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Alexander G. Tumanov, Peter West *

Department of Mathematics, King's College, London WC2R 2LS, UK

A R T I C L E I N F O

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A B S T R A C T

We construct the non-linear realisation of the semi-direct product of E_{11} and its vector representation in eleven dimensions and find the dynamical equations it predicts at low levels. These equations are completely determined by the non-linear realisation and when restricted to contain only the usual fields of supergravity and the usual spacetime we find precisely the equations of motion of eleven dimensional supergravity. This paper extends the results announced in hep-th/1512.01644 and in particular it contains the contributions to the equations of motion that involve derivatives with respect to the level one generalised coordinates.

1. Introduction

Quite some time ago it was conjectured that the low energy effective action for strings and branes is the non-linear realisation of the semi-direct product of E_{11} and its vector (l_1) representation, denoted $E_{11} \otimes_s l_1$ [1,2]. This theory has an infinite number of fields, associated with E_{11}, which live on a generalised spacetime, associated with the vector representation l_1.

The fields obey equations of motion that follow from the symmetries of the non-linear realisation. Although it was clear from the beginning [1] that the fields at low levels were just those of the maximal supergravity theories the unfamiliar nature of spacetime discouraged the construction of the equations of motion. The earliest attempts often used only the usual coordinates of spacetime and only the Lorentz part of the $I_c(E_{11})$ local symmetry. As a result the full power of the symmetries of the non-linear realisation was not exploited and the results were incomplete. A more systematic approach was used to constructing the equations of motion of the $E_{11} \otimes_s l_1$ non-linear realisation in eleven [3] and four [4] dimensions by including both the higher level generalised coordinates and local symmetries in $I_c(E_{11})$. These papers did find the equations of motion of the form fields but found only partial results for the gravity equation.

Recently the equations of motion of the non-linear realisation $E_{11} \otimes_s l_1$ were found for all the usual fields in the maximal supergravity fields, including gravity, in five and eleven dimensions [5]. The equations of motion in eleven dimensions were completely determined and in agreement with those of eleven dimensional supergravity when one keeps only the usual supergravity fields and the usual coordinates of spacetime. In this paper we will give some of the details of this calculation as well as giving the terms in the equations of motion that contain derivatives to the level one generalised coordinate. We will also complete the variation of the gravity equation under the symmetries of the non-linear realisation to show that it varies in the three form equation.

In this section we will also review the main features of the non-linear realisation. In section 2 we will formulate the eleven dimensional theory including the explicit forms of the Cartan forms and generalised vielbein. Section 3 derives the variations of the Cartan form under the symmetries of the non-linear realisation and in particular discusses an important subtilty associated with the fixing of the group element of the non-linear realisation using its local symmetry. Using these results in section 4 we find the equations of motion for the three form and gravity and show that they vary into each other. Finally we discuss some of the consequences of the result in section 5.

To fix the notation, and as it is still not that well understood, we recall from previous papers the main features of the non-linear realisation of $E_{11} \otimes_s l_1$ which is constructed from the group element $g \in E_{11} \otimes_s l_1$ that can be written as

$$g = g_l g_E \tag{1.1}$$

* Corresponding author.
E-mail addresses: peter.west540@gmail.com, peter.west@kcl.ac.uk (P. West).

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In this equation g_E is a group element of E_{11} and so can be written in the form $g_E = e^{A_{\underline{\alpha}} R^{\underline{\alpha}}}$ where the $R^{\underline{\alpha}}$ are the generators of E_{11} and $A_{\underline{\alpha}}$ are the fields in the non-realisation. The group element g_l is formed from the generators of the vector (l_1) representation and so has the form $e^{z^A L_A}$ where z^A are the coordinates of the generalised spacetime. The fields $A_{\underline{\alpha}}$ depend on the coordinates z^A. The non-linear realisation is, by definition, invariant under the transformations

$$g \to g_0 g, \quad g_0 \in E_{11} \otimes_s l_1, \quad \text{as well as} \quad g \to gh, \quad h \in I_c(E_{11}) \tag{1.2}$$

The group element $g_0 \in E_{11}$ is a rigid transformation, that is, it is a constant. The group element h belongs to the Cartan involution invariant subalgebra of E_{11}, denoted $I_c(E_{11})$; it is a local transformation meaning that it depends on the generalised spacetime. The action of the Cartan involution can be taken to be $I_c(R^{\underline{\alpha}}) = -R^{-\underline{\alpha}}$ for any root α and so the Cartan involution invariant subalgebra is generated by $R^{\underline{\alpha}} - R^{-\underline{\alpha}}$.

As the generators in g_l form a representation of E_{11} the above transformations for $g_0 \in E_{11}$ can be written as

$$g_l \to g_0 g_l g_0^{-1}, \quad g_E \to g_0 g_E \quad \text{and} \quad g_E \to g_E h \tag{1.3}$$

The dynamics of the non-linear realisation is just an action, or set of equations of motion, that are invariant under the transformations of equation (1.2). We now recall how to construct the dynamics of the $E_{11} \otimes_s l_1$ non-linear realisation using the Cartan forms which are given by

$$\mathcal{V} \equiv g^{-1} dg = \mathcal{V}_E + \mathcal{V}_l, \tag{1.4}$$

where

$$\mathcal{V}_E = g_E^{-1} dg_E \equiv dz^\Pi G_{\Pi,\underline{\alpha}} R^{\underline{\alpha}}, \quad \text{and} \quad \mathcal{V}_l = g_E^{-1}(g_l^{-1} dg_l) g_E = g_E^{-1} dz \cdot l g_E \equiv dz^\Pi E_\Pi{}^A l_A \tag{1.5}$$

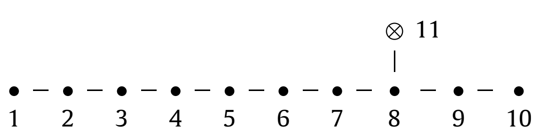
Clearly $\mathcal{V}_E$ belongs to the E_{11} algebra and it is the Cartan form of E_{11} while $\mathcal{V}_l$ is in the space of generators of the l_1 representation and one can recognise $E_\Pi{}^A = (e^{A_{\underline{\alpha}} D^{\underline{\alpha}}})_\Pi{}^A$ as the vielbein on the generalised spacetime.

Both $\mathcal{V}_E$ and $\mathcal{V}_l$ are invariant under rigid transformations, but under the local $I_c(E_{11})$ transformations of equation (1.3) they change as

$$\mathcal{V}_E \to h^{-1} \mathcal{V}_E h + h^{-1} dh \quad \text{and} \quad \mathcal{V}_l \to h^{-1} \mathcal{V}_l h \tag{1.6}$$

2. The eleven dimensional theory

The theory in eleven dimensions is found by deleting the node labelled 11 of the E_{11} Dynkin diagram and decomposing the $E_{11} \otimes_s l_1$ algebra into representations of the resulting algebra which is GL(11).



The E_{11} generators can be classified by a level which is the number of up minus down indices divided by three. This level is preserved by the E_{11} commutation relations. The decomposition of E_{11} into representations of SL(11) up to level four can be found in the book [6]. The positive level generators are [1]

$$K^a{}_b, \; R^{a_1 a_2 a_3}, \; R^{a_1 a_2 \ldots a_6} \text{ and } R^{a_1 a_2 \ldots a_8, b}, \ldots \tag{2.1}$$

where the generator $R^{a_1 a_2 \ldots a_8, b}$ obeys the condition $R^{[a_1 a_2 \ldots a_8, b]} = 0$ and the indices $a, b, \ldots = 1, 2 \ldots 11$. The negative level generators are given by

$$R_{a_1 a_2 a_3}, \; R_{a_1 a_2 \ldots a_6}, \; R_{a_1 a_2 \ldots a_8, b}, \ldots \tag{2.2}$$

The vector (l_1) representation decomposes into representations of $GL(11)$ as [2,3]

$$P_a, \; Z^{ab}, \; Z^{a_1 \ldots a_5}, \; Z^{a_1 \ldots a_7, b}, \; Z^{a_1 \ldots a_8}, \; Z^{b_1 b_2 b_3, a_1 \ldots a_8}, \ldots \tag{2.3}$$

For the eleven dimensional theory the group element of $E_{11} \otimes_s l_1$ is of the form $g = g_l g_E$ where

$$g_E = \ldots e^{h_{a_1 \ldots a_8, b} R^{a_1 \ldots a_8, b}} e^{A_{a_1 \ldots a_6} R^{a_1 \ldots a_6}} e^{A_{a_1 a_2 a_3} R^{a_1 a_2 a_3}} e^{h_a{}^b K^a{}_b} \tag{2.4}$$

and

$$g_l = e^{x^a P_a} e^{x_{ab} Z^{ab}} e^{x_{a_1 \ldots a_5} Z^{a_1 \ldots a_5}} \ldots = e^{z^A L_A} \tag{2.5}$$

The parameters of the group element g_E will become the fields of the theory while the parameters of the group element g_l will become the coordinates of the generalised spacetime on which the fields depend. The above parameterisation differs slightly from that used in reference [3] and this will lead to corresponding differences in some of the later equations in this paper.

The Cartan forms of E_{11} were defined in equation (1.4) and those of the E_{11} part can be written in the form

$$\mathcal{V}_E = G_a{}^b K^a{}_b + G_{c_1 \ldots c_3} R^{c_1 \ldots c_3} + G_{c_1 \ldots c_6} R^{c_1 \ldots c_6} + G_{c_1 \ldots c_8, b} R^{c_1 \ldots c_8, b} + \ldots \tag{2.6}$$

We now evaluate these E_{11} Cartan form in terms of the field that parameterise the group element of equation (2.4), one finds that [3]

$$G_a{}^b = (e^{-1}de)_a{}^b, \qquad G_{a_1\ldots a_3} = e_{a_1}{}^{\mu_1}\ldots e_{a_3}{}^{\mu_3} dA_{\mu_1\ldots\mu_3},$$
$$G_{a_1\ldots a_6} = e_{a_1}{}^{\mu_1}\ldots e_{a_6}{}^{\mu_6}(dA_{\mu_1\ldots\mu_6} - A_{[\mu_1\ldots\mu_3}dA_{\mu_4\ldots\mu_6]})$$
$$G_{a_1\ldots a_8,b} = e_{a_1}{}^{\mu_1}\ldots e_{a_8}{}^{\mu_8} e_b{}^{\nu}(dh_{\mu_1\ldots\mu_8,\nu} - A_{[\mu_1\ldots\mu_3}dA_{\mu_4\mu_5\mu_6}A_{\mu_7\mu_8]\nu} + 3A_{[\mu_1\ldots\mu_6}dA_{\mu_7\mu_8]\nu}$$
$$+ A_{[\mu_1\ldots\mu_3}dA_{\mu_4\mu_5\mu_6}A_{\mu_7\mu_8\nu]} - 3A_{[\mu_1\ldots\mu_6}dA_{\mu_7\mu_8\nu]}) \tag{2.7}$$

where $e_\mu{}^a \equiv (e^h)_\mu{}^a$.

The generalised vielbein $E_\Pi{}^A$, can be evaluated from its definition of equation (1.5) to be given as a matrix by [3,7]

$$E = (\det e)^{-\frac{1}{2}} \begin{pmatrix} e_\mu{}^a & -3e_\mu{}^c A_{cb_1b_2} & 3e_\mu{}^c A_{cb_1\ldots b_5} + \frac{3}{2}e_\mu{}^c A_{[b_1b_2b_3}A_{|c|b_4b_5]} \\ 0 & (e^{-1})_{[b_1}{}^{\mu_1}(e^{-1})_{b_2]}{}^{\mu_2} & -A_{[b_1b_2b_3}(e^{-1})_{b_4}{}^{\mu_1}(e^{-1})_{b_5]}{}^{\mu_2} \\ 0 & 0 & (e^{-1})_{[b_1}{}^{\mu_1}\ldots(e^{-1})_{b_5]}{}^{\mu_5} \end{pmatrix} \tag{2.8}$$

3. The transformations of the Cartan forms

The Cartan forms are inert under the rigid transformations of equation (1.2) but under the local Cartan invariant involution transformation $h \in I_c(E_{11})$ they transform as in equation (1.6). As the Cartan involution invariant subalgebra of SL(11) is SO(11) they transform under SO(11) for the lowest level transformations. At the next level they transform under a group element h which involves the generators at levels ± 1 and it is of the form

$$h = 1 - \Lambda_{a_1a_2a_3}S^{a_1a_2a_3}, \quad \text{where } S^{a_1a_2a_3} = R^{a_1a_2a_3} - \eta^{a_1b_1}\eta^{a_2b_2}\eta^{a_3b_3}R_{b_1b_2b_3} \tag{3.1}$$

Under this transformation the Cartan forms of equation (1.6) change as

$$\delta\mathcal{V}_E = \left[S^{a_1a_2a_3}\Lambda_{a_1a_2a_3}, \mathcal{V}_E\right] - S^{a_1a_2a_3}d\Lambda_{a_1a_2a_3}. \tag{3.2}$$

These local variations of the Cartan forms are straightforward to compute, using the E_{11} algebra and they are given by [3]

$$\delta G_a{}^b = 18\Lambda^{c_1c_2b}G_{c_1c_2a} - 2\delta_a^b\Lambda^{c_1c_2c_3}G_{c_1c_2c_3}, \tag{3.3}$$

$$\delta G_{a_1a_2a_3} = -\frac{5!}{2}G_{b_1b_2b_3a_1a_2a_3}\Lambda^{b_1b_2b_3} - 3G^c{}_{[a_1}\Lambda_{|c|a_2a_3]}, -d\Lambda_{a_1a_2a_3} \tag{3.4}$$

$$\delta G_{a_1\ldots a_6} = 2\Lambda_{[a_1a_2a_3}G_{a_4a_5a_6]} - 8.7.2G_{b_1b_2b_3[a_1\ldots a_5,a_6]}\Lambda^{b_1b_2b_3} + 8.7.2G_{b_1b_2a_1\ldots a_5a_6,b_3}\Lambda^{b_1b_2b_3} \tag{3.5}$$

$$\delta G_{a_1\ldots a_8,b} = -3G_{[a_1\ldots a_6}\Lambda_{a_7a_8]b} + 3G_{[a_1\ldots a_6}\Lambda_{a_7a_8b]} \tag{3.6}$$

When carrying out the local $I_c(E_{11})$ transformations one must take into account the fact that we have also used the local symmetry to fix the group element to have a simpler form, as we have done in equation (2.4). In most past applications this matter has usually been resolved by computing the compensating local subgroup transformation h required for a given rigid transformation g_0 to restore the group element into the chosen form. This involves manipulating group elements and is often very long and cumbersome. In this paper we will use an alternative approach which was used in the calculations of reference [5]. The new method applies to any non-linear realisation for which the local subgroup is the Cartan involution invariant subalgebra, however, to be concrete we will explain it for the case of interest to us here, that is, the $E_{11} \otimes_s l_1$ non-linear realisation.

The non-linear realisation $E_{11} \otimes_s l_1$ has a group element $g = g_l g_E$ that is subject to the two types of transformations of equation (1.2) which are required to be symmetries of the dynamics. It is often very useful to use the local transformation to gauge away some of the fields in the group element g_E. When the local subgroup is the Cartan involution invariant sub algebra, $I_c(E_{11})$, we can use the local symmetry to gauge away all the fields associate with negative root generators. In fact it is desirable to keep the level zero symmetries, such as Lorentz symmetry, manifest and so we only remove all the fields associated with the negative roots except for those at level zero. Put another way we use the gauge transformations to remove all the negative level fields from the group element and then the only remaining local symmetries are those of level zero. We now assume we have made such a choice of group element g_E.

As the group element has only positive level fields and generators, it follows that the Cartan forms constructed from it will contain only positive level generators. Although the Cartan forms are inert under the rigid transformations, their form is not preserve by the local transformations of equation (1.2), other than by the transformations of level zero. The local transformations which involves level plus and minus one level generators can be written in the form $h = 1 - \Lambda \cdot (R^{(1)} - R^{(-1)})$ and this will not preserve the form of the Cartan form. The precise form of this transformation is given in equation (3.1) for the case of eleven dimensions. Such a transformation will result in a change in the Cartan forms that has a level minus one contribution. To preserve the form of the Cartan forms we set this contribution to zero and so find the equation

$$[\Lambda \cdot R^{(-1)}, \mathcal{V}^{(0)}] - d\Lambda \cdot R^{(-1)} = 0 \tag{3.7}$$

where the superscript denotes the level.

Equation (3.7) should be thought of as a constraint on the spacetime dependent parameter Λ and it can be solved by taking

$$\Lambda \cdot R^{(-1)} = (g_E^{(0)})^{-1}\Lambda_c \cdot R^{(-1)}g_E^{(0)} \tag{3.8}$$

where Λ_c is a constant parameter.

For the case of eleven dimensions $g_E^{(0)} = e^{h_a{}^b K_a{}^b}$ and we find that equation (3.7) takes the form

$$d\Lambda^{a_1a_2a_3} - 3G_b{}^{[a_1|}\Lambda^{b|a_2a_3]} = 0 \tag{3.9}$$

The solution to this equation is given by

$$\Lambda^{a_1a_2a_3} = \Lambda_c^{\tau_1\tau_2\tau_3} e_{\tau_1}{}^{a_1} e_{\tau_2}{}^{a_2} e_{\tau_3}{}^{a_3} \tag{3.10}$$

where $\Lambda_c^{\tau_1\tau_2\tau_3}$ is a constant. The reader may verify that this is the same result as solving equation (3.8). We note that the local transformation is really only a rigid transformation as should indeed be the case as we have fixed the form of the group element using the local transformation.

We can use equation (3.9) to eliminate $d\Lambda^{a_1a_2a_3}$ from the transformations of equations (3.3), (3.5), (3.6). In fact this only affects the transformation of the Cartan form for the three form of equation (3.4) which now becomes

$$\delta G_{a_1a_2a_3} = -\frac{5!}{2} G_{b_1b_2b_3a_1a_2a_3}\Lambda^{b_1b_2b_3} - 6G_{(c[a_1|)}\Lambda_{c|a_2a_3]} \tag{3.11}$$

In carrying out the variation of the equations of motion we will encounter the derivative of the parameter of equation (3.10) and in processing these terms we must take account of the fact that only $\Lambda_c^{\tau_1\tau_2\tau_3}$ is independent of the generalised spacetime. This observation plays a crucial role in the calculations in this paper and those of reference [5].

The above method is equivalent to carrying out a rigid E_{11} transformation and finding the compensating local transformation that preserves the form of the group element. However, it is very much simpler and easier to implement than the old method.

The Cartan forms, discussed above, were written as forms and so they are written as $G_{\underline{\alpha}}$ where $G_{\underline{\alpha}} \equiv dz^{\Pi} G_{\Pi,\underline{\alpha}}$ and $G_{\Pi,\underline{\alpha}}$ are the components. The first index Π is associated with the vector representation (l_1) while the second index is associated with E_{11}. Although the Cartan forms when written in form notation are invariant under the rigid transformations of equation (1.2) once written in terms of components they are not invariant. We can remedy this by taking the first index to be a tangent index, that is, $G_{A,\underline{\alpha}} = (E^{-1})_A{}^{\Pi} G_{\Pi,\underline{\alpha}}$ which is inert under the rigid E_{11} transformations, but transforms under the local $I_c(E_{11})$ transformations. This latter transformation just being that for the inverse vielbein of equation (1.6). One finds that the Cartan forms, when referred to the tangent space, transform on their l_1 index as [3]

$$\delta G_{a,\bullet} = -3G^{b_1b_2}{}_{,\bullet}\Lambda_{b_1b_2a}, \qquad \delta G^{a_1a_2}{}_{,\bullet} = 6\Lambda^{a_1a_2b} G_{b,\bullet} \tag{3.12}$$

These transformations are to be combined with the local transformations on the second E_{11} index given earlier in this section.

4. Eleven dimensional equations of motion

The non-linear realisation of $E_{11} \otimes_s l_1$ was computed at low levels in [3] where one found the equation of motion that relates the three form and six form fields. It will be instructive to rederive this equation so as to make clear the origin of terms in the equations of motion that contain derivatives with respect to the higher level coordinates.

We seek a set of equations which are first order in derivatives and are invariant under the symmetries of the non-linear realisation. For the equation for the three form we should consider one that has four indices as it also includes one derivative. We can also built the equations out of the Cartan forms of equation (2.7) as these are invariant under the rigid transformations of equation (1.2) leaving only the problem of finding equations that are invariant under the local transformations. At lowest level these latter transformations are just local Lorentz transformations. On grounds of Lorentz invariance the only equation which is first order in the Cartan forms for the three and six form and has four Lorentz indices must be of the generic form

$$G_{[a_1,a_2a_3a_4]} - c\epsilon_{a_1a_2a_3a_4}{}^{b_1\dots b_7} G_{[b_1,b_2\dots b_7]} = 0 \tag{4.1}$$

where c is a constant. We note that the Cartan forms do appear with all their indices totally antisymmetrised although this was not an initial requirement.

The real test for the equation (4.1) is if it is invariant under the transformation of $I_c(E_{11})$ at the next levels and so we consider the variation of this equation under the transformations of equation (3.1), or equivalently equations (3.3), (3.5), (3.6) and equation (3.11). To find this equation it will suffice to keep only the terms that involve the three form and six form. The variation of $G_{[a_1,a_2a_3a_4]}$ of equation (3.11) leads to the Cartan form for the six form but it is not totally antisymmetrised in all its indices. Clearly we can only obtain an invariant equation if we have such a total antisymmetry. The way to resolve this problem is to consider the object

$$\mathcal{G}_{a_1,a_2a_3a_4} \equiv G_{[a_1,a_2a_3a_4]} + \frac{15}{2} G^{b_1b_2}{}_{,b_1b_2a_1\dots a_4} \tag{4.2}$$

The variation of this object can be written in the form

$$\delta\mathcal{G}_{a_1,a_2a_3a_4} = -\frac{1}{2.4!.4!}\epsilon_{a_1,a_2a_3a_4b_1b_2b_3c_1\dots c_4}\epsilon^{c_1\dots c_4e_1\dots e_7} G_{e_1\dots e_7}\Lambda^{b_1b_2b_3} \tag{4.3}$$

It is then straightforward to find that, up to the level we are working, the invariant equation is given by [3]

$$E_{a_1\dots a_4} \equiv \mathcal{G}_{[a_1,a_2a_3a_4]} - \frac{1}{2.4!}\epsilon_{a_1a_2a_3a_4}{}^{b_1\dots b_7} G_{b_1,b_2\dots b_7} = 0 \tag{4.4}$$

its variation being given by

$$\delta E_{a_1\dots a_4} = \frac{1}{4!}\epsilon_{a_1\dots a_4}{}^{b_1\dots b_7}\Lambda_{b_1b_2b_3} E_{b_4\dots b_7} + \dots \tag{4.5}$$

where $+\ldots$ denote gravity and higher level contributions.

The fact that equation (4.4) is invariant under the transformations of the non-linear realisation up to the level demanded justifies the one assumption made, namely that there does exist a equation that is first order in derivatives.

Rather than vary the three form equation (4.4) to find the gravity equation we will take the derivative of this equation in such a way as to eliminate the dual six form gauge field and then vary this equation to find the gravity equation. The variation of the first order equation has been given in a previous paper [3], but its unfamiliar form and derivation have meant that it has not been properly evaluated.

Taking the derivative of equation (4.4) we find the result

$$\partial_\nu((\det e)^{\frac{1}{2}} G^{[\nu,\mu_1\mu_2\mu_3]}) + \frac{1}{2.4!}(\det e)^{-1}\epsilon^{\mu_1\mu_2\mu_3\tau_1\ldots\tau_8} G_{[\tau_1,\tau_2\tau_3\tau_4]} G_{[\tau_5,\tau_6\tau_7\tau_8]} = 0 \tag{4.6}$$

which is the familiar second order equation of motion for the three form. We have discarded the terms which contain derivatives with respect to the higher level generalised coordinates as we will recover such terms when we vary equation (4.6). When we converted the first (l_1) index on the Cartan form to be a tangent index we used the inverse vielbein computed from the vielbein given in equation (2.8). We notice that the inverse vielbein contains a factor of $(\det e)^{\frac{1}{2}}$ compared to what one might normally expect. This explains the unusual factors of $(\det e)^{\frac{1}{2}}$ that populate the following equations.

In order to vary equation (4.6) under the $I_c(E_{11})$ transformation it is best to rewrite it in terms of the Cartan form of E_{11} using the expressions of equation (2.7). We find that it is equivalent to the equation

$$\begin{aligned} E^{a_1a_2a_3} &\equiv E^{a_1a_2a_3(1)} + E^{a_1a_2a_3(2)} \\ &\equiv \frac{1}{2} G_{b,d}{}^{d} G^{[b,a_1a_2a_3]} - 3G_{b,d}{}^{[a_1|} G^{[b,d|a_2a_3]]} - G_{c,b}{}^{c} G^{[b,a_1a_2a_3]} + (\det e)^{\frac{1}{2}} e_b{}^{\mu}\partial_\mu G^{[b,a_1a_2a_3]} \\ &\quad + \frac{1}{2.4!}\epsilon^{a_1a_2a_3b_1\ldots b_8} G_{[b_1,b_2b_3b_4]} G_{[b_5,b_6b_7b_8]} = 0 \end{aligned} \tag{4.7}$$

The expression $E^{a_1a_2a_3(1)}$ contains all terms that do not involve the epsilon symbol while $E^{a_1a_2a_3(2)}$ involves the one term that does.

We will vary the equations of motion so as to keep in the variations all terms that contain ordinary spacetime derivatives. This ensures that we will find all such terms in the equations of motion. However, we must also find all terms in the equations of motion we are varying that contain derivatives with respect to the level one generalised spacetime. Indeed, if we have a term in the variation of the form

$$\Lambda^{\tau\mu_1\mu_2} G_{\tau,\bullet} f^{\bullet}{}_{\mu_1\mu_2} \tag{4.8}$$

then using equation (3.12) we can cancel this by adding the term

$$-\frac{1}{6} G^{\mu_1\mu_2}{}_{,\bullet} f^{\bullet}{}_{\mu_1\mu_2} \tag{4.9}$$

to the equation of motion that is being varied. We will refer to such terms as l_1 terms. The other variations of this term are of a higher level than we are keeping. Hence when varying a given equation of motion we will find l_1 terms in this equation, but not in the new equation that results from the variation. The latter are then found by varying the new equation. The equations in this paper are given with the understanding that they have been computed up to these levels in the derivatives with respect to the generalised coordinates. Of course one can only add an l_1 term to an equation if it has the required index structure and symmetries.

So that the reader can get a feel for the intricate way in which the calculation works we now give some indications of how the variations of equation (4.7) under the local $I_c(E_{11})$ transformation of equations (3.3), (3.5), (3.6) and equation (3.11) are carried out. Varying the Cartan form $G^{[a_1,a_2a_3a_4]}$ contained in $E^{a_1a_2a_3(1)}$ under the $I_c(E_{11})$ transformation of equation (3.11) and converting the result back to carry world indices we find the expression

$$e_{\mu_1}{}^{[a_1} e_{\mu_2}{}^{a_2} e_{\mu_3}{}^{a_3]}\{3\partial_\nu\left((\det e)\omega_{\tau,}{}^{[\nu\mu_1|} - (\det e)^{\frac{1}{2}} G_{\tau,}{}^{[\nu\mu_1|}\right)\Lambda^{\tau|\mu_2\mu_3]} + \frac{5!}{2}(\det e)^{\frac{1}{2}} G^{[\nu,\mu_1\mu_2\mu_3]}{}_{\tau_1\tau_2\tau_3}\Lambda^{\tau_1\tau_2\tau_3}\} \tag{4.10}$$

When carrying out the variation it is important to recall the discussion of section three and, in particular, the fact that only the parameter $\Lambda^{\mu_1\mu_2\mu_3}$ is a constant.

By undoing the antisymmetrisation of the four indices we can rewrite the first term as

$$3e_{\mu_1}{}^{[a_1} e_{\mu_2}{}^{a_2} e_{\mu_3}{}^{a_3]}\partial_\nu\left(\det e\,\omega_{\tau,}{}^{[\nu\mu_1|}\right)\Lambda^{\tau|\mu_2\mu_3]} = \frac{3}{2} e_{\mu_1}{}^{[a_1} e_{\mu_2}{}^{a_2} e_{\mu_3}{}^{a_3]}\{\partial_\nu\left(\det e\,\omega_{\tau,}{}^{\nu\mu_1}\right)\Lambda^{\tau\mu_2\mu_3} + \partial_\nu\left(\det e\,\omega_{\tau,}{}^{\mu_1\mu_2}\right)\Lambda^{\tau\nu\mu_4}\} \tag{4.11}$$

In order to process the first term of equation (4.11) we note that

$$e_\mu{}^a\partial_\nu\left(\det e\,\omega_{\tau,}{}^{\nu\mu}\right) = \det e\left(e_b{}^\nu\partial_\nu\omega_{\tau,}{}^{ba} + \left(e_\mu{}^a\partial_\nu e_c{}^\mu\right)\omega_{\tau,}{}^{\nu c} + \left(e_c{}^\lambda\partial_\nu e_\lambda{}^c\right)\omega_{\tau,}{}^{\nu a} + \partial_\nu e_b{}^\nu\omega_{\tau,}{}^{ba}\right), \tag{4.12}$$

the relations

$$(\det e)\,\omega_{\mu,}{}^{ab}\omega_{\lambda,}{}^{c\lambda} = -\omega_{\mu,}{}^{ab}\partial_\lambda\left((\det e)\,e_b{}^\lambda\right) \tag{4.13}$$

and that

$$-\omega_{\nu,}{}^{a}{}_{c}\omega_{\mu,}{}^{c\nu} = (\det e)^{-\frac{1}{2}}\left(G_{a,(c\nu)} - G_{c,(a\nu)} - G_{\nu,[ac]}\right)\omega_{\mu,}{}^{c\nu} = -(\det e)^{-\frac{1}{2}} G_{c,\,\nu a}\omega_{\mu,}{}^{c\nu} = -e_b{}^\lambda\partial_\nu e_\lambda{}^a\omega_{\mu,}{}^{\nu b} \tag{4.14}$$

The Ricci tensor is given by

$$R_\mu{}^a = \partial_\mu \omega_{\nu,}{}^{ab} e_b{}^\nu - \partial_\nu \omega_{\mu,}{}^{ab} e_b{}^\nu + \omega_{\mu,}{}^a{}_c \omega_{\nu,}{}^{cb} e_b{}^\nu - \omega_{\nu,}{}^a{}_c \omega_{\mu,}{}^{cb} e_b{}^\nu, \tag{4.15}$$

whereupon we recognise that the first term in equation (4.11) is just the first, third and fourth terms of the Ricci tensor and as a result we can write this term as

$$\frac{3}{2} \det e \{R_\tau{}^{[a_1|} - \partial_\tau(\omega_{\nu,}{}^{[a_1|b}) e_b{}^\nu\} \Lambda^{\tau|a_1 a_2]} \tag{4.16}$$

However, the second term in equation (4.11) is of the form of equation (4.9) and so it can be introduced by adding an l_1 term to the three form equation of motion. Of course we can not from these considerations determine the coefficient of this term to be exactly as required to find the full Ricci tensor. However, this coefficient is fixed to the desired result once we vary the resulting gravity equation as was shown at the linearised level in [5]. For simplicity of presentation we will take the coefficient to be as required. The reader who wishes to insert an arbitrary coefficient and follow it through the remaining calculations, including the non-linear variation of the gravity equation, is encouraged to do so.

The second terms in both equations (4.10) and (4.11) are of the form $G_{\tau,\bullet}\Lambda^{\tau\mu\nu}$ and so they can be cancelled by adding l_1 terms to the three form equation of motion.

The variation of the second term in equation (4.7), that is, the terms in $E^{a_1a_2a_3(2)}$ can be processed by using equation (4.4) to swap the seven form field strength for the four form field strength. One finds the terms associated with the energy momentum tensor, further l_1 terms and terms which are cancelled by the variation of $E^{a_1a_2a_3(1)}$.

As explained, above when carrying out the variation of the three form equation we find the l_1 terms that we must add to this equation. The result of all these calculations is that the three form equation of motion, up to the level we are calculating, now takes the form

$$\begin{aligned}
\mathcal{E}^{a_1a_2a_3} \equiv\ & \frac{1}{2} G_{b,d}{}^d G^{[b,a_1a_2a_3]} - 3G_{b,d}{}^{[a_1|} G^{[b,d|a_2a_3]]} - G_{c,b}{}^c G^{[b,a_1a_2a_3]} + (\det e)^{\frac{1}{2}} e_b{}^\mu \partial_\mu G^{[b,a_1a_2a_3]} \\
& + \frac{1}{2.4!} \epsilon^{a_1a_2a_3b_1\ldots b_8} G_{[b_1,b_2b_3b_4]} G_{[b_5,b_6b_7b_8]} - 9G^{ca_1}{}_{,cd_1d_2} G^{[d_1,d_2a_2a_3]} + \frac{5}{16} \varepsilon^{a_1a_2a_3b_1\ldots b_8} G_{b_1,b_2b_3b_4} G^{c_1c_2}{}_{,c_1c_2b_5\ldots b_8} \\
& + \frac{1}{4} e_{\mu_1}{}^{[a_1} e_{\mu_2}{}^{a_2} e_{\mu_3}{}^{a_3]} \partial_\nu \left((\det e)^{\frac{1}{2}} G^{\mu_1\mu_2}{}_{,}{}^{\nu\mu_3} \right) + \frac{1}{4} (\det e)^{\frac{1}{2}} \omega_{\nu,}{}^{[a_1|b} G^{a_2a_3]}{}_{,b}{}^\nu \\
& + \frac{1}{4} G^{[a_1a_2|}{}_{,d}{}^d (G^{|a_3]}{}_{,c}{}^c - G_{c,}{}^{|a_3]c}) + \frac{1}{4} \partial_\nu \left((\det e)^{\frac{1}{2}} (G^{[a_1a_2|}{}_{,d}{}^d e^{\nu|a_3]} - G^{[a_1a_2|}{}_{,}{}^{|a_3]\nu}) \right) \\
& + \frac{1}{2} (G^{[a_1}{}_{,c}{}^{a_2} G^{|c|a_3]}{}_{,d}{}^d - G^{c[a_1}{}_{,}{}^{a_2|e|} G_{e,c}{}^{a_3]}) + \frac{15}{2} e_{\mu_1}{}^{[a_1} e_{\mu_2}{}^{a_2} e_{\mu_3}{}^{a_3]} \partial_\nu \left((\det e)^{\frac{1}{2}} G^{d_1d_2}{}_{,d_1d_2}{}^{\nu\mu_1\mu_2\mu_3} \right) \\
& + e_{\mu_1}{}^{[a_1} e_{\mu_2}{}^{a_2} e_{\mu_3}{}^{a_3]} \Big(\frac{1}{2} (\det e)^{\frac{1}{2}} g_{\tau\sigma} g^{\mu_1\lambda} \partial_\lambda G^{\tau\mu_2,(\sigma\mu_3)} - \frac{1}{2} G^{\tau\mu_1}{}_{,d}{}^d G^{\mu_2,(\mu_3}{}_{\tau)} \\
& - \frac{1}{4} G^{\tau\mu_1,(\mu_2}{}_{\tau)} G^{\mu_3}{}_{,d}{}^d - G^{\tau\mu_1}{}_{,(\tau\sigma)} G^{\mu_2,(\mu_3\sigma)} + G^{\tau\mu_1,(\mu_2\sigma)} G_{\sigma,(\tau}{}^{\mu_3)} \Big) = 0
\end{aligned} \tag{4.17}$$

Under the variation of the local transformations of equation (3.1) this equation of motion transforms as

$$\begin{aligned}
\delta\mathcal{E}^{a_1a_2a_3} =\ & \frac{3}{2} E_b{}^{[a_1|} \Lambda^{b|a_2a_3]} + \frac{1}{24} e_{\mu_1}^{a_1} e_{\mu_2}^{a_2} e_{\mu_3}^{a_3} \epsilon^{\mu_1\mu_2\mu_3\nu\lambda_1\ldots\lambda_4\tau_1\tau_2\tau_3} \partial_\nu \left((\det e)^{-\frac{1}{2}} E_{\lambda_1\ldots\lambda_4} g_{\tau_1\kappa_1} g_{\tau_2\kappa_2} g_{\tau_3\kappa_3} \right) \Lambda^{\kappa_1\kappa_2\kappa_3} \\
& + \frac{1}{24.4!} \epsilon^{a_1a_2a_3b_1\ldots b_8} \epsilon_{b_1\ldots b_4c_1c_2c_3e_1\ldots e_4} E_{b_5\ldots b_8} G^{[e_1,e_2\ldots e_4]} \Lambda^{c_1c_2c_3}
\end{aligned} \tag{4.18}$$

where

$$E_a{}^b \equiv (\det e) R_a{}^b - 12.4 G_{[a,c_1c_2c_3]} G^{[b,c_1c_2c_3]} + 4\delta_a^b G_{[c_1,c_2c_3c_4]} G^{[c_1,c_2c_3c_4]} = 0 \tag{4.19}$$

and $E_{\lambda_1\ldots\lambda_4}$ is the first order in derivatives duality relation of equation (4.4). Clearly, the graviton equation of motion is equation (4.19).

We will now carry out the variation of the gravity equation under the $I_c(E_{11})$ transformation of equations (3.3), (3.5), (3.6) and equation (3.11). This calculation requires the variation of the spin connection which we have defined to be given by

$$(\det e)^{\frac{1}{2}} \omega_{c,ab} = -G_{a,(bc)} + G_{b,(ac)} + G_{c,[ab]} \tag{4.20}$$

Since the $G_{A,\underline{\alpha}}$ contain a factor of $(\det e)^{\frac{1}{2}}$ this is the standard expression for the spin connection. The variation will result in only the four form field strength $G_{[c_1,c_2c_3c_4]}$ if we add to the spin connection certain l_1 terms. Indeed if we define

$$(\det e)^{\frac{1}{2}} \Omega_{c,ab} = (\det e)^{\frac{1}{2}} \omega_{c,ab} - 3G^{dc}{}_{,dab} - 3G^d{}_{b,dac} + 3G^d{}_{a,dbc} - \eta_{bc} G^{d_1d_2}{}_{,d_1d_2a} + \eta_{ac} G^{d_1d_2}{}_{,d_1d_2b} \tag{4.21}$$

one then finds that

$$\begin{aligned}
\delta((\det e)^{\frac{1}{2}} \Omega_{c,ab}) =\ & -18.2\Lambda^{d_1d_2}{}_c G_{[a,bd_1d_2]} - 18.2\Lambda^{d_1d_2}{}_b G_{[a,cd_1d_2]} - 18.2\Lambda^{d_1d_2}{}_a G_{[c,bd_1d_2]} \\
& + 8\eta_{bc} \Lambda^{d_1d_2d_3} G_{[a,d_1d_2d_3]} - 8\eta_{ac} \Lambda^{d_1d_2d_3} G_{[b,d_1d_2d_3]}
\end{aligned} \tag{4.22}$$

Substituting the spin connection $\Omega_{c,ab}$ for the standard spin connection $\omega_{c,ab}$ in the Riemann tensor we define

$$\mathcal{R}_a{}^b = e_a{}^\mu \partial_\mu \Omega_{\nu,}{}^{bd} e_d{}^\nu - e_a{}^\mu \partial_\nu \Omega_{\mu,}{}^{bd} e_d{}^\nu + \Omega_{a,}{}^b{}_c \Omega_{d,}{}^{cd} - \Omega_{d,}{}^b{}_c \Omega_{a,}{}^{cd}. \tag{4.23}$$

In fact $\mathcal{R}_a{}^b$ is no longer symmetric in a and b interchange when we consider the terms that have level one derivatives in the generalised coordinates. We replace the Ricci tensor by the object of equation (4.23) in the equation of motion of equation (4.19). We will then require its variation which is given by

$$\begin{aligned}\delta\{(\det e)\mathcal{R}_{ab}\} = \{&36\Lambda^{d_1d_2c}(\omega_{c,a}{}^e + \omega^e{}_{,ac})G_{[b,ed_1d_2]} - 36\Lambda^{d_1d_2}{}_b\omega^{c,}{}_a{}^e G_{[c,ed_1d_2]} + 18G^e{}_{,c}{}^c\Lambda^{d_1d_2}{}_b G_{[a,ed_1d_2]} \\ &- 36G_{c,}{}^{ec}\Lambda^{d_1d_2}{}_b G_{[a,ed_1d_2]} + (a \leftrightarrow b)\} + \eta_{ab}(-8G_{e,}{}^{ce} + 4G^c{}_{,e}{}^e)\Lambda^{d_1d_2d_3}G_{[c,d_1d_2d_3]} - 18G_{e,c}{}^c\Lambda^{d_1d_2e}G_{[a,bd_1d_2]} \\ &+ 36G_{c,e}{}^c\Lambda^{d_1d_2e}G_{[a,bd_1d_2]} + 18\omega_{a,b}{}^e\Lambda^{d_1d_2c}G_{c,ed_1d_2} + 6\omega_{e,b}{}^e\Lambda^{d_1d_2d_3}G_{d_1,d_2d_3a} \\ &+ 8\,(\det e)^{\frac{1}{2}}\, e_a{}^\mu\partial_\mu\left(G_{[b,\tau_1\tau_2\tau_3]}\Lambda^{\tau_1\tau_2\tau_3}\right) \\ &+ (\det e)^{\frac{1}{2}}\, e^{c\nu}\partial_\nu\Big(36\Lambda^{d_1d_2}{}_a G_{[b,cd_1d_2]} + 36\Lambda^{d_1d_2}{}_c G_{[b,ad_1d_2]} + 36\Lambda^{d_1d_2}{}_b G_{[a,\,cd_1d_2]} \\ &- 8\eta_{ac}G_{[b,d_1d_2d_3]}\Lambda^{d_1d_2d_3} + 8\eta_{ab}G_{[c,\,d_1d_2d_3]}\Lambda^{d_1d_2d_3}\Big)\end{aligned} \tag{4.24}$$

Calculating the other variation of the other terms in the gravity equation of motion (4.19) we find that its variation is given by

$$\begin{aligned}\delta\mathcal{E}_{ab} = &-36\Lambda^{d_1d_2}{}_a E_{bd_1d_2} - 36\Lambda^{d_1d_2}{}_b E_{ad_1d_2} + 8\eta_{ab}\Lambda^{d_1d_2d_3}E_{d_1d_2d_3} \\ &- 2\epsilon_a{}^{c_1c_2c_3e_1\ldots e_4f_1f_2f_3}\Lambda_{f_1f_2f_3}E_{e_1\ldots e_4}G_{[b,c_1c_2c_3]} - 2\epsilon_b{}^{c_1c_2c_3e_1\ldots e_4f_1f_2f_3}\Lambda_{f_1f_2f_3}E_{e_1\ldots e_4}G_{[a,c_1c_2c_3]} \\ &+ \frac{1}{3}\eta_{ab}\epsilon^{c_1\ldots c_4e_1\ldots e_4f_1f_2f_3}E_{e_1\ldots e_4}G_{[c_1,c_2c_3c_4]}\Lambda_{f_1f_2f_3}\end{aligned} \tag{4.25}$$

where

$$\begin{aligned}\mathcal{E}_{ab} \equiv\, &(\det e)\mathcal{R}_{ab} - 12.4G_{[a,c_1c_2c_3]}G^{[e,c_1c_2c_3]}\eta_{eb} + 4\eta_{ab}G_{[c_1,c_2c_3c_4]}G^{[c_1,c_2c_3c_4]} - 3.5!G^{d_1d_2}{}_{,d_1d_2a}{}^{c_1c_2c_3}G_{[b,c_1c_2c_3]} \\ &- 3.5!G^{d_1d_2}{}_{,d_1d_2b}{}^{c_1c_2c_3}G_{[a,c_1c_2c_3]} + \frac{5!}{2}\eta_{ab}G^{d_1d_2}{}_{,d_1d_2c_1\ldots c_4}G^{[c_1,c_2c_3c_4]} - 12G^{c_1c_2}{}_{,a}{}^{c_3}G_{[b,c_1c_2c_3]} + 3G^{c_1c_2}{}_{,e}{}^e G_{[a,bc_1c_2]} \\ &- 6\,(\det e)\, e_b{}^\mu e_a{}^\lambda\partial_{[\mu|}\left[(\det e)^{-\frac{1}{2}}\, G^{\tau_1\tau_2}{}_{,|\lambda\tau_1\tau_2]}\right] - (\det e)^{\frac{1}{2}}\,\omega_{c,b}{}^c G^{d_1d_2}{}_{,d_1d_2a} - 3\,(\det e)^{\frac{1}{2}}\,\omega_{a,b}{}^c G^{d_1d_2}{}_{,d_1d_2c} = 0\end{aligned} \tag{4.26}$$

In carrying out the variation we find the l_1 terms we must add to the gravity equation which are now included above. We note that some of the terms in equation (4.26) are not symmetric under the interchange of a and b so compensating the same lack of symmetry in $\mathcal{R}_{ab}$.

Thus we have found that, up to the level at which we are working, the second order in derivatives three form and gravity equations (4.17) and (4.26) respectively rotate into each other as well as the first order duality equation (4.4). However, once we vary the latter equation we will find equations of motion for the higher level fields in the $E_{11} \otimes_s l_1$, non-linear realisation and hence the higher order fields can only be eliminated from the complete system by truncating in a way that destroys the E_{11} symmetry.

We recognise equation (4.17) and equation (4.26) as the equations of motion of the bosonic sector of eleven dimensional supergravity [12] once we throw away the terms that have derivatives with respect to the level one generalised coordinates.

5. Conclusion

In this paper we have constructed the dynamics that follow from the non-linear realisation of $E_{11} \otimes_s l_1$ in eleven dimensions for the low level fields and generalised coordinates. The result is unique and when we truncate it to contain only the usual fields of supergravity, that is, the graviton and the three form, and also take only the usual coordinates of spacetime we find the equations of motion of eleven dimensional supergravity. Thus we have a very direct path from the Dynkin diagram of E_{11} to the eleven dimensional supergravity theory. It is inevitable that one will find the analogous results in other dimensions. Indeed the five dimensional theory was found in reference [5] except for some coefficients which were undetermined, however, these can be fixed to the required values from the eleven dimensional theory using dimensional reduction.

The $E_{11} \otimes_s l_1$ realisation is a unified theory in that it contains all the maximal supergravities in one theory. The theory in D dimensions appears by deleting node D in the E_{11} Dynkin digram and decomposing $E_{11} \otimes_s l_1$ with respect to the resulting $GL(D) \times E_{11-D}$ algebra [8–10]. The $E_{11} \otimes_s l_1$ also includes the gauged supergravities [9–11]. Furthermore, it includes effects that are beyond the usual supergravity description and are know to be present in the theory of strings and branes. Since the supergravity theories themselves contain many of the low energy properties of strings and branes it would seem inevitable that one should replace the many different supergravity theories by the $E_{11} \otimes_s l_1$ realisation as the low energy effective theory of strings and branes.

The E_{11} theory is very predictive in that one can, at least as a matter of principle find how the higher level fields and coordinates enter into the equations of motion. It would be very interesting to find what are the physical meaning of the higher level fields and coordinates. Reference [5] mentions a number of avenues that one can explore in future work.

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References

[1] P. West, E_{11} and M theory, Class. Quantum Gravity 18 (2001) 4443, arXiv:hep-th/0104081.
[2] P. West, E_{11}, SL(32) and central charges, Phys. Lett. B 575 (2003) 333–342, arXiv:hep-th/0307098.
[3] P. West, Generalised geometry, eleven dimensions and E_{11}, J. High Energy Phys. 1202 (2012) 018, arXiv:1111.1642.
[4] P. West, E11, generalised space–time and equations of motion in four dimensions, J. High Energy Phys. 1212 (2012) 068, arXiv:1206.7045.
[5] A. Tumanov, P. West, E11 must be a symmetry of strings and branes, arXiv:1512.01644.
[6] P. West, Introduction to Strings and Branes, Cambridge University Press, 2012.
[7] A. Tumanov, P. West, Generalised vielbeins and non-linear realisations, arXiv:1405.7894.
[8] P. West, The IIA, IIB and eleven dimensional theories and their common E_{11} origin, Nucl. Phys. B 693 (2004) 76–102, arXiv:hep-th/0402140.
[9] F. Riccioni, P. West, The E_{11} origin of all maximal supergravities, J. High Energy Phys. 0707 (2007) 063; arXiv:0705.0752.
[10] F. Riccioni, P. West, E(11)-extended spacetime and gauged supergravities, J. High Energy Phys. 0802 (2008) 039, arXiv:0712.1795.
[11] E. Bergshoeff, I. De Baetselier, T. Nutma, E(11) and the embedding tensor, J. High Energy Phys. 0709 (2007) 047, arXiv:0705.1304.
[12] E. Cremmer, B. Julia, J. Scherk, Supergravity theory in eleven-dimensions, Phys. Lett. B 76 (1978) 409.