Almost Hermitian Geometry on Six Dimensional Nilmanifolds

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Abstract. The fundamental 2–form of an invariant almost Hermitian structure on a 6–dimensional Lie group is described in terms of an action by $SO(4)\times U(1)$ on complex projective 3–space. This leads to a combinatorial description of the classes of almost Hermitian structures on the Iwasawa and other nilmanifolds.

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Introduction

Let (M,g,J) be an almost Hermitian manifold, so that J is an almost complex structure orthogonal relative to the Riemannian metric g. The decomposition of ∇J (∇ is the Levi–Civita connection) into irreducible components under the action of the unitary group determines the Gray–Hervella class of the almost Hermitian structure [7]. In real dimension $2n$ with $n\geqslant 3$, ∇J has four components W_1,W_2,W_3,W_4, and one is interested in structures for which one or more of these vanishes. For example, (M,J) is a complex manifold if and only if $W_1=W_2=0$, and (M,ω) is a symplectic manifold (ω is the fundamental 2–form determined by g and J) if and only if $W_1=W_3=W_4=0$.

In this paper, we are concerned with left–invariant tensors g,J on a Lie group G of real dimension 6. Indeed, we fix a metric g and consider the space $\mathcal{Z}$ of *all* left–invariant almost complex structures J compatible with g and an orientation. This reduces many questions to properties of the Lie algebra $\mathfrak{g}$ of G. The manifold $\mathcal{Z}$ is isomorphic to the complex projective space $\mathbb{CP}^3$, and a choice of standard coordinates allows us to visualize it in terms of a tetrahedron, in which the edges and faces represent projective subspaces $\mathbb{CP}^1$ and $\mathbb{CP}^2$. This technique and was first used in [1], which identified the

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subsets $\mathcal{C}$, $\mathcal{S}$ of $\mathcal{Z}$ corresponding to complex and symplectic structures compatible with a given metric on the complex Heisenberg group $G_{\rm H}$.

In order to develop the theory in a more systematic way, we show in §3 that every single component of ∇J (equivalently, $\nabla\omega$) can be readily extracted by means of wedging with appropriate differential forms. Some of the integrability equations (such as those of Lemma 2) interact effectively with the nilpotency condition on

$$d:\mathfrak{g}^*\to\textstyle\bigwedge^2\mathfrak{g}^* \tag{1}$$

that arises in the theory of minimal models, and our approach is particularly suited to the case in which the Lie group G is nilpotent. On the other hand, our investigation is also designed to illustrate aspects of the theory of differential forms and complex structures on 6–manifolds that can be applied to non–invariant settings.

The non–existence of a Kähler metric on a nilmanifold $\Gamma\backslash G$ (other than a torus) will imply that there are no compatible complex and symplectic structures, so $\mathcal{C}\cap\mathcal{S}=\emptyset$. This relatively deep fact pervades the descriptions of subsets of $\mathcal{Z}$ in our examples, and renders the combinatorial aspect of the results all the more striking. In six dimensions, a fundamental 2–form ω satisfies $W_4=0$ if and only if $\omega\wedge\omega$ is closed, and this is an especially natural condition in our context. We point out that the set of such 2–forms is orthogonal to the image of (1) and always non–empty. The corresponding class of cosymplectic structures is an intersection of real hypersurfaces of $\mathcal{Z}$, and typically has dimension b_1 (the first Betti number of $\mathfrak{g}$).

The kernel of (1) gives rise to a real 4–dimensional space $\mathbb{D}$ of invariant closed 1–forms on $G_{\rm H}$, and the action of a corresponding subgroup $SO(4)$ on $\mathcal{Z}$ provides symmetry that can be exploited to simplify the equations. In fact, we parametrize $\omega(P;a,b)\in\mathcal{Z}$ by means of $P\in SO(4)$ and (a,b) in a unit circle, and seek solutions $P;a,b$ to the problem at hand, exploiting the concept of self–duality and a 'conjugated' exterior derivative d^P. Most of the calulations were done by hand, and then checked with MAPLE's differential form package. The most interesting part is to interpret the conditions geometrically as subsets of $\mathcal{Z}$, and there is a sufficiently rich class of examples to illustrate many contrasting features of the theory.

Once one has determined the four classes for which W_i vanishes (with $i=1,2,3,4$ in turn), all the other classes may be obtained by taking intersections. The resulting inclusions can be summarized by means of a quotient lattice, which is illustrated in §4 for G_H. In this case, we obtain a complete description of the 16 Gray–Hervella classes, in terms of faces, edges, and vertices of the standard tetrahedron. The upshot is that every class (for which at least one W_i vanishes) is contained in the union of two particular faces, and has at most two connected components.

The techniques are equally applicable to a class of similar examples, and we carry out the same process for the two other irreducible nilpotent Lie algebras with $b_1=4$ (indeed, $b_1\geqslant 4$). In these cases, we give a complete description of the classes for which one of the 'larger' components W_2,W_3 is zero, and these classes get progressively smaller in the three cases. We also identify enough subsets (unions of points, circles and 2-spheres) with $W_1=0$ or $W_4=0$ to determine most of the classes for which two W_i vanish. In §5, we show that the other 2–step example has exactly four Hermitian structures, but that two of these can be made to coincide by modifying the inner product. In §6, we are able to detect the non–existence of both complex and symplectic structures in the 3–step case, for which the various classes are best visualized using a 'similar' tetrahedron arising from a change of basis.

1. Invariant tensors in 6 dimensions

The complex Heisenberg group is given by the set of matrices

$$G_{\mathrm{H}} = \left\{ \begin{pmatrix} 1 & \zeta_1 & \zeta_3 \\ 0 & 1 & \zeta_2 \\ 0 & 0 & 1 \end{pmatrix} : \ \zeta_i \in \mathbb{C},\ i = 1, 2, 3 \right\} \tag{2}$$

under multiplication. The 1–forms $\alpha^1 = d\zeta_1$, $\alpha^2 = d\zeta_2$, $\alpha^3 = -d\zeta_3 + \zeta_1 d\zeta_2$ are left–invariant on G_{H}. The Lie bracket is determined by the equations

$$\begin{cases} d\alpha^i = 0, \quad i = 1, 2, \\ d\alpha^3 = \alpha^1 \wedge \alpha^2, \end{cases} \tag{3}$$

and the α^i are holomorphic relative to the natural complex structure J_0 on G_{H}. Mapping the above matrix to (ζ_1, ζ_2) determines a homomorphism $\mu\colon G_H \to (\mathbb{C}^2, +)$ of complex Lie groups.

In this paper, we shall treat G_{H} as a *real* Lie group, and setting

$$\alpha^1 = e^1 + ie^2, \quad \alpha^2 = e^3 + ie^4, \quad \alpha^3 = e^5 + ie^6$$

provides a real basis (e^i) of $\mathfrak{g}_{\mathrm{H}}^*$ where $\mathfrak{g}_{\mathrm{H}}$ denotes the corresponding Lie algebra. The real version of (3) is therefore

$$\begin{cases} de^i = 0, \quad 1 \leqslant i \leqslant 4, \\ de^5 = e^1 \wedge e^3 + e^4 \wedge e^2 \\ de^6 = e^1 \wedge e^4 + e^2 \wedge e^3. \end{cases} \tag{4}$$

The kernel of (1) is the 4–dimensional subspace $\mathbb{D}$ spanned by e^1, e^2, e^3, e^4, and this coincides with the image of μ^*. Moreover, the image of (1) lies in $\ ^2\mathbb{D}$. The fundamental role played by $\mathbb{D}$ in the theory of invariant structures on G_{H} was emphasized in [1], and we now set about generalizing this situation.

Let G be a real 6–dimensional nilpotent Lie group, and let Γ be a discrete subgroup of G for which the set $\Gamma\backslash G$ of right cosets is a compact manifold M. A classification of corresponding Lie algebras was given in [9] (see the tables in [5, 14]). Given that G necessarily has rational structure constants, such a Γ must exist [10]. For example, the *Iwasawa manifold* is the compact quotient space $M = \Gamma\backslash G_{\mathrm{H}}$ where Γ is the subgroup defined by restricting ζ_i in (2) to be Gaussian integers. The existence of a compact quotient enables one to apply Nomizu's theorem to compute the cohomology of $\Gamma\backslash G$, and to assert the non–existence of a Kähler metric on M unless G is abelian (see [8, 3, 11, 4, 15]).

We shall suppose that a 4–dimensional subspace $\mathbb{D}$ of $\mathfrak{g}^*$, the dual of the Lie algebra of G, is chosen. Let $(e^1,\ldots,e^6)$ a basis of $\mathfrak{g}^*$ such that $\mathbb{D}=\langle e^1,e^2,e^3,e^4\rangle$. Orientations on $\mathfrak{g}^*$ and $\mathbb{D}$ are determined by the forms

$$\upsilon = e^{123456}, \quad \upsilon' = e^{1234} \tag{5}$$

respectively. (From now on we abbreviate $e^j \wedge e^j \wedge \cdots$ to $e^{ij\cdots}$.) Each e^i is a left–invariant 1–form on G, and therefore passes to the quotient M. We shall denote the corresponding form on M by the same symbol; for example, υ determines an orientation of M. A tensor on M will be called *invariant* if it can be expressed in terms of the basis (e^i) using constant coefficients. In particular, a differential form is invariant if and only if its pull–back to G is left–invariant by G. Declaring the chosen 1–forms e^i to be orthonormal determines an invariant Riemannian metric

$$g=\sum_{i=1}^{6} e^i\otimes e^i. \tag{6}$$

In the following study, we shall regard the choice of υ, $\mathbb{D}$, υ', g as fixed, rather than a particular basis (e^i).

The symmetry group of $(\mathfrak{g},\upsilon,g)$ is $SO(6)$ and that of $(\mathbb{D},\upsilon',g)$ is $SO(4)$. The decomposition

$${}^2\mathbb{D} = {}^2_+\mathbb{D}\oplus {}^2_-\mathbb{D} \tag{7}$$

into eigenvalues of the $*$ operator gives rise to a double covering $SO(4)\to SO(3)\times SO(3)$. It follows that $P\in SO(4)$ can be represented by the 6×6 matrix

$$\begin{pmatrix} P_+ & 0\\ 0 & P_- \end{pmatrix}, \tag{8}$$

where $P_+ :\ {}^2_+\mathbb{D}\to {}^2_+\mathbb{D}$ and $P_- :\ {}^2_-\mathbb{D}\to {}^2_-\mathbb{D}$. With respect to the bases

$$\begin{array}{ll} (e^{12}+e^{34},e^{13}+e^{42},e^{14}+e^{23}) & \text{of}\quad {}^2_+\mathbb{D},\\ (e^{12}-e^{34},e^{13}-e^{42},e^{14}-e^{23}) & \text{of}\quad {}^2_-\mathbb{D}, \end{array}$$

we may write

$$P_+=\begin{pmatrix} r & u & x\\ s & v & y\\ t & w & z\end{pmatrix},\quad P_-=\begin{pmatrix} r' & u' & x'\\ s' & v' & y'\\ t' & w' & z'\end{pmatrix}\in SO(3). \tag{9}$$

An invariant almost complex structure J on M is an endomorphism of $\mathfrak{g}$ (or of $\mathfrak{g}^*$) such that $J^2=-1$. It is uniquely determined by writing

$$Je^j=\sum_{i=1}^{6} a_i^j e^i,\quad j=1,\ldots,6$$

where (a_i^j) is a constant matrix. Following the notation of [1], we denote by $\mathcal{Z}$ the set of invariant positively–oriented orthogonal almost complex structures on M. Thus, $J\in\mathcal{Z}$

if and only if (a_i^j) is both orthogonal and skew–symmetric with positive determinant. The group $SO(6)$ acts on $\mathcal{Z}$ by conjugation, and the stabilizer of the almost complex structure J_0, for which (like the one defined by (3))

$$J_0 e^1 = -e^2, \quad J_0 e^3 = -e^4, \quad J_0 e^5 = -e^6, \tag{10}$$

can be identified with $U(3)$. Thus, $\mathcal{Z}$ is the Hermitian symmetric space $SO(6)/U(3)$. The double covering $SU(4) \to SO(6)$ allows one to regard $SO(6)$ as a transitive subgroup of projective transformations of $\mathbb{CP}^3$, and the following is well known.

Proposition 1. $\mathcal{Z}$ is isomorphic to the complex projective space $\mathbb{CP}^3$.

We describe the isomorphism explicitly, referring the reader to [1, 2] for more details. Let $V \cong \mathbb{C}^4$ denote the standard representation of $SU(4)$ and let (v^0, v^1, v^2, v^3) be a unitary basis of V. Then $\;^2V$ is the complexification of a real vector space which we identify with $\mathfrak{g}^*$. This identification can be chosen in such a way that

$$\begin{aligned} 2v^{01} &= e^1 + ie^2, \qquad & 2v^{23} &= e^1 - ie^2, \\ 2v^{02} &= e^3 + ie^4, \qquad & 2v^{31} &= e^3 - ie^4, \\ 2v^{03} &= e^5 + ie^6, \qquad & 2v^{12} &= e^5 - ie^6 \end{aligned} \tag{11}$$

(remember that v^{ij} denotes $v^i \wedge v^j$). A point $J \in \mathcal{Z}$ corresponds to a totally isotropic subspace of the complexification of $\mathfrak{g}^*$, namely the i–eigenspace of J, and any such subspace equals

$$V_u = \{u \wedge v : v \in V\} \subset \;^2V,$$

which depends on $[u] \in \mathbb{P}(V) \cong \mathbb{CP}^3$. For example, (10) corresponds to the point $[v^0] = [1, 0, 0, 0]$ of $\mathbb{CP}^3$. We shall also consider the almost complex structures

$$J_1 = [0, 1, 0, 0], \quad J_2 = [0, 0, 1, 0], \quad J_3 = [0, 0, 0, 1].$$

Notice that an almost complex structure with the 'wrong' orientation corresponds not to a subspace of type V_u, but to one of type $\;^2W$ where W is a 3–dimensional subspace of V. For example, $-J_0$ corresponds to $W = \langle v^1, v^2, v^3 \rangle$. This is an aspect of the well-known $\alpha \leftrightarrow \beta$ duality of the Klein correspondence [13]. In fact many of the constructions below illustrate concepts from elementary projective geometry.

2. An action of $SO(4)$ on $\mathbb{CP}^3$

The *fundamental 2–form* ω of the almost Hermitian structure (g, J) is defined by

$$\omega(X, Y) = g(JX, Y).$$

Associated to J_0, J_1, J_2, J_3 are the fundamental 2–forms

$$\begin{array}{lcl} \bullet \quad \omega_0 &=& e^{12} + e^{34} + e^{56}, \\ \bullet \quad \omega_1 &=& e^{12} - e^{34} - e^{56}, \\ \bullet \quad \omega_2 &=& -e^{12} + e^{34} - e^{56}, \\ \bullet \quad \omega_3 &=& -e^{12} - e^{34} + e^{56}. \end{array} \tag{12}$$

Since the metric g will always be fixed, one may equally well label points of $\mathcal{Z}$ by their corresponding fundamental forms.

Given the coordinate system on $\mathbb{CP}^3$ determined by (v^i), it is helpful to visualize $\mathcal{Z}$ as a *solid* tetrahedron with vertices (12), and to consider

$\mathcal{E}_{ij}$, the 'edge' ($\cong \mathbb{CP}^1 \cong S^2$) containing ω_i, ω_j;

$\mathcal{F}_i$, the 'face' ($\cong \mathbb{CP}^2$) opposite ω_i.

Whilst this picture may be geometrically inaccurate, it provides a powerful tool for analysing some of the singular varieties that arise in the classification of almost Hermitian classes.

For future reference, we define a number of additional objects:

(i) The *equatorial circle* in the edge $\mathcal{E}_{ij}$ is the set

$$C_{ij} = \{\omega \in \mathcal{E}_{ij} : g(\omega, \omega_i) = g(\omega, \omega_j)\} \tag{13}$$

where g here denotes the metric induced on 2–forms. For example,

$$c(\theta) = e^{56} + \cos\theta(e^{13} + e^{42}) + \sin\theta(e^{14} + e^{23}) \tag{14}$$

is a typical element of C_{03}.

(ii) The *generalized edge* determined by a decomposable unit 2–form $\sigma = e \wedge f$ is

$$\lceil \sigma \rfloor = \{\sigma + \tau \in \mathcal{Z} : \tau \in \quad {}^2\langle e, f\rangle^{\perp}\}. \tag{15}$$

Consistency with the orientation (see (5)) requires that τ be chosen from a 2-sphere, and $\lceil \sigma \rfloor$ is a typical complex projective line in $\mathcal{Z}$. From (11), $\mathcal{E}_{03} = \lceil e^{56} \rfloor$ is the line $\mathbb{P}(\langle v^0, v^3\rangle)$.

(iii) The *polar set* of an arbitrary non–zero 2–form σ is

$$\langle \sigma \rangle^{\perp}_{\mathcal{Z}} = \{\omega \in \mathcal{Z} : g(\omega, \sigma) = 0\};$$

this is simply the intersection of the hyperplane $\langle \sigma \rangle^{\perp}$ of $\quad {}^2\mathbb{R}^6$ with the submanifold $\mathcal{Z}$ of fundamental forms with respect to g.

We shall now describe the action of $SO(4) = \mathrm{Aut}(\mathbb{D}, \upsilon', g)$ on $\mathcal{Z}$. Let J be an almost complex structure with fundamental 2–form $\omega \in \mathcal{Z}$. Since $-Je^5$ is a unit 1–form orthogonal to e^5, it has the form $ae^6 + bf^1$ where $f^1 \in \mathbb{D}$, $\|f^1\| = 1$, and $a^2 + b^2 = 1$. The unit 1–form $af^1 - be^6$ is then orthogonal to both e^5 and Je^5. The next result is a consequence of this observation.

Proposition 2. The fundamental 2–form defined by an arbitrary point of $\mathcal{Z}$ has the form

$$\omega = e^5 \wedge (ae^6 + bf^1) - f^2 \wedge (af^1 - be^6) + f^3 \wedge f^4, \tag{16}$$

where (f^1, f^2, f^3, f^4) is an oriented orthonormal basis of $\mathbb{D}$ and $a^2 + b^2 = 1$.

Given (f^i), we may define $P \in SO(4)$ by setting $P(e^i) = f^i$ for $1 \quad i \quad 4$. Relative to the original basis (e^i), it therefore makes sense to write (16) as $\omega(P; a, b)$. For example, $\omega(I; 1, 0) = \omega_0$ and $\omega(I; -1, 0) = \omega_2$ (I denotes the identity) are two of the vertices of the tetrahedron, and

$$\omega(I; a, b) = e^{34} + \tau, \qquad \tau \in \ {}^2\langle e^1, e^2, e^5, e^6 \rangle \tag{17}$$

lies in $\mathcal{E}_{02} = \lceil e^{34} \rfloor$ for any $(a, b) \in S^1$. In particular, the points

$$\begin{aligned} \varpi_0 &= \omega(I, 0, 1) = -e^{15} + e^{26} + e^{34}, \\ \varpi_2 &= \omega(I, 0, -1) = e^{15} - e^{26} + e^{34} \end{aligned} \tag{18}$$

lie in the circle C_{02}.

What the Proposition is really saying is that there is a transitive action of $K = SO(4) \times U(1)$ on $\mathcal{Z}$, where $SO(4)$ is described in (8), and $U(1)$ acts by rotating points (a, b) of the circle. Replacing b by $-b$ in (16) has the same effect as changing the sign of every f^i, so the element $(-I, -1) \in K$ acts trivially, and there is an effective action of $K/\mathbb{Z}_2$ on $\mathcal{Z}$. Consider the possible stabilizers of $SO(4)$ on $\mathcal{Z}$ and the corresponding orbits:

Exceptional case. If $b = 0$ then $a = \pm 1$, $\mathbb{D}$ is J–invariant and the subgroup of $SO(4)$ stabilizing $\omega(P; \pm 1, 0)$ is isomorphic to $U(2)$. This gives rise to two exceptional orbits: as P varies in $SO(4)$,

$\omega(P; 1, 0)$ spans the edge $\mathcal{E}_{03} = \lceil e^{56} \rfloor$, and

$\omega(P; -1, 0)$ spans the edge $\mathcal{E}_{12} = \lceil -e^{56} \rfloor$.

Generic case. If $b \neq 0$, the stabilizer of $\omega(P; a, b)$ is the subgroup $SO(2)$ of $SO(4)$ fixing f^1, f^2, f^{34}, and acting by the rotation

$$\begin{aligned} f^3 &\mapsto \ \cos\theta\, f^3 + \sin\theta\, f^4, \\ f^4 &\mapsto -\sin\theta\, f^3 + \cos\theta\, f^4. \end{aligned} \tag{19}$$

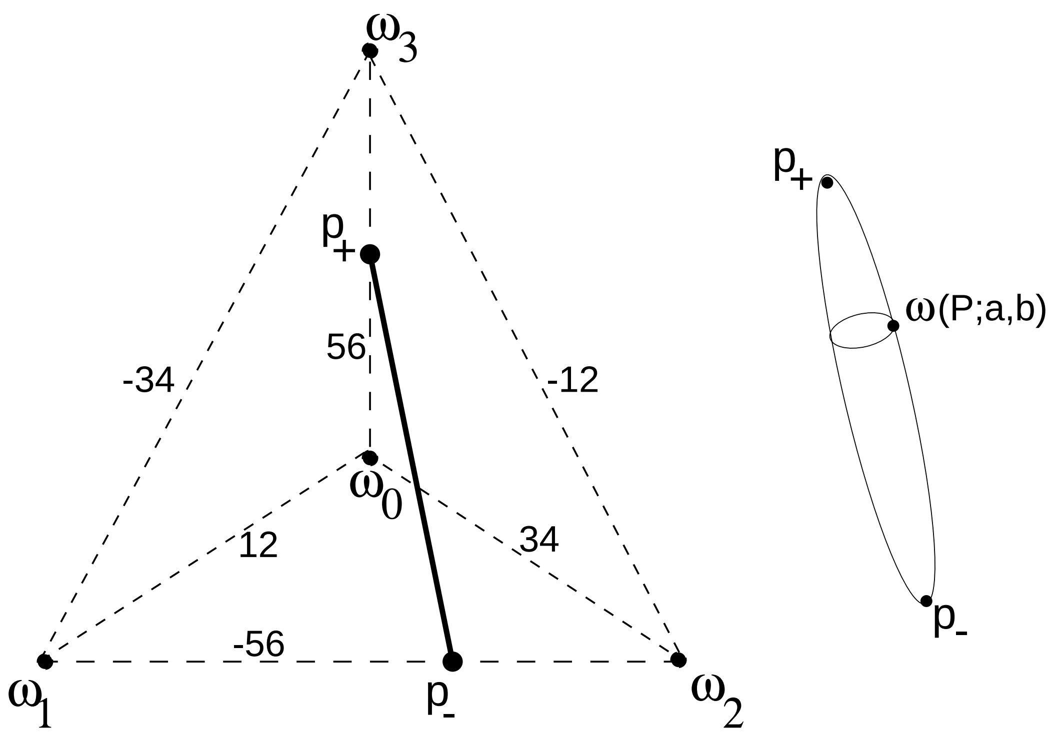
The generic orbit of $SO(4)$ on $\mathcal{Z}$ is therefore the 5–dimensional space $SO(4)/SO(2)$. An example of such an orbit is

$$\langle e^{56} \rangle^{\perp}_{\mathcal{Z}} = \{ P(\varpi_0) = \omega(P; 0, 1) : P \in SO(4) \}; \tag{20}$$

this is because the points in (18) are orthogonal to the 2–form e^{56}, which is fixed by $SO(4)$.

One can now visualize the action of K filling out the whole tetrahedron, and the relevance of (9). Each edge represents a complex projective line, that is, a 2-sphere. As a, b vary, (16) describes a great circle on another such line or 2-sphere Σ, corresponding to the edge shown bold in the figure. We view it as an elongated elliptical wire, one point $p_+ = \omega(P; 1, 0)$ of which lies on $\mathcal{E}_{03}$ and the diametrically opposite point $p_- = \omega(P; -1, 0)$ on $\mathcal{E}_{12}$. Varying P_+ moves p_+ 'up and down' $\mathcal{E}_{03}$ ('around' would be more accurate), and varying P_- moves p_- around $\mathcal{E}_{12}$.

The stabilizer of the pair (p_+,p_-) in $SO(4)$ is isomorphic to $U(1)$, and this acts to rotate the wire to fill out the 2-sphere Σ with poles p_+,p_-. This is what is represented by the right-hand side of the Figure. The forms (17) may be regarded as the wire's 'position of rest', in which the bold edge coincides with $\mathcal{E}_{02}$. The label '$\pm ij$' on an edge indicates that it equals $\lceil \pm e^{ij} \rfloor$ in the notation of (15).



Example. Consider the subset

$$\mathcal{S} = \{\omega(P;0,1) : P_+ \text{ has } r=1,\ P_- \text{ is arbitrary}\} \tag{21}$$

of (20).The condition $r=1$ implies that P_+ leaves fixed $e^{12}+e^{34}$, and so $P(\omega_0)=\omega_0$. A point of $\mathcal{S}$ therefore lies on a 'wire' joining ω_0 to an arbitrary point of $\mathcal{E}_{12}$, and is contained in the affine plane $\mathcal{F}_3 \setminus \mathcal{E}_{12} \cong \mathbb{C}^2$. Fix $e^5\wedge f^1 + f^2\wedge e^6 + f^{34} \in \mathcal{S}$, and let J be the associated almost complex structure. The fact that J lies in the face opposite ω_3 implies that J commutes with J_3 [2], so

$$f^2 = Je^6 = -JJ_3e^5 = -J_3Je^5 = J_3f^1$$

is determined by f^1, which is a unit vector in $\mathbb{D}$. It follows that $\mathcal{S}$ is a 3-sphere, and the mapping $\omega(P;0,1)\mapsto p_-$ is a Hopf fibration $\mathcal{S}\to\mathcal{E}_{12}\cong S^2$. It was shown in [1] that $\mathcal{S}$ parametrizes the set of almost Kähler structures on the Iwasawa manifold.

3. Detecting classes by exterior differentiation

Let ∇ denote the Levi-Civita connection. The covariant derivative $\nabla\omega$ of the fundamental 2–form of an almost Hermitian manifold M has various symmetry properties. If W is a real vector space of dimension $2n \quad 6$ with an almost complex structure J and a real positive definite inner product compatible with J, let $\mathcal{W}$ denote the subspace of $W^* \otimes \ {}^2W^*$ of the tensors with the same symmetry properties of $\nabla\omega$.

A decomposition

$$\mathcal{W} = \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4 \tag{22}$$

is given in [7], in which the four summands are invariant under the action of the unitary group $U(n)$ and irreducible. The complete classification of almost Hermitian manifolds into $2^4 = 16$ classes is obtained by taking W to be the tangent space. Ignoring the generic case, in which no component of $\nabla\omega$ vanishes, we consider the subsets

$$\begin{aligned} \mathcal{Z}_{ijk} &= \{J \in \mathcal{Z} : \nabla\omega \in \mathcal{W}_i \oplus \mathcal{W}_j \oplus \mathcal{W}_k\}, \\ \mathcal{Z}_{ij} &= \{J \in \mathcal{Z} : \nabla\omega \in \mathcal{W}_i \oplus \mathcal{W}_j\}, \\ \mathcal{Z}_{i} &= \{J \in \mathcal{Z} : \nabla\omega \in \mathcal{W}_i\}, \end{aligned} \tag{23}$$

where i, j, k are distinct elements of $\{1, 2, 3, 4\}$. Observe that, in this notation, $\mathcal{Z}_i \subseteq \mathcal{Z}_{ij} \subseteq \mathcal{Z}_{ijk}$, whereas $\mathcal{Z}_i \cap \mathcal{Z}_j$ is the set of Kähler structures if $i = j$.

In particular,

$\mathcal{Z}_1$	is the set of	nearly-Kähler structures,
$\mathcal{Z}_2$		almost Kähler structures
$\mathcal{Z}_3$		cosymplectic Hermitian structures,
$\mathcal{Z}_4$		locally conformal Kähler structures.

The last statement makes use of the hypothesis $n \quad 3$ in order that $d\omega = \omega \wedge \theta$, with θ a *closed* 1–form. However, we shall be more interested in determining the 'maximal' classes $\mathcal{Z}_{234}$, $\mathcal{Z}_{134}$, $\mathcal{Z}_{124}$, $\mathcal{Z}_{123}$, since all the others can be obtained as intersections of these four. The only one of these that has a special name is the class $\mathcal{Z}_{123}$ of 'semi-Kähler' or 'cosymplectic' structures, characterized by the condition $d\omega \wedge \omega^{n-2} = 0$.

In [6] the decomposition (22) is described in terms of differential forms, and we review this approach. Given J, the complexified cotangent space at any point m of M is given by

$$T^*_m M \otimes_{\mathbb{R}} \mathbb{C} = \quad {}^{1,0} \oplus \quad {}^{0,1} \tag{24}$$

where the two summands are the $+1$, -1 eigenspaces of J , respectively. This leads to the usual decomposition of forms into types, whereby the $(p+q)$th exterior power of (24) contains a subspace ^{p,q} isomorphic to ${}^{p}\ {}^{1,0} \otimes \ {}^{q}\ {}^{0,1}$. Both ${}^{p,q} \oplus \ {}^{q,p}$, $p = q$, and ^{p,p} are complexifications of real vector spaces which are denoted by $[\![\ {}^{p,q}]\!]$ and $[\ {}^{p,p}]$ respectively in [6]. Then

Proposition 3. There are isomorphisms

$$\mathcal{W}_1 \cong [\![\ {}^{3,0}]\!], \quad \mathcal{W}_2 \cong [\![V(2,1,0,\ldots,0)]\!], \quad \mathcal{W}_3 \cong [\![\ {}^{2,1}_{0}]\!], \quad \mathcal{W}_4 \cong [\![\ {}^{1,0}]\!] \cong T^*_m M.$$

Here, $V(\lambda)$ denotes that irreducible complex representation of $U(n)$ with dominant weight λ. Also, ${}^{p,q}_{0}$ is the space of 'effective' forms, equivalently the Hermitian complement of the image of ${}^{p-1,q-1}$ under wedging with ω.

It is well known that the covariant derivative $\nabla\omega$ of the fundamental 2–form determines the Nijenhuis tensor of the almost complex structure J, which is essentially the real part of the tensorial operator

$$d : {}^{1,0} \to {}^{0,2}, \tag{25}$$

and lies in $\mathcal{W}_1 \oplus \mathcal{W}_2$. As observed in [6], the skew-symmetric part of $\nabla\omega$, which lies in

$$^3T^*_m M = [\![{}^{3,0}]\!] \oplus [\![{}^{2,1}_{0}]\!] \oplus [\![{}^{1,0}]\!] \cong \mathcal{W}_1 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4,$$

is proportional to $d\omega$. These algebraic properties lead to an easier way of computing the 16 Gary-Hervella classes, which is especially fruitful in the case of $n = 3$.

Suppose from now on that $M = G$ is a Lie group of real dimension 6. Let (α, β, γ) be a basis of invariant $(1, 0)$–forms, so that the fundamental 2–form may be written

$$\omega = \tfrac{1}{2} i(\alpha \wedge \bar{\alpha} + \beta \wedge \bar{\beta} + \gamma \wedge \bar{\gamma}). \tag{26}$$

We omit the wedging symbol '$\wedge$' for the remainder of this section.

Lemma 1. The class $\mathcal{Z}_{234}$ is characterized by the equation

$$(d\alpha)\beta\gamma\alpha\bar{\alpha} + (d\beta)\gamma\alpha\beta\bar{\beta} + (d\gamma)\alpha\beta\gamma\bar{\gamma} = 0. \tag{27}$$

Proof. Recall that $\mathcal{W}_1 \cong [\![{}^{3,0}]\!]$. Since $\alpha\beta\gamma$ spans the space of invariant $(3, 0)$–forms and $\nabla\omega$ is *real*, the component of $\nabla\omega$ in $\mathcal{W}_1$ vanishes if and only if $(d\omega)\alpha\beta\gamma = 0$. The result follows from (26). □

Lemma 2. The class $\mathcal{Z}_{134}$ is characterized by the equations

$$\begin{gathered}
(d\alpha)\gamma\alpha\beta\bar{\beta} = 0 \\
(d\alpha)\alpha\beta\gamma\bar{\gamma} = 0 \\
(d\beta)\alpha\beta\gamma\bar{\gamma} = 0 \\
(d\beta)\beta\gamma\alpha\bar{\alpha} = 0 \\
(d\gamma)\beta\gamma\alpha\bar{\alpha} = 0 \\
(d\gamma)\gamma\alpha\beta\bar{\beta} = 0 \\
(d\alpha)\beta\gamma\alpha\bar{\alpha} = (d\beta)\gamma\alpha\beta\bar{\beta} = (d\gamma)\alpha\beta\gamma\bar{\gamma}.
\end{gathered}$$

Proof. The Nijenhuis tensor N can be identified with the component of $\nabla\omega$ in $\mathcal{W}_1 \oplus \mathcal{W}_2$, so we seek that component of N belonging to $\mathcal{W}_2$. With reference to (25), we may write

$$N = ((d\alpha)^{0,2}, (d\beta)^{0,2}, (d\gamma)^{0,2}).$$

Now, $(d\alpha)^{0,2} = 0$ if and only if

$$(d\alpha)\beta\gamma\alpha\bar{\alpha} = (d\alpha)\gamma\alpha\beta\bar{\beta} = (d\alpha)\alpha\beta\gamma\bar{\gamma} = 0.$$

The vanishing of N would give 9 such complex–valued equations altogether, though a linear combination of these is represented by Lemma 1. The proof is completed by the observation that the difference of any two of the three terms in the last line of equations is orthogonal to the left–hand side of (27). □

Lemma 3. The class $\mathcal{Z}_{124}$ is characterized by the equations $(d\omega)\eta_i = 0$ for $1 \le i \le 6$, where

$$\begin{aligned} \eta_1 &= \alpha\beta\bar\gamma \\ \eta_2 &= \beta\gamma\bar\alpha \\ \eta_3 &= \alpha\gamma\bar\beta \\ \eta_4 &= \alpha\beta\bar\beta - \alpha\gamma\bar\gamma \\ \eta_5 &= \beta\alpha\bar\alpha - \beta\gamma\bar\gamma \\ \eta_6 &= \gamma\alpha\bar\alpha - \gamma\beta\bar\beta. \end{aligned}$$

Proof. Recall that $\mathcal{W}_3 \cong [\![\Lambda_0^{2,1}]\!]$, and $(d\omega)\eta_i$ represents the component of $d\omega$ parallel to the effective $(2,1)$–form $\bar\eta_i$. $\square$

These equations can also be re–written in terms of $d\alpha$, $d\beta$, $d\gamma$, but to no apparent advantage.

Lemma 4. If the Lie group G is *nilpotent*, the class $\mathcal{Z}_{123}$ is characterized by the equations

$$\begin{aligned} (d\alpha)\omega\omega &= 0, \\ (d\beta)\omega\omega &= 0, \\ (d\gamma)\omega\omega &= 0. \end{aligned} \tag{28}$$

Proof. Given that $n = 3$, $\mathcal{Z}_{123}$ is determined by the condition

$$0 = d(\omega^2) = 2\omega\, d\omega.$$

The nilpotency condition implies that the exterior derivative of any invariant 5–form is zero. This can be seen either by considering the action of d on a basis of the space of invariant 5-forms, or by appealing to Nomizu's theorem [12] on an associated nilmanifold $\Gamma\backslash G$. Since

$$\mathbb{R} \cong H^6(\Gamma\backslash G, \mathbb{R}) \cong \frac{\Lambda^6\mathfrak{g}^*}{d(\Lambda^5\mathfrak{g}^*)},$$

it cannot be the case that d of any 5-form is non–zero. The identity

$$0 = d(\omega\omega\alpha) = 2(d\omega)\omega\alpha + \omega\omega(d\alpha)$$

now gives the result. $\square$

The equations (28) can be interpreted as a linear (rather than quadratic) constraint on $\omega \in \mathcal{Z}$. The theory of the $*$ operator on an oriented inner product space permits us to write

$$\sigma \wedge (*\omega) = g(\sigma, \omega)\upsilon,$$

where g is the induced inner product on forms of the same degree, and υ is the volume form (5). But (up to a universal constant) $*\omega = \omega\omega$, and (28) becomes

$$0 = g(\omega, d\alpha) = g(\omega, d\beta) = g(\omega, d\gamma).$$

It follows that M is cosymplectic if and only if ω is orthogonal to the image of d. In the notation (20), if $(\sigma_1, \dots, \sigma_k)$, $k = 6 - b_1$, is a real basis of the image of (1), then

$$\mathcal{Z}_{123} = \bigcap_{i=1}^{k} \langle\sigma_i\rangle_{\mathcal{Z}}^{\perp}. \tag{29}$$

Proposition 4. Let G be a 6–dimensional nilpotent Lie group. Then, relative to any left–invariant metric, $\mathcal{Z}_{123} = \emptyset$.

Proof. The nilpotency of G means that there is a basis (e^i) of $\mathfrak{g}^*$ such that $de^1 = de^2 = 0$ and $de^k \in \ {}^2\langle e^1, \dots, e^{k-1}\rangle$ for $k \quad 3$. Applying Gram–Schmidt, we may assume that this basis is orthonormal. Since e^4 appears only in de^5, de^6, and e^5 only in de^6, there exists an orthonormal basis (f^1, f^2, f^3) of $\langle e^1, e^2, e^3\rangle$ such that $e^4 \wedge f^1$ and $e^5 \wedge f^2$ are orthogonal to $d(\mathfrak{g}^*)$. Then $e^4 \wedge f^1 + e^5 \wedge f^2 \pm e^6 \wedge f^3 \in \mathcal{Z}_{123}$. □

4. Sixteen classes of the Iwasawa manifold

In this section we shall compute the almost Hermitian structures on the Iwasawa manifold by combining Proposition 2 of §2 and Lemmas 1–4 of §3. We work in terms of the matrix P given by (8),(9). In order to state our main result, we regard a point of $\mathcal{Z} \cong \mathbb{CP}^3$ as a fundamental 2-form of the corresponding almost complex structure, and use freely the notation $\mathcal{E}_{ij}$, $\mathcal{F}_i$ (introduced after (12)) and $\mathcal{S}$ (from (21)).

Theorem 1. Let M be the Iwasawa manifold endowed with the metric (6). The classes of almost Hermitian structures defined in (23) are given by the following subsets of $\mathcal{Z}$.

$$\begin{array}{rcl} \mathcal{Z}_{234} & = & \mathcal{F}_3, \\ \mathcal{Z}_{134} & = & \{\omega_0\} \cup \mathcal{E}_{12}, \\ \mathcal{Z}_{124} & = & \{\omega_3\} \cup \mathcal{S}, \\ \mathcal{Z}_{123} & = & \mathcal{F}_0 \cup \mathcal{F}_3. \end{array}$$

Corollary 1. The classes (23) for M are given by the lattice

$$\begin{tikzcd}
\mathcal{Z}_{123} = & & \mathcal{F}_0 \cup \mathcal{F}_3 \arrow[ddl, dash] \arrow[dr, dash] & \\
\mathcal{Z}_{23} = \mathcal{Z}_{234} = & & & \mathcal{F}_3 \arrow[ddll, dash] \arrow[ddd, dash] \\
\mathcal{Z}_{12} = \mathcal{Z}_{124} = & \{\omega_3\} \cup \mathcal{S} \arrow[d, dash] & & \\
\mathcal{Z}_{2} = \mathcal{Z}_{24} = & \mathcal{S} \arrow[ddr, dash] & & \\
\mathcal{Z}_{3} = \mathcal{Z}_{13} = \mathcal{Z}_{34} = \mathcal{Z}_{134} = & & & \{\omega_0\} \cup \mathcal{E}_{12} \arrow[dl, dash] \\
\mathcal{Z}_{1} = \mathcal{Z}_{4} = \mathcal{Z}_{14} = & & \emptyset &
\end{tikzcd}$$

The fact that $\mathcal{Z}_{34} = \mathcal{Z}_3$ means that all the Hermitian structures on M are automatically cosymplectic. We shall have more to say in §6 regarding other equalities between the classes.

The elements

$$\begin{cases} \alpha &= ae^6 + be^1 - ie^5, \\ \beta &= ae^1 - be^6 + ie^2, \\ \gamma &= e^3 + ie^4 \end{cases} \tag{30}$$

are $(1,0)$–forms relative to the fundamental 2–form (16) in which $P = I$, and $\omega(I; a, b)$ is then given by (26). The problem is to determine $P \in SO(4) \subset SO(6)$ and $(a, b) \in S^1$ such that $\omega(P; a, b) = P(\omega(I; a, b))$ satisfies the appropriate condition. Now $P\alpha, P\beta, P\gamma$ are $(1,0)$–forms relative to $\omega(P; a, b)$, so (as an example) the first equation of Lemma 2 becomes

$$(d(P\alpha))(P\gamma)(P\alpha)(P\beta)(P\bar\beta) = 0,$$

or equivalently

$$(d^P\alpha)\gamma\alpha\beta\bar\beta = 0, \tag{31}$$

where $d^P = P^{-1} \circ d \circ P$. Our method then consists of applying Lemmas 1,2,3,4 to (30), but using the operator d^P in place of d.

The fact that $\mathbb{D} = \ker d$ ensures that $d \circ P = d$, and matters are even simpler in the case of the complex Heisenberg group, for then $d^P = P_+^{-1} \circ d$. Indeed,

$$d(P\alpha) = a\,de^6 - ide^5, \quad d(P\beta) = -b\,de^6, \quad d(P\gamma) = 0, \tag{32}$$

and we may assume that

$$\begin{array}{rcl} d^Pe^5 &=& s(e^{12} + e^{34}) + v(e^{13} + e^{42}) + y(e^{14} + e^{23}), \\ d^Pe^6 &=& t(e^{12} + e^{34}) + w(e^{13} + e^{42}) + z(e^{14} + e^{23}), \end{array} \tag{33}$$

in accordance with (4) and (9) (noting that P_+^{-1} is represented by the transpose matrix).

With these preliminaries, we may proceed to the proof of Theorem 1.

The class $\mathcal{Z}_{234}$. From (30),

$$\begin{aligned} \alpha\beta\gamma = (&e^{136} - ae^{246} - be^{124} + ae^{145} + e^{235} - be^{456}) \\ &+ i(e^{146} + ae^{236} + be^{123} - ae^{135} + be^{356} + e^{245}). \end{aligned}$$

The equation defining $\mathcal{Z}_{234}$ can now be obtained from Lemma 1, with the aid of (33). In the case of the Iwasawa manifold, it is

$$(a+1)[(y+w) + i(z-v)] = 0.$$

The solutions are
(1i) $a = -1$, giving the entire edge $\mathcal{E}_{12}$ of the tetrahedron.
(1ii) $\{y = -w,\ z = v\}$, giving

$$P_+ = \begin{pmatrix} 1 & 0 & 0 \\ 0 & v & -w \\ 0 & w & v \end{pmatrix}, \quad v^2 + w^2 = 1, \tag{34}$$

bearing in mind that the 3×3 matrix must be orthogonal with determinant 1. For arbitrary $P \in SO(4)$, $\omega(P; 1, 0)$ generates the edge $\mathcal{E}_{03}$, but here, $e^{12} + e^{34}$ is fixed by P, so $\omega(P; 1, 0) = \omega_0$. Since $\omega(P; -1, 0)$ is still free to move on $\mathcal{E}_{12}$, we obtain the whole face $\mathcal{F}_3$ (compare (21)) as a, b vary.
Note that (1i) is now subsumed under (1ii).

The class $\mathcal{Z}_{134}$. Combining (32) with Lemma 2 gives the respective equations

$$\begin{cases} b[(v-az)+i(aw+y)]=0, \\ b[at-is]=0, \\ b^2t=0, \\ b(1+a)[z-iw]=0, \\ (a+1)[(-2az+v+z)+i(2aw-w+y)]=0, \\ (a+1)[(v-az)+i(aw+y)]=0. \end{cases}$$

Possible solutions are:
(2i) $\{b=0,\, a=-1\}$, giving the edge $\mathcal{E}_{12}$.
(2ii) $\{b=0,\, a=1,\, v=z,\, w=-y\}$, giving the vertex ω_0 as a special case of (34).
(2iii) If $b=0$ then the middle two equations give $t=w=z=0$. This would imply that P_+ is singular, which is impossible.

The class $\mathcal{Z}_{124}$. For this, one first computes the η_i in Lemma 3. For example,

$$\begin{aligned}\eta_1 = e^{136} + ae^{246} + be^{124} - ae^{145} + be^{456} + e^{235} \\ + i(-e^{146} + ae^{236} + be^{123} - ae^{135} + be^{356} - e^{245}).\end{aligned}$$

Lemma 3 now gives a total of 11 real equations:

$$\begin{cases} (a-1)[(y+w)+i(v-z)]=0, \\ (a+1)[(2aw-y-w)+i(2az-z+v)]=0, \\ (a-1)[(2aw+y+w)+i(2az+z-v)]=0, \\ (a-1)[s+iat]=0, \\ ib(a-1)t=0, \\ ab(w+iz)=0. \end{cases}$$

The solutions are
(3i) $\{a=1,\, b=0,\, y=w,\, v=-z\}$, giving

$$P_+ = \begin{pmatrix} -1 & 0 & 0 \\ 0 & v & w \\ 0 & w & -v \end{pmatrix}, \tag{35}$$

and the vertex ω_3.
(3ii) $b=0$, forcing $s=t=0$ and $y=-w$, $v=z$. If $a=0$ we obtain $w=z=0$, which is impossible on top of $t=0$. Whence, $a=0$, $b=\pm1$, and P_+ satisfies (34). It follows that $\omega(P;0,\pm1)$ describes the 3-sphere (21).

The class $\mathcal{Z}_{123}$. We know from Corollary 1 that this is the intersection of $\langle e^{13}+e^{42}\rangle^\perp_{\mathcal{Z}}$ and $\langle e^{14}+e^{23}\rangle^\perp_{\mathcal{Z}}$, though we shall use the $SO(4)$ action to determine it geometrically. Observe that

$$\begin{aligned} d\omega\wedge\omega &= de^5\wedge(e^{126}+ae^{346}+be^{134}) - de^6\wedge(e^{125}+ae^{345}+be^{234}) \\ &= (a+1)s\, e^{12346} - (a+1)t\, e^{12345}. \end{aligned}$$

This can vanish in one of two ways:
(4i) $a = -1$, giving the edge $\mathcal{E}_{12}$.
(4ii) $s = t = 0$, which reduces P_+ to one of (34), (35). In the former case ($r = 1$), the vertex ω_0 is fixed on $\mathcal{E}_{03}$ and we obtain the face $\mathcal{F}_3$, as explained in (1ii). In the latter case ($r = -1$), $P(\omega_0) = \omega_3$, and we obtain the face $\mathcal{F}_0$ opposite to ω_0.
In conclusion, $\mathcal{Z}_{123} = \mathcal{F}_0 \cup \mathcal{F}_3$.

The proof of Theorem 1 is now complete.

5. An example with discrete Hermitian structures

We have explained (after (31)) the importance of the condition that the 4–dimensional $SO(4)$ module $\mathbb{D}$ used to define $\omega(P; a, b)$ lie in the kernel of $d \colon \mathfrak{g}^* \to \quad {}^2\mathfrak{g}^*$. According to the classification, there are 7 isomorphism classes of nilpotent Lie algebras that, in common with $\mathfrak{g}_{\mathrm{H}}$, have first Betti number b_1 (the dimension of the kernel of (1)) equal to 4. Of these, only three are irreducible, namely $\mathfrak{g}_1 = \mathfrak{g}_{\mathrm{H}}$ given by (4), another 2–step Lie algebra $\mathfrak{g}_2$, and a 3–step algebra $\mathfrak{g}_3$ (see for example [14]). There is no particular reason to restrict to irreducible algebras, except that one would expect the reducible cases to be easier to describe.

The structure equations

$$\begin{cases} de^i = 0, \quad 1 \quad i \quad 4 \\ de^5 = e^{12}, \\ de^6 = e^{14} + e^{23}. \end{cases} \tag{36}$$

of $\mathfrak{g}_2$ are very similar to (4). In applying the methods of §4, the real only difference is that now

$$\begin{aligned} d^P e^5 &= P^{-1}\Big(\tfrac{1}{2}(e^{12} + e^{34}) + \tfrac{1}{2}(e^{12} - e^{34})\Big) \\ &= \tfrac{1}{2}\Big[r(e^{12} + e^{34}) + u(e^{13} + e^{42}) + x(e^{14} + e^{23}) \\ &\qquad + r'(e^{12} - e^{34}) + u'(e^{13} - e^{42}) + x'(e^{14} - e^{23})\Big] \end{aligned} \tag{37}$$

(compare (33)).

As usual, we consider the e^i as invariant forms on an associated Lie group G_2, or nilmanifold M_2, and we choose the metric g (as in (6)) that renders them orthonormal.

Theorem 2. Let M be a nilmanifold associated to (36), with the metric g. The classes of invariant almost Hermitian structures defined in (23) satisfy

$$\begin{aligned} \mathcal{Z}_{234} &\supset \{\omega_1, \omega_2, c(\tfrac{2\pi}{3}), c(-\tfrac{2\pi}{3})\} \cup C, \\ \mathcal{Z}_{134} &= \{\omega_1, \omega_2, c(\tfrac{2\pi}{3}), c(-\tfrac{2\pi}{3})\}, \\ \mathcal{Z}_{124} &= C, \\ \mathcal{Z}_{123} &\supset \{c(0), c(\pi)\} \cup C \cup C' \cup C_{02} \cup C_{13} \cup C_{12}, \end{aligned}$$

(notation as in (13),(14)), where C, C' are other circles and the unions are disjoint.

Provided we use (37), the computations required for the proof of Theorem 2 are very similar to those of §4, and we omit verification of facts (2i) and (3i) below that are used to determine the 'larger classes'.

The class $\mathcal{Z}_{234}$. Lemma 1 gives

$$(a+1)(x+2w)+(a-1)x'-i[(a+1)(u-2z)+(a-1)u']=0.$$

(1i) When $a=-1$ we get $x'=0=u'$ giving the two points ω_1,ω_2.
(1ii) When $a=1$ we get $x=-2w$ and $u=2z$. This implies that one of r,t vanishes, and the only possibility is that $r=0$ and $4z^2+4w^2=1$. This forces $t=\pm\frac{1}{2}\sqrt{3}$ and $s=-\frac{1}{2}$ (the sign of s is fixed by the condition that $\det P_+=1$), giving solutions

$$\begin{array}{rcl} c(\frac{2\pi}{3}) & = & -\frac{1}{2}(e^{13}+e^{42})+\frac{\sqrt{3}}{2}(e^{14}+e^{23})+e^{56}, \\ c(-\frac{2\pi}{3}) & = & -\frac{1}{2}(e^{13}+e^{42})-\frac{\sqrt{3}}{2}(e^{14}+e^{23})+e^{56}. \end{array}$$

(1iii) When $a=0$ we get $x'=x+2w$ and $u'=u-2z$, that include the solutions in (3ii) below.

The class $\mathcal{Z}_{134}$. Lemma 2 gives
(2i) If $b=0$ then $\omega(P;a,b)$ cannot satisfy $W_2=0$.
(2ii) If $a=1$ the equations are the same as in (1ii).
(2iii) If $a=-1$ we get $u'=0=x'$, giving ω_1,ω_2.

The class $\mathcal{Z}_{124}$.
(3i) If $a=0$ then $\omega(P;a,b)$ cannot satisfy $W_3=0$.
(3ii) If $a=0$, Lemma 3 gives

$$u'=u-2z,\quad x'=x+2w,\quad r'=r,\quad t=0. \tag{38}$$

Let ξ denote the unit vector (r,u,x). Then we see that

$$|\xi-\xi'|^2=(r-r')^2+(u-u')^2+(x-x')^2=4(w^2+z^2)=4.$$

This forces $\xi'=-\xi$, whence $r=0$ and

$$\begin{array}{l} P^{-1}(e^{12})=ue^{42}+xe^{23}=e^2\wedge(xe^3-ue^4), \\ P^{-1}(e^{34})=xe^{14}+ue^{13}=e^1\wedge(xe^4+ue^3). \end{array}$$

Since the top row of P_+ is determined by $P^{-1}(e^{12}+e^{34})$, and $\det P_+=1$, we obtain

$$P_+=\begin{pmatrix} 0 & u & x \\ 1 & 0 & 0 \\ 0 & x & -u \end{pmatrix} \quad\text{or}\quad \begin{pmatrix} 0 & u & x \\ -1 & 0 & 0 \\ 0 & -x & u \end{pmatrix}.$$

However, the first matrix gives $x'=3x$, $u'=3u$ and $|\xi|\quad 3$, which is impossible.

In computing $\omega(P;0,1)=P(\varpi_0)$, we are free to apply an element of $SO(2)$ that rotates e^3,e^4 so that $x=1$ and $u=0$ (see (18), (19)). It follows that $P^{-1}\langle e^1,e^2\rangle=\langle e^2,e^3\rangle$ and $P^{-1}\langle e^3,e^4\rangle=\langle e^1,e^4\rangle$, and

$$P^{-1}:\quad \begin{array}{ll} e^1\mapsto Ae^2+Be^3 & e^2\mapsto -Be^2+Ae^3,\\ e^3\mapsto Ae^1-Be^4 & e^4\mapsto\ Be^1+Ae^4, \end{array}$$

where $A^2+B^2=1$, and one can check that (38) holds. The inverse mapping is given by

$$P:\quad \begin{array}{c} e^1\mapsto Ae^3+Be^4 \qquad e^2\mapsto Ae^1-Be^2,\\ e^{34}\mapsto (Be^1+Ae^2)(-Be^3+Ae^4), \end{array}$$

and it follows that $\mathcal{Z}_{124}$ is the circle

$$C=\{-(Ae^3+Be^4)\wedge e^5+(Ae^1-Be^2)\wedge e^6+(Be^1+Ae^2)\wedge(-Be^3+Ae^4)\} \quad (39)$$

in the 'middle' of the tetrahedron.

The class $\mathcal{Z}_{123}$. In order to visualize this using Corollary 1, it is helpful to realize that the hypersurface $\langle e^{12}\rangle^\perp_{\mathcal{Z}}$ is 'suspended' half way between the edges $\mathcal{E}_{01}=\lceil e^{12}\rfloor$, $\mathcal{E}_{23}=\lceil -e^{12}\rfloor$ of the tetrahedron (labelled '12' and '−12' in the figure in §2), with which it has empty intersection. Modulo a change of basis, this description follows from (20).

The additional orthogonality to $e^{14}+e^{23}$ implies the following:

(4i) $\mathcal{Z}_{123}\cap\mathcal{E}_{03}$ consists of the points

$$c(0)=e^{13}+e^{42}+e^{56},\quad c(\pi)=-e^{13}-e^{42}+e^{56}.$$

(4ii) $\mathcal{Z}_{123}$ contains the full equators (13) in $\mathcal{E}_{12},\mathcal{E}_{02},\mathcal{E}_{13}$.

(4iii) Any point on the circle (39) is obviously orthogonal to both e^{12} and $e^{14}+e^{23}$. The same is true for the circle

$$C'=\{(Ae^3+Be^4)\wedge e^5+(Ae^1-Be^2)\wedge e^6-(Be^1+Ae^2)\wedge(-Be^3+Ae^4)\}.$$

This completes the proof of Theorem 2, which provides enough information to determine most of the classes $\mathcal{Z}_i$, $\mathcal{Z}_{ij}$.

Corollary 2. The set $\mathcal{Z}_{34}$ of invariant Hermitian structures on (M_2,g) consists of four distinct points on C_{03}, and the set $\mathcal{Z}_2$ of compatible symplectic structures is the circle C.

According to [14, Theorem 3.3], *any* invariant complex structure J on M_2 has the property that $\mathbb{D}=\langle e^1,e^2,e^3,e^4\rangle$ is J-invariant. Let ${}^{p,q}\mathbb{D}={}^{p,q}\mathfrak{g}^*\cap\mathbb{D}_{\mathbb{C}}$. Any $(1,0)$-form not in ${}^{1,0}D$ will be a multiple of $\alpha=e^5+ce^6+\beta$, with $c\in\mathbb{C}$ and $\beta\in{}^{1,0}\mathbb{D}$. The classification of complex structures then amounts to determining which almost complex structures on $\mathbb{D}$ have the property that

$$d\alpha=e^{12}+c(e^{14}+e^{23})\in{}^{1,1}\mathbb{D}\oplus{}^{2,0}\mathbb{D}.$$

This becomes a more subtle problem with the imposition of an arbitrary metric.

Example. Let $g' = \frac{1}{4}(e^1 \otimes e^1 + e^2 \otimes e^2) + \sum_{i=3}^{6} e^i \otimes e^i$, so that $\{\frac{1}{2}e^1, \frac{1}{2}e^2, e^3, e^4, e^5, e^6\}$ is orthonormal relative to g'. We denote the space of g'-orthogonal almost complex structures by $\mathcal{Z}'$. Since any complex structure $J \in \mathcal{Z}'$ satisfies $J(\mathbb{D}) = \mathbb{D}$, it must be that $Je^5 = \pm e^6$ and J lies in $\mathcal{E}'_{12}$ $(c = i)$ or $\mathcal{E}'_{03}$ $(c = -i)$. Two obvious candidates in $\mathcal{E}'_{12}$, namely

$$\begin{aligned} \omega'_1 &= \tfrac{1}{4}e^{12} - e^{34} - e^{56}, \\ \omega'_2 &= -\tfrac{1}{4}e^{12} + e^{34} - e^{56}, \end{aligned}$$

are the counterparts to $\omega_1, \omega_2 \in \mathcal{Z}_{34}$. But this time, the equation

$$d(e^5 + ie^6) = e^{12} + i(e^{14} + e^{23}) = e^1(\tfrac{1}{2}e^2 + ie^4) + e^2(-\tfrac{1}{2}e^1 + ie^3)$$

leads to a *unique* solution

$$c'(\pi) = -\tfrac{1}{2}e^{13} - \tfrac{1}{2}e^{42} + e^{56} \in C'_{03}.$$

Thus, $\mathcal{Z}'_{34} = \{\omega'_1, \omega'_2, c'(\pi)\}$ consists of exactly *three* points.

This example is significant, because it shows that the homotopy class of the set of Hermitian structures is not independent of the choice of metric. One would expect the set of invariant Hermitian structures on M_2 to be discrete relative to any inner product on $\mathfrak{g}_2$. In this connection, the maximum number of isolated orthogonal complex structures that can co–exist on 6-manifold will depend on properties of its Weyl conformal curvature tensor.

6. Treatment of the 3–step case

A greater contrast with the calculations of §4 is provided by $\mathfrak{g}_3$, whose structure equations are

$$\begin{cases} de^i = 0, \quad 1 \quad i \quad 4 \\ de^5 = e^{12}, \\ de^6 = e^{15} + e^{34}. \end{cases} \tag{40}$$

This example is of special interest, since the associated group G_3 or nilmanifold M_3 possesses *no* invariant complex *or* symplectic forms, relative to *any* metric [14]. We shall detect and generalize this non–existence for the structures compatible with the standard metric (6).

Equation (37) is again applicable. In order to compute $d^P e^6$, recall that $\omega(P; a, b) = \omega(PQ; a, b)$ if Q belongs to the subgroup $SO(2)$ of $SO(4)$ which fixes e^1, e^2, and rotates e^3, e^4. By left–multiplying P by a suitable Q, we may suppose that

$$P^{-1}(e^1) = \sum_{i=1}^{3} \lambda_i e^i,$$

where $\lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 1$. With this choice of P,

$$\begin{aligned} d^P e^6 &= P^{-1}(e^1) \wedge e^5 + P^{-1}(e^{34}) \\ &= (\lambda_1 e^{15} + \lambda_2 e^{25} + \lambda_3 e^{35}) + \tfrac{1}{2}\Big[r(e^{12} + e^{34}) + u(e^{13} + e^{42}) + x(e^{14} + e^{23}) \\ &\qquad - r'(e^{12} - e^{34}) - u'(e^{13} - e^{42}) - x'(e^{14} - e^{23})\Big]. \end{aligned}$$

The presence of the λ_i complicates the situation, though we shall give a partial description of classes of almost Hermitian structures in terms of the generalized edges (15).

The class $\mathcal{Z}_{234}$. According to Lemma 1, $\omega(P;a,b)$ has $W_1=0$ if and only if

$$\begin{aligned}&x(1+a)-x'(1-a)+u(1+a)-u'(1-a)\\&\qquad+i\Big[x(1+a)-x'(1-a)-u(1+a)+u'(1-a)+2b\lambda_3\Big]=0.\end{aligned}$$

In the special case $\lambda_3=0$, the equations become

$$\begin{aligned}x(1+a)&=x'(1-a),\\u(1+a)&=u'(1-a).\end{aligned}\tag{41}$$

From (37),

$$P^{-1}(e^{12})=(\lambda_1e^1+\lambda_2e^2)\wedge P^{-1}(e^2)=\tfrac12(r+r')e^{12}+\tfrac12(r-r')e^{34}+\cdots$$

Since the left-hand side has no term in e^{34}, we obtain $r=r'$. Thus, $u^2+x^2=(u')^2+(x')^2$, and (41) gives two cases:
(1i) $u=x=u'=x'=0$. Either $r=r'=1$, giving the edge $\mathcal{E}_{02}$, or $r=r'=-1$, giving the edge $\mathcal{E}_{13}$.
(1ii) $a=0$. We have $r=r'$, $u=u'$ and $x=x'$. From (37) we obtain

$$P^{-1}(e^{12})=re^{12}+ue^{13}+xe^{14}=e^1\wedge(re^2+ue^3+xe^4).\tag{42}$$

It follows that $P^{-1}(e^1)$ is a linear combination of e^1 and $re^2+ue^3+xe^4$, but from above this can only happen if $u=x=0$ (case (1i)) or $P^{-1}(e^1)=\pm e^1$. In the latter case, the fundamental 2–form belongs to the disjoint union $\lceil e^{15}\rfloor\cup\lceil -e^{15}\rfloor$ of two spheres, parametrized by (r,u,x) and the choice of sign.

In view of these results, we perform a rotation of 90° in the $\langle e^2,e^5\rangle$ plane to transform (12) into

$$\begin{array}{lrcl}\bullet & \varpi_0 &=& -e^{15}+e^{26}+e^{34},\\ \bullet & \varpi_1 &=& -e^{15}-e^{26}-e^{34},\\ \bullet & \varpi_2 &=& e^{15}-e^{26}+e^{34},\\ \bullet & \varpi_3 &=& e^{15}+e^{26}-e^{34},\end{array}\tag{43}$$

extending the notation (18). We now state

Theorem 3. Let M_3 be a nilmanifold associated to (40), with a standard metric (6). The classes of almost Hermitian structures defined in (23) are given by

$$\begin{array}{rcl}\mathcal{Z}_{234}&\supset&\mathcal{E}_{02}\cup\mathcal{E}_{13}\cup\lceil e^{15}\rfloor\cup\lceil -e^{15}\rfloor,\\ \mathcal{Z}_{134}&=&\emptyset,\\ \mathcal{Z}_{124}&=&\{\varpi_1,\varpi_2\},\\ \mathcal{Z}_{123}&\supset&C_{02}\cup C_{13}\cup\lceil e^{26}\rfloor,\end{array}$$

where $\varpi_1\in C_{13}$ and $\varpi_2\in C_{02}$.

The first inclusion follows from (1i),(1ii), and is certainly strict. We proceed to investigate the class $\mathcal{Z}_{134}$ that in general contains the set $\mathcal{C}=\mathcal{Z}_{34}$ of Hermitian structures.

The class $\mathcal{Z}_{134}$. Lemma 2 gives

$$\begin{cases} 2a\lambda_3-(u+u')b+(x-x')ab-i[(x+x')b+(u-u')ab]=0,\\ a\,[2a\lambda_1+b(r-r')]+i[2a\lambda_2-b(r+r')]=0,\\ b\,[2a\lambda_1+b(r-r')]-2i\lambda_2 b=0,\\ b\,[(1+a)x+(1-a)x']-ib[(1+a)u+(1-a)u']=0,\\ -(1+a)u+(1-a)u'+(a+a^2-b^2)x+(a-a^2+b^2)x'-2\lambda_3 b\\ \quad -i\Big[(1+a)x-(1-a)x'+(a+a^2-b^2)u+(a-a^2+b^2)u'\Big]=0,\\ -(1+a)u+(1-a)u'+(a+a^2)x+(a-a^2)x'\\ \quad -i\Big[(1+a)x-(1-a)x'+(a+a^2)u+(a-a^2)u'\Big]=0. \end{cases}$$

We obtain the following cases.
(2i) $a=0$, $b=\pm1$ implies $r=r'=u=u'=x=x'=0$, which is impossible since $P_\pm$ must be non–singular.
(2ii) $b=0$, $a=\pm1$ implies that $\lambda_i=0$ for all i, which is equally absurd.
(2iii) $ab=0$ implies that $\lambda_2=\lambda_3=0$ and $u=u'=x=x'=0$. Another consequence of the equations is that $r=-r'$, but this contradicts Lemma 5.

Thus, $\mathcal{Z}_{134}$ is indeed empty.

The class $\mathcal{Z}_{124}$. Lemma 3 gives

$$\begin{cases} (a-1)(u+x)+(a+1)(u'+x')\\ \quad +i[(a-1)(u-x)+(a+1)(u'-x')-2\lambda_3 b]=0,\\ -(a+1)x-(a-1)x'+(a+2a^2-1)u+(a-2a^2+1)u'\\ \quad +i[(a+1)u+(a-1)u'+(a+2a^2-1)x+(a-2a^2+1)x'+2\lambda_3 b]=0,\\ (a-1)x+(a+1)x'-(a-2a^2+1)u-(a+2a^2-1)u'\\ \quad -i[(a-1)u+(a+1)u'+(a-2a^2+1)x+(a+2a^2-1)x'+2\lambda_3 b]=0,\\ (a-1)r+(a+1)r'-ia[2\lambda_1 b-(a-1)r+(a+1)r']=0\\ b\,[2\lambda_1 b-r(a-1)+r'(a+1)]=0\\ ab\,[(u-u')+i(x-x')]=0. \end{cases}$$

Possible solutions are
(3i) $a=0$, $b=\pm1$ implies that $r=r'=\mp\lambda_1$, $u=u'$, $x=x'$, and $\lambda_3=0$. From (42), we deduce that $r=\pm1$ *and* $u=x=0$. All solutions lie on $\mathcal{E}_{02}$ (if $r=r'=1$) or $\mathcal{E}_{13}$ (if $r=r'=-1$). However, the condition that $P(e^1)=-bre^1$ reduces solutions to ϖ_1,ϖ_2.
(3ii) $b=0$, $a=\pm1$ implies that $r=u=x=r'=u'=x'=0$, which is impossible.
(3iii) $ab=0$ implies $u=x=u'=x'=0$, whence $r^2=(r')^2=1$. But this contradicts $(1-a)r=(1+a)r'$, which is another consequence of the equations.

The class $\mathcal{Z}_{123}$. According to (29), this is a singular intersection of $\langle e^{12}\rangle^\perp_{\mathcal{Z}}$ and $\langle e^{15}+e^{34}\rangle^\perp_{\mathcal{Z}}$. Lemma 4 gives

$$\begin{cases} (a+1)r+(a-1)r'=0\\ (a+1)r+r'(1-a)=2b\lambda_1, \end{cases}$$

and thus $\lambda_1 b = r(a+1) = r'(1-a)$. We can use this to describe some subsets of $\mathcal{Z}_{123}$:
(4i) $b = 0$ gives $r = 0$ and solution set C_{03}, or $r' = 0$ and C_{12}.
(4ii) $a = 0$ implies $r = r' = \lambda_1$, and this gives $\lceil e^{26} \rfloor$.

The subset $\mathcal{Z}_{123}$ intersects the edges $\bigcup \mathcal{E}_{ij}$ of the original tetrahedron in the circles C_{02}, C_{13} and the points $\varpi_0, \varpi_3 \in \lceil e^{26} \rfloor$. It intersects each face $\mathcal{F}_i$ in a 'cone' joining one of these two points to one of the two circles. On the other hand, Theorem 3 is saying is that $\mathcal{Z}_{234}$ contains four of the six edges

$$\mathcal{E}_{02} = \lceil e^{34} \rfloor, \quad \mathcal{E}_{13} = \lceil -e^{34} \rfloor, \quad \lceil \pm e^{15} \rfloor, \quad \lceil \pm e^{26} \rfloor$$

of the new tetrahedron with vertices (43). The new edge $\lceil e^{26} \rfloor$ is *not* one of these four, and is disjoint from ϖ_1, ϖ_2, that one can check do not lie in $\mathcal{Z}_{123}$. It follows that the class $\mathcal{Z}_2$ of symplectic structures is empty, and moreover

Corollary 3. The following classes (23) for (M_3, g) are empty: $\mathcal{Z}_1, \mathcal{Z}_2, \mathcal{Z}_3, \mathcal{Z}_4, \mathcal{Z}_{12}, \mathcal{Z}_{13}, \mathcal{Z}_{14}, \mathcal{Z}_{34}, \mathcal{Z}_{134}$.

As already remarked, the vanishing of $\mathcal{C} = \mathcal{Z}_{34}$ and $\mathcal{S} = \mathcal{Z}_2$ does not depend on the choice of metric on M_3; it would be interesting to know whether this is true of the other classes listed above. A more positive feature of M_3 is that (at least with respect to g) $\mathcal{Z}_{24} \setminus \mathcal{Z}_2$ is non–empty. Indeed, the equations

$$d\varpi_1 = -\varpi_1 \wedge e_2, \quad d\varpi_2 = \varpi_2 \wedge e_2$$

show that $\mathcal{Z}_{24}$ contains ϖ_1, ϖ_2, and Theorem 3 implies that it contains no other points.

The examples studied in §4, §5, §6 all have $\mathcal{Z}_{14} = \emptyset$ (see for example Corollary 1), and it is natural to ask to what extent this is a general phenomenon. In particular, we conjecture that the classes $\mathcal{Z}_1, \mathcal{Z}_4$ are both empty for any Riemannian metric on any compact 6–dimensional nilmanifold other than a torus. This would generalize the non–existence of a Kähler metric on such manifolds.

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