I^{ss} , the Tate form cannot in general be achieved globally, however a generalized Tate form presented in [23] can. For this paper we will concentrate on the simply-laced, i.e. ADE type singularities, which in particular do not have monodromy³. The only case in which our starting point is thus not the most general one is the $SU(m)$ with $m = 6, 7, 8, 9$.

³When referring to Kodaira fibers of type I_n^s , I_n^{*s} , IV^{*s} , III^* and II^* , respectively, we sometimes will denote them by A , D or E type fibers, where it is understood, that the multiplicities are as in the Kodaira classification.

With this minor caveat in mind, consider now the elliptically fibered CY defined by (2.2), which has a singularity along $\zeta_0 = 0$, whose type is specified by the vanishing orders i_m as summarized in appendix A. For most purposes it will be sufficient to work in the patch $w = 1$. The various sections have the following classes

Section	Class	
x	$\sigma + 2c_1$	
y	$\sigma + 3c_1$	
w	σ	
ζ_0	S_2	(2.5)
b_n	$nc_1 - i_n S_2$	

The codimension 1 singularities are along components of the vanishing locus of the discriminant

$$\Delta = 4f^3 + 27g^2 = O(\zeta_0^d), \quad (2.6)$$

where d is the vanishing order as specified in appendix A. The higher codimension loci are defined below.

The resolution of the singularity in codimension 1 in the base, i.e. along a surface S_2 specified by the section ζ_0 , is well-known to give rise to fibers with exceptional \mathbb{P}^1 s, whose intersections are represented by the affine Dynkin diagrams of the ADE type of the singularity. From the point of view of F-theory [1–3], this corresponds to realizing the gauge degrees of freedom on 7-branes wrapping the surface S_2 . The abelian part of the gauge bosons has its origin in the M-theory 3-form C_3 , decomposed into the $(1, 1)$ forms $\omega^{(1,1)}$ of the exceptional \mathbb{P}^1 s

$$C_3 = A^i \wedge \omega_i^{(1,1)}, \quad (2.7)$$

whereas the remaining gauge bosons arise from M2-branes wrapping the exceptional \mathbb{P}^1 s in the fiber. We will denote the divisors obtained by fibering the exceptional \mathbb{P}^1 s over the surface S_2 by Cartan divisors $D_{-\alpha_i}$, labeled by the simple roots α_i of the gauge group. The section ζ_0 will correspond to the affine root $-\alpha_0$. In summary:

Codimension 1

The fibers in codimension 1 (i.e. along $\zeta_0 = 0$) have intersections given by the affine Dynkin diagram of the ADE type of the singularity. The Cartan divisors $D_{-\alpha_i}$ are obtained as the irreducible exceptional divisors of the resolution.

Less explored is the structure of the fibers in higher codimension. For specific singularity types in Calabi-Yau 3- and 4-folds this was studied in [7–10, 25, 30, 31, 33]. First consider the

higher order terms in the expansion of the discriminant with respect to ζ_0

$$\Delta = \zeta_0^d \Delta_d + \zeta_0^{d+i} \Delta_{d+i} + O(\zeta_0^{d+i+1}), \quad (2.8)$$

where $\zeta_0 = 0$ intersected with $\Delta_d = 0$ yields a higher vanishing order, and we would expect new components to emerge in the fiber. To clarify what happens in the fiber, we resolve the space in codimension 1 but also 2 and 3, where the latter are defined as

$$\begin{aligned} \text{Codimension 2 : } & \zeta_0 = \Delta_d = 0 \\ \text{Codimension 3 : } & \zeta_0 = \Delta_d = \Delta_{d+i} = 0. \end{aligned} \quad (2.9)$$

The following table 2.13 lists each of the codimension 2 and 3 loci for the Tate forms in the table in appendix A. From an F-theoretic point of view, codimension 2 corresponds to the matter loci, and codimension 3 to the points, where Yukawa couplings are generated.

In summary we find the following fiber structure in higher codimensions: along minimal singularities the resolved geometry has Kodaira fibers for A_n and D_n singularities in codimension 2 and 3, including multiplicities. For non-minimal loci the fibers do not intersect in the affine Dynkin diagram of a simple Lie algebra. We will study the latter following the logic put forward in [9], by considering the splitting of the lower codimension fibers.

Codimension 2

Along the codimension 2 loci the exceptional \mathbb{P}^1 in the fiber either remain irreducible, or split, generating new components that are labeled by weights of a highest weight representation of the gauge group, i.e. their intersections with the irreducible components of the fibers in codimension 1 are Dynkin labels of representations. Generically, this splitting gives rise to one new effective curve in the fiber. Along the codimension 2 loci, the Cartan divisors $D_{-\alpha_i}$ will restrict to surfaces, the irreducible components of which will be denoted by S_{v_i} , where v_i is either a weight of a highest weight representation, or in case $D_{-\alpha_i}$ remains irreducible, a root. We will refer to these as matter surfaces.

Codimension 3

Along the codimension 3 loci, either, the fibers remain irreducible, or a \mathbb{P}^1 labeled by a weight, becomes homologous to the sum of two others, compatible with the generation of the Yukawa coupling. Put differently, these three curves are connected by a 3-chain, which shrinks to zero size at the codimension 3 locus. The restriction of the matter surfaces S_{v_i} will be denoted by the curves Σ_{v_i} .

2.2 Higher Codimension Loci and Minimality

In the following table 2.13 we provide the codimension 2 and 3 loci for the Tate forms of appendix A. The table contains loci, which are non-minimal. We marked those accordingly by putting them into brackets $[\dots]$. A singularity is non-minimal along a codimension 1 locus $\zeta_0 = 0$ if the vanishing orders of f , g and Δ all exceed $(3, 5, 11)$, i.e. it is non-minimal if

$$f = O(\zeta_0^4), \quad g = O(\zeta_0^6), \quad \Delta = O(\zeta_0^{12}). \quad (2.10)$$

The sections f and g of the Weierstrass form are related to the sections in the Tate form by

$$\begin{aligned} f &= \frac{1}{48} \left(- (a_1^2 + 4a_2)^2 + 24a_1a_3 + 48a_4 \right) \\ g &= \frac{1}{864} \left(- (a_1^2 + 4a_2)^3 + 36(a_1a_3 + 2a_4)(a_1^2 + 4a_2) - 216(a_3^2 + 4a_6) \right), \end{aligned} \quad (2.11)$$

where $a_n = b_n \zeta_0^{i_n}$ in (2.2). Along the codimension 2 and 3 locus, we apply the same criterion of non-minimality to $f|_{\Delta_d=0}$, $g|_{\Delta_d=0}$ and $\Delta|_{\Delta_d=0}$, and $f|_{\Delta_d=\Delta_{d+1}=0}$, $g|_{\Delta_d=\Delta_{d+1}=0}$ and $\Delta|_{\Delta_d=\Delta_{d+1}=0}$, respectively in the notation of (2.9). For matter this question was addressed in a Tate algorithmic way in [10]. An example for non-minimal codimension 3 singularity and its resolution appeared in [25].

There are a few low rank outlier cases that are minimal:

- $Sp(k)$ for $k = 2$: $b_1^2 + 4b_2 = 0$ and $b_1^2 + 4b_2 = b_1b_3 + 2b_4 = 0$ are minimal
- $Sp(k)$ for $k = 3$: $b_1^2 + 4b_2 = 0$ is minimal
- $SU(2k)$ for $k = 3$: $b_1 = b_2 = 0$ is minimal

Finally, note that the following abbreviations were used in the table:

$$\begin{aligned} SU(3) : \quad & P \equiv 4b_4^3 + 4b_2^3b_6 - b_2^2b_4^2 - 18b_2b_4b_6 - 27b_6^2 \\ G_2 : \quad & P \equiv 4b_6b_2^3 - b_4^2b_2^2 - 18b_4b_6b_2 + 4b_4^3 + 27b_6^2 \\ & Q \equiv 2b_3^2b_2^3 - 2b_1b_3b_4b_2^2 + 6b_1^2b_6b_2^2 - b_1^2b_4^2b_2 - 9b_3^2b_4b_2 \\ & \quad - 18b_1b_3b_6b_2 + 12b_1b_3b_4^2 + 27b_3^2b_6 - 9b_1^2b_4b_6 \\ SO(11) : \quad & Q \equiv b_1b_2b_3b_4 - b_2^2b_3^2 - b_1^2b_2b_6 + 2b_4b_6 \\ SO(13) : \quad & Q \equiv b_6b_1^6 - b_3b_4b_1^5 - b_4^2b_1^4 - (b_3^3 + 36b_6b_3) b_1^3 \\ & \quad + 6b_4(5b_3^2 - 12b_6) b_1^2 + 96b_3b_4^2b_1 + 64b_4^3 + 27(b_3^2 + 4b_6)^2 \\ SO(14) : \quad & P \equiv (b_1^3 - 27b_3) b_3^3 + b_1^2(b_1^3 - 30b_3) b_4b_3 - 64b_4^3 + b_1(b_1^3 - 96b_3) b_4^2 \\ E_8 : \quad & Q \equiv b_3b_4b_1^5 - b_2b_3^2b_1^4 + b_4^2b_1^4 + b_3^3b_1^3 + 8b_2b_3b_4b_1^3 - 8b_2^2b_3^2b_1^2 + 8b_2b_4^2b_1^2 \\ & \quad - 30b_2^2b_4b_1^2 + 36b_2b_3^3b_1 - 96b_3b_4^2b_1 + 16b_2^2b_3b_4b_1 - 27b_3^4 - 64b_4^3 \\ & \quad - 16b_2^3b_3^2 + 16b_2^2b_4^2 + 72b_2b_3^2b_4 \end{aligned} \quad (2.12)$$

Type	Group	Codim 2	Codim 3
I_2	$SU(2)$	$b_1^2 + 4b_2 = 0$ $P \equiv b_1^2 b_6 + b_2 b_3^2 + 4b_2 b_6 - b_4^2 - b_1 b_3 b_4 = 0$	$b_1^2 + 4b_2 = b_1 b_3 + 2b_4 = 0$ $P = b_3^2 + 4b_6 = 0$ $P = b_1^2 + 4b_2 = 0$ $P = b_1 b_3 + 2b_4 = 0$
I_{2k}^{ns}	$Sp(k)$	$[b_1^2 + 4b_2 = 0]$ $P \equiv b_1^2 b_6 + b_2(b_3^2 + 4b_6) - b_4(b_1 b_3 + b_4) = 0$	$[b_1^2 + 4b_2 = b_1 b_3 + 2b_4 = 0]$ $P = b_1 b_3 + 2b_4 = 0$ $P = 4b_4^2 + 4b_1 b_3 b_4 + b_1^2 b_3^2 = 0$
I_4^s	$SU(4)$	$b_1 = 0$ $P \equiv b_1 b_3 b_4 + b_4^2 - b_1^2 b_6 = 0$	$b_1 = 4b_4^2(b_2^2 - 4b_4) + b_2^2(b_3^2 + 6b_6) = 0$ $P = b_2 = 0$ $P = b_3^2 + 4b_6 = 0$
I_{2k}^s	$SU(2k)$	$b_1 = 0$ $P \equiv b_1 b_3 b_4 + b_4^2 - b_1^2 b_6 = 0$	$[b_1 = b_2 = 0]$ $b_1 = b_4 = 0$ $P = b_2 = 0$ $P = b_3^2 + 4b_6 = 0$
I_5^s	$SU(5)$	$b_1 = 0$ $P \equiv b_2 b_3 + b_1(b_1 b_6 - b_3 b_4) = 0$	$b_1 = b_2 = 0$ $b_1 = b_3 = 0$ $P = b_3^2 + b_4^2 b_1 + 4b_1 b_2 b_6 = 0$
I_{2k+1}^s	$SU(2k+1)$	$b_1 = 0$ $P \equiv b_2 b_3^2 + b_1(b_1 b_6 - b_3 b_4) = 0$	$[b_1 = b_2 = 0]$ $b_1 = b_3 = 0$ $P = b_4^2 - 4b_2 b_6 = 0$
III	$SU(2)$	$b_4 = 0$	$b_4 = b_3^2 + 4b_6 = 0$
IV^s	$SU(3)$	$b_3 = 0$	$b_3 = P = 0$
I_0^{*ns}	G_2	$P = 0$	$P = Q = 0$
I_0^{*ss}	$SO(7)$	$b_4 = 0$ $P \equiv b_2^2 - 4b_4 = 0$	$b_4 = b_3^2 + 4b_6 = 0$ $P = (b_1 b_2 - 2b_3)^2 - 16b_6 = 0$
I_1^{*ns}	$SO(9)$	$b_2 = 0$ $b_3^2 + 4b_6 = 0$	$b_2 = b_3^2 + 4b_6 = 0$ $b_3^2 + 4b_6 = b_1 b_3 + 2b_4 = 0$
I_1^{*s}	$SO(10)$	$b_2 = 0$ $b_3 = 0$	$b_2 = b_3 = 0$ $b_3 = 4b_2 b_6 - b_4^2 = 0$
I_2^{*ns}	$SO(11)$	$b_2 = 0$ $P \equiv b_4^2 - 4b_2 b_6 = 0$	$b_2 = b_4 = 0$ $P = Q = 0$

(2.13)

Type	Group	Codim 2	Codim 3
I_2^{*s}	$SO(12)^*$	$b_2 = 0$ $b_4 = 0$	$[b_2 = b_4 = 0]$ $b_3 = b_4 = 0$
I_3^{*ns}	$SO(13)$	$[b_2 = 0]$ $b_3^2 + 4b_6 = 0$	$[b_2 = Q = 0]$ $b_3^2 + 4b_6 = b_1b_3 + 2b_4 = 0$
I_3^{*s}	$SO(14)$	$[b_2 = 0]$ $b_3 = 0$	$[b_2 = P = 0]$ $b_3 = b_4^2 - 4b_2b_6 = 0$
I_3^{*ns}	$SO(15)$	$[b_2 = 0]$ $P \equiv b_4^2 - 4b_2b_6 = 0$	$[b_2 = b_4 = 0]$ $[b_2 = b_1^4 - 64b_4 = 0]$ $P = b_2b_3^2 + b_1^2b_6 - b_1b_3b_4 = 0$
I_3^{*s}	$SO(16)^*$	$[b_2 = 0]$ $b_4 = 0$	$[b_2 = b_4 = 0]$ $[b_2 = b_1^4 - 64b_4 = 0]$ $b_3 = b_4 = 0$
I_{2k-3}^{*ns}	$SO(4k+1)$	$[b_2 = 0]$ $b_3^2 + 4b_6 = 0$	$[b_2 = b_1 = 0]$ $[b_2 = b_1b_3b_4 + b_4^2 - b_1^2b_6 = 0]$ $b_3^2 + 4b_6 = b_1b_3 + 2b_4 = 0$
I_{2k-3}^{*s}	$SO(4k+2)$	$[b_2 = 0]$ $b_3 = 0$	$[b_2 = b_1 = 0]$ $[b_2 = b_3 = 0]$ $[b_2 = b_1b_3 + b_4 = 0]$ $[b_2 = b_4 = 0]$ $b_3 = 4b_2b_6 - b_4^2 = 0$
I_{2k-2}^{*ns}	$SO(4k+3)$	$[b_2 = 0]$ $P \equiv b_4^2 - 4b_2b_6 = 0$	$[b_2 = b_4 = 0]$ $[b_2 = b_1 = 0]$ $P = b_2b_3^2 + b_1^2b_6 - b_1b_3b_4 = 0$
I_{2k-2}^{*s}	$SO(4k+4)^*$	$[b_2 = 0]$ $b_4 = 0$	$[b_2 = b_4 = 0]$ $[b_2 = b_1 = 0]$ $b_3 = b_4 = 0$
IV^{*ns}	F_4	$b_3^2 + 4b_6 = 0$	$[b_3^2 + 4b_6 = b_1b_3 + 2b_4 = 0]$
IV^{*s}	E_6	$b_3 = 0$	$b_3 = b_4 = 0$
III^*	E_7	$b_4 = 0$	$[b_4 = b_6 = 0]$
II^*	E_8	$[b_6 = 0]$	$[b_6 = Q = 0]$

(2.14)

2.3 General Resolution Procedure

The Tate form (2.2) is singular at

$$x = y = \zeta_0 = 0, \quad (2.15)$$

in particular, all the derivatives vanish at this locus. The singularity is resolved by blowing up, which introduces a new exceptional \mathbb{P}^2 , specified by a section $\zeta = 0$ and with projective coordinates $[X_1, Y_1, Z_1]$, where

$$x = X_1\zeta, \quad y = Y_1\zeta, \quad \zeta_0 = Z_1\zeta. \quad (2.16)$$

We use the shorthand for this blowup

$$(x, y, \zeta_0; \zeta). \quad (2.17)$$

Applying this repeatedly for all singular loci, given by the vanishing of three sections, resolves the singularity in codimension 1, for A_n , but for D_n and E_n , resolutions along loci of the type $y = \zeta = 0$ at the end fully resolves the singularity in codimension 1. The canonical class under the blowup (2.17) changes by $K_Y \rightarrow K_Y + 2E$, where E is the class of the exceptional divisor specified by $\zeta = 0$.

The resulting space (after proper transformation) however is still singular in higher codimensions. Small resolutions along these loci resolve the space completely. For instance, let $B = 0$ inside Y_4 specify a codimension 2 singularity, and let the space be singular along $B = Y = Z = 0$. A small resolution can remedy this: introduce an exceptional \mathbb{P}^1 given by $\delta = 0$ and having projective coordinates $[Y_1, Z_1]$, which are related to the original coordinates by $(Y, Z) = (\delta Y_1, \delta Z_1)$. For this resolution we shall use the shorthand notation

$$(Y, Z; \delta). \quad (2.18)$$

The canonical class changes by $K_Y \rightarrow K_Y + E$, where E is the class of exceptional divisor $\delta = 0$. Repeating this process yields eventually a fully resolved CY fourfold.

As observed already in [8] for $SU(5)$, there are various choices for the small resolutions, which are all birationally equivalent. We shall find that the general structure of the geometries after resolving the codimension 1 singularities takes the form

$$y(y + V) = \zeta_1 \cdots \zeta_\ell U, \quad (2.19)$$

where U, V depend on all sections (including the exceptional sections and the b_n). For $SU(5)$, $\ell = 2$ and the geometry is of a so-called binomial type, and was resolved by choosing the two

loci for the small resolutions to be $y = A = 0$ and $(y + V) = B = 0$ where $A, B = \zeta_1, \zeta_2$ or U . Generically $\ell > 2$, and this method for small resolutions does not generalize. We will show that a lot of simplifications in this procedure arise by performing all small resolutions with $y = 0$, i.e. for $SU(5)$ as we iterate in appendix B, the two small resolutions are along $y = \zeta_1 = 0$ and $y = \zeta_2 = 0$. Generally, the small resolutions will be of the type

$$(y, \zeta_i; \delta_i), \quad i = 1, \dots, \ell. \quad (2.20)$$

Note, that we could also include small resolutions along U instead of one of the ζ_i , however, we will consider here only the resolutions of the type (2.20). Indeed, there is an interesting web of small resolutions [8, 38]. The small resolutions we will consider have the property that the exceptional divisors all become irreducible, i.e. to each section in the resolved space, there is a corresponding irreducible exceptional divisor, i.e. these are Cartier. In the following we apply this resolution to the Tate forms with ADE fibers with the aim to study the structure of the higher codimension fibers.

For a concise notation of the geometries, it will be useful to define the following abbreviations

$$\begin{aligned} A(z) &= \prod_{i=1}^{k-1} z_i^{i-1} \\ C(z) &= \prod_{i=1}^{k-1} z_i^{k-(i+1)} \\ B(z) &= \prod_{i=1}^{k-1} z_i. \end{aligned} \quad (2.21)$$

Note that e.g. $A(\zeta\delta) = \prod_{i=1}^{k-1} (\zeta_i \delta_i)^{i-1}$.

2.4 Fibers in Higher Codimension

As a summary of our results we now list the properties of the fibers in codimension 2 and 3 for A_n and D_n , n even and odd. We will be mainly concerned with the representations that occur in codimension 2, and the Yukawa couplings that are generated in codimension 3, as this will generalize to the non-minimal cases. We tabulate how the Cartan divisors split into matter representations of the gauge group, and how these further split along the Yukawa points. These are the data that are most relevant in order to understand the physical interpretation in terms of F-theory and 7-brane effective theories. The specific intersections of the fibers, which may be of interest from a geometric point of view are shown in each of the following sections. In codimension 2 most intersections give rise to affine Dynkin diagrams, and for

Group	Codim 2/3	Representation/Yukawa
$SU(2k+1)$	$b_1 = 0$ $P = 0$	$\Lambda^2 V, \Lambda^2 \bar{V}$ V, \bar{V}
	$b_1 = b_3 = 0$ $b_1 = b_2 = 0$ $P = Q = 0$	$V \otimes V \otimes \Lambda^2 V$ $[\Lambda^2 \bar{V} \otimes \Lambda^2 \bar{V} \otimes \Lambda^4 V]$ $V \otimes \bar{V} \otimes \mathbf{1}$
$SU(2k)$	$b_1 = 0$ $P = 0$	$\Lambda^2 V, \Lambda^2 \bar{V}$ V, \bar{V}
	$b_1 = b_4 = 0$ $b_1 = b_2 = 0$ $P = b_2 = 0$	$V \otimes V \otimes \Lambda^2 V$ $[\Lambda^2 \bar{V} \otimes \Lambda^2 \bar{V} \otimes \Lambda^4 V]$ $V \otimes \bar{V} \otimes \mathbf{1}$
$SO(4k+2)$	$b_3 = 0$ $b_2 = 0$	V $[S^\pm]$
	$b_3 = b_4^2 - 4b_2b_6 = 0$ $b_2 = b_3 = 0$	$V \otimes V \otimes \mathbf{1}$ $[S^+ \otimes S^- \otimes V]$
$SO(4k+4)$	$b_2 = 0$ $b_4 = 0$	$[S^+]$ V
	$b_3 = b_4 = 0$ $b_2 = b_4 = 0$	$V \otimes V \otimes \mathbf{1}$ $[V \otimes V \otimes \Lambda^2 V]$

(2.22)

Table 1: Matter and Yukawa couplings that arise at the codimension 2 and 3 singularities for ADE gauge groups. Entries in brackets indicate non-minimal singularities, and we will explain in what way the entries are justified in those cases.

minimal singularities have multiplicities as required for a Kodaira fiber. For the non-minimal cases, the fibers are non-Kodaira throughout.

Note that at the loci, where the singularity is non-minimal, marked by parenthesis in table 2.13, we do not expect a standard field theoretic description at low energies. Nevertheless we can follow what happens to the Cartan divisors along those loci. In codimension 2, this occurs for the $SO(2n)$ groups along the matter locus $b_2 = 0$, where we show that the fiber splits such that the irreducible components carry the Cartan charges of spin representations. Similarly, for the $SU(n)$ groups, the codimension 3 loci, which would generalize the E-type enhancements at low n , yield non-minimal singularities. Nevertheless, as elaborated upon before, these can be studied in terms of the splitting of the matter surfaces, which is consistent with the generation of certain Yukawa couplings (generalizing the top Yukawa for $SU(5)$ for instance). Matter and couplings that arise at such non-minimal loci are put in parenthesis in the table.

3 Resolution of A Type Singularities

Consider the Tate forms for A type singularities. For $SU(n)$ we need to discuss the case of n even and odd separately

$$\begin{aligned} n = 2k : \quad & 0 = y^2 - x^3 + b_1xy - b_2x^2\zeta_0 + b_3y\zeta_0^k - b_4x\zeta_0^k - b_6\zeta_0^{2k} \\ n = 2k + 1 : \quad & 0 = y^2 - x^3 + b_1xy - b_2x^2\zeta_0 + b_3y\zeta_0^k - b_4x\zeta_0^{k+1} - b_6\zeta_0^{2k+1}, \end{aligned} \quad (3.1)$$

as the higher codimension structure is different. Note further, that for the low k values, some degeneracies may occur. We provide, for some of these outlier cases in appendix B, the explicit resolutions.

3.1 $SU(2k + 1)$

The Tate form for $SU(2k + 1)$ is

$$0 = y^2 - x^3 + b_1xy - b_2x^2\zeta_0 + b_3y\zeta_0^k - b_4x\zeta_0^{k+1} - b_6\zeta_0^{2k+1}. \quad (3.2)$$

To resolve this in codimension 1, we apply the procedure outlined in section 2.3. The blowups, in the notation (2.17), are

$$(x, y, \zeta_i; \zeta_{i+1}), \quad i = 0, \dots, k - 1. \quad (3.3)$$

After proper transformation, the resulting space is smooth in codimension 1 and takes the form

$$\begin{aligned} & y(y + b_1x + b_3\zeta_0^k B(\zeta)C(\zeta)) \\ & = B(\zeta) [\zeta_0\zeta_k b_2x^2 + x^3 A(\zeta)\zeta_k^k + b_4x\zeta_0^{k+1}\zeta_k B(\zeta)C(\zeta) - b_6\zeta_0^{2k+1}\zeta_k B(\zeta^2)C(\zeta^2)]. \end{aligned} \quad (3.4)$$

This has the general form

$$yY = \zeta_1 \cdots \zeta_k V. \quad (3.5)$$

In particular, this geometry is still singular in higher codimension. We choose the following small resolutions to remedy this

$$(y, \zeta_i; \delta_i), \quad i = 1, \dots, k. \quad (3.6)$$

As explained earlier, there is a choice in the set of small resolutions, and their structure will be studied elsewhere [38]. Note also, that the small resolution in [9] is different from the one chosen here. This is mainly due to the fact that the one chosen here, is easily generalized to

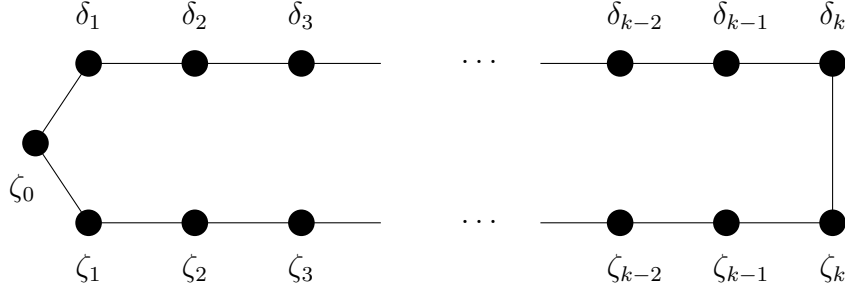


Figure 1: Intersection graph of the fibers in codimension 1, resulting in the affine A_{2k} Dynkin diagram. The labels denote the sections which give rise to the Cartan divisors $D_{-\alpha_i}$ as is spelled out in (3.9).

all values of k , (3.5) ensures that there is a one to one correspondence between irreducible exceptional divisors and sections. The fully resolved geometry for $SU(2k + 1)$ is

$$\begin{aligned} \tilde{T}_{SU(2k+1)} : \\ y^2 B(\delta) \delta_k - x^3 B(\zeta) A(\zeta \delta) \zeta_k^k \delta_k^{k-1} + b_1 x y - b_2 x^2 \zeta_0 B(\zeta) \zeta_k + b_3 y \zeta_0^k B(\zeta \delta) C(\zeta \delta) \\ - b_4 x \zeta_0^{k+1} B(\zeta^2 \delta) C(\zeta \delta) \zeta_k - b_6 \zeta_0^{2k+1} B(\zeta^3 \delta^2) C(\zeta^2 \delta^2) \zeta_k = 0, \end{aligned} \quad (3.7)$$

with the abbreviations (2.21). The exceptional sections are

$$\begin{aligned} \zeta_i, \quad i = 0, \dots, k, \\ \delta_i, \quad i = 1, \dots, k. \end{aligned} \quad (3.8)$$

Their classes as well as the projective relations among these are listed in appendix C.1. In the following sections we will study the fibers in codimension 1, 2 and 3.

3.1.1 Codimension 1

The resolved geometry is given in terms of projective coordinates, which have to satisfy projective relations, which we list in appendix C.1. The small resolutions were chosen so that the vanishing locus of each section is irreducible. For each section the equation of the resolved

Tate form Y_4 restricted to the vanishing locus of the section is listed in the following table.

Divisor	Section	Equation in Y_4
$D_{-\alpha_0}$	ζ_0	$0 = y^2\delta_1 - x^3\zeta_1 + b_1xy$
$D_{-\alpha_1}$	ζ_1	$0 = \delta_1 + b_1x$
$D_{-\alpha_i}$ $i = 2, \dots, k-1$	ζ_i	$0 = \delta_{i-1}\delta_i + b_1x$
$D_{-\alpha_k}$	ζ_k	$0 = \delta_{k-1}\delta_k + b_1x + b_3\zeta_{k-1}\delta_{k-1}$
$D_{-\alpha_{k+1}}$	δ_k	$0 = b_1yx - b_2x^2\zeta_k + b_3y\delta_{k-1} - b_4x\zeta_k\delta_{k-1} - b_6\zeta_k\delta_{k-1}^2$
$D_{-\alpha_{2k+1-i}}$ $i = 2, \dots, k-1$	δ_i	$0 = b_1y - b_2\zeta_i\zeta_{i+1}$
$D_{-\alpha_{2k}}$	δ_1	$0 = b_1y - b_2\zeta_0\zeta_1\zeta_2 - \zeta_1\zeta_2^2\delta_2$

(3.9)

These irreducible exceptional divisors are the *Cartan divisors* $D_{-\alpha_i}$, and are naturally labeled them by the simple roots of $SU(2k+1)$, with α_0 labeling the affine root. Note that the Cartan divisors in Y_4 can be thought of as fibering \mathbb{P}^1 s in the resolved fiber over the surface S_2 . Computing the intersection of the Cartan divisors with the exceptional \mathbb{P}^1 s in the resolved fiber reproduces the (extended) Cartan matrix of the group $SU(2k+1)$, as depicted in figure 1. To each Cartan divisor, we can associate a Cartan charge vector computed from the intersections with the \mathbb{P}^1 s given by

Cartan Divisor	j th Cartan charge
$D_{-\alpha_0}$	$(-\alpha_0)_j = \delta_{j,1} + \delta_{j,2k} - 2\delta_{j,0}$
$D_{-\alpha_i}$	$(-\alpha_i)_j = \delta_{i,j+1} + \delta_{j,i+1} - 2\delta_{i,j}$
$D_{-\alpha_{2k}}$	$(-\alpha_{2k})_j = \delta_{j,2k-1} + \delta_{j,0} - 2\delta_{j,2k}$

(3.10)

3.1.2 Codimension 2

Along higher codimension loci of enhanced symmetry, which we summarized in table 2.13, the exceptional divisors will split. The particular splitting has an interesting structure, and depends on the choice of small resolutions. There are two codimension 2 matter loci for $SU(2k+1)$: $b_1 = 0$ and $P = 0$. The vanishing order $O(\zeta_0^{2k+1})$ of the discriminant increases along these loci as follows

$$\begin{aligned}
b_1 = 0 : \quad \Delta|_{b_1=0} &= O(\zeta_0^{2k+3}) \\
P = 0 : \quad \Delta|_{P=0} &= O(\zeta_0^{2k+2}),
\end{aligned}
\tag{3.11}$$

which has a clear representation theoretic interpretation: the matter that is localized corresponds to decomposition of the adjoint of $SO(4k+2)$ and $SU(2k+2)$, respectively, with respect to $SU(2k+1)$. We confirm this from the intersection diagrams in codimension 2, which for A_n singularities give rise to D and A type Dynkin diagrams and the fibers have the correct multiplicities for Kodaira fibers.

Denote the matter surfaces by S_v , where v is the Cartan charge obtained by intersecting the surface with the Cartan divisors. Along $b_1 = 0$ the matter surfaces, which are irreducible components of the restriction of the Cartan divisors, are

Matter	Section	Equation in $Y_4 _{b_1=0}$	j th Cartan charge
$S_{-\alpha_0}$	ζ_0	$0 = y^2\delta_1 - x^3\zeta_1$	$\delta_{j,1} + \delta_{j,2k} - 2\delta_{j,0}$
$S_{-\alpha_1}$	ζ_1	$0 = \delta_1$	$\delta_{j,2} + \delta_{j,0} - 2\delta_{j,1}$
$S_{v_i} \quad i = 2, \dots, k-1$	ζ_i	$0 = \delta_i$	$\delta_{j,i+1} - \delta_{j,i} + \delta_{j,2k+2-i} - \delta_{j,2k+1-i}$
$S_{v_{k+i}} \quad i = 2, \dots, k-1$	ζ_i	$0 = \delta_{i-1}$	$\delta_{j,i-1} - \delta_{j,i} - \delta_{j,2k+2-i} + \delta_{j,2k+1-i}$
S_{v_k}	ζ_k	$0 = b_3\zeta_{k-1} + \delta_k$	$-\delta_{j,k} + \delta_{j,k+2}$
$S_{v_{2k}}$	δ_1	$0 = b_2\zeta_0 + \zeta_2\delta_2$	$\delta_{j,1} - \delta_{j,2k}$
$S_{-\alpha_{k+1}}$	δ_k	$0 = \delta_{k-1}b_3y - b_2x^2\zeta_k$ $-\delta_{k-1}\zeta_k(b_4x + b_6\delta_{k-1})$	$\delta_{j,k} + \delta_{j,k+2} - 2\delta_{j,k+1}$
$S_{v_{k+1}}$	ζ_k	$0 = \delta_{k-1}$	$\delta_{j,k-1} - \delta_{j,k} + \delta_{j,k+1} - \delta_{j,k+2}$

(3.12)

Put differently, as we pass to the codimension 2 locus $b_1 = 0$, the following Cartan divisors become reducible and split into matter surfaces

$$\begin{aligned}
D_{-\alpha_i} &\longrightarrow S_{v_i} + S_{v_{k+i}} \\
D_{-\alpha_{2k+1-i}} &\longrightarrow S_{v_i} + S_{v_{k+i+1}} \\
D_{-\alpha_{2k}} &\longrightarrow S_{v_1} + S_{v_{k+2}} + S_{v_{2k}} \\
D_{-\alpha_k} &\longrightarrow S_{v_k} + S_{v_{k+1}}.
\end{aligned}
\tag{3.13}$$

All other Cartan divisors remain irreducible. The splitting is depicted in figure 2. Note that this graph also allows to read off the multiplicities: the green nodes are the initial affine A_{2k} Dynkin diagram and the green lines indicate how the simple roots split (not their intersections). The multiplicities are in particular not always one. The intersection graph of the irreducible components labeled by weights⁴ is depicted in blue, and is given by the affine D_{2k+1} Dynkin diagram.

The Cartan charges can be identified with Dynkin labels of weights of representations of $SU(2k+1)$. Along the codimension 2 locus $b_1 = 0$, the weights are in the representations

$$\mathbf{k}(2\mathbf{k} + \mathbf{1}) = \Lambda^2 V \quad \text{and} \quad \overline{\mathbf{k}(2\mathbf{k} + \mathbf{1})} = \Lambda^2 \overline{V}, \tag{3.14}$$

where V is the fundamental $2\mathbf{k} + \mathbf{1}$ of $SU(2k+1)$. Denote the highest weights by

$$\mu_{\Lambda^2 V} = (0, 1, 0, \dots, 0), \quad \mu_{\Lambda^2 \overline{V}} = (0, \dots, 0, 1, 0). \tag{3.15}$$

⁴Fiberizing those curves over the codimension 2 locus yields the matter surfaces S_v .

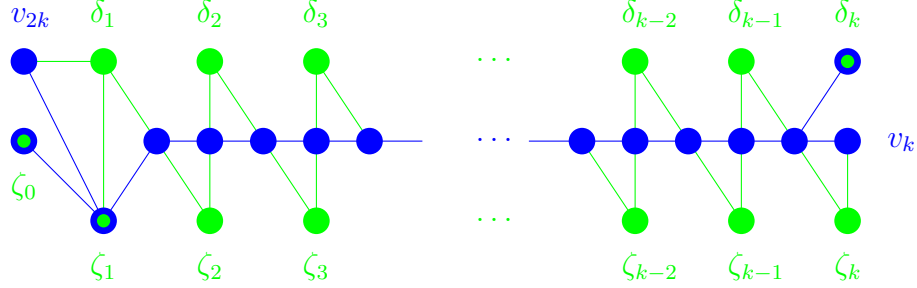


Figure 2: Splitting of the Cartan divisors $D_{-\alpha_i}$ along the codimension 2 locus $b_1 = 0$ for $SU(2k+1)$. \bullet are the simple roots of A_{2k} , green lines indicate how they split along $b_1 = 0$. \bullet are the irreducible components of the fiber over the codimension 2 locus, the blue lines give their intersection graph, which reproduces the affine D_{2k+1} . Curves that remain irreducible are bicolored.

then the weights carried by the matter surfaces in codimension 2 are

Matter Surface	Weights
$S_{-\alpha_0}$	$-\alpha_0$
$S_{-\alpha_1}$	$-\alpha_1$
S_{v_i}	$\mu_{\Lambda^2 V} - \left(\alpha_1 + 2 \sum_{\ell=1}^i \alpha_\ell + \sum_{\ell=i+1}^{2k+1-i} \alpha_\ell \right)$
$S_{v_{k+i}}$	$\mu_{\Lambda^2 \bar{V}} - \left(\alpha_{2k} + 2 \sum_{\ell=2k-i+2}^{2k-1} \alpha_{2k-\ell} + \sum_{\ell=i+1}^{2k+1-i} \alpha_\ell \right)$
$S_{v_{k+1}}$	$\mu_{\Lambda^2 \bar{V}} - \left(\alpha_{2k} + \alpha_k + \alpha_{k+1} + 2 \sum_{\ell=k+2}^{2k-1} \alpha_\ell \right)$
S_{v_k}	$\mu_{\Lambda^2 V} - \left(\alpha_1 + \alpha_{k+1} + 2 \sum_{\ell=2}^k \alpha_\ell \right)$
$S_{v_{2k}}$	$\mu_{\Lambda^2 V} - \left(\sum_{\ell=2}^{2k} \alpha_\ell \right)$
$S_{-\alpha_{k+1}}$	$-\alpha_{k+1}$

(3.16)

The second codimension 2 locus, where the discriminant has increased vanishing order, is along

$$P = b_2 b_3^2 + b_1^2 b_6 - b_1 b_3 b_4 = 0. \quad (3.17)$$

The geometry restricted to $P = 0^5$

$Y_4 \cap (P = 0)$:

$$\begin{aligned} & b_1^2 y^2 B(\delta) \delta_k - b_1^2 x^3 B(\zeta) A(\zeta \delta) \zeta_k^k \delta_k^{k-1} + b_1^3 x y - b_1^2 b_2 x^2 \zeta_0 B(\zeta) \zeta_k + b_1^2 b_3 y \zeta_0^k B(\zeta \delta) C(\zeta \delta) \\ & - b_1^2 b_4 x \zeta_0^{k+1} B(\zeta^2 \delta) C(\zeta \delta) \zeta_k - (b_1 b_3 b_4 - b_2 b_3^2) \zeta_0^{2k+1} B(\zeta^3 \delta^2) C(\zeta^2 \delta^2) \zeta_k = 0. \end{aligned} \quad (3.18)$$

⁵At this locus we are explicitly not interested in the situation where $b_1 = 0$ since $b_1 = P = 0$ provides a greater enhancement of the vanishing order. For this reason we can scale the Tate form by b_1^2 without loss of generality and do the substitution for the $P = 0$ condition by replacing the $b_1^2 b_6$ term.

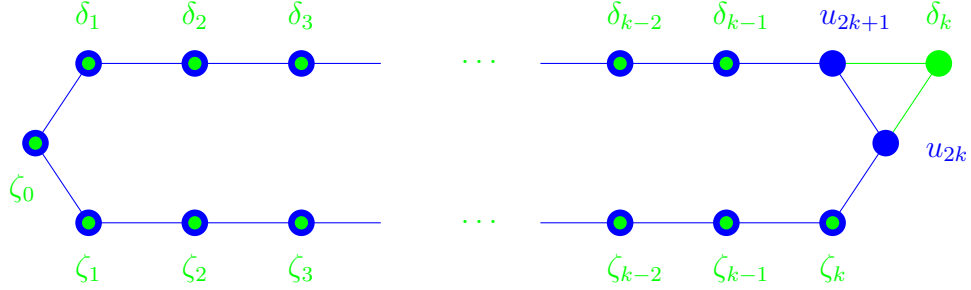


Figure 3: Codimension 2 splitting for $SU(2k + 1)$ along $P = 0$. Bicolored nodes correspond to Cartan divisors that remain irreducible when passing to $P = 0$. The only one that splits is δ_k , which splits according to the green lines. The irreducible components of the fibers are blue and their intersections are given by the blue lines. Bicolored nodes remain irreducible.

The exceptional divisors are again obtained by setting the sections ζ_i and δ_i to zero. These all remain irreducible, except for $\delta_k = 0$, which restricted to the $P = 0$ locus has two components

$$\delta_k = 0 : \quad (b_1x + b_3\delta_{k-1})(b_1^2y + b_2b_3\zeta_k\delta_{k-1} - b_1\zeta_k(b_2x + b_4\delta_{k-1})) = 0. \quad (3.19)$$

Let us first write out the irreducible matter surfaces, i.e. the irreducible components of the exceptional divisors restricted to $P = 0$

Matter Surface	Section	Equation in $Y_4 _{P=0}$
$S_{-\alpha_0}$	ζ_0	$0 = y^2\delta_1 - x^3\zeta_1 + b_1xy$
$S_{-\alpha_1}$	ζ_1	$0 = \delta_1 + b_1x$
$S_{-\alpha_i} \quad i = 2, \dots, k-1$	ζ_i	$0 = \delta_{i-1}\delta_i + b_1x$
$S_{-\alpha_k}$	ζ_k	$0 = \delta_{k-1}\delta_k + b_1x + b_3\zeta_{k-1}\delta_{k-1}$
$S_{-\alpha_{2k}}$	δ_1	$0 = b_1y - b_2\zeta_0\zeta_1\zeta_2 - \zeta_1\zeta_2^2\delta_2$
$S_{-\alpha_{2k+1-i}} \quad i = 2, \dots, k-1$	δ_i	$0 = b_1y - b_2\zeta_i\zeta_{i+1}$
$S_{u_{2k}}$	δ_k	$0 = b_1x + b_3\delta_{k-1}$
$S_{u_{2k+1}}$	δ_k	$0 = b_1^2y + b_2b_3\zeta_k\delta_{k-1} - b_1\zeta_k(b_2x + b_4\delta_{k-1})$

The Cartan charges, worked out from the intersection relations in appendix C.1, are

Matter Surface	j th Cartan Charge	Weights
$S_{-\alpha_0}$	$\delta_{j,1} + \delta_{j,2k} - 2\delta_{j,0}$	$-\alpha_0$
$S_{-\alpha_1}$	$\delta_{j,2} + \delta_{j,0} - 2\delta_{j,1}$	$-\alpha_1$
$S_{-\alpha_i} \quad i = 2, \dots, k-1$	$\delta_{j,i+1} + \delta_{j,i-1} - 2\delta_{j,i}$	$-\alpha_i$
$S_{-\alpha_k}$	$\delta_{j,k+1} + \delta_{j,k-1} - 2\delta_{j,k}$	$-\alpha_k$
$S_{-\alpha_{2k}}$	$\delta_{j,0} + \delta_{j,2k-1} - 2\delta_{j,2k}$	$-\alpha_{2k}$
$S_{-\alpha_{k+i}} \quad i = 2, \dots, k-1$	$\delta_{j,k+i+1} + \delta_{j,k+i-1} - 2\delta_{j,k+i}$	$-\alpha_{k+i}$
$S_{u_{2k}}$	$\delta_{j,k} - \delta_{j,k+1}$	$\mu_{\nabla} - \sum_{\ell=k+1}^{2k} \alpha_\ell$
$S_{u_{2k+1}}$	$\delta_{j,k+2} - \delta_{j,k+1}$	$\mu_V - \sum_{\ell=k+1}^{2k} \alpha_\ell$

These Cartan charges either correspond to roots, or to weights of the (anti-)fundamental representation of $SU(2k+1)$

$$(\mathbf{2k+1}) = V \quad \text{and} \quad \overline{(\mathbf{2k+1})} = \overline{V}, \quad (3.22)$$

whose highest weights we will label by

$$\mu_V = (1, 0, \dots, 0), \quad \mu_{\overline{V}} = (0, \dots, 0, 1). \quad (3.23)$$

In summary, along the matter locus $P = 0$ the Cartan divisor $\delta_k = 0$ splits as in (3.19), into two matter surfaces associated to the (anti-) fundamental representation

$$D_{-\alpha_{k+1}} \longrightarrow S_{u_{2k}} + S_{u_{2k+1}}. \quad (3.24)$$

The structure of the fiber along $P = 0$ is depicted in figure 3. Note that in this case the fiber is of Kodaira type corresponding to an $SU(2k+2)$ affine Dynkin diagram, with the correct multiplicities. The fact that the representations, which arise here is the (anti-) fundamental one is in accord with decomposing the adjoint of $SU(2k+2)$ with respect to $SU(2k+1)$.

3.1.3 Codimension 3

Finally we discuss the codimension three loci. These occur along

$$b_1 = b_3 = 0, \quad P = Q = 0, \quad b_1 = b_2 = 0. \quad (3.25)$$

It is useful to follow the $b_1 = 0$ matter surfaces (i.e. codimension two fibers) through to vanishing b_2 or b_3 . Along these loci, the vanishing order of the discriminant increases as follows

$$\begin{aligned} b_1 = b_3 = 0 : \quad \Delta|_{b_1=b_3=0} &= O(\zeta_0^{2k+4}) \\ P = Q = 0 : \quad \Delta|_{P=Q=0} &= O(\zeta_0^{3k}) \\ b_1 = b_2 = 0 : \quad \Delta|_{b_1=b_2=0} &= O(\zeta_0^{3k+3}), \end{aligned} \quad (3.26)$$

where the first locus will be shown to have a simple interpretation in terms of a local enhancement to $SO(2k+4)$, which will be clear from the intersections of the fibers. The case $b_1 = b_2 = 0$ is more subtle, as the singularity is non-minimal and the intersections are not of a standard Dynkin type. These generalize, what in the low k examples, such as $SU(5)$, corresponds to exceptional enhancements. We will discuss them in the next subsection.

Consider first $b_1 = b_3 = 0$, along which the equation for the matter surface $S_{-\alpha_{k+1}}$ becomes reducible

$$S_{-\alpha_{k+1}} \cdot [b_3] : \quad b_1 = b_3 = \delta_k = \zeta_k(\delta_{k-1}(b_4x + b_6\delta_{k-1}) + b_2x^2) = 0. \quad (3.27)$$

One component is precisely $\Sigma_{v_k} = S_{v_k} \cdot [b_3]$, and the other component is a quadratic whose roots will be denoted by $\Sigma_{u_{2k}}^{(i)}$, which have j th Cartan charge $(\delta_{j,k} - \delta_{j,k+1})$. Note that these two components do not intersect, as depicted in figure 4. In summary, the splitting is

$$S_{-\alpha_{k+1}} \longrightarrow \Sigma_{v_k} + \Sigma_{u_{2k}}^{(1)} + \Sigma_{u_{2k}}^{(2)}, \quad (3.28)$$

where we use the notation $\Sigma_v = S_v \cdot [b_3]$. Note that the weight corresponding to u_{2k} is in the anti-fundamental representation

$$S_{u_{2k}} : \quad \mu_{\bar{V}} - \left(\sum_{\ell=k+1}^{2k} \alpha_\ell \right). \quad (3.29)$$

Put differently, (3.28) implies that three matter surfaces with weights v_k and two with u_{2k} become homologous to each other, consistent with the group theoretic coupling

$$b_1 = b_3 = 0 : \quad \bar{V} \otimes \bar{V} \otimes \Lambda^2 V. \quad (3.30)$$

Next consider the codimension 3 locus $P = Q = 0$, see (2.13). These conditions are equivalent to

$$P = Q = 0 : \quad \begin{aligned} b_1^2 b_6 - b_1 b_3 b_4 + b_2 b_3^2 &= 0 \\ b_1 b_4 - 2b_2 b_3 &= 0. \end{aligned} \quad (3.31)$$

We can apply the second equation to rewrite the Tate form (3.18), where $P = 0$ was already implemented. In particular, using the second equation on (3.19), we see that along $P = Q = 0$ the two irreducible components of $\delta_k = P = 0$, $S_{u_{2k}}$ and $S_{u_{2k+1}}$ intersect, and the equation for δ_k can be rewritten in an unfactored form along $P = Q = 0$. This is consistent with the $S_{u_{2k}}$ and $S_{u_{2k+1}}$ being (anti-) fundamental representations, forming a coupling with a singlet along $P = Q = 0$

$$P = Q = 0 : \quad V \otimes \bar{V} \otimes \mathbf{1}. \quad (3.32)$$

3.1.4 Non-minimal Loci

Finally, consider the codimension 3 locus $b_1 = b_2 = 0$. For $k = 2$ for instance this is the locus corresponding to an “exceptional enhancement” to E_6 . However, observe that, from table 2.13, for general values of k the singularity becomes non-minimal, i.e. along $b_1 = b_2 = 0$ the sections f and g defined in (2.11) of the Weierstrass form have vanishing orders

$$\begin{aligned} f|_{b_1=b_2=0} &= O(\zeta_0^4) \\ g|_{b_1=b_2=0} &= O(\zeta_0^6) \\ \Delta|_{b_1=b_2=0} &= O(\zeta_0^{12}). \end{aligned} \quad (3.33)$$

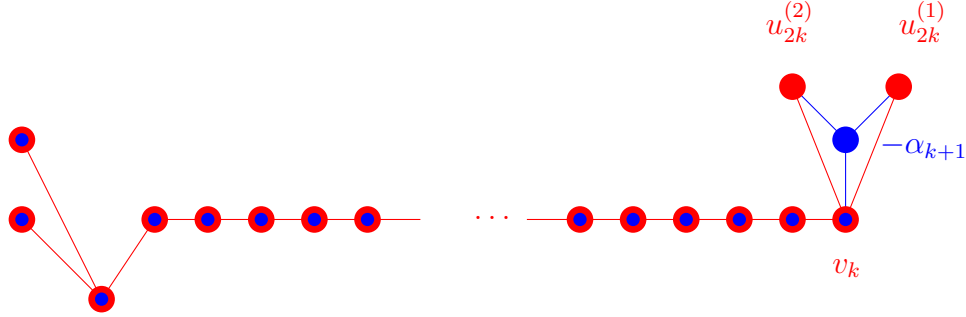


Figure 4: Codimension 3 splitting along $b_1 = b_3 = 0$ of the $b_1 = 0$ codimension 2 locus for $SU(2k + 1)$. The only node that splits is α_{k+1} , which has three components $u_{2k}^{(1,2)}$ and v_k . The irreducible components in codimension 3 are \bullet or bi-colored, and the red lines indicate the intersections of these. Bicolored nodes remain irreducible.

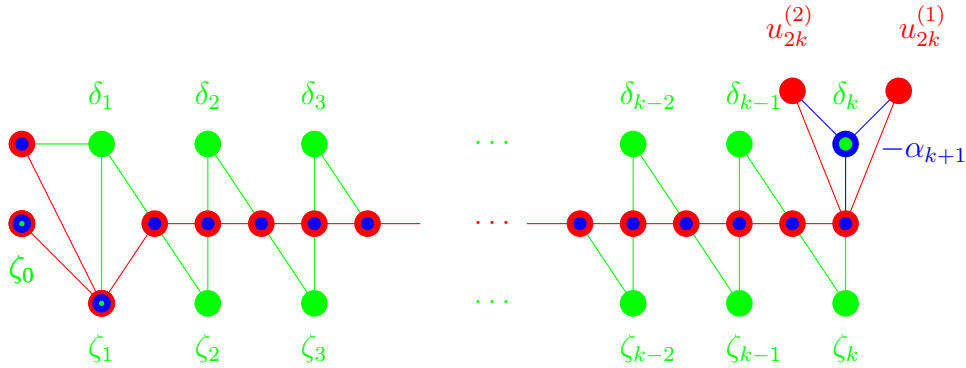


Figure 5: Summary graph: splitting along $b_1 = b_3 = 0$ for $SU(2k + 1)$. \bullet are initial Cartan divisors, green lines indicate how they split in codimension 2 along $b_1 = 0$ into \bullet . Blue lines indicate the splitting of these when passing to $b_3 = 0$. \bullet are the irreducible components in codimension 3, and red lines indicate their intersections.

One way to see this explicitly is to consider the resolved Tate form (3.7) intersected with $b_1 = b_2 = 0$

$$\begin{aligned} & \tilde{T}_{SU(2k+1)}|_{b_1=b_2=0} : \\ & \delta_2 \cdots \delta_{k-1} \left(y^2 \delta_1 \delta_k - x^3 \left(\prod_{i=3}^{k-2} \delta_i^{i-2} \right) B(\zeta) A(\zeta) \zeta_k^k \delta_k^{k-1} + b_3 y \zeta_0^k \delta_1 B(\zeta) C(\zeta \delta) \right. \\ & \quad \left. - b_4 x \zeta_0^{k+1} \delta_1 B(\zeta^2) C(\zeta \delta) \zeta_k - b_6 \zeta_0^{2k+1} \delta_1 B(\zeta^3 \delta) C(\zeta^2 \delta^2) \zeta_k \right) = 0, \end{aligned} \quad (3.34)$$

which makes it clear that the surfaces corresponding to

$$\mathfrak{N}_i : \quad b_1 = b_2 = \delta_i = 0, \quad i = 2, \dots, k-1, \quad (3.35)$$

are completely contained inside the fiber! This is to be expected for a non-minimal singularity, and indicates the breakdown of a purely field theoretic description of the effective theory. The same kind of non-minimal codimension 3 locus was analyzed in [25] for E_7 .

Nevertheless, we dare to study these codimension 3 loci. One open question is to understand the effective theory at such points. In the present paper, our main concern is the structure of the fibers. Our philosophy was so far to follow the codimension d loci to the codimension $d+1$ loci and understand how irreducible components become reducible. As we shall see, following the matter curves at $b_1 = 0$ to the point $b_1 = b_2 = 0$, they split in a fashion that is consistent with the generation of Yukawa couplings for the matter representations at hand. We can then ask what the intersections of the irreducible components of the fibers are. For this we provide two ways to depict them: the first type of graph is similar to the ones we obtained from the splitting along minimal singular loci, showing the irreducible components of the fiber and their intersections – without including the surfaces (3.35), these are depicted in figures 6 and 7. For the non-minimal case it makes sense to also depict the surface components (3.35), and how they intersect the remainder of the fiber, as shown in figure 8.

As discussed, let us consider the splitting of the $b_1 = 0$ matter surfaces along the locus $b_1 = b_2 = 0$. There are two matter surfaces that split

$$\begin{aligned} S_{v_{2k}} \cdot [b_2] : & \quad \delta_1 = \delta_2 \zeta_2 = 0 \\ S_{-\alpha_{k+1}} \cdot [b_2] : & \quad \delta_k = \delta_{k-1} (b_3 y - \zeta_k (b_4 x + b_6 \delta_{k-1})) = 0. \end{aligned} \quad (3.36)$$

For the first one it is clear that one of the components is the curve obtained by restricting $S_{v_{k+2}}$, $S_{v_{k+2}} \cdot [b_2] \equiv \Sigma_{v_{k+2}}$, so that

$$S_{v_{2k}} \longrightarrow \Sigma_{v_{k+2}} + \Sigma_{w'}, \quad (3.37)$$

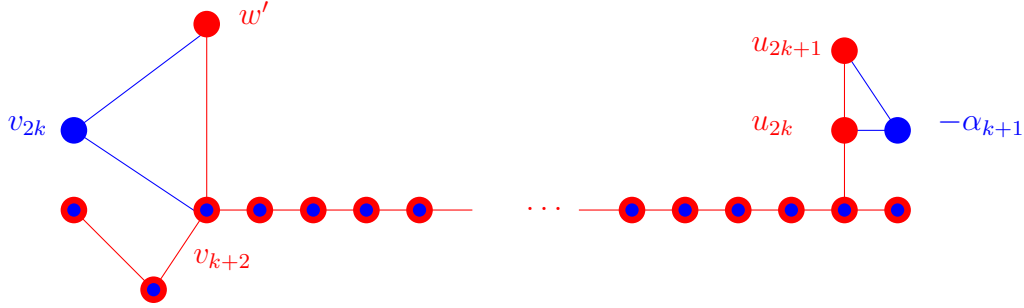


Figure 6: Splitting along the $b_1 = b_2 = 0$ codimension 3 locus for $SU(2k + 1)$. \bullet represent the fiber components along the matter locus $b_1 = 0$. Passing to the locus $b_1 = b_2 = 0$ two of the fiber components split, as indicated by the blue lines. \bullet labels the irreducible fiber components in codimension 3, and the red lines indicate their intersections, which are clearly not of Kodaira type. The splittings are explained in (3.37, 3.40).

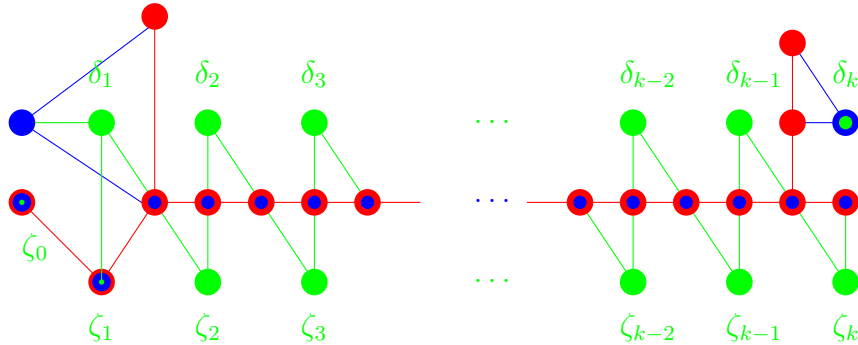


Figure 7: The complete splitting process for $SU(2k + 1)$ to get from the irreducible fiber components labeled by roots \bullet in codimension 1, to the irreducible fiber components in codimension 2, i.e. matter \bullet along $b_1 = 0$ to the ones along codimension 3, i.e. the Yukawa \bullet locus $b_1 = b_2 = 0$. Green and blue lines indicate how the fibers split going from codimension 1 to 2 to 3, whereas red lines indicate the intersections in codimension 3.

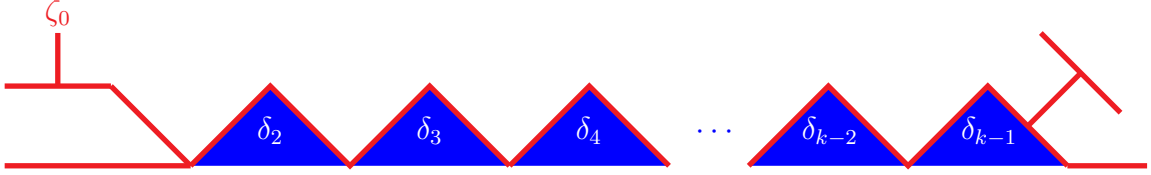


Figure 8: Fiber along the codimension 3 locus $b_1 = b_2 = 0$. Here, the red line is the dual graph to the intersection graph in figure 6. The blue triangles are the 1 dimension higher (i.e. surface) components \aleph_i given by the vanishing of δ_i as in (3.35) that enter the fiber along this non-minimal locus. The two edges that are part of the red intersection graph arise from the intersection of the components \aleph_i with ζ_i and ζ_{i+1} .

where the Cartan charge of the additional component is

$$(w')_j = \delta_{j,2} - \delta_{j,2k-1}, \quad (3.38)$$

which is a weight in the representation $\Lambda^4 V$. The corresponding Yukawa coupling corresponds to

$$b_1 = b_2 = 0 : \quad \Lambda^2 \bar{V} \otimes \Lambda^2 \bar{V} \otimes \Lambda^4 V. \quad (3.39)$$

Finally, $S_{-\alpha_{k+1}}$, which as we discussed descends to codimension 2 from a root, splits as

$$S_{-\alpha_{k+1}} \longrightarrow \Sigma_{u_{2k}} + \Sigma_{u_{2k+1}}, \quad (3.40)$$

where the charges are computed from the intersection relations in the appendix

$$\begin{aligned} u_{2k} &= -(-\delta_{j,k} + \delta_{j,k+1}) = \mu_{\bar{V}} - \left(\sum_{\ell=k+1}^{2k} \alpha_\ell \right) \\ u_{2k+1} &= +(-\delta_{j,k+1} + \delta_{j,k+2}) = \mu_V - \left(\sum_{\ell=1}^k \alpha_\ell \right). \end{aligned} \quad (3.41)$$

This corresponds to a root splitting into a fundamental and anti-fundamental representation. The splittings are depicted in figure 6.

As we explained earlier, these diagrams are in no way expected to be a full description of the fiber. Their intersection topology is not of Dynkin type and seem to be a generalization of exceptional Dynkin diagrams to higher rank. The algebras associated to this are not

finite dimensional Lie algebras. We do not expect a simple Higgs mechanism, like in the minimal case, to describe the effective theory. As a final point, it may be useful to depict the intersections including the additional surface components (3.35), which can be seen in figure 8.

We should add a remark about $SU(5)$, which is discussed in detail in appendix B, which like many low k cases is not quite following the splitting pattern as given in figure 6. In particular, the fiber component $\delta_1 = \delta_2 = 0$ joins with the components that split off from $\delta_{k=2} = 0$, and give rise to the intersections of an E_6 diagram. Note however, that no affine E_6 fiber is observed, as already well known from [8, 9]. This Yukawa is consistent with w' in (3.37) being a weight of $\Lambda^4 V \cong V$ in the case of $SU(5)$.

3.2 $SU(2k)$

The procedure for $SU(2k)$ is quite similar to the $SU(2k + 1)$, however, as the fibers split slightly differently, we will give a brief discussion of the codimension 1, 2, and 3 structure. The Tate form for $SU(2k)$ is

$$0 = y^2 - x^3 + b_1xy - b_2x^2\zeta_0 + b_3y\zeta_0^k - b_4x\zeta_0^k - b_6\zeta_0^{2k}. \quad (3.42)$$

The ordered set of resolutions required to resolve this geometry in all codimensions, is

$$\begin{aligned} (x, y, \zeta_i; \zeta_{i+1}) & \quad i = 0, \dots, k-1 \\ (y, \zeta_i; \delta_i) & \quad i = 1, \dots, k-1, \end{aligned} \quad (3.43)$$

Note that after the blowups the geometry takes again the form (3.5) like for $SU(2k + 1)$, and we applied the same class of small resolutions. The fully resolved geometry, after proper transformation is then

$$\begin{aligned} y^2B(\delta) - x^3B(\zeta)A(\zeta\delta)\zeta_k^k + b_1xy - b_2x^2\zeta_0B(\zeta)\zeta_k + b_3y\zeta_0^kB(\zeta\delta)C(\zeta\delta) \\ - b_4x\zeta_0^kB(\zeta)C(\zeta\delta) - b_6\zeta_0^{2k}B(\zeta^2\delta)C(\zeta^2\delta^2) = 0. \end{aligned} \quad (3.44)$$

The projectivity relations are again included in appendix C.1.

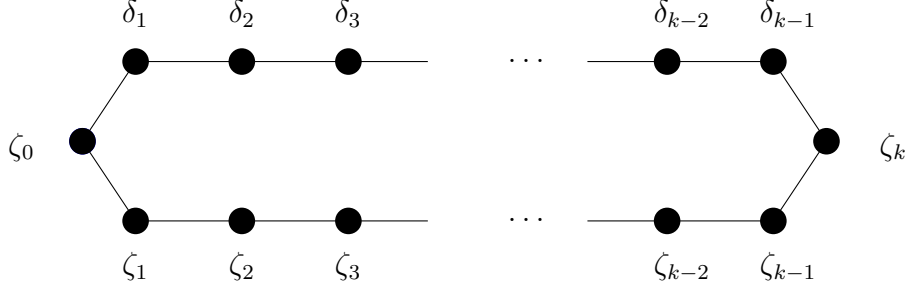


Figure 9: Intersection graph of the fibers in codimension 1, resulting in the affine A_{2k-1} Dynkin diagram. The labels are explained in (3.45).

3.2.1 Codimension 1

The irreducible exceptional divisors are

Divisor	Section	Equation in Y_4
$D_{-\alpha_0}$	ζ_0	$0 = y^2\delta_1 - x^3\zeta_1 + b_1xy$
$D_{-\alpha_1}$	ζ_1	$0 = \delta_1 + b_1x$
$D_{-\alpha_i}$ $i = 2, \dots, k-1$	ζ_i	$0 = \delta_{i-1}\delta_i + b_1x$
$D_{-\alpha_k}$	ζ_k	$0 = y^2\delta_{k-1} + b_1xy + b_3y\zeta_{k-1}\delta_{k-1} - b_4x\zeta_{k-1} - b_6\zeta_{k-1}^2\delta_{k-1}$
$D_{-\alpha_{k+1}}$	δ_{k-1}	$0 = b_1y - b_2\zeta_k\zeta_{k-1} - b_4\zeta_{k-1}\delta_{k-2}$
$D_{-\alpha_{2k-i}}$ $i = 2, \dots, k-2$	δ_i	$0 = b_1y - b_2\zeta_i\zeta_{i+1}$
$D_{-\alpha_{2k-1}}$	δ_1	$0 = b_1y - b_2\zeta_0\zeta_1\zeta_2 - \zeta_1\zeta_2^2\delta_2$

(3.45)

The intersections follow from the relations in appendix C.1, and reproduce the affine $SU(2k)$ Dynkin diagram, where ζ_0 corresponds to the divisors $D_{-\alpha_0}$. The intersection of the sections ζ_i and δ_i are as in figure 9.

3.2.2 Codimension 2

The codimension 2 enhancements are along $b_1 = 0$ or $P = b_1b_3b_4 + b_4^2 - b_1^2b_6 = 0$. The vanishing of the discriminant, which is generically $O(\zeta_0^{2k})$, increases as follows

$$\begin{aligned}
 b_1 = 0 : \quad \Delta|_{b_1=0} &= O(\zeta_0^{2k+2}) \\
 P = 0 : \quad \Delta|_{P=0} &= O(\zeta_0^{2k+1}),
 \end{aligned}
 \tag{3.46}$$

corresponding to a local enhancement to $SO(4k)$ and $SU(2k+1)$, as will be demonstrated from the intersection graph of the fiber in codimension 2.

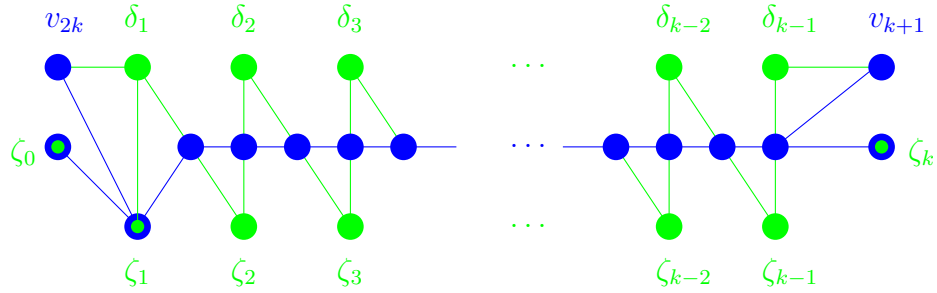


Figure 10: Splitting of the Cartan divisors $D_{-\alpha_i}$ along the codimension 2 locus $b_1 = 0$ for $SU(2k)$. \bullet are the simple roots of A_{2k-1} , green lines indicate how they split along $b_1 = 0$. \bullet are the irreducible components of the fiber over the codimension 2 locus, the blue lines give their intersection graph. Curves that remain irreducible are bicolored. The intersections and multiplicities are those of a D_{2k} Kodaira fiber.

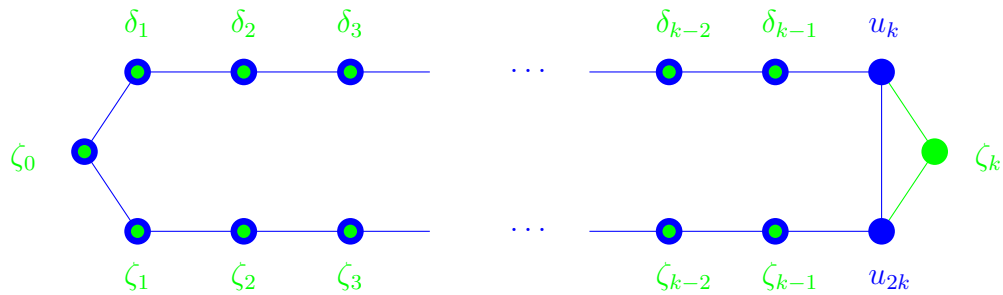


Figure 11: Intersection graph of the fibers for $SU(2k)$ in codimension 2 along $P = 0$. The root α_k splits into weights u_k and u_{2k} . The intersections of the irreducible components \bullet in codimension 2 give the affine A_{2k} Dynkin diagram.

The irreducible divisors for the $b_1 = 0$ matter locus are

Matter	Section	Equation in $Y_4 _{b_1=0}$	j th Cartan charge
$S_{-\alpha_0}$	ζ_0	$0 = y^2\delta_1 - x^3\zeta_1$	$\delta_{j,1} + \delta_{j,2k-1} - 2\delta_{j,0}$
$S_{-\alpha_1}$	ζ_1	$0 = \delta_1$	$\delta_{j,0} + \delta_{j,2} - 2\delta_{j,1}$
S_{v_i} $i = 2, \dots, k-1$	ζ_i	$0 = \delta_i$	$\delta_{j,i+1} - \delta_{j,i} + \delta_{j,2k+1-i} - \delta_{j,2k-i}$
$S_{-\alpha_k}$	ζ_k	$0 = \delta_{k-1}y^2 - b_4x\zeta_{k-1} + \delta_{k-1}(b_3y\zeta_{k-1} - b_6\zeta_{k-1}^2)$	$\delta_{j,k+1} + \delta_{j,k-1} - 2\delta_{j,k}$
$S_{v_{k+1}}$	δ_{k-1}	$0 = b_2\zeta_k + b_4\delta_{k-2}$	$\delta_{j,k-1} - \delta_{j,k+1}$
$S_{v_{k+i}}$ $i = 2, \dots, k-1$	ζ_i	$0 = \delta_{i-1}$	$\delta_{j,i-1} - \delta_{j,i} + \delta_{j,2k-i} - \delta_{2k+1-i}$
$S_{v_{2k}}$	δ_1	$0 = b_2\zeta_0 + \zeta_2\delta_2$	$\delta_{j,1} - \delta_{j,2k-1}$

(3.47)

The weights that appear are either of the $\Lambda^2 V$ or $\Lambda^2 \bar{V}$, where $V = \mathbf{2k}$ is the fundamental representation. In summary, along $b_1 = 0$ the following Cartans split

$$\begin{aligned}
D_{-\alpha_i} &\longrightarrow S_{v_i} + S_{v_{k+i}} \\
D_{-\alpha_{2k-i}} &\longrightarrow S_{v_i} + S_{v_{k+i+1}} \\
D_{-\alpha_{k+1}} &\longrightarrow S_{v_{k-1}} + S_{v_{k+1}} \\
D_{-\alpha_{2k-1}} &\longrightarrow S_{v_1} + S_{v_{k+2}} + S_{v_{2k}}.
\end{aligned}
\tag{3.48}$$

The other codimension 2 locus at which the discriminant enhances is

$$P = b_1 b_3 b_4 + b_4^2 - b_1^2 b_6 = 0, \tag{3.49}$$

without b_1 vanishing. Again, we can, without loss of generality, scale the Tate form by b_1^2 and perform the $P = 0$ substitution, resulting in

$$\begin{aligned}
&b_1^2 y^2 B(\delta) - b_1^2 x^3 B(\zeta) A(\zeta\delta)\zeta_k^k + b_1^3 xy - b_1^2 b_2 x^2 \zeta_0 B(\zeta)\zeta_k + b_1^2 b_3 y \zeta_0^k B(\zeta\delta) C(\zeta\delta) \\
&- b_1^2 b_4 x \zeta_0^k B(\zeta) C(\zeta\delta) - (b_1 b_3 b_4 + b_4^2) \zeta_0^2 B(\zeta^2\delta) C(\zeta^2\delta^2) = 0.
\end{aligned}
\tag{3.50}$$

The irreducible exceptional divisors along $P = 0$ are

Matter	Section	Equation in $Y_4 _{P=0}$	j th Cartan charge
$S_{-\alpha_0}$	ζ_0	$0 = y^2\delta_1 - x^3\zeta_1 + b_1xy$	$\delta_{j,1} + \delta_{j,2k-1} - 2\delta_{j,0}$
$S_{-\alpha_1}$	ζ_1	$0 = \delta_1 + b_1x$	$\delta_{j,0} + \delta_{j,2} - 2\delta_{j,1}$
$S_{-\alpha_i}$, $i = 2, \dots, k-1$	ζ_i	$0 = \delta_{i-1}\delta_i + b_1x$	$\delta_{j,i-1} + \delta_{j,i+1} - 2\delta_{j,i}$
S_{u_k}	ζ_k	$0 = b_1y - b_4\zeta_{k-1}$	$-\delta_{j,k} + \delta_{j,k+1}$
$S_{-\alpha_{2k-1}}$	δ_1	$0 = b_1y - b_2\zeta_0\zeta_1\zeta_2 - \zeta_1\zeta_2^2\delta_2$	$\delta_{j,0} + \delta_{j,2k-2} - 2\delta_{j,2k-1}$
$S_{-\alpha_{2k-i}}$, $i = 2, \dots, k-2$	δ_i	$0 = b_1y - b_2\zeta_i\zeta_{i+1}$	$\delta_{j,2k-i-1} + \delta_{j,2k-i+1} - 2\delta_{j,2k-i}$
$S_{-\alpha_{k+1}}$	δ_{k-1}	$0 = b_1y - b_2\zeta_{k-1}\zeta_k - b_4\zeta_{k-1}\delta_{k-2}$	$\delta_{j,k} + \delta_{j,k+2} - 2\delta_{j,k+1}$
$S_{u_{2k}}$	ζ_k	$0 = b_1^2x + b_1y\delta_{k-1} + \delta_{k-1}(b_4\zeta_{k-1} + b_1b_3\zeta_{k-1})$	$\delta_{j,k-1} - \delta_{j,k}$

(3.51)

Using the calculations in the appendix C.1 the Cartan charges of the components that split along the locus $P = 0$ are found to be of the fundamental representation $V = \mathbf{2k}$. In particular, the Cartan divisor associated to the vanishing of the section ζ_k becomes reducible and splits

$$D_{-\alpha_k} \longrightarrow S_{u_k} + S_{u_{2k}}. \quad (3.52)$$

3.2.3 Codimension 3

Along the codimension 3 loci, as in the $SU(2k+1)$ case, only a few of the matter surfaces split further. The vanishing order of the discriminant increases to

$$\begin{aligned} b_1 = b_4 = 0 : \quad \Delta|_{b_1=b_3=0} &= O(\zeta_0^{2k+3}) \\ P = b_2 = 0 : \quad \Delta|_{P=b_2=0} &= O(\zeta_0^{3k}) \\ b_1 = b_2 = 0 : \quad \Delta|_{b_1=b_2=0} &= O(\zeta_0^{3k}). \end{aligned} \quad (3.53)$$

Along $b_1 = b_4 = 0$ note that $P = 0$ and indeed, we observe the following matter splitting

$$S_{-\alpha_k} \longrightarrow \Sigma_{v_{k+1}} + \Sigma_{u_k^{(1)}} + \Sigma_{u_k^{(2)}}, \quad (3.54)$$

where $u_k^{(i)}$ label the two curves, which are obtained as the two solutions of $\zeta_k = b_1 = b_4 = 0$, which both have charge u_k , and intersect along $\zeta_{k-1} = 0$. This is consistent with the Yukawa coupling

$$b_1 = b_4 = 0 : \quad \bar{V} \otimes \bar{V} \otimes \Lambda^2 V. \quad (3.55)$$

Figure 12 shows the intersection graph of the fiber along this codimension 3 locus, which is of affine D type, and the multiplicities which can be read off from the splitting of the nodes in codimension 2, are consistent with a Kodaira fiber. Similarly, $P = b_2 = 0$ yields the coupling

$$P = b_2 = 0 : \quad V \otimes \bar{V} \otimes \mathbf{1}. \quad (3.56)$$

Finally, there is the non-minimal locus $b_1 = b_2 = 0$ which is very similar to the $SU(2k+1)$ case. Again, new surface components enter the fiber, like the \aleph_i defined in (3.35). We again consider the splitting of the matter surfaces that we discussed along $b_1 = 0$. The matter surface that splits is $S_{v_{2k}}$, which corresponds to the $b_2\zeta_0 + \zeta_2\delta_2$ component of $\delta_1 = b_1 = 0$ inside Y_4 . Along $b_2 = 0$ this splits further into two components

$$S_{v_{2k}} \cdot [b_2] : \quad \delta_1 = \delta_2\zeta_2 = 0. \quad (3.57)$$

The Cartan charges are readily determined from the appendix C.1 and show that along $b_1 = b_2 = 0$ we generate a new component in the fiber, given by the curve

$$\Sigma_{w'} : \quad b_1 = b_2 = \delta_1 = \delta_2 = 0, \quad (3.58)$$

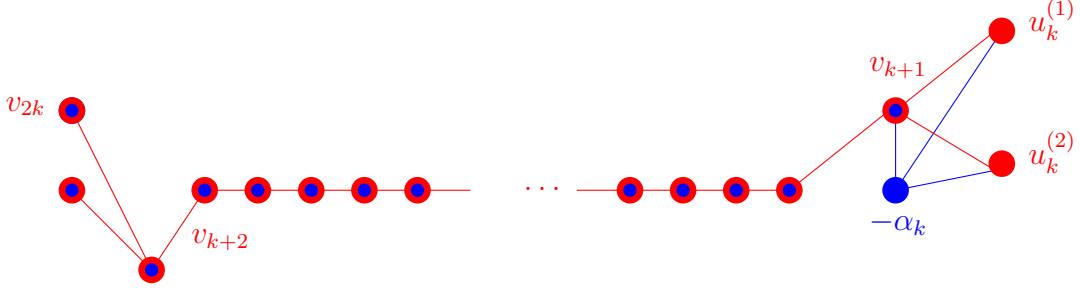


Figure 12: The fibers along the codimension 3 locus $b_1 = b_4 = 0$ for $SU(2k)$. The irreducible components in codimension 3 are depicted in terms of red and bicolored dots. Red lines give their intersections. The blue dots are the codimension 2 $b_1 = 0$ fibers, most of which remain irreducible (specified by the bicoloring), except for $-\alpha_k$ and the blue line indicates how $-\alpha_k$ splits. The intersections are of D type Kodaira. The multiplicities are read off from the splitting from one codimension lower.

the Cartan charges of which are

$$(w')_j = \delta_{j,2} - \delta_{2k-1,j}, \quad (3.59)$$

and is the Dynkin label of a weight of the $\Lambda^4 V$ representation. The splitting of $S_{v_{2k}}$ along $b_2 = 0$ is then

$$S_{v_{2k}} \longrightarrow \Sigma_{w'} + \Sigma_{v_{k+2}}, \quad (3.60)$$

where $\Sigma_{v_{k+2}} = S_{v_{k+2}} \cdot [b_2]$. The intersections of the irreducible components in the fiber are depicted in figures 13 and 14. This is consistent with the Yukawa coupling

$$b_1 = b_2 = 0 : \quad \Lambda^2 \bar{V} \otimes \Lambda^2 \bar{V} \otimes \Lambda^4 V, \quad (3.61)$$

keeping of course in mind the non-minimality of this locus.

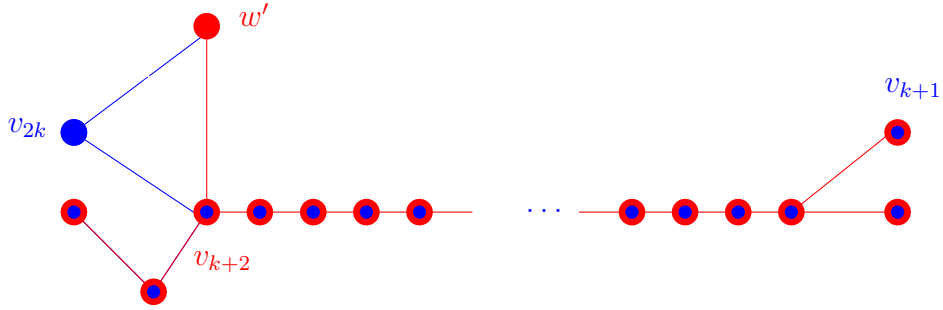


Figure 13: Codimension 3 locus $b_1 = b_2 = 0$ for $SU(2k)$. The irreducible components in codimension 3 are depicted in terms of red and bicolored dots. Red lines give their intersections. The blue dots are the codimension 2 $b_1 = 0$ fibers, most of which remain irreducible (specified by the bicoloring) and the blue line indicates how v_{2k} splits.

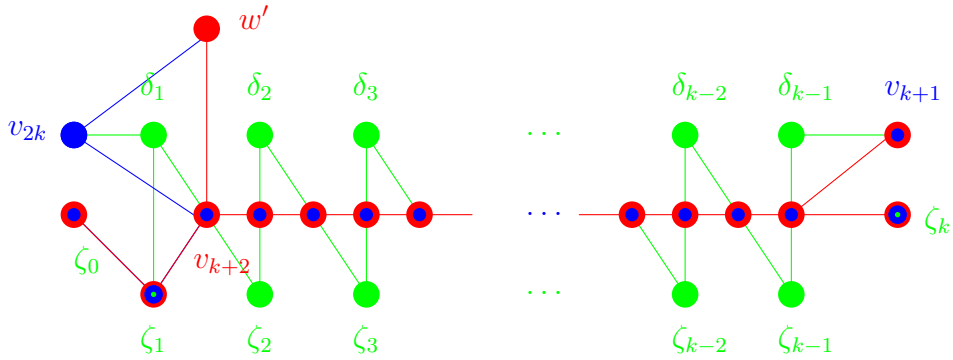


Figure 14: Summary of splittings for codimension 3 locus $b_1 = b_2 = 0$ for $SU(2k)$. Green dots are the fiber components in codimension 1, blue dots in codimension 2, and red in codimension 3. Red lines are the intersections in codimension 3, whereas blue and green lines indicate how the codimension d splits into the codimension $d + 1$ fibers. In particular tricolored nodes remain irreducible in all codimension.

4 Resolution of D Type Singularities

Similarly to the A_n case the D_n splits into odd and even n , and we shall study these in turn. The main difference to the $SU(n)$ case is that for $SO(2\ell)$ there are non-minimal loci in codimension 2.

4.1 $SO(4k + 2)$

The Tate form for an $SO(4k + 2)$ singularity above $\zeta_0 = 0$ is

$$y^2 - x^3 + b_1xy\zeta_0 - b_2x^2\zeta_0 + b_3y\zeta_0^k - b_4x\zeta_0^{k+1} - b_6\zeta_0^{2k+1} = 0. \quad (4.1)$$

The singularity in the fiber can be resolved in all codimensions using the notation of (2.17) by the following resolutions

$$\begin{aligned} (x, y, \zeta_i; \zeta_{i+1}) & \quad i = 0, \dots, k-1 \\ (y, \zeta_i; \delta_i) & \quad i = 1, \dots, k \\ (\zeta_i, \delta_i; \kappa_i) & \quad i = 1, \dots, k-1 \\ (\zeta_{i+1}, \delta_i; \epsilon_i) & \quad i = 1, \dots, k-1. \end{aligned} \quad (4.2)$$

Note that after the first two sets of resolutions, i.e. $(x, y, \zeta_i; \zeta_{i+1})$ and $(y, \zeta_i; \delta_i)$, the geometry takes the form

$$B(\delta)\delta_k U = B(\zeta)\zeta_k V, \quad (4.3)$$

which again makes the different choices of small resolutions explicit. We will, as for $SU(n)$, pick one such small resolution and leave the study of the network of flops to future work. The geometry obtained after applying all the resolutions (4.2) on the space defined by (4.1) is

$$\begin{aligned} & y^2 B(\delta)\delta_k - x^3 \zeta_k^k \delta_k^{k-1} B(\zeta\epsilon) A(\zeta\delta\kappa^2\epsilon^2) + b_1xy\zeta_0 B(\zeta\delta\kappa\epsilon)\zeta_k\delta_k \\ & - b_2x^2\zeta_0 B(\zeta)\zeta_k + b_3y\zeta_0^k B(\zeta\delta\kappa) C(\zeta\delta\kappa^2\epsilon^2) \\ & - b_4x\zeta_0^{k+1}\zeta_k B(\zeta^2\delta\kappa^2\epsilon) C(\zeta\delta\kappa^2\epsilon^2) - b_6\zeta_0^{2k+1}\zeta_k B(\zeta^3\delta^2\kappa^4\epsilon^2) C(\zeta^2\delta^2\kappa^4\epsilon^4) = 0, \end{aligned} \quad (4.4)$$

along with the projective relations between the sections that are documented in appendix C.2. We use the convenient shorthand (2.21).

4.1.1 Codimension 1

The projective relations in appendix C.2 allow us to write the equation in Y_4 corresponding to the vanishing of the exceptional sections in a particularly simple form. The irreducible

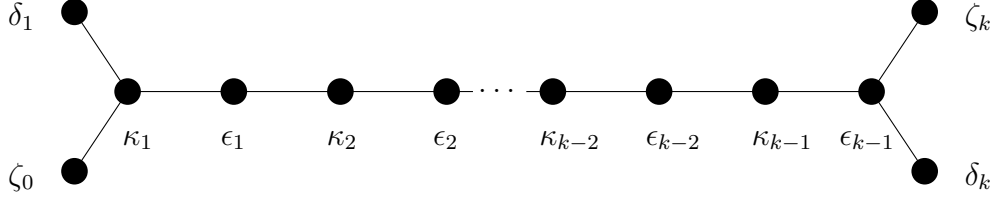


Figure 15: Intersection graph of the fibers in codimension 1, resulting in the affine D_{2k+1} Dynkin diagram.

exceptional divisors are

Divisor	Section	Equation in Y_4
$D_{-\alpha_0}$	ζ_0	$0 = y^2\delta_1 - x^3\zeta_1$
$D_{-\alpha_1}$	δ_1	$0 = \delta_2\kappa_2^2\epsilon_1 + b_2\zeta_0$
$D_{-\alpha_2}$	κ_1	$0 = \delta_1 - \zeta_1\zeta_2(b_2\zeta_0 + \zeta_2\epsilon_1)$
$D_{-\alpha_{2i+1}}$ $i = 1, \dots, k-2$	ϵ_i	$0 = \delta_i - b_2\zeta_{i+1}$
$D_{-\alpha_{2i}}$ $i = 2, \dots, k-1$	κ_i	$0 = \delta_{i-1}\delta_i - b_2\zeta_i\zeta_{i+1}$
$D_{-\alpha_{2k-1}}$	ϵ_{k-1}	$0 = \delta_{k-1}(b_3\kappa_{k-1} + \delta_k) - b_2\zeta_k$
$D_{-\alpha_{2k}}$	ζ_k	$0 = \delta_k + b_3\zeta_{k-1}\kappa_{k-1}$
$D_{-\alpha_{2k+1}}$	δ_k	$0 = b_3\delta_{k-1}y - \zeta_k b_2 x^2 - \delta_{k-1}\epsilon_{k-1}\zeta_k(b_4x + b_6\delta_{k-1}\epsilon_{k-1})$

(4.5)

The $D_{-\alpha_i}$ s are the Cartan divisors, the label α_i refers to the affine simple $SO(4k+2)$ roots. Computing the intersections of these Cartan divisors with each other will allow us to construct a Cartan charge vector. The procedure for calculating the intersections of these divisors is explained in appendix C.2. To each Cartan divisor we construct the j th element of the Cartan charge vector by intersecting that Cartan divisor with $D_{-\alpha_j}$. The Cartan charge vectors are

Cartan divisor	j th Cartan charge
$D_{-\alpha_0}$	$\delta_{j,2} - 2\delta_{j,0}$
$D_{-\alpha_1}$	$\delta_{j,2} - 2\delta_{j,1}$
$D_{-\alpha_2}$	$\delta_{j,0} + \delta_{j,1} + \delta_{j,3} - 2\delta_{j,2}$
$D_{-\alpha_{2i+1}}$	$\delta_{j,2i} + \delta_{j,2i+2} - 2\delta_{j,2i+1}$
$D_{-\alpha_{2i}}$	$\delta_{j,2i-1} + \delta_{j,2i+1} - 2\delta_{j,2i}$
$D_{-\alpha_{2k-1}}$	$\delta_{j,2k-2} + \delta_{j,2k} + \delta_{j,2k+1} - 2\delta_{j,2k-1}$
$D_{-\alpha_{2k}}$	$\delta_{j,2k-1} - 2\delta_{j,2k}$
$D_{-\alpha_{2k+1}}$	$\delta_{j,2k-1} - 2\delta_{j,2k+1}$

(4.6)

which are the (affine) simple roots α_i of $SO(4k+2)$ in the canonical ordering. Figure 15 summarizes the intersection graph with the labels referring to the sections.

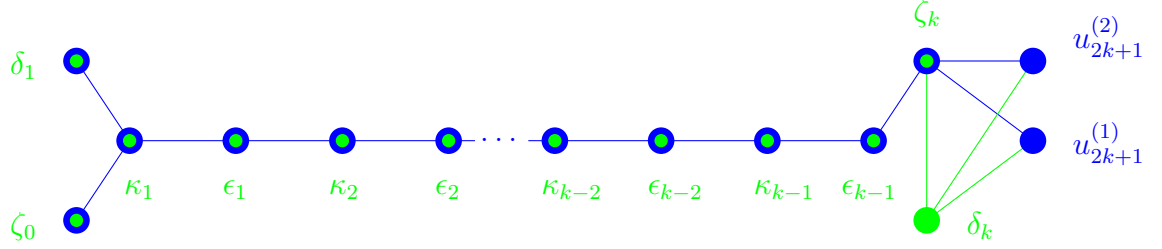


Figure 16: Intersection graph of the fibers in codimension 2 along $b_3 = 0$ for $SO(4k + 2)$. \bullet (and bicolored nodes) are the irreducible components in codimension 2, green lines indicate the splitting, in particular $\delta_k = 0$ splits off $u_{2k+1}^{(i)}$. Blue lines are the intersections of the irreducible components in codimension 2. From the latter we see that the intersection graph is an affine D type Dynkin diagram, and the multiplicities are those of a Kodaira fiber.

4.1.2 Codimension 2

Along the higher codimension loci, as listed in table 2.13, the equation in Y_4 for the vanishing of the exceptional sections can become reducible. Along the two matter loci $b_2 = 0$ and $b_3 = 0$ the vanishing order of the discriminant, which is $O(\zeta_0^{2k+3})$ increases to

$$\begin{aligned} b_3 = 0 : \quad \Delta|_{b_3=0} &= O(\zeta_0^{2k+4}) \\ b_2 = 0 : \quad \Delta|_{b_2=0} &= O(\zeta_0^{2k+6}). \end{aligned} \tag{4.7}$$

For low rank cases, such as $SO(10)$, $b_2 = 0$ corresponds to an exceptional enhancement, e.g. E_6 , in particular this gives rise to spin representations. For higher values of k , $b_2 = 0$ is non-minimal. The matter surfaces at a particular codimension 2 locus will be the irreducible components of (4.5) at that locus. The specifics of the intersections of these with the Cartan divisors are again given in appendix C.2.

Along the codimension 2 locus $b_3 = 0$ the irreducible components of the restrictions of the

Cartan divisors, i.e. matter surfaces, and the corresponding Cartan charges are

Matter surface	Section	Equation in $Y_{b_3=0}$	j th Cartan charge
$S_{-\alpha_0}$	ζ_0	$0 = y^2\delta_1 - x^3\zeta_1$	$\delta_{j,2} - 2\delta_{j,0}$
$S_{-\alpha_1}$	δ_1	$0 = \delta_2\kappa_2^2\epsilon_1 + b_2\zeta_0$	$\delta_{j,2} - 2\delta_{j,1}$
$S_{-\alpha_2}$	κ_1	$0 = \delta_1 - \zeta_1\zeta_2(b_2\zeta_0 + \zeta_2\epsilon_1)$	$\delta_{j,0} + \delta_{j,1} + \delta_{j,3} - 2\delta_{j,2}$
$S_{-\alpha_{2i+1}}$ $i = 1, \dots, k-2$	ϵ_i	$0 = \delta_i - b_2\zeta_{i+1}$	$\delta_{j,2i} + \delta_{j,2i+2} - 2\delta_{j,2i+1}$
$S_{-\alpha_{2i}}$ $i = 2, \dots, k-1$	κ_i	$0 = \delta_{i-1}\delta_i - b_2\zeta_i\zeta_{i+1}$	$\delta_{j,2i-1} + \delta_{j,2i+1} - 2\delta_{j,2i}$
$S_{-\alpha_{2k-1}}$	ϵ_{k-1}	$0 = \delta_{k-1}\delta_k - b_2\zeta_k$	$\delta_{j,2k-2} + \delta_{j,2k} + \delta_{j,2k+1} - 2\delta_{j,2k-1}$
$S_{-\alpha_{2k}}$	ζ_k	$0 = \delta_k$	$\delta_{j,2k-1} - 2\delta_{j,2k}$
$S_{-\alpha_{2k+1}}$	δ_k	$0 = \zeta_k b_2 x^2$	$\delta_{j,2k-1} - 2\delta_{j,2k+1}$
$S_{u_{2k+1}^{(i)}}$ $i = 1, 2$	δ_k	$0 = b_2 x^2$ $+ \delta_{k-1}\epsilon_{k-1}\zeta_k(b_4x + b_6\delta_{k-1}\epsilon_{k-1})$ $+ \delta_{k-1}\epsilon_{k-1}(b_4x + b_6\delta_{k-1}\epsilon_{k-1})$	$\delta_{j,2k} - \delta_{j,2k+1}$

(4.8)

Note that $u_{2k+1}^{(1,2)}$ labels the divisors corresponding to the roots of the quadratic polynomial in the last line, which are generically distinct curves, as its discriminant does not generically vanish along $b_3 = 0$. The only non-trivial splitting occurs for $D_{-\alpha_{2k+1}}$, as is depicted in figure 16

$$D_{-\alpha_{2k+1}} \longrightarrow S_{-\alpha_{2k}} + S_{u_{2k+1}^{(1)}} + S_{u_{2k+1}^{(2)}}. \quad (4.9)$$

The charges are related to the highest weight $(\mu_V)_j = \delta_{j,1}$ of the fundamental representation by

$$S_{u_{2k+1}^{(i)}} : \quad \mu_V - \left(\sum_{i=1}^{2k-1} \alpha_i + \alpha_{2k+1} \right). \quad (4.10)$$

We discuss the non-minimal matter locus $b_2 = 0$ in section 4.1.4.

4.1.3 Codimension 3

For a generic $SO(4k+2)$ the codimension 3 loci of symmetry enhancement are along

$$\begin{aligned} b_3 &= b_4^2 - 4b_2b_6 = 0 \\ b_2 &= b_{1,3,4} = 0 \\ b_2 &= b_1b_3 + b_4 = 0. \end{aligned} \quad (4.11)$$

Only along $b_2 = b_3 = 0$ do the matter surfaces split further, however, this is a non-minimal locus, which will be discussed in the next subsection.

The only minimal singularity enhancement occurs along $b_3 = b_4^2 - 4b_2b_6 = 0$. Consider the matter surfaces $S_{u_{2k+1}^{(i)}}$. We can assume, as this is codimension 3, that without loss of generality, b_2 does not vanish in addition. The discriminant of the quadratic equation defining

$S_{u_{2k+1}}^{(i)}$ is $b_4^2 - 4b_2b_6 = 0$, so that along this codimension 3 locus, the equation for these matter surfaces reduces to

$$(2b_2x + b_4\delta_{k-1}\epsilon_{k-1})^2 = 0. \quad (4.12)$$

I.e. the two distinct matter surfaces become degenerate, corresponding to a Yukawa coupling of the two $4k + 2$ dimensional fundamental V representations in $S_{u_{2k+1}}^{(i)}$

$$b_3 = b_4^2 - 4b_2b_6 = 0 : \quad V \otimes V \otimes \mathbf{1}. \quad (4.13)$$

4.1.4 Non-minimal Loci

Finally consider the matter locus $b_2 = 0$, which is non-minimal. This is easily seen that the resolved geometry (4.4) along $b_2 = 0$ has the form

$$\delta_2 \cdots \delta_{k-1} (\cdots) = 0, \quad (4.14)$$

which shows that the following loci are completely contained inside the fiber

$$\aleph_i : \quad b_2 = \delta_i = 0, \quad i = 2, \cdots, k-1. \quad (4.15)$$

As in the case for the non-minimal codimension 3 locus for A_n , we do not capitulate, but see what we can learn from the splitting of the Cartan divisors in codimension 1 along this locus. There are two observations which we will arrive at: first of all, the Cartan divisors split into surfaces which carry Cartan charges corresponding to spin representations, and in addition, depending on the specific small resolution, in addition there are matter surfaces in $\Lambda^i V$, for V the fundamental representation. We consider here the resolution (4.2) and give another example resolution of the D_{2k+1} singularities in appendix D, where the matter surfaces arising along this non-minimal locus are different.

Consider the Cartan divisors (4.5) along $b_2 = 0$. These split into irreducible matter

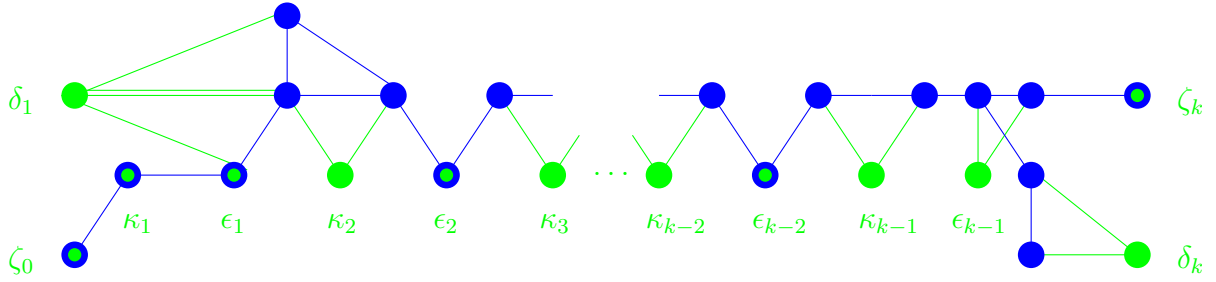


Figure 17: Intersection graph of the fibers in codimension 2 along $b_2 = 0$ for $SO(4k + 2)$. \bullet are the irreducible components in codimension 2, bicolored nodes remain irreducible when passing to codimension 2. Green lines indicate the splitting, blue lines the intersections.

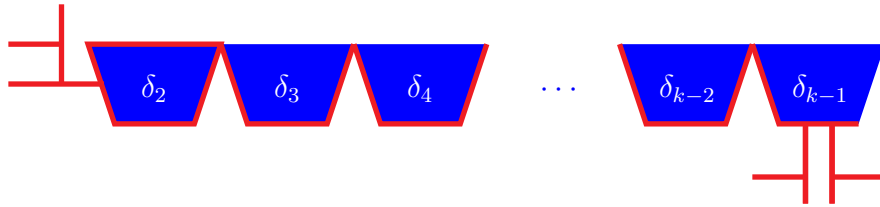


Figure 18: Fiber along the non-minimal codimension 2 locus $b_2 = 0$. The red line is the dual graph to the (blue line) intersection graph in figure 17. The blue semi-hexagons are the 1 dimension higher components \aleph_i given by the vanishing of δ_i as in (4.15) that enter the fiber along $b_2 = 0$. The edges that are part of the red intersection graph arise from the intersection of the components \aleph_i with κ_i , ϵ_i and κ_{i+1} .

surfaces, which have Cartan charges given in the following table

Matter surface	Section	Equation in $Y _{b_2=0}$	j th Cartan charge	Rep
$S_{-\alpha_0}$	ζ_0	$0 = y^2\delta_1 - x^3\zeta_1$	$\delta_{j,2} - 2\delta_{j,0}$	Adj
S_{v_0}	δ_1	$0 = \delta_2$	$\delta_{j,4}$	$\Lambda^4 V$
S_{v_1}	δ_1	$0 = \kappa_2$	$-\delta_{j,1} + \delta_{j,3} - \delta_{j,4}$	Adj
$S_{-\alpha_2}$	κ_1	$0 = \delta_1 - \zeta_1\zeta_2^2\epsilon_1$	$\delta_{j,0} + \delta_{j,1} + \delta_{j,3} - 2\delta_{j,2}$	Adj
$S_{-\alpha_{2i+1}}$ $i = 1, \dots, k-2$	ϵ_i	$0 = \delta_i$	$\delta_{j,2i} + \delta_{j,2i+2} - 2\delta_{j,2i+1}$	Adj
$S_{-\alpha_{2k}}$	ζ_k	$0 = \delta_k + b_3\zeta_{k-1}\kappa_{k-1}$	$\delta_{j,2k-1} - 2\delta_{j,2k}$	Adj
$S_{v_{3i}}$ $i = 2, \dots, k-1$	κ_i	$0 = \delta_{i-1}$	$\delta_{j,2i-1} - \delta_{j,2i}$	V
$S_{v_{3i+1}}$ $i = 2, \dots, k-1$	κ_i	$0 = \delta_i$	$\delta_{j,2i+1} - \delta_{j,2i}$	V
$S_{s_{3k-1}}$	ϵ_{k-1}	$0 = \delta_{k-1}$	$\delta_{j,2k-2} - \delta_{j,2k-1} + \delta_{2k+1}$	S^-
$S_{s_{3k}}$	ϵ_{k-1}	$0 = b_3\kappa_{k-1} + \delta_k$	$-\delta_{j,2k-1} + \delta_{2k}$	S^+
$S_{s_{3k+1}}$	δ_k	$0 = b_3y$	$-\delta_{j,2k+1}$	S^-
$S_{s_{3k+2}}$	δ_k	$-\epsilon_{k-1}\zeta_k(b_4x + b_6\delta_{k-1}\epsilon_{k-1})$ $0 = \delta_{k-1}$	$\delta_{j,2k-1} - \delta_{j,2k+1}$	S^+

(4.16)

The last column indicates the representation. In summary, when passing to the codimension 2 locus $b_2 = 0$ the following Cartan divisors split into matter surfaces

$$\begin{aligned}
D_{-\alpha_1} &\longrightarrow S_{-\alpha_3} + 2 \times S_{v_0} + S_{v_1} \\
D_{-\alpha_{2i}} &\longrightarrow S_{v_{3i}} + S_{v_{3i+1}} \\
D_{-\alpha_{2k-1}} &\longrightarrow S_{s_{3k-1}} + S_{s_{3k}} \\
D_{-\alpha_{2k+1}} &\longrightarrow S_{s_{3k+1}} + S_{s_{3k+3}}.
\end{aligned}
\tag{4.17}$$

The Cartan charge vector can be associated with Dynkin labels of weights of certain representations of $SO(4k+2)$. Denote the highest weights of the spin representations by

$$\begin{aligned}
(\mu_{S^+})_j &= \delta_{2k+1,j} \\
(\mu_{S^-})_j &= \delta_{2k,j}.
\end{aligned}
\tag{4.18}$$

Each matter surface whose Cartan charge vector is not a simple root is listed below, along with the Cartan charge vector.

Matter surface	Weight
$S_{s_{3k-1}}$	$\mu_{S^-} - (\alpha_{2k-1} + \alpha_{2k})$
$S_{s_{3k}}$	$\mu_{S^+} - \left(\sum_{i=1}^{2k-1} i\alpha_i + (k-1)\alpha_{2k} + k\alpha_{2k+1} \right)$
$S_{s_{3k+1}}$	$\mu_{S^-} - \left(\sum_{i=1}^{2k-1} i\alpha_i + k\alpha_{2k} + k\alpha_{2k+1} \right)$
$S_{s_{3k+2}}$	$\mu_{S^+} - \alpha_{2k+1}$

(4.19)

Along this non-minimal locus, it is clear that we do not expect a Dynkin diagram like intersection of the irreducible fiber components. As for codimension 3 for $SU(n)$ we can draw the

intersection graph of the irreducible fiber components above $b_2 = 0$, which are depicted in figure 17. The graph is not a standard Dynkin diagram, however, for low k values it reduces to E type Dynkin diagrams. Alternatively we can include the additional components of the fiber \aleph_i and how they intersect with the remaining components, as in figure 18.

The codimension 3 non-minimal locus, which results in further splitting of the $b_2 = 0$ matter is along $b_2 = b_3 = 0$, where $S_{v_{3k+1}}$ splits as

$$\delta_k = 0, \quad \epsilon_{k-1}\zeta_k(b_4x + b_6\delta_{k-1}\epsilon_{k-1}) = 0, \quad (4.20)$$

which translates into

$$S_{s_{3k+1}} \longrightarrow \Sigma_{s_{3k}} + \Sigma_{u_{2k+1}} + \Sigma_{-\alpha_{2k}}, \quad (4.21)$$

which is consistent with a coupling

$$b_2 = b_3 = 0: \quad S^+ \otimes S^- \otimes V. \quad (4.22)$$

In summary, considering the non-minimal loci results in higher dimensional components in the fiber. However, we can nevertheless study some properties of these loci in codimension 2 and 3 by following the splitting of the Cartan divisors that are present in codimension 1. Interestingly, they generate representations, and couplings, that one would not expect from an ordinary field theoretic Higgsing of a higher rank A or D type gauge group.

4.2 $SO(4k + 4)$

An elliptically fibered Calabi-Yau fourfold with an $SO(4k + 4)$ singularity along $\zeta_0 = 0$ can be put into Tate form

$$y^2 - x^3 + b_1xy\zeta_0 - b_2x^2\zeta_0 + b_3y\zeta_0^{k+1} - b_4x\zeta_0^{k+1} - b_6\zeta_0^{2k+1} = 0, \quad (4.23)$$

along with the additional condition that $b_2x^2 + b_4x + b_6$ factors. We shall choose to satisfy this extra condition by defining our $SO(4k + 4)$ singular Calabi-Yau by

$$y^2 - x^3 + b_1xy\zeta_0 - b_2x^2\zeta_0 + b_3y\zeta_0^{k+1} - b_4x\zeta_0^{k+1} = 0. \quad (4.24)$$

The discriminant of this Tate form is

$$\Delta(T_{D_{\text{even}}}) = 256b_2^2b_4^2\zeta_0^{2k+4} + O(\zeta_0^{2k+5}), \quad (4.25)$$

which has the correct order noted in appendix A. In particular, we have not inadvertently increased the order into a different type of singularity by setting $b_6 = 0$. This choice for

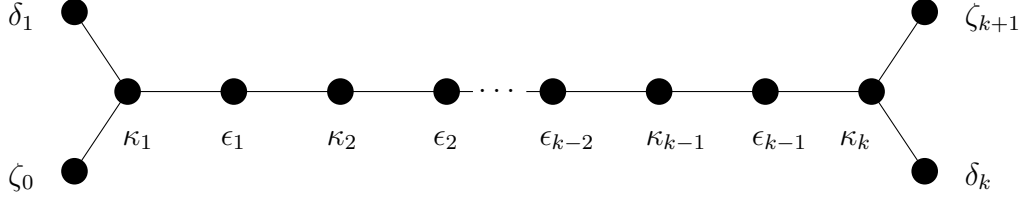


Figure 19: Intersection graph of the fibers in codimension 1, resulting in the affine D_{2k+2} Dynkin diagram.

solving the additional factorization condition is not the most general such solution, as already noted in [22]. This space can be resolved by performing the following blowups, expressed in the notation of (2.17)

$$\begin{aligned}
(x, y, \zeta_i; \zeta_{i+1}) & \quad i = 0, \dots, k \\
(y, \zeta_i; \delta_i) & \quad i = 1, \dots, k \\
(\zeta_i, \delta_i; \kappa_i) & \quad i = 1, \dots, k \\
(\zeta_{i+1}, \delta_i; \epsilon_i) & \quad i = 1, \dots, k - 1.
\end{aligned} \tag{4.26}$$

The geometry obtained after proper transformation, corresponding to a Calabi-Yau with a resolved $SO(4k + 4)$ singularity, is then

$$\begin{aligned}
& y^2 B(\delta) \delta_k - x^3 \zeta_k^k \zeta_{k+1}^{k+1} \delta_k^{k-1} \kappa_k^{2k-2} B(\zeta \epsilon) A(\zeta \delta \kappa^2 \epsilon^2) + b_1 x y \zeta_0 \zeta_{k+1} B(\zeta \delta \kappa \epsilon) \\
& - b_2 x^2 \zeta_0 B(\zeta) \zeta_k \zeta_{k+1} + b_3 y \zeta_0^{k+1} \zeta_k B(\zeta^2 \delta^2 \kappa^2 \epsilon^2) C(\zeta \delta \kappa^2 \epsilon^2) \\
& - b_4 x \zeta_0^{k+1} \zeta_k B(\zeta^2 \delta \kappa^2 \epsilon) C(\zeta \delta \kappa^2 \epsilon^2) = 0,
\end{aligned} \tag{4.27}$$

along with the projectivity conditions between the sections given in appendix C.3.

4.2.1 Codimension 1

The irreducible exceptional divisors for $SO(4k+4)$ are obtained as

Divisor	Section	Equation in Y_4	j th Cartan charge
$D_{-\alpha_0}$	ζ_0	$0 = y^2\delta_1 - x^3\zeta_1$	$\delta_{j,2} - 2\delta_{j,0}$
$D_{-\alpha_1}$	δ_1	$0 = b_2\zeta_0 + \delta_2\kappa_2^2\epsilon_1$	$\delta_{j,2} - 2\delta_{j,1}$
$D_{-\alpha_2}$	κ_1	$0 = \delta_1 - \zeta_1\zeta_2(b_2\zeta_0 + \zeta_2\epsilon_1)$	$\delta_{j,0} + \delta_{j,1} + \delta_{j,3} - 2\delta_{j,2}$
$D_{-\alpha_{2i+1}}$ $i = 1, \dots, k-1$	ϵ_i	$0 = \delta_i - b_2\zeta_{i+1}$	$\delta_{j,2i} + \delta_{j,2i+2} - 2\delta_{j,2i+1}$
$D_{-\alpha_{2i}}$ $i = 2, \dots, k-1$	κ_i	$0 = \delta_{i-1}\delta_i - b_2\zeta_i\zeta_{i+1}$	$\delta_{j,2i-1} + \delta_{j,2i+1} - 2\delta_{j,2i}$
$D_{-\alpha_{2k}}$	κ_k	$0 = \delta_{k-1}(\delta_k - b_4\zeta_k\epsilon_{k-1}) - b_2\zeta_k\zeta_{k+1}$	$\delta_{j,2k-1} + \delta_{j,2k+1} + \delta_{j,2k+2} - 2\delta_{j,2k}$
$D_{-\alpha_{2k+1}}$	δ_k	$0 = b_2\zeta_{k+1} + b_4\delta_{k-1}$	$\delta_{j,2k} - 2\delta_{j,2k+1}$
$D_{-\alpha_{2k+2}}$	ζ_{k+1}	$0 = \delta_k y(b_3\zeta_k\kappa_k + y) - b_4\zeta_k x$	$\delta_{j,2k} - 2\delta_{j,2k+2}$

(4.28)

The intersections are reproduced in figure 19, where the simple roots of the standard $SO(4k+4)$ Lie group are the Cartan charge vector associated to the labelled section above each node.

4.2.2 Codimension 2

As can be seen from (4.25) the codimension 2 loci are $b_2 = 0$ or $b_4 = 0$, along which the Cartan divisors descend to matter surfaces, with some of the divisors becoming reducible. Each matter surface will also have a Cartan charge vector, obtained by intersecting that matter surface with the Cartan divisors, in the manner of appendix C.3.

When $b_4 = 0$ the only Cartan divisor that becomes reducible is $\zeta_{k+1} = 0$, which yields three components, one of which is $S_{-\alpha_{2k+1}}$ and the remaining ones are

Matter surface	Section	Equation in $Y_4 _{b_4=0}$	j th Cartan charge	Rep
$S_{u_1^{(1)}}$	ζ_{k+1}	$0 = y$	$\delta_{j,2k+1} - \delta_{j,2k+2}$	V
$S_{u_1^{(2)}}$	ζ_{k+1}	$0 = (y + b_3\zeta_k\kappa_k)$	$\delta_{j,2k+1} - \delta_{j,2k+2}$	V

(4.29)

In summary along $b_4 = 0$ the splitting is

$$D_{-\alpha_{2k+2}} \longrightarrow S_{-\alpha_{2k+1}} + S_{u_1^{(1)}} + S_{u_1^{(2)}}. \quad (4.30)$$

The Cartan charge vector associated to each matter surface is a weight in some representation of $SO(4k+4)$. The representations each is associated to are as follows

Matter surface	Weight
S_{u_1}	$\mu_V - \left(\sum_{i=1}^{2k-1} \alpha_i + \alpha_{2k+1} \right)$
$S_{S_{2k}}$	$\mu_{S^+} - \alpha_{2k} - \alpha_{2k+2}$
$S_{S_{2k+1}}$	$\mu_{S^+} - \left(\sum_{i=1}^{2k} i\alpha_i + k\alpha_{2k+1} + k\alpha_{2k+2} \right)$

(4.31)

At $b_2 = 0$ the singularity is, like for $SO(4k+2)$, non-minimal. We determine the irreducible matter surfaces from the restriction of the Cartan divisors as

Matter surface	Section	Equation in $Y_4 _{b_2=0}$	j th Cartan charge	Rep
$S_{-\alpha_0}$	ζ_0	$0 = y^2\delta_1 - x^3\zeta_1$	$\delta_{j,2} - 2\delta_{j,0}$	Adj
S_{v_0}	δ_1	$0 = \delta_2$	$\delta_{j,4}$	$\Lambda^4 V$
S_{v_6}	κ_2	$0 = \delta_1$	$-\delta_{j,1} + \delta_{j,3} - \delta_{j,4}$	Adj
S_{v_7}	κ_2	$0 = \delta_2$	$\delta_{j,1} - \delta_{j,4} + \delta_{j,5}$	Adj
$S_{-\alpha_2}$	κ_1	$0 = \delta_1 - \zeta_1\zeta_2^2\epsilon_1$	$\delta_{j,0} + \delta_{j,1} + \delta_{j,3} - 2\delta_{j,2}$	Adj
$S_{-\alpha_{2i+1}}$ $i = 1, \dots, k-1$	ϵ_i	$0 = \delta_i$	$\delta_{j,2i} + \delta_{j,2i+2} - 2\delta_{j,2i+1}$	Adj
$S_{-\alpha_{2k+1}}$	δ_k	$0 = b_4\delta_{k-1}$	$\delta_{j,2k} - 2\delta_{j,2k+1}$	Adj
$S_{-\alpha_{2k+2}}$	ζ_{k+1}	$0 = \delta_k y (b_3\zeta_k\kappa_k + y) - b_4\zeta_k x$	$\delta_{j,2k} - 2\delta_{j,2k+2}$	Adj
$S_{v_{3i}}$ $i = 3, \dots, k-1$	κ_i	$0 = \delta_{i-1}$	$\delta_{j,2i-1} - \delta_{j,2i}$	V
$S_{v_{3i+1}}$ $i = 3, \dots, k-1$	κ_i	$0 = \delta_i$	$\delta_{j,2i+1} - \delta_{j,2i}$	V
$S_{s_{2k}}$	κ_k	$0 = \delta_{k-1}$	$\delta_{j,2k-1} - \delta_{j,2k} + \delta_{j,2k+1}$	S^+
$S_{s_{2k+1}}$	κ_k	$0 = \delta_k - b_4\zeta_k\epsilon_{k-1}$	$\delta_{j,2k+2} - \delta_{j,2k}$	S^+

(4.32)

where the Cartan divisors that split into matter surfaces nontrivially along $b_2 = 0$ are

$$\begin{aligned}
D_{-\alpha_1} &\longrightarrow S_{-\alpha_3} + 2 \times S_{v_6} + S_{v_0} \\
D_{-\alpha_{2i}} &\longrightarrow S_{v_{3i}} + S_{v_{3i+1}} \\
D_{-\alpha_{2k}} &\longrightarrow S_{s_{2k}} + S_{s_{2k+1}}.
\end{aligned}
\tag{4.33}$$

The fibers for both codimension 2 loci are depicted in figures 20 and 21.

4.2.3 Codimension 3

$SO(4k+4)$ has 3 codimension 3 enhancements of the discriminant. These are

$$\begin{aligned}
b_4 = b_3 = 0 \\
b_2 = b_{1,4} = 0.
\end{aligned}
\tag{4.34}$$

The first locus is the only one that is minimal. The two distinct matter surfaces $S_{u_1^{(i)}}$ become degenerate at this locus, both are given by the equation $b_3 = b_4 = \zeta_{k+1} = y = 0$, so that this codimension 3 locus allows the generation of the Yukawa coupling

$$b_3 = b_4 = 0 : \quad V \otimes V \otimes \mathbf{1}.
\tag{4.35}$$

Along the non-minimal locus $b_2 = b_4 = 0$, the $b_2 = 0$ matter surface splits further as

$$S_{-\alpha_{2k+2}} \longrightarrow \Sigma_{-\alpha_{2k+1}} + \Sigma_{u_1^{(1)}} + \Sigma_{u_1^{(2)}}.
\tag{4.36}$$

Along the other locus no further splittings occur.

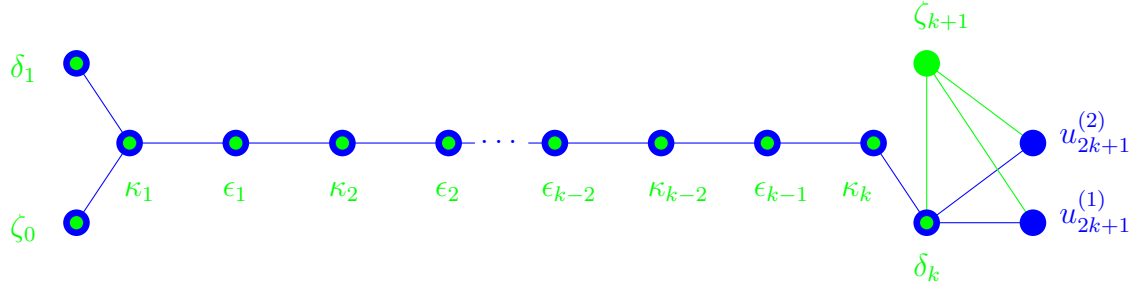


Figure 20: Intersection graph of the fibers in codimension 2 along $b_4 = 0$ for $SO(4k + 4)$. \bullet (and bicolored nodes) are the irreducible components in codimension 2, green lines indicate the splitting.

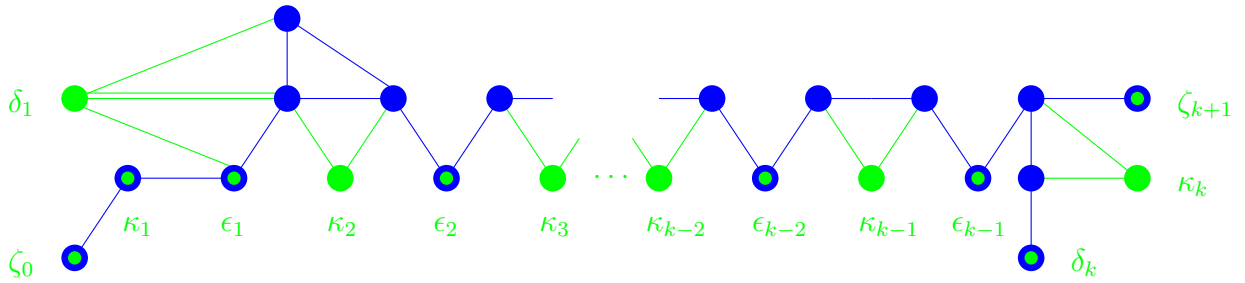


Figure 21: Intersection graph of the fibers in codimension 2 along $b_2 = 0$ for $SO(4k + 4)$. \bullet are the irreducible components in codimension 2, bicolored nodes remain irreducible when passing to codimension 2. Green lines indicate the splitting, blue lines the intersections. From the latter we see that the intersection graph is not an ADE Dynkin diagram.

5 Resolution of E Type Singularities

Finally, there are three exceptional cases left, which we will discuss in turn, including their higher codimension structure.

5.1 E_6

For E_6 all singularities up and including codimension 3 are minimal. The Tate form for E_6 is

$$y^2 - x^3 - b_6\zeta_0^5 - b_2\zeta_0^2x^2 - b_4\zeta_0^3x + b_1\zeta_0xy + b_3\zeta_0^2y = 0, \quad (5.1)$$

and applying the following resolutions to the geometry resolves it fully

$$\begin{aligned} &(x, y, \zeta_0; \zeta_1) \\ &(x, y, \zeta_1; \zeta_2) \\ &(y, \zeta_1; \delta_1) \\ &(y, \zeta_2; \delta_2) \\ &(\delta_1, \zeta_1; \kappa_1) \\ &(\delta_1, \zeta_2; \epsilon_1) \\ &(\delta_1, \delta_2; \epsilon_2) \\ &(\delta_1, \epsilon_1; \epsilon_3). \end{aligned} \quad (5.2)$$

After taking the proper transform yields

$$\begin{aligned} &\delta_1\delta_2\epsilon_2y^2 + \delta_1\zeta_1\kappa_1y\zeta_0(b_3\zeta_0 + b_1\delta_2\zeta_2x\epsilon_1\epsilon_2\epsilon_3) \\ &= \zeta_1\zeta_2\epsilon_1(b_6\delta_1^2\zeta_1^2\kappa_1^4\epsilon_1\epsilon_2\epsilon_3^3\zeta_0^5 + \delta_1\zeta_1\kappa_1^2x\epsilon_3\zeta_0^2(b_4\zeta_0 + b_2\delta_2\zeta_2x\epsilon_1\epsilon_2\epsilon_3) + \delta_2\zeta_2x^3). \end{aligned} \quad (5.3)$$

The irreducible exceptional divisors, which intersect in an affine E_6 Dynkin diagram, are

Divisor	Section	Equation in Y
$D_{-\alpha_6}$	ζ_0	$0 = \delta_1y^2 - \zeta_1x^3$
$D_{-\alpha_1}$	ζ_2	$0 = b_3\zeta_1\kappa_1 + \delta_2$
$D_{-\alpha_0}$	δ_2	$0 = b_3y - \zeta_2\epsilon_1(b_4x + b_6\epsilon_1\epsilon_2)$
$D_{-\alpha_5}$	κ_1	$0 = \delta_1 - \zeta_1\zeta_2^2\epsilon_1$
$D_{-\alpha_3}$	ϵ_1	$0 = b_3\kappa_1 + \delta_2\epsilon_2$
$D_{-\alpha_2}$	ϵ_2	$0 = b_3\delta_1y - \epsilon_1(b_4\delta_1\epsilon_3 + \delta_2)$
$D_{-\alpha_4}$	ϵ_3	$0 = \delta_1(b_3\kappa_1 + \epsilon_2) - \epsilon_1$

The higher codimension structure was discussed in [30] with the result that along $b_3 = 0$ the following Cartan divisors split

$$\begin{aligned} D_{-\alpha_3} &\longrightarrow S_{(-1,1,1,-1,0,0,0)} + S_{(1,0,-1,-1,1,0,0)} \\ D_{-\alpha_2} &\longrightarrow S_{(0,0,-1,1,0,0,0)} + S_{(1,0,-1,-1,1,0,0)} \\ D_{-\alpha_0} &\longrightarrow S_{-\alpha_1} + S_{(-1,1,1,-1,0,0,0)} + S_{(-1,1,0,0,0,0,0)}, \end{aligned} \quad (5.5)$$

which are all weights of the **27** representation. At the codimension 3 locus $b_3 = b_4 = 0$ the matter surface $S_{(-1,1,0,0,0,0)}$, given by $\delta_2 = b_4x + b_6\epsilon_1\epsilon_2 = 0$ becomes reducible

$$S_{(-1,1,0,0,0,0)} \longrightarrow \Sigma_{(-1,1,1,-1,0,0,0)} + \Sigma_{(1,0,-1,-1,1,0,0)}. \quad (5.6)$$

This is consistent with the **27**³ Yukawa coupling in E_6 .

5.2 E_7

For E_7 there is one codimension 2 locus of symmetry enhancement, and one codimension 3 one, which is non-minimal, and has been discussed in [25]. The Tate form for an E_7 singularity at $\zeta_0 = 0$ is

$$y^2 - x^3 + b_1xy\zeta_0 - b_2x^2\zeta_0^2 + b_3y\zeta_0^3 - b_4x\zeta_0^3 - b_6\zeta_0^5. \quad (5.7)$$

To resolve the space, we apply the following blowups

$$\begin{aligned} & (x, y, \zeta_0; \zeta_1) \\ & (x, y, \zeta_1; \zeta_2) \\ & (y, \zeta_1; \delta_1) \\ & (y, \zeta_2; \delta_2) \\ & (\zeta_2, \delta_1; \epsilon_1) \\ & (\zeta_1, \delta_1; \epsilon_2) \\ & (\zeta_2, \delta_2; \epsilon_3) \\ & (\delta_1, \delta_2; \epsilon_4) \\ & (\delta_2, \epsilon_1; \epsilon_5) \\ & (\epsilon_1, \epsilon_4; \epsilon_6), \end{aligned} \quad (5.8)$$

where the notation is that defined above (2.17). Note that the last three small resolutions ensure, in particular, that the exceptional divisors are all irreducible. The exceptional sections are

$$\{\zeta_0, \epsilon_2, \epsilon_4, \delta_1, \epsilon_6, \epsilon_5, \epsilon_3, \delta_2\}, \quad (5.9)$$

in terms of which the resolved geometry is

$$\begin{aligned} & \delta_1\delta_2\zeta_1\zeta_2y\epsilon_1\epsilon_2\epsilon_3\epsilon_4\epsilon_5\epsilon_6\zeta_0 (b_3\delta_1\zeta_1\epsilon_1\epsilon_2^2\epsilon_4\epsilon_5\epsilon_6^2\zeta_0^2 + b_1x) + \delta_1\delta_2y^2\epsilon_4 = \\ & \zeta_1\zeta_2\epsilon_1 (\delta_2\zeta_2x^2\epsilon_3^2\epsilon_5 (b_2\delta_1\zeta_1\epsilon_1\epsilon_2^2\epsilon_4\epsilon_5\epsilon_6^2\zeta_0^2 + x) + \delta_1\zeta_1\epsilon_2^2\zeta_0^3 (b_6\delta_1\zeta_1\epsilon_1\epsilon_2^2\epsilon_4\epsilon_5\epsilon_6^2\zeta_0^2 + b_4x)) \end{aligned} \quad (5.10)$$

along with the projective relations given in appendix C.4. The exceptional divisors in (5.9) intersect in an affine E_7 Dynkin diagram

Cartan Divisor	Section	Equation in Y_4	Cartan charges
$D_{-\alpha_0}$	ζ_0	$0 = \delta_1 y^2 - \zeta_1 x^3$	$(-2, 1, 0, 0, 0, 0, 0)$
$D_{-\alpha_1}$	ϵ_2	$0 = \delta_1 - \zeta_1 \epsilon_1$	$(1, -2, 0, 1, 0, 0, 0)$
$D_{-\alpha_2}$	ϵ_4	$0 = -b_4 \delta_1 - \delta_2 \epsilon_5$	$(0, 0, -2, 0, 1, 0, 0)$
$D_{-\alpha_3}$	δ_1	$0 = \epsilon_1$	$(0, 1, 0, -2, 1, 0, 0)$
$D_{-\alpha_4}$	ϵ_6	$0 = -b_4 \delta_1 \epsilon_1 + \delta_1 \epsilon_4 - \epsilon_5 \epsilon_1$	$(0, 0, 1, 1, -2, 1, 0)$
$D_{-\alpha_5}$	ϵ_5	$0 = \delta_2 \epsilon_4 - b_4 \epsilon_1$	$(0, 0, 0, 0, 1, -2, 1)$
$D_{-\alpha_6}$	ϵ_3	$0 = \delta_2 - \zeta_2 \epsilon_1 (b_4 x + b_6 \epsilon_1 \epsilon_5)$	$(0, 0, 0, 0, 0, 1, -2)$
$D_{-\alpha_7}$	δ_2	$0 = b_4 x + b_6 \epsilon_4 \epsilon_5$	$(0, 0, 0, 0, 0, 0, 1)$

(5.11)

At the codimension 2 enhancement locus $b_4 = 0$, it is clear that $D_{-\alpha_5}$ and $D_{-\alpha_6}$ becomes reducible. The splitting is as follows

$$\begin{aligned}
D_{-\alpha_5} &\longrightarrow S_{(0,0,-1,0,1,-1,0,1)} + S_{(0,0,-1,0,0,1,0,-1)} \\
D_{-\alpha_6} &\longrightarrow S_{(0,0,-1,0,1,-1,0,1)} + S_{(0,0,1,0,0,-1,1,-1)},
\end{aligned}
\tag{5.12}$$

where the weights of the matter surfaces S_v correspond to the **56** representation. Finally, the codimension 3 locus at $b_4 = b_6 = 0$ is non-minimal. Indeed, $\delta_2 = 0$ is completely contained in the fiber along this locus.

5.3 E_8

The singular Tate form for E_8 is

$$y^2 - x^3 + b_1 x y \zeta_0 - b_2 x^2 \zeta_0^2 + b_3 y \zeta_0^3 - b_4 x \zeta_0^4 - b_6 \zeta_0^5 = 0. \tag{5.13}$$

Again, $\zeta_0 = 0$ denotes the locus above which the singular E_8 fiber is. The following combination of blowups and small resolutions resolve the space in all codimensions

$$\begin{aligned}
& (x, y, \zeta_0; \zeta_1) \\
& (x, y, \zeta_1; \zeta_2) \\
& (y, \zeta_1; \delta_1) \\
& (y, \zeta_2; \delta_2) \\
& (\zeta_2, \delta_1; \epsilon_1) \\
& (\zeta_1, \delta_1; \epsilon_2) \\
& (\zeta_2, \delta_2; \epsilon_3) \\
& (\delta_1, \delta_2; \epsilon_4) \\
& (\delta_2, \epsilon_1; \epsilon_5) \\
& (\epsilon_1, \epsilon_4; \epsilon_6) \\
& (\delta_2, \epsilon_4; \epsilon_7) \\
& (\delta_2, \epsilon_5; \epsilon_8) \\
& (\epsilon_4, \epsilon_5; \epsilon_9) \\
& (\epsilon_5, \epsilon_7; \epsilon_{10})
\end{aligned} \tag{5.14}$$

Resolved geometry is then

$$\begin{aligned}
& \delta_1 \delta_2 \epsilon_4 \epsilon_7 y \left(y + \zeta_0 \zeta_1 \zeta_2 \epsilon_1 \epsilon_2 \epsilon_3 \epsilon_5 \epsilon_6 \epsilon_8 \epsilon_9 \epsilon_{10} \left(b_3 \delta_1 \zeta_1 \zeta_0^2 \epsilon_1 \epsilon_2^2 \epsilon_4 \epsilon_5 \epsilon_6^2 \epsilon_7 \epsilon_8 \epsilon_9^2 \epsilon_{10}^2 + b_1 x \right) \right) \\
& = \zeta_1 \zeta_2 \epsilon_1 \epsilon_5 \times \\
& \left(\delta_1 \zeta_0^2 \zeta_1 \epsilon_1 \epsilon_2^2 \epsilon_4 \epsilon_6^2 \epsilon_9 \left(b_6 \delta_1 \zeta_1 \zeta_0^3 \epsilon_2^2 + \delta_2 \zeta_2 x \epsilon_3^2 \epsilon_5 \epsilon_7 \epsilon_8^2 \epsilon_9 \epsilon_{10}^2 \right) \left(b_4 \delta_1 \zeta_1 \zeta_0^2 \epsilon_1 \epsilon_2^2 \epsilon_4 \epsilon_5 \epsilon_6^2 \epsilon_7 \epsilon_8 \epsilon_9^2 \epsilon_{10}^2 + b_2 x \right) \right) + \delta_2 \zeta_2 x^3 \epsilon_3^2 \epsilon_8
\end{aligned} \tag{5.15}$$

The exceptional sections are

$$\{ \epsilon_3, \epsilon_7, \epsilon_8, \epsilon_{10}, \epsilon_9, \epsilon_6, \delta_1, \epsilon_2, \zeta_0 \} , \tag{5.16}$$

and the intersection of the associated Cartan divisors is as follows

Cartan Divisor	Section	Equation in Y_4	
$D_{-\alpha_1}$	ϵ_3	$0 = \delta_2 - b_6 \zeta_2 \epsilon_1^2 \epsilon_5$	(-2 0 1 0 0 0 0 0 0)
$D_{-\alpha_2}$	ϵ_7	$0 = b_6 \epsilon_4 \epsilon_9 + \delta_2 \epsilon_8$	(0 -2 0 1 0 0 0 0 0)
$D_{-\alpha_3}$	ϵ_8	$0 = \delta_2 \epsilon_7 - b_6 \epsilon_5$	(1 0 -2 1 0 0 0 0 0)
$D_{-\alpha_4}$	ϵ_{10}	$0 = \epsilon_7 - \epsilon_5 (b_6 \epsilon_9 + \epsilon_8)$	(0 1 1 -2 1 0 0 0 0)
$D_{-\alpha_5}$	ϵ_9	$0 = \epsilon_5 - \epsilon_4 \epsilon_7$	(0 0 0 1 -2 1 0 0 0)
$D_{-\alpha_6}$	ϵ_6	$0 = \delta_1 \epsilon_4 - \epsilon_1 \epsilon_5$	(0 0 0 0 1 -2 1 0 0)
$D_{-\alpha_7}$	δ_1	$0 = \epsilon_1$	(0 0 0 0 0 1 -2 1 0)
$D_{-\alpha_8}$	ϵ_2	$0 = \delta_1 - \zeta_1 \epsilon_1$	(0 0 0 0 0 0 1 -2 1)
$D_{-\alpha_0}$	ζ_0	$0 = \delta_1 y^2 - \zeta_1 x^3$	(0 0 0 0 0 0 0 1 -2)

(5.17)

The intersection graph is, as expected, the affine E_8 Dynkin diagram.

E_8 has only non-minimal loci of symmetry enhancement. In particular, along the matter locus $b_6 = 0$, δ_2 becomes a component of the fiber. As in other instances, we nevertheless can consider the splitting of the Cartan divisors, which is summarized by

$$\begin{aligned}
D_{-\alpha_2} &\rightarrow S_{(0,-1,1,0,0,0,0,0)} + S_{(0,-1,-1,1,0,0,0,0)} \\
D_{-\alpha_3} &\rightarrow S_{(1,1,-1,0,0,0,0,0)} + S_{(0,-1,-1,1,0,0,0,0)},
\end{aligned}
\tag{5.18}$$

where the matter surfaces all carry Cartan charges corresponding to weights of the **248** representation of E_8 . Along the (non-minimal) Yukawa coupling locus $b_6 = Q = 0$ from table 2.13, we do not find any further splitting.

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A Tate forms

Type	Group	a_1	a_2	a_3	a_4	a_6	Δ
I_0	—	0	0	0	0	0	0
I_1	—	0	0	1	1	1	1
I_2	$SU(2)$	0	0	1	1	2	2
I_3^{ns}	$Sp(1)$	0	0	2	2	3	3
I_3^s	$SU(3)$	0	1	1	2	3	3
I_{2n}^{ns}	$Sp(n)$	0	0	n	n	$2n$	$2n$
I_{2n}^s	$SU(2n)$	0	1	n	n	$2n$	$2n$
I_{2n+1}^{ns}	$Sp(n)$	0	0	$n+1$	$n+1$	$2n+1$	$2n+1$
I_{2n+1}^s	$SU(2n+1)$	0	1	n	$n+1$	$2n+1$	$2n+1$
II	—	1	1	1	1	1	2
III	$SU(2)$	1	1	1	1	2	3
IV^{ns}	$Sp(1)$	1	1	1	2	2	4
IV^s	$SU(3)$	1	1	1	2	3	4
I_0^{*ns}	G_2	1	1	2	2	3	6
I_0^{*ss}	$SO(7)$	1	1	2	2	4	6
I_0^{*s}	$SO(8)^*$	1	1	2	2	4	6
I_1^{*ns}	$SO(9)$	1	1	2	3	4	7
I_1^{*s}	$SO(10)$	1	1	2	3	5	7
I_2^{*ns}	$SO(11)$	1	1	3	3	5	8
I_2^{*s}	$SO(12)^*$	1	1	3	3	5	8
I_{2n-3}^{*ns}	$SO(4n+1)$	1	1	n	$n+1$	$2n$	$2n+3$
I_{2n-3}^{*s}	$SO(4n+2)$	1	1	n	$n+1$	$2n+1$	$2n+3$
I_{2n-2}^{*ns}	$SO(4n+3)$	1	1	$n+1$	$n+1$	$2n+1$	$2n+4$
I_{2n-2}^{*s}	$SO(4n+4)^*$	1	1	$n+1$	$n+1$	$2n+1$	$2n+4$
IV^{*ns}	F_4	1	2	2	3	4	8
IV^{*s}	E_6	1	2	2	3	5	8
III^*	E_7	1	2	3	3	5	9
II^*	E_8	1	2	3	4	5	10
non-min	—	1	2	3	4	6	12

Table 2: Vanishing order of the sections a_n and Δ in the Tate forms from [22].

B Example Resolutions

B.1 $SU(5)$ Reloaded

The resolution of the $SU(5)$ Tate form has been well understood [8, 9]. Nevertheless, it may be useful for the reader, in order to get familiar with our general analysis and the structures introduced, to readdress this resolution. The low k cases are generally somewhat pathological, which in particular will become clear in codimension 3 for $SU(5)$. The Tate form of $SU(5)$ is

$$y^2 + b_1xy + b_3\zeta_0^2y = x^3 + b_2\zeta_0x^2 + b_4\zeta_0^3x + b_6\zeta_0^5, \quad (\text{B.1})$$

where the $SU(5)$ singular fiber is located along $\zeta_0 = 0$. The singularity in codimension 1 can be resolved by two blowups

$$\begin{aligned} (x, y, \zeta_0; \zeta_1) \\ (x, y, \zeta_1; \zeta_2), \end{aligned} \quad (\text{B.2})$$

using the notation defined in (2.17). After the proper transform this results in

$$y(y + b_1x + b_3\zeta_1\zeta_0^2) = \zeta_1\zeta_2(b_2x^2\zeta_0 + \zeta_1\zeta_0^3(b_6\zeta_1\zeta_0^2 + b_4x) + \zeta_2x^3). \quad (\text{B.3})$$

The space (B.3) is resolved in codimension 1, however not in higher codimension. There is a network of small resolutions whose structure will be discussed in [38]. For the current purposes of resolving the space, we pick one of these, for instance

$$\begin{aligned} (y, \zeta_1; \delta_1) \\ (y, \zeta_2; \delta_2). \end{aligned} \quad (\text{B.4})$$

Note that this small resolution is different from the one studied in [9]. The resulting geometry, after proper transformation, is smooth and given by

$$y(\delta_1(b_3\zeta_1\zeta_0^2 + \delta_2y) + b_1x) = \zeta_1\zeta_2(b_2x^2\zeta_0 + \delta_1\zeta_1\zeta_0^3(b_6\delta_1\zeta_1\zeta_0^2 + b_4x) + \delta_2\zeta_2x^3). \quad (\text{B.5})$$

The projectivity relations are, as detailed in appendix C.1 for the general case,

$$\begin{aligned} [\delta_2\zeta_2x, \delta_1\delta_2^2\zeta_2y, \zeta_0] \\ [x, \delta_1\delta_2y, \delta_1\zeta_1] \\ [\delta_2y, \zeta_1] \\ [y, \zeta_2] \end{aligned} \quad (\text{B.6})$$

The classes are

Section	Class
x	$\sigma + 2c_1 - E_1 - E_2$
y	$\sigma + 3c_1 - E_1 - E_2 - E_3 - E_4$
ζ_0	$S - E_1$
ζ_1	$E_1 - E_2 - E_3 - E_4$
ζ_2	$E_2 - E_4$
δ_1	E_3
δ_2	E_4

(B.7)

Each of the exceptional sections gives rise to an irreducible exceptional divisor

Divisor	Section	Equation in Y_4
$D_{-\alpha_0}$	ζ_0	$0 = -\zeta_1 x^3 + b_1 y_4 x + y_4^2 \delta_1$
$D_{-\alpha_1}$	ζ_1	$0 = x b_1 + \delta_1$
$D_{-\alpha_2}$	ζ_2	$0 = x b_1 + \delta_1 (\delta_2 + b_3 \zeta_1)$
$D_{-\alpha_4}$	δ_1	$0 = b_1 y_4 - \zeta_1 \zeta_2 (\delta_2 \zeta_2 + b_2 \zeta_0)$
$D_{-\alpha_3}$	δ_2	$0 = y_4 (x b_1 + b_3 \delta_1) - (b_2 x^2 + \delta_1 (x b_4 + b_6 \delta_1)) \zeta_2$

(B.8)

The divisors intersect in the affine A_4 Dynkin diagram. We compute the intersections as explained in appendix C.1. To spell this out, note that the relations in the intersection ring, contained already in projective relations (B.6) are

$$\begin{aligned}
\sigma(\sigma + 2c_1)(\sigma + 3c_1) &= 0 \\
(\sigma + 2c_1 - E_1)(\sigma + 3c_1 - E_1)(S - E_1) &= 0 \\
(\sigma + 2c_1 - E_1 - E_2)(\sigma + 3c_1 - E_1 - E_2)(E_1 - E_2) &= 0 \\
(\sigma + 3c_1 - E_1 - E_2 - E_3)(E_1 - E_2 - E_3) &= 0 \\
(\sigma + 3c_1 - E_1 - E_2 - E_3 - E_4)(E_2 - E_4) &= 0.
\end{aligned}$$
(B.9)

From these we derive (C.10).

The codimension 2 matter loci correspond to $\mathbf{10}$ localized at $b_1 = 0$ and $\bar{\mathbf{5}}$ matter, which is at $P = 0$. From the general analysis in section 3.1.2, in particular (3.13), we infer that along $b_1 = 0$ the splittings are⁶

$$\begin{aligned}
D_{-\alpha_2} &\rightarrow S_{v_2} + S_{v_{k+2}} \\
D_{-\alpha_4} &\rightarrow S_{v_{2k}} + S_{v_{k+2}} + D_{-\alpha_1} \cdot [b_1],
\end{aligned}$$
(B.10)

where $S_{v_1} = D_{-\alpha_1} \cdot [b_1]$ is irreducible, and the weights are as in the Cartan charge table (3.12)

$$\begin{aligned}
v_2 &= (0, 0, -1, 0, 1) \\
v_{2k} &= (0, 1, 0, 0, -1) \\
v_{k+2} &= (0, 1, -1, 1, -1).
\end{aligned}$$
(B.11)

⁶The divisors $S_{v_{k+2}}$ and $S_{v_{2k}}$ would have the same labels for $k = 2$, however, they are distinct, and defined as in section 3.1.2.

Along $P = b_2 b_3^2 + b_1 (b_1 b_6 - b_3 b_4) = 0$, which is the locus of the $\bar{\mathbf{5}}$ matter, the only divisor that becomes reducible is $\delta_{k=2} = 0$. In particular its equation along $P = 0$ factorizes

$$\delta_2 = P = 0 : \quad (b_3 \delta_3 + b_1 x_2) (b_2 b_3 \delta_3 \zeta_2 - b_1 \zeta_2 (b_4 \delta_3 + b_2 x_2) + b_1^2 y) = 0. \quad (\text{B.12})$$

As of section 3.1.2, or by direct computation, the split is

$$D_{-\alpha_3} \quad \rightarrow \quad S_{u_4} + S_{u_5}, \quad (\text{B.13})$$

where

$$\begin{aligned} u_4 &= (0, 0, 1, -1, 0) \\ u_5 &= (0, 0, 0, -1, 1). \end{aligned} \quad (\text{B.14})$$

In codimension 3 there is the $b_1 = b_3 = 0$ $SO(12)$ point and the $b_1 = b_2 = 0$ or E_6 point, which generate bottom and top Yukawas in a GUT model. Along $b_1 = b_3 = 0$ our general analysis implies

$$S_{v_5} = D_{-\alpha_4} \cdot [b_1] \quad \longrightarrow \quad \Sigma_{v_2} + 2 \times \Sigma_{u_4}. \quad (\text{B.15})$$

Or put differently, the restriction of the root $\Sigma_{v_5} = S_{v_5} \cdot [b_3] = D_{-\alpha_4} \cdot [b_1] \cdot [b_3]$ splits to $\Sigma_{v_2} + 2\Sigma_{u_4}$, which all become homologous at $b_1 = b_3 = 0$. This is consistent with a bottom type Yukawa coupling. At $b_1 = b_2 = 0$ the following splitting occurs, see (3.37, 3.40)

$$\begin{aligned} S_{v_{2k}} &\longrightarrow \Sigma_{v_{k+2}} + \Sigma_{w'} \\ S_{-\alpha_{k+1}} &\longrightarrow \Sigma_{u_{2k}} + \Sigma_{u_{2k+1}}. \end{aligned} \quad (\text{B.16})$$

Note that w' was a weight of $\Lambda^4 V \cong V$ for $SU(5)$, and this is therefore consistent with the top Yukawa coupling. The second splitting is an adjoint splitting into fundamental and anti-fundamental. Note that the intersection graph is that of an E_6 (not affine) Dynkin diagram, with multiplicities $(1, 2, 3, 2, 1, 1)$. Note that this fiber does not appear in [8]. Likewise, along $b_1 = b_3 = 0$ the fibers are Kodaira D type with the correct multiplicities, which appears to be different from the findings in [8], however, is in agreement with all the multiplicities obtained in [9].

B.2 $SO(10)$

The resolution of $SO(10)$ was done in [31], however as for $SU(5)$ we present it in the framework of the current paper, so as to have one explicit example using the general analysis for $SO(4k+2)$ with $k = 2$. Furthermore, the low k cases exhibit some pathologies, which will be explained. The singular Tate form is

$$y^2 - x^3 + b_1 x y \zeta_0 - b_2 x^2 \zeta_0 + b_3 y \zeta_0^2 - b_4 x \zeta_0^3 - b_6 \zeta_0^5 = 0. \quad (\text{B.17})$$

This is resolved as in (4.2) with

$$\begin{aligned}
& (x, y, \zeta_0; \zeta_1) \\
& (x, y, \zeta_1; \zeta_2) \\
& (y, \zeta_1; \delta_1) \\
& (y, \zeta_2; \delta_2) \\
& (\delta_1, \zeta_1; \kappa_1) \\
& (\delta_1, \zeta_2; \epsilon_1)
\end{aligned} \tag{B.18}$$

The fully resolved Tate form is

$$y\delta_1(y\delta_2 + \zeta_0\zeta_1(b_3\zeta_0 + b_1x\delta_2\epsilon_1\zeta_2)\kappa_1) = \zeta_1\zeta_2(b_2x^2\zeta_0 + x^3\delta_2\epsilon_1\zeta_2 + \delta_1\epsilon_1\zeta_0^3\zeta_1\kappa_1^2(b_4x + b_6\delta_1\epsilon_1\zeta_0^2\zeta_1\kappa_1^2)). \tag{B.19}$$

The irreducible exceptional divisors are labeled by the affine roots of $SO(10)$, and reproducing the corresponding Dynkin diagram from the intersections in section 4.1

Divisor	Section	Equation in Y_4
$D_{-\alpha_0}$	ζ_0	$0 = y^2\delta_1 - x^3\zeta_1$
$D_{-\alpha_1}$	δ_1	$0 = \delta_2\epsilon_1 + b_2\zeta_0$
$D_{-\alpha_2}$	κ_1	$0 = \delta_1 - \zeta_1\zeta_2(b_2\zeta_0 + \epsilon_1\zeta_2)$
$D_{-\alpha_3}$	ϵ_1	$0 = \delta_1(\delta_2 + b_3\kappa_1) - b_2\zeta_2$
$D_{-\alpha_4}$	ζ_2	$0 = \delta_2 + b_3\zeta_1\kappa_1$
$D_{-\alpha_5}$	δ_2	$0 = b_3y\delta_1 - \zeta_2(b_2x^2 + \delta_1\epsilon_1(b_4x - b_6\delta_1\epsilon_1))$

(B.20)

Note that compared with (4.5) there is an ambiguity as $D_{-\alpha_3}$ has a definition both as $D_{-\alpha_{2i+1}}$ for $i = 1$ and as $D_{-\alpha_{2k-1}}$. Along the codimension two locus $b_2 = 0$ (which is minimal for this low rank example) the Cartan divisors either remain irreducible or split,

$$\begin{aligned}
D_{-\alpha_0} &\longrightarrow S_{-\alpha_0} \\
D_{-\alpha_1} &\longrightarrow S_{v_1} + S_{v_2} \\
D_{-\alpha_2} &\longrightarrow S_{-\alpha_2} \\
D_{-\alpha_3} &\longrightarrow S_{v_1} + S_{v_3} \\
D_{-\alpha_4} &\longrightarrow S_{-\alpha_4} \\
D_{-\alpha_5} &\longrightarrow S_{v_2} + S_{v_4},
\end{aligned} \tag{B.21}$$

where the the restricting of a Cartan divisor that remains irreducible is denoted by the matter surface $S_{-\alpha_i} \equiv D_{-\alpha_i} \cdot [b_2]$, and the additional new components that arise from the splitting are weights of the spin representations **16** and $\overline{\mathbf{16}}$.

Similarly, along the matter locus $b_3 = 0$ the only nontrivial splitting is

$$D_{-\alpha_5} \longrightarrow S_{-\alpha_4} + 2 \times S_{u_5}, \tag{B.22}$$

where $S_{-\alpha_4} = D_{-\alpha_4} \cdot [b_3]$ and u_5 is a weight of the vector representation **10**.

Finally, consider the codimension 3 loci. Along $b_2 = b_3 = 0$ we expect the Yukawa coupling $\mathbf{16}^2\mathbf{10}$. Indeed the matter surfaces restrict as in (4.21)

$$S_{v_4} \longrightarrow \Sigma_{v_3} + \Sigma_{-\alpha_4} + \Sigma_{u_5} \quad (\text{B.23})$$

where this is precisely the coupling $\mathbf{16}^2\mathbf{10}$. Along the other loci, in particular $b_3 = b_4^2 - 4b_2b_6 = 0$ no further splitting occurs, however as in the codimension 3 locus $P = Q = 0$ case for $SU(2k+1)$ the coupling generated is realizing $\mathbf{10}^2\mathbf{1}$.

C Details of the Resolution

C.1 $SU(n)$ and general computational Methods

We consider the $SU(n)$ singularities with resolutions as explained in (3.3, 3.6, 3.43). After the resolution, the sections in the smooth Tate forms (3.4) and (3.44) are in the following classes:

Section	Class
x	$\sigma + 2c_1 - E_1 - \dots - E_k$
y	$\sigma + 3c_1 - E_1 - \dots - E_{2k}$
ζ_0	$S_2 - E_1$
ζ_i	$E_i - E_{i+1} - E_{k+i} \quad i = 1, \dots, k-1$
ζ_k	$E_k - E_{2k}$
δ_i	$E_{k+i} \quad i = 1, \dots, k$

(C.1)

Section	Class
x	$\sigma + 2c_1 - E_1 - \dots - E_k$
y	$\sigma + 3c_1 - E_1 - \dots - E_{2k-1}$
ζ_0	$S_2 - E_1$
ζ_i	$E_i - E_{i+1} - E_{k+i} \quad i = 1, \dots, k-1$
ζ_k	E_k
δ_i	$E_{k+i} \quad i = 1, \dots, k-1$

We consider $n = 2k + 1$ as well as $n = 2k$. After all the blowups the projective relations are

$$\begin{aligned}
SU(2k+1) : \quad & \begin{aligned} & (i) \quad [xG_2^k(\zeta\delta), yG_2^k(\zeta\delta)B_1^k(\delta), \zeta_0] \\ & (ii) \quad [xG_{i+2}^k(\zeta\delta), yG_{i+2}^k(\zeta\delta)B_1^k(\delta), \zeta_i\delta_i] \quad i = 1, \dots, k-1 \\ & (iii) \quad [yB_{i+1}^k(\delta), \zeta_i] \quad i = 1, \dots, k-1 \\ & (iv) \quad [y, \zeta_k] \end{aligned} \\
SU(2k) : \quad & \begin{aligned} & (i) \quad [xG_2^k(\zeta)G_2^{k-1}(\delta), yG_2^k(\zeta)G_2^{k-1}(\delta)B_1^{k-1}(\delta), \zeta_0] \\ & (ii) \quad [xG_{i+2}^k(\zeta)G_{i+2}^{k-1}(\delta), yG_{i+2}^k(\zeta)G_{i+2}^{k-1}(\delta)B_1^{k-1}(\delta), \zeta_i\delta_i] \quad i = 1, \dots, k-1 \\ & (iii) \quad [yB_{i+1}^{k-1}(\delta), \zeta_i] \quad i = 1, \dots, k-1 \end{aligned}
\end{aligned} \tag{C.2}$$

where, in the usual way, $[a, b, c]$ for instances means, that not all three entries can vanish at the same time. Recall the notation

$$\begin{aligned}
A(z) &= \prod_{i=1}^{k-1} z_i^{i-1} \\
C(z) &= \prod_{i=1}^{k-1} z_i^{k-(i+1)} \\
B(z) &= \prod_{i=1}^{k-1} z_i,
\end{aligned} \tag{C.3}$$

where it is understood that e.g. $A(\zeta\delta) = \prod_{i=1}^{k-1} (\zeta_i\delta_i)^{i-1}$. Futhermore define

$$\begin{aligned}
G_i^n(\zeta) &= \prod_{j=i}^n \zeta_j^{j-i+1} \\
B_i^n(\zeta) &= \prod_{j=i}^n \zeta_j,
\end{aligned} \tag{C.4}$$

where it is understood that $B_i^n(x) = G_i^n(x) = 1$ when $i > n$. For each singularity type we compute two types of intersections:

- Intersections of the Cartan divisors

$$D_{-\alpha_i} \cdot_{Y_4} C_{-\alpha_j} = D_{-\alpha_i} \cdot_{X_5} D_{-\alpha_j} \cdot_{X_5} [Y_4] \cdot_{X_5} D_1 \cdot_{X_5} D_2 \tag{C.5}$$

whith $C_{-\alpha_j}$ being the component of the fiber associated to the root $-\alpha_j$, and $D_{-\alpha_i}$ is the Cartan divisor obtained by fibering $C_{-\alpha_i}$ over S_2 . In the last equality we have chosen D_1, D_2 such that $D_1 \cdot_{B_3} D_2 \cdot_{B_3} S_2 = 1$. The only non-trivial intersection in the fiber is σ^2 , so that from the above intersection we need to extract the $\sigma^2 S_2$ coefficient in the fivefold intersection.

- Intersection of Cartan divisors with Matter surfaces. The matter surfaces are of the type, e.g. for $b_1 = 0$ matter

$$S_v = D_{-\alpha_i} \cdot [b_1] \quad (\text{C.6})$$

To determine the weights (“Cartan charges”) v we need to intersect with the original Cartans:

To compute these intersections, we need to take into account the relations that result from the blowups. These allow us to replace certain combinations of intersections and rewrite everything in terms of independent intersections. The blowup relations are

$$\begin{aligned} (i) \quad & 0 = (\sigma + 2c_1 - E_1)(\sigma + 3c_1 - E_1)(S_2 - E_1) \\ (ii) \quad & 0 = (\sigma + 2c_1 - E_1 - \cdots - E_{i+1})(\sigma + 3c_1 - E_1 - \cdots - E_{i+1})(E_i - E_{i+1}) \quad i = 1, \dots, k-1 \\ (iii) \quad & 0 = (\sigma + 3c_1 - E_1 - \cdots - E_{k+i})(E_i - E_{i+1} - E_{k+i}) \quad i = 1, \dots, k-1 \\ (iv) \quad & 0 = (\sigma + 3c_1 - E_1 - \cdots - E_{2k})(E_k - E_{2k}) \end{aligned} \quad (\text{C.7})$$

Where (iv) only holds in the $SU(2k+1)$ case. The relations (i) and (ii), which arise from the blowups, can be solved for E_i^3 , whereas the ones from the small resolutions (iii) and (iv) are solved for E_{k+i}^2 . We also have the blowup relation coming from the original \mathbb{P}^2 bundle, i.e. the projective relation $[x, y, w]$,

$$\sigma^3 = -5c_1\sigma^2 - 6c_1^2\sigma. \quad (\text{C.8})$$

Futhermore, the blowups were all in the patch $w = 1$, which means that the divisor $w = 0$ does not intersect the exceptional divisors, i.e.

$$\sigma \cdot E_i = \sigma \cdot E_{k+i} = 0. \quad (\text{C.9})$$

The computational strategy is now as follows: as explained above, the only non-trivial intersection in the fiber is σ^2 . In order to obtain a non-trivial intersection for Cartan divisors, we thus need to determine the coefficient of $\sigma^2 S_2$ in (C.5). So, given a fivefold intersection in X_5 , we first use the the blowup relations to reduce the expressions to independent intersection products, as well as (C.9), to extract the coefficient of $\sigma^2 S_2$, which intersects once with $D_1 D_2$. We will now first use the blowup relations to determine an independent set of triple intersections.

From the blowup relations we can determine the following triple intersections, which allow us to replace all E_i^3 and E_{k+i}^2 in all intersection computations

$$\begin{aligned} E_i^3 &= \sigma^2 S_2 + \cdots & i = 1, \dots, k \\ E_j E_{k+i}^2 &= (\delta_{ij} - \delta_{j,i+1}) \sigma^2 S_2 + \cdots & j = 1, \dots, k; i = 1, \dots, k-1 \\ E_{k+j} E_{k+i}^2 &= ((3 - \delta_{i1}) \delta_{ij} - \delta_{i,j+1}) \sigma^2 S_2 + \cdots & i, j = 1, \dots, k-1 \end{aligned} \quad (\text{C.10})$$

For $SU(2k+1)$ we have in addition the following relations

$$\begin{aligned} E_j E_{2k}^2 &= \delta_{jk} \sigma^2 S_2 + \dots & j = 1, \dots, k \\ E_{k+j} E_{2k}^2 &= (\delta_{jk} - \delta_{k+j+1}) \sigma^2 S_2 + \dots & j = 1, \dots, k. \end{aligned} \quad (\text{C.11})$$

Note that the \dots indicate terms not involving $\sigma^2 S_2$. For instance we can compute the intersections of the Cartan divisors which have classes (C.1) with this information. Consider the product (C.5) and expand it in terms of E_i , and apply the blowup relations (C.10), extracting the coefficient of $\sigma^2 S_2$ and making use of (C.9).

C.2 $SO(4k+2)$

After performing the blowups (4.2) on the geometry (4.1) then the classes of the sections in the new, resolved geometry are

Section	Class
x	$\sigma + 2c_1 - E_1 - \dots - E_k$
y	$\sigma + 3c_1 - E_1 - \dots - E_{2k}$
ζ_0	$S_2 - E_1$
ζ_1	$E_1 - E_2 - E_{k+1} - E_{2k+1}$
ζ_i	$E_i - E_{i+1} - E_{k+i} - E_{2k+i} - E_{3k+i-1}$
ζ_k	$E_k - E_{2k} - E_{4k-1}$
δ_i	$E_{k+i} - E_{2k+i} - E_{3k+i}$
δ_k	E_{2k}
κ_i	E_{2k+i}
ϵ_i	E_{3k+i}
Y_4	$3\sigma + 6c_1 - 2E_1 - \dots - 2E_k - E_{k+1} - \dots - E_{3k-1} - E_{3k+1} - \dots - E_{4k-1}$

(C.12)

The projective relations in the resolved geometry are

$$\begin{aligned} (i) & \left[x G_2^k (\delta \zeta) G_2^{k-1} (\kappa^2 \epsilon) G_1^{k-1} (\epsilon) B_1^{k-1} (\epsilon), y B_1^{k-1} (\delta \kappa \epsilon^2) \delta_k G_2^k (\zeta \delta) G_2^{k-1} (\kappa^2 \epsilon) G_1^{k-1} (\epsilon), \zeta_0 \right] \\ (ii) & \left[x G_3^k (\zeta \delta) G_3^{k-1} (\kappa^2 \epsilon) G_2^{k-1} (\epsilon) B_2^{k-1} (\epsilon), y B_1^{k-1} (\delta \kappa \epsilon) \delta_k x G_3^k (\zeta \delta) G_3^{k-1} (\kappa^2 \epsilon) G_2^{k-1} (\epsilon) B_2^{k-1} (\epsilon), \zeta_1 \delta_1 \kappa_1^2 \epsilon_1 \right] \\ (iii) & \left[x G_{i+2}^k (\zeta \delta) G_{i+2}^{k-1} (\kappa^2 \epsilon) G_{i+1}^{k-1} (\epsilon) B_{i+1}^{k-1} (\epsilon), y B_1^{k-1} (\delta \kappa \epsilon) \delta_k x G_{i+2}^k (\zeta \delta) G_{i+2}^{k-1} (\kappa^2 \epsilon) G_{i+1}^{k-1} (\epsilon) B_{i+1}^{k-1} (\epsilon), \zeta_i \delta_i \kappa_i^2 \epsilon_i \epsilon_{i-1} \right] \\ & \hspace{15em} i = 2, \dots, k-1 \\ (iv) & \left[y B_2^{k-1} (\delta \kappa \epsilon) \delta_k, \zeta_1 \kappa_1 \right] \\ (v) & \left[y B_{i+1}^{k-1} (\delta \kappa \epsilon) \delta_k, \zeta_i \kappa_i \epsilon_{i-1} \right] \hspace{15em} i = 2, \dots, k-1 \\ (vi) & \left[y, \zeta_k \epsilon_{k-1} \right] \\ (vii) & \left[\zeta_1, \delta_1 \epsilon_1 \right] \\ (viii) & \left[\zeta_i \epsilon_{i-1}, \delta_i \epsilon_i \right] \hspace{15em} i = 2, \dots, k-1 \\ (ix) & \left[\delta_i, \zeta_{i+1} \right] \hspace{15em} i = 1, \dots, k-1 \\ (x) & \left[\delta_{k-1}, \zeta_k \right] \end{aligned} \quad (\text{C.13})$$

The projective relations give rise to the blowup relations

$$\begin{array}{l|l}
(i) & (\sigma + 2c_1 - E_1)(\sigma + 3c_1 - E_1)(S_2 - E_1) = 0 \\
(ii) & (\sigma + 2c_1 - E_1 - E_2)(\sigma + 3c_1 - E_1 - E_2)(E_1 - E_2) = 0 \\
(iii) & (\sigma + 2c_1 - E_1 - \dots - E_{i+1})(\sigma + 3c_1 - E_1 - \dots - E_{i+1})(E_i - E_{i+1}) = 0 \\
(iv) & (\sigma + 3c_1 - E_1 - \dots - E_{k+1})(E_1 - E_2 - E_{k+1}) = 0 \\
(v) & (\sigma + 3c_1 - E_1 - \dots - E_{k+i})(E_i - E_{i+1} - E_{k+i}) = 0 \\
(vi) & (\sigma + 3c_1 - E_1 - \dots - E_{2k})(E_k - E_{2k}) = 0 \\
(vii) & (E_1 - E_2 - E_{k+1} - E_{2k+1})(E_{k+1} - E_{2k+1}) = 0 \\
(viii) & (E_i - E_{i+1} - E_{k+i} - E_{2k+i})(E_{k+i} - E_{2k+i}) = 0 \\
(ix) & (E_{k+i} - E_{2k+i} - E_{3k+i})(E_{i+1} - E_{i+2} - E_{k+i+1} - E_{2k+i+1} - E_{3k+i}) = 0 \\
(x) & (E_{2k-1} - E_{3k-1} - E_{4k-1})(E_1 - E_{2k} - E_{4k-1}) = 0
\end{array} \tag{C.14}$$

Again we solve these in terms of an independent set of triple intersections, noting only the $\sigma^2 S_2$ component, where \dots represent terms that are not of the form $A\sigma^2 S_2$.

$$\begin{array}{l}
E_i^3 = \sigma^2 S_2 + \dots \\
E_j E_{k+i}^2 = (\delta_{i,j} - \delta_{j,i+1})\sigma^2 S_2 + \dots \\
E_{k+j} E_{k+i}^2 = ((3 - \delta_{i,1})\delta_{i,j} - \delta_{i,j+1})\sigma^2 S_2 + \dots \\
E_j E_{2k}^2 = \delta_{j,k}\sigma^2 S_2 + \dots \\
E_{k+j} E_{2k}^2 = -\delta_{k,j+1}\sigma^2 S_2 + \dots \\
E_{2k}^3 = \sigma^2 S_2 + \dots \\
E_j E_{2k+i}^2 = (\delta_{i,j} - \delta_{j,i+1})\sigma^2 S_2 + \dots \\
E_{k+j} E_{2k+i}^2 = ((1 - \delta_{i,1})\delta_{i,j} - \delta_{i,j+1})\sigma^2 S_2 + \dots \\
E_{2k} E_{2k+i}^2 = 0 + \dots \\
E_{2k+j} E_{2k+i}^2 = 2\delta_{i,j}\sigma^2 S_2 + \dots \\
E_j E_{3k+i}^2 = \dots \\
E_{k+j} E_{3k+i}^2 = (\delta_{i,j} - \delta_{j,i+1})\sigma^2 S_2 + \dots \\
E_{2k} E_{3k+i}^2 = -\delta_{k,i+1}\sigma^2 S_2 + \dots \\
E_{2k+j} E_{3k+i}^2 = -(\delta_{i,j} + \delta_{j,i+1})\sigma^2 S_2 + \dots \\
E_{3k+j} E_{3k+i}^2 = 4\delta_{i,j}\sigma^2 S_2 + \dots \\
E_j E_{4k-1}^2 = 0 + \dots \\
E_{k+j} E_{4k-1}^2 = \delta_{k,j+1}\sigma^2 S_2 + \dots \\
E_{2k} E_{4k-1}^2 = -\sigma^2 S_2 + \dots \\
E_{2k+j} E_{4k-1}^2 = -\delta_{k,j+1}\sigma^2 S_2 + \dots \\
E_{3k+j} E_{4k-1}^2 = 3\delta_{k,j+1}\sigma^2 S_2 + \dots
\end{array} \tag{C.15}$$

Using the blowup relations and the classes of the exceptional sections we find the non-zero

intersections of the Cartans with each other and Y_4 to be

$$\begin{aligned}
[\zeta_0] \cdot_{X_5} [\zeta_0] \cdot_{X_5} [Y_4] &= -2\sigma^2 S_2 + \dots \\
[\zeta_0] \cdot_{X_5} [\kappa_i] \cdot_{X_5} [Y_4] &= \delta_{i,1} \sigma^2 S_2 + \dots \\
[\delta_1] \cdot_{X_5} [\delta_1] \cdot_{X_5} [Y_4] &= -2\sigma^2 S_2 + \dots \\
[\delta_1] \cdot_{X_5} [\kappa_i] \cdot_{X_5} [Y_4] &= \delta_{i,1} \sigma^2 S_2 + \dots \\
[\kappa_i] \cdot_{X_5} [\kappa_j] \cdot_{X_5} [Y_4] &= -2\delta_{i,j} \sigma^2 S_2 + \dots \\
[\kappa_i] \cdot_{X_5} [\epsilon_j] \cdot_{X_5} [Y_4] &= (\delta_{i,j} + \delta_{i,j+1}) \sigma^2 S_2 + \dots \\
[\epsilon_i] \cdot_{X_5} [\epsilon_j] \cdot_{X_5} [Y_4] &= -2\delta_{i,j} \sigma^2 S_2 + \dots \\
[\epsilon_i] \cdot_{X_5} [\zeta_k] \cdot_{X_5} [Y_4] &= \delta_{k,i+1} \sigma^2 S_2 + \dots \\
[\epsilon_i] \cdot_{X_5} [\delta_k] \cdot_{X_5} [Y_4] &= \delta_{k,i+1} \sigma^2 S_2 + \dots \\
[\zeta_k] \cdot_{X_5} [\zeta_k] \cdot_{X_5} [Y_4] &= -2\sigma^2 S_2 + \dots \\
[\delta_k] \cdot_{X_5} [\delta_k] \cdot_{X_5} [Y_4] &= -2\sigma^2 S_2 + \dots
\end{aligned} \tag{C.16}$$

With all other intersections of this form between any 2 exceptional sections and Y_4 having no factor like $A\sigma^2 S_2$. Note that the coefficients of $\sigma^2 S_2$ here reproduce the negative of the Cartan matrix for $SO(4k+2)$.

When we go to the codimension 2 loci certain divisors become reducible and we need to calculate the Cartan charge vector associated to each irreducible component. To calculate these we need to know the following intersections (where \dots are terms not of the type $A\sigma^2 S_2$)

$$\begin{aligned}
[\kappa_i] \cdot_{X_5} [\delta_i] \cdot_{X_5} [\zeta_0] &= \dots \\
[\kappa_i] \cdot_{X_5} [\delta_i] \cdot_{X_5} [\delta_1] &= \delta_{i,2} \sigma^2 S_2 + \dots \\
[\kappa_i] \cdot_{X_5} [\delta_i] \cdot_{X_5} [\kappa_j] &= -\delta_{i,j} \sigma^2 S_2 + \dots \\
[\kappa_i] \cdot_{X_5} [\delta_i] \cdot_{X_5} [\epsilon_j] &= \delta_{i,j} \sigma^2 S_2 + \dots \\
[\kappa_i] \cdot_{X_5} [\delta_i] \cdot_{X_5} [\zeta_k] &= \dots \\
[\kappa_i] \cdot_{X_5} [\delta_i] \cdot_{X_5} [\delta_k] &= \dots \\
[\epsilon_{k-1}] \cdot_{X_5} [\delta_{k-1}] \cdot_{X_5} [\zeta_0] &= \dots \\
[\epsilon_{k-1}] \cdot_{X_5} [\delta_{k-1}] \cdot_{X_5} [\delta_1] &= \dots \\
[\epsilon_{k-1}] \cdot_{X_5} [\delta_{k-1}] \cdot_{X_5} [\kappa_i] &= \delta_{k,i+1} \sigma^2 S_2 + \dots \\
[\epsilon_{k-1}] \cdot_{X_5} [\delta_{k-1}] \cdot_{X_5} [\epsilon_i] &= -\delta_{k,i+1} \sigma^2 S_2 + \dots \\
[\epsilon_{k-1}] \cdot_{X_5} [\delta_{k-1}] \cdot_{X_5} [\zeta_k] &= \dots \\
[\epsilon_{k-1}] \cdot_{X_5} [\delta_{k-1}] \cdot_{X_5} [\delta_k] &= \sigma^2 S_2 + \dots \\
[\delta_k] \cdot_{X_5} [\delta_{k-1}] \cdot_{X_5} [\zeta_0] &= \dots \\
[\delta_k] \cdot_{X_5} [\delta_{k-1}] \cdot_{X_5} [\delta_1] &= \dots \\
[\delta_k] \cdot_{X_5} [\delta_{k-1}] \cdot_{X_5} [\kappa_i] &= \dots \\
[\delta_k] \cdot_{X_5} [\delta_{k-1}] \cdot_{X_5} [\epsilon_i] &= \delta_{k,i+1} \sigma^2 S_2 + \dots \\
[\delta_k] \cdot_{X_5} [\delta_{k-1}] \cdot_{X_5} [\zeta_k] &= \dots \\
[\delta_k] \cdot_{X_5} [\delta_{k-1}] \cdot_{X_5} [\delta_k] &= -\sigma^2 S_2 + \dots
\end{aligned} \tag{C.17}$$

C.3 $SO(4k + 4)$

The classes of the various sections in the resolved geometry are

Section	Class		
x	$\sigma + 2c_1 - E_1 - \dots - E_{k+1}$		
y	$\sigma + 3c_1 - E_1 - \dots - E_{2k+1}$		
ζ_0	$S_2 - E_1$		
ζ_1	$E_1 - E_2 - E_{k+2} - E_{2k+2}$		
ζ_i	$E_i - E_{i+1} - E_{k+1+i} - E_{2k+1+i} - E_{3k+i}$	$i = 2, \dots, k$	(C.18)
ζ_{k+1}	E_{k+1}		
δ_i	$E_{k+1+i} - E_{2k+1+i} - E_{3k+1+i}$	$i = 1, \dots, k-1$	
δ_k	$E_{2k+1} - E_{3k+1}$		
κ_i	E_{2k+1+i}	$i = 1, \dots, k$	
ϵ_i	E_{3k+1+i}	$i = 1, \dots, k-1$	
Y_4	$3\sigma + 6c_1 - 2E_1 - \dots - 2E_{k+1} - E_{k+2} - \dots - E_{4k}$		

And the projective relations are

(i)	$[xG_2^{k+1}(\zeta)G_2^k(\delta)G_2^{k-1}(\kappa^2\epsilon)G_1^{k-1}(\epsilon)B_1^{k-1}(\epsilon), yB_1^{k-1}(\delta\kappa\epsilon)\delta_k G_2^{k+1}(\zeta)G_2^k(\delta)G_2^{k-1}(\kappa^2\epsilon)G_1^{k-1}(\epsilon)B_1^{k-1}(\epsilon), \zeta_0]$	
(ii)	$[xG_3^{k+1}(\zeta)G_3^k(\delta)G_3^{k-1}(\kappa^2\epsilon)G_2^{k-1}(\epsilon)B_2^{k-1}(\epsilon), yB_1^{k-1}(\delta\kappa\epsilon)\delta_k G_3^{k+1}(\zeta)G_3^k(\delta)G_3^{k-1}(\kappa^2\epsilon)G_2^{k-1}(\epsilon)B_2^{k-1}(\epsilon), \zeta_1\delta_1\kappa_1^2\epsilon_1]$	
(iii)	$[xG_{i+2}^{k+1}(\zeta)G_{i+2}^k(\delta)G_{i+2}^{k-1}(\kappa^2\epsilon)G_{i+1}^{k-1}(\epsilon)B_{i+1}^{k-1}(\epsilon), yB_1^{k-1}(\delta\kappa\epsilon)\delta_k G_{i+2}^{k+1}(\zeta)G_{i+2}^k(\delta)G_{i+2}^{k-1}(\kappa^2\epsilon)G_{i+1}^{k-1}(\epsilon)B_{i+1}^{k-1}(\epsilon), \zeta_i\delta_i\kappa_i^2\epsilon_{i-1}\epsilon_i], \quad i = 2, \dots, k-1$	
(iv)	$[x, yB_1^{k-1}(\delta\kappa\epsilon)\delta_k, \zeta_k\delta_k\kappa_k^2\epsilon_{k-1}]$	
(v)	$[yB_2^{k-1}(\delta\kappa\epsilon)\delta_k, \zeta_1\kappa_1]$	
(vi)	$[yB_{i+1}^{k-1}(\delta\kappa\epsilon)\delta_k, \zeta_i\kappa_i\epsilon_{i-1}]$	$i = 2, \dots, k-1$
(vii)	$[\zeta_1, \delta_1\epsilon_1]$	
(viii)	$[\zeta_i\epsilon_{i-1}, \delta_i\epsilon_i]$	$i = 2, \dots, k-1$
(ix)	$[\zeta_k\epsilon_{k-1}, \delta_k]$	
(x)	$[\zeta_{i+1}, \delta_i]$	$i = 1, \dots, k-2$
(xi)	$[\zeta_k, \delta_{k-1}]$	

(C.19)

The projective relations imply blowup relations

$$\begin{aligned}
&(\sigma + 2c_1 - E_1)(\sigma + 3c_1 - E_1)(S_2 - E_1) = 0 \\
&(\sigma + 2c_1 - E_1 - \dots - E_{i+1})(\sigma + 3c_1 - E_1 - \dots - E_{i+1})(E_i - E_{i+1}) = 0 \quad i = 1, \dots, k \\
&(\sigma + 3c_1 - E_1 - \dots - E_{k+1+i})(E_i - E_{i+1} - E_{k+1+i}) = 0 \quad i = 1, \dots, k \\
&(E_i - E_{i+1} - E_{k+1+i} - E_{2k+1+i})(E_{k+1+i} - E_{2k+1+i}) = 0 \quad i = 1, \dots, k \\
&(E_{i+1} - E_{i+2} - E_{k+2+i} - E_{2k+2+i} - E_{3k+1+i})(E_{k+1+i} - E_{2k+1+i} - E_{3k+1+i}) = 0 \quad i = 1, \dots, k-1
\end{aligned}$$

(C.20)

We eliminate the non-independent 3-way intersections. keeping terms of the type $\sigma^2 S_2$ only

$$\begin{aligned}
E_i^3 &= \sigma^2 S_2 + \dots \\
E_j E_{k+1+i}^2 &= (\delta_{i,j} - \delta_{j,i+1}) \sigma^2 S_2 + \dots \\
E_{k+1+j} E_{k+1+i}^2 &= ((3 - \delta_{i,1}) \delta_{i,j} - \delta_{i,j+1}) \sigma^2 S_2 + \dots \\
E_j E_{2k+1+i}^2 &= (\delta_{i,j} - \delta_{j,i+1}) \sigma^2 S_2 + \dots \\
E_{k+1+j} E_{2k+1+i}^2 &= ((1 - \delta_{i,1}) \delta_{i,j} - \delta_{i,j+1}) \sigma^2 S_2 + \dots \\
E_{2k+1+j} E_{2k+1+i}^2 &= 2\delta_{i,j} \sigma^2 S_2 + \dots \\
E_j E_{3k+1+i}^2 &= \dots \\
E_{k+1+j} E_{3k+1+i}^2 &= (\delta_{i,j} - \delta_{j,i+1}) \sigma^2 S_2 + \dots \\
E_{2k+1+j} E_{3k+1+i}^2 &= -(\delta_{i,j} + \delta_{j,i+1}) \sigma^2 S_2 + \dots \\
E_{3k+1+j} E_{3k+1+i}^2 &= 4\delta_{i,j} \sigma^2 S_2 + \dots
\end{aligned} \tag{C.21}$$

The non-zero (in the coefficient of $\sigma^2 S_2$ sense) intersections involving two of the exceptional sections and Y_4 are

$$\begin{aligned}
[\zeta_0] \cdot_{X_5} [\zeta_0] \cdot_{X_5} [Y_4] &= -2\sigma^2 S_2 + \dots \\
[\zeta_0] \cdot_{X_5} [\kappa_i] \cdot_{X_5} [Y_4] &= \delta_{i,1} \sigma^2 S_2 + \dots \\
[\delta_1] \cdot_{X_5} [\delta_1] \cdot_{X_5} [Y_4] &= -2\sigma^2 S_2 + \dots \\
[\delta_1] \cdot_{X_5} [\kappa_i] \cdot_{X_5} [Y_4] &= \delta_{i,1} \sigma^2 S_2 + \dots \\
[\kappa_i] \cdot_{X_5} [\kappa_j] \cdot_{X_5} [Y_4] &= -2\delta_{i,j} \sigma^2 S_2 + \dots \\
[\kappa_i] \cdot_{X_5} [\epsilon_j] \cdot_{X_5} [Y_4] &= (\delta_{i,j} - \delta_{i,j+1}) \sigma^2 S_2 + \dots \\
[\kappa_i] \cdot_{X_5} [\zeta_{k+1}] \cdot_{X_5} [Y_4] &= \delta_{i,k} \sigma^2 S_2 + \dots \\
[\kappa_i] \cdot_{X_5} [\delta_k] \cdot_{X_5} [Y_4] &= \delta_{i,k} \sigma^2 S_2 + \dots \\
[\epsilon_i] \cdot_{X_5} [\epsilon_j] \cdot_{X_5} [Y_4] &= -2\delta_{i,j} \sigma^2 S_2 + \dots \\
[\zeta_{k+1}] \cdot_{X_5} [\zeta_{k+1}] \cdot_{X_5} [Y_4] &= -2\sigma^2 S_2 + \dots \\
[\delta_k] \cdot_{X_5} [\delta_k] \cdot_{X_5} [Y_4] &= -2\sigma^2 S_2 + \dots
\end{aligned} \tag{C.22}$$

Additionally for higher codimension it will be necessary to know the following intersections to work out the Cartan charge vectors of the irreducible matter surfaces

$$\begin{aligned}
[\kappa_i] \cdot_{X_5} [\delta_i] \cdot_{X_5} [\zeta_0] &= \dots \\
[\kappa_i] \cdot_{X_5} [\delta_i] \cdot_{X_5} [\delta_1] &= \delta_{i,2} \sigma^2 S_2 + \dots \\
[\kappa_i] \cdot_{X_5} [\delta_i] \cdot_{X_5} [\kappa_j] &= -\delta_{i,j} \sigma^2 S_2 + \dots \\
[\kappa_i] \cdot_{X_5} [\delta_i] \cdot_{X_5} [\epsilon_j] &= \delta_{i,j} \sigma^2 S_2 + \dots \\
[\kappa_i] \cdot_{X_5} [\delta_i] \cdot_{X_5} [\zeta_{k+1}] &= \dots \\
[\kappa_i] \cdot_{X_5} [\delta_i] \cdot_{X_5} [\delta_k] &= \dots \\
[\kappa_k] \cdot_{X_5} [\delta_{k-1}] \cdot_{X_5} [\zeta_0] &= \dots \\
[\kappa_k] \cdot_{X_5} [\delta_{k-1}] \cdot_{X_5} [\delta_1] &= \dots \\
[\kappa_k] \cdot_{X_5} [\delta_{k-1}] \cdot_{X_5} [\kappa_i] &= -\delta_{i,k} \sigma^2 S_2 + \dots \\
[\kappa_k] \cdot_{X_5} [\delta_{k-1}] \cdot_{X_5} [\epsilon_i] &= \delta_{k,i+1} \sigma^2 S_2 + \dots \\
[\kappa_k] \cdot_{X_5} [\delta_{k-1}] \cdot_{X_5} [\zeta_{k+1}] &= \dots \\
[\kappa_k] \cdot_{X_5} [\delta_{k-1}] \cdot_{X_5} [\delta_k] &= \sigma^2 S_2 + \dots
\end{aligned} \tag{C.23}$$

C.4 E_7

In this section we provide a few details on the resolution of the E_7 singularity. The coordinates have to satisfy the following projective relations

$$\begin{aligned}
& [x\delta_2\varepsilon_1\varepsilon_3^3\varepsilon_4\varepsilon_5^2\varepsilon_6^2\zeta_2, y\delta_1\delta_2^2\varepsilon_1^2\varepsilon_2\varepsilon_3^4\varepsilon_4^3\varepsilon_5^4\varepsilon_6^5\zeta_2, \zeta_0] \\
& [x, y\delta_1\delta_1\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4^2\varepsilon_5^2\varepsilon_6^3, \delta_1\varepsilon_1\varepsilon_2^2\varepsilon_4\varepsilon_5\varepsilon_6^2\zeta_1] \\
& [y\delta_1\varepsilon_3\varepsilon_4\varepsilon_5\varepsilon_6, \varepsilon_2\zeta_1] \\
& [y, \varepsilon_1\varepsilon_3\varepsilon_5\varepsilon_6\zeta_2] \\
& [\varepsilon_3\zeta_2, \delta_1\varepsilon_2\varepsilon_4\varepsilon_6] \\
& [\zeta_1, \delta_1\varepsilon_4\varepsilon_6] \\
& [\zeta_2, \delta_2\varepsilon_4\varepsilon_5\varepsilon_6] \\
& [\delta_1, \delta_2\varepsilon_5] \\
& [\delta_2, \varepsilon_1\varepsilon_6] \\
& [\varepsilon_1, \varepsilon_4]
\end{aligned} \tag{C.24}$$

The various sections have the following classes

Section	Class
x	$\sigma + 2c_1 - E_1 - E_2$
y	$\sigma + 3c_1 - E_1 - E_2 - E_3 - E_4$
ζ_0	$S_2 - E_1$
ζ_1	$E_1 - E_2 - E_3 - E_6$
ζ_2	$E_2 - E_4 - E_5 - E_7$
δ_1	$E_3 - E_5 - E_6 - E_8$
δ_2	$E_4 - E_7 - E_8 - E_9$
ε_1	$E_5 - E_9 - E_{10}$
ε_2	E_6
ε_3	E_7
ε_4	$E_8 - E_{10}$
ε_5	E_9
ε_6	E_{10}
Y_4	$3\sigma + 6c_1 - 2E_1 - 2E_2 - E_3 - E_4 - E_5 - E_6 - E_7 - E_8 - E_9 - E_{10}$

(C.25)

C.5 E_8

In section 5.3 we resolved the E_8 singularity. The projectivity relations for the coordinates that were used to resolve the geometry are

$$\begin{aligned}
& [x\delta_2\varepsilon_1\varepsilon_{10}^4\varepsilon_3^2\varepsilon_4\varepsilon_5^2\varepsilon_6^2\varepsilon_7^2\varepsilon_8^3\varepsilon_9^3\zeta_2, y\delta_1\delta_2^2\varepsilon_1^2\varepsilon_{10}^9\varepsilon_2\varepsilon_3^3\varepsilon_4^3\varepsilon_5^4\varepsilon_6^5\varepsilon_7^5\varepsilon_8^6\varepsilon_9^7\zeta_2, \zeta_0] \\
& [x, y\delta_1\delta_2\varepsilon_1\varepsilon_{10}^5\varepsilon_2\varepsilon_3\varepsilon_4^2\varepsilon_5^4\varepsilon_6^3\varepsilon_7^3\varepsilon_8^3\varepsilon_9^4, \delta_1\varepsilon_1\varepsilon_{10}^2\varepsilon_2^2\varepsilon_4\varepsilon_5\varepsilon_6^2\varepsilon_7\varepsilon_8\varepsilon_9^2\zeta_1] \\
& [y, \delta_2\varepsilon_{10}^3\varepsilon_3\varepsilon_4\varepsilon_5\varepsilon_6\varepsilon_7^2\varepsilon_8^2\varepsilon_9^2, \varepsilon_2\zeta_1] \\
& [y, \varepsilon_1\varepsilon_{10}\varepsilon_3\varepsilon_5\varepsilon_6\varepsilon_8\varepsilon_9\zeta_2] \\
& [\varepsilon_3\zeta_2, \delta_1\varepsilon_{10}\varepsilon_2\varepsilon_4\varepsilon_6\varepsilon_7\varepsilon_9] \\
& [\zeta_1, \delta_1\varepsilon_{10}\varepsilon_4\varepsilon_6\varepsilon_7\varepsilon_9] \\
& [\zeta_2, \delta_2\varepsilon_{10}^3\varepsilon_4\varepsilon_5\varepsilon_6\varepsilon_7^2\varepsilon_8^2\varepsilon_9^2] \\
& [\delta_1, \delta_2\varepsilon_{10}^2\varepsilon_5\varepsilon_7\varepsilon_8^2\varepsilon_9] \\
& [\delta_2\varepsilon_{10}\varepsilon_7\varepsilon_8, \varepsilon_1\varepsilon_6] \\
& [\varepsilon_1, \varepsilon_4\varepsilon_7\varepsilon_9\varepsilon_{10}] \\
& [\delta_2\varepsilon_8, \varepsilon_4\varepsilon_9] \\
& [\delta_2, \varepsilon_5\varepsilon_9\varepsilon_{10}] \\
& [\varepsilon_4, \varepsilon_5\varepsilon_{10}] \\
& [\varepsilon_5, \varepsilon_7]
\end{aligned} \tag{C.26}$$

The various sections have the following classes

Section	Class
x	$\sigma + 2c_1 - E_1 - E_2$
y	$\sigma + 3c_1 - E_1 - E_2 - E_3 - E_4$
ζ_0	$S_2 - E_1$
ζ_1	$E_1 - E_2 - E_3 - E_6$
ζ_2	$E_2 - E_4 - E_5 - E_7$
δ_1	$E_3 - E_5 - E_6 - E_8$
δ_2	$E_4 - E_7 - E_8 - E_9 - E_{10} - E_{11}$
ε_1	$E_5 - E_9 - E_{11}$
ε_2	E_6
ε_3	E_7
ε_4	$E_8 - E_{10} - E_{11} - E_{13}$
ε_5	$E_9 - E_{12} - E_{13} - E_{14}$
ε_6	E_{10}
ε_7	$E_{11} - E_{14}$
ε_8	E_{12}
ε_9	E_{13}
ε_{10}	E_{14}

Finally, the class of the fourfold after the resolutions is given by

$$[Y_4] = 3\sigma + 6c_1 - 2E_1 - 2E_2 - E_3 - E_4 - E_5 - E_6 - E_7 - E_8 - E_9 - E_{10} - E_{11} - E_{12} - E_{13} - E_{14}. \tag{C.28}$$

D Alternative Resolution of $SO(4k+2)$ and Non-Minimality

In this appendix we discuss an alternative resolution of $SO(4k+2)$, which we believe is related by flops to the one in the main text. The structure of the minimal singular loci is unchanged, however, something interesting happens along the non-minimal matter locus $b_2 = 0$, which we will discuss here.

Again, our starting point is the Tate form for $SO(4k+2)$

$$y^2 - x^3 + b_1xy\zeta_0 - b_2x^2\zeta_0 + b_3y\zeta_0^k - b_4x\zeta_0^{k+1} - b_6\zeta_0^{2k+1} = 0. \quad (\text{D.1})$$

We consider here the resolution, using the notation of (2.17), where we first apply as many blowups as possible, and then additional small resolutions to make the exceptional divisors irreducible

$$\begin{aligned} (x, y, \zeta_i; \zeta_{i+1}) & \quad i = 0, \dots, k-1 \\ (y, \zeta_i, \zeta_{i+1}; \gamma_i) & \quad i = 1, \dots, k-1 \\ (y, \zeta_i; \delta_i) & \quad i = 1, \dots, k \\ (\delta_1, \zeta_1; \delta_0) & \end{aligned} \quad (\text{D.2})$$

The last blowup is not needed to resolve the space, merely to make each of our exceptional sections irreducible. The resolved geometry is

$$\begin{aligned} & y^2B(\delta)\delta_k - x^3B(\zeta\delta)A(\zeta\delta\gamma^2)\zeta_k^k\delta_k^{k-1}\delta_0 + b_1xy\zeta_0B(\zeta\delta\gamma)\zeta_k\delta_k\delta_0 \\ & - b_2x^2\zeta_0B(\zeta)\zeta_k + b_3y\zeta_0^kB(\zeta\delta\gamma)C(\zeta\delta\gamma^2)\delta_0^{2k-3} \\ & - b_4x\zeta_0^{k+1}B(\zeta^2\delta\gamma)C(\zeta\delta\gamma^2)\zeta_k\delta_0^{2k-2} - b_6\zeta_0^{2k+1}B(\zeta^3\delta^2\gamma^2)C(\zeta^2\delta^2\gamma^4)\zeta_k\delta_0^{4k-4}, \end{aligned} \quad (\text{D.3})$$

along with the projective relations which are readily obtained from the set of resolutions (D.2).

In codimension 1, the Cartan divisors are now given in terms of

Divisor	Section	Equation in Y_4
$D_{-\alpha_0}$	ζ_0	$0 = y^2\delta_1 - x^3\zeta_1$
$D_{-\alpha_1}$	δ_1	$0 = \gamma_1 + b_2\zeta_0$
$D_{-\alpha_2}$	δ_0	$0 = \delta_1 - (\gamma_1 + b_2\zeta_0)\zeta_1$
$D_{-\alpha_{2i+1}} \quad i = 1, \dots, k-2$	γ_i	$0 = y^2\delta_i\delta_{i+1} - b_2\zeta_i\zeta_{i+1}$
$D_{-\alpha_{2i}} \quad i = 2, \dots, k-1$	ζ_i	$0 = \delta_i$
$D_{-\alpha_{2k-1}}$	γ_{k-1}	$0 = y^2\delta_{k-1}\delta_k + b_3y\zeta_{k-1}\delta_{k-1} - b_2\zeta_{k-1}\zeta_k$
$D_{-\alpha_{2k}}$	ζ_k	$0 = \delta_k + b_3\zeta_{k-1}$
$D_{-\alpha_{2k+1}}$	δ_k	$0 = b_3y - b_2x^2\zeta_k - b_4x\gamma_{k-1}\zeta_k - b_6\gamma_{k-1}^2\zeta_k$

Let us consider the codimension 2 loci. Along $b_3 = 0$ the splitting into matter surfaces is completely analogous to the one in the main text, and the same Cartan charge for $u_{2k+1}^{(i)}$ appears.

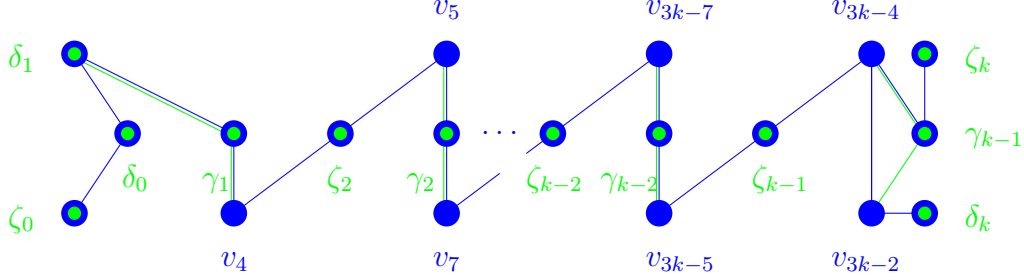


Figure 22: Intersection graph of the fibers in codimension 2 along $b_2 = 0$ for $SO(4k + 2)$ for the alternative resolution (D.2). \bullet are the irreducible components in codimension 2, bicolored nodes either remain irreducible when passing to the codimension 2, or have a remaining component, e.g. γ_i split off into three components, generically, which is indicated by two new blue nodes and one bicolored one. Green lines indicate the splitting, blue lines the intersections.

The interesting things happen along the $b_2 = 0$ non-minimal locus. Restricting the Cartan divisors to $b_2 = 0$, the irreducible components are now as follows – in particular, the last column in the table, which indicates the Cartan charges, makes it clear that the weights that appear are different from the ones in the resolution in the main text:

Matter surface	Section	Equation in $Y _{b_2=0}$	j th Cartan charge
$S_{-\alpha_0}$	ζ_0	$0 = y^2\delta_1 - x^3\zeta_1$	$\delta_{j,2} - 2\delta_{j,0}$
$S_{-\alpha_1}$	δ_1	$0 = \gamma_1$	$\delta_{j,2} - 2\delta_{j,1}$
$S_{-\alpha_2}$	δ_0	$0 = \delta_1 - \gamma_1\zeta_1$	$\delta_{j,0} + \delta_{j,1} + \delta_{j,3} - 2\delta_{j,2}$
S_{v_3}	γ_1	$0 = y$	$\delta_{j,1} - \delta_{j,3}$
$S_{v_{3i}}$, $i = 2, \dots, k-2$	γ_i	$0 = y$	$-\delta_{j,2i+1}$
$S_{v_{3i-1}}$, $i = 2, \dots, k-1$	γ_i	$0 = \delta_i$	$\delta_{j,2i}$
$S_{v_{3i+1}}$, $i = 1, \dots, k-2$	γ_i	$0 = \delta_{i+1}$	$\delta_{j,2i+2}$
$S_{v_{3k-3}}$	γ_{k-1}	$0 = y$	$\delta_{j,2k+1} - \delta_{j,2k-1}$
$S_{v_{3k-2}}$	γ_{k-1}	$0 = y\delta_k + b_3\zeta_{k-1}$	$\delta_{j,2k} - \delta_{j,2k-1}$
$S_{-\alpha_{2k}}$	ζ_k	$0 = \delta_k + b_3\zeta_{k-1}$	$\delta_{j,2k-1} - 2\delta_{j,2k}$
$S_{-\alpha_{2k+1}}$	δ_k	$0 = b_3y - \gamma_{k-1}\zeta_k(b_4x + b_6\gamma_{k-1})$	$\delta_{j,2k-1} - 2\delta_{j,2k+1}$
$S_{-\alpha_{2i}}$, $i = 2, \dots, k-1$	ζ_i	$0 = \delta_i$	$\delta_{j,2i-1} + \delta_{j,2i+1} - 2\delta_{j,2i}$

(D.5)

In addition to the spin representations, in this case we find weights of the other fundamental representations $\Lambda^i V$. The splitting of the Cartan divisors can be summarized as follows

$$\begin{aligned}
D_{-\alpha_3} &\longrightarrow S_{-\alpha_1} + 2 \times S_{v_3} + S_{v_4} \\
D_{-\alpha_{2i+1}} &\longrightarrow S_{v_{3i-1}} + 2 \times S_{v_{3i}} + S_{v_{3i+1}} \\
D_{-\alpha_{2k-1}} &\longrightarrow S_{v_{3k-4}} + S_{v_{3k-3}} + S_{v_{3k-2}},
\end{aligned}
\tag{D.6}$$

Denote the highest weights of the fundamental representations by $(\mu_{\Lambda^i V})_j = \delta_{i,j}$, then the weights appering in the fiber along $b_2 = 0$ are summarized as follows

Matter surface	Weight
S_{v_3}	$\mu_{\Lambda^{2V}} - \left(\alpha_1 + 3\alpha_2 + \sum_{j=3}^{2k-1} 4\alpha_j + 2\alpha_{2k} + 2\alpha_{2k+1} \right)$
$S_{v_{3i-1}}$	$\mu_{\Lambda^{2iV}}$
$S_{v_{3i+1}}$	$\mu_{\Lambda^{2i+2V}}$
$S_{v_{3i}}$	$\mu_{\Lambda^{2i+1}} - \left(\sum_{j=1}^{2i} 2j\alpha_j + 2(2i+1) \sum_{j=2i+1}^{2k-1} \alpha_j + (2i+1)\alpha_{2k} + (2i+1)\alpha_{2k+1} \right)$
$S_{v_{3k-3}}$	$\mu_{S^+} - \left(\sum_{i=1}^{2k-1} i\alpha_i + k\alpha_{2k} + (k-1)\alpha_{2k+1} \right)$
$S_{v_{3k-2}}$	$\mu_{S^+} - \left(\sum_{i=1}^{2k-1} i\alpha_i + 2k\alpha_{2k} + 2k\alpha_{2k+1} \right)$
$S_{u_{2k+1}}$	$\mu_V - \left(\sum_{i=1}^{2k-1} \alpha_i + \alpha_{2k+1} \right)$

(D.7)

It would be interesting to understand the mathematics behind which representations appear when passing to a non-minimal locus, and how this is encoded in the choice of small resolutions.

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