Research Article

Optimal Bounds for the Variance of Self-Intersection Local Times

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Abstract

We will denote the corresponding quantities for simple random walk with nonsingular covariance matrix, was treated in [3] where it obtained for $d = 1$. The case $d = 2$, with $X$ centered with nonsingular covariance matrix, was treated in [3] where it

1. Introduction and Main Results

Let $X, X_1, X_2, \ldots$ be independent, identically distributed, $\mathbb{Z}^d$-valued random variables, and define the random walk $S_0 = 0$, $S_n = \sum_{j=1}^{n} X_j$, for $n \geq 1$. The special case with $P(X_j = e) = 1/(2d)$, for all $e \in \mathbb{Z}^d$ with $|e| = 1$, is known as the simple random walk in $\mathbb{Z}^d$ and will be denoted by $(SRW_n)_{n \in \mathbb{N}}$. Let $l(n, x)$ be the local time of $(S_n)_{n \in \mathbb{N}}$ at the site $x \in \mathbb{Z}^d$, and define for a positive integer $\alpha$ the $\alpha$-fold self-intersection local time

$$L_n = L_n(\alpha) = \sum_{x \in \mathbb{Z}^d} l(n, x)^\alpha$$

$$= \sum_{i_1, \ldots, i_\alpha \geq 0} \mathbb{1}(S_{i_1} = \cdots = S_{i_\alpha}).$$

(1)

We will denote the corresponding quantities for simple random walk in $\mathbb{Z}^d$ by $L^{SRW}_n(\alpha, d)$ or simply $L^{SRW}_n(\alpha)$ when the dimension is clear from the context.

Let $R^+$ and $R^-$ be, respectively, the semigroup and the group generated by the support of $X$,

$$R^+ = \{ x \in \mathbb{Z}^d \mid P(S_n = x) > 0 \text{ for some } n \geq 0 \},$$

$$R^- = \{ x \in \mathbb{Z}^d \mid x = y - z \text{ for some } x, y \in R^+ \}. \quad (2)$$

Following Spitzer [1], we call the random variable $X$ and the random walk it generates genuinely $d$-dimensional if the group $R$ is $d$-dimensional.

The quantity $L_n(\alpha)$ has received considerable attention in the literature due to its relation to self-avoiding walks and random walks in random scenery. In particular let the random scenery $\{\xi_e, e \in \mathbb{Z}^d\}$ be a collection of i.i.d. random variables, independent of $(S_n)_{n \geq 0}$, and define the process $Z_0 = 0$, $Z_n = \sum_{j=1}^{n} \xi_S$. Then $(Z_n)_{n \geq 0}$ is commonly referred to as random walk in random scenery and was introduced in Kesten and Spitzer [2], where functional limit theorems were obtained for $Z_{[nt]}$ under appropriate normalization for the case $d = 1$. The case $d = 2$, with $X$ centered with nonsingular covariance matrix, was treated in [3] where it
was shown that $\widetilde{Z}|_{n!}/\sqrt{n \log n}$ converges weakly to Brownian motion. As is obvious from the identities $Z_n = \sum_{x \in \mathbb{Z}^d} l(n, x)K_n(x)$ and $\text{var}(Z_n) = \var[L_n(2)] \var(\tilde{X})$, limit theorems for $(Z_n)_n$ usually require asymptotic results for the local times of the random walk $(S_n)_n$.

Such asymptotic results are usually obtained from Fourier techniques applied to the characteristic function $f(t) = \mathbb{E}^{\exp(it \cdot X)}$, under the additional assumption of a Taylor expansion of the form $f(t) = 1 - \langle \Sigma t, t \rangle + o(|t|^2)$, where $\Sigma$ is a positive definite covariance matrix $[3-7]$, which further requires that $\mathbb{E}|X|^2 < \infty$ and $\mathbb{E}X = 0$. Similar restrictions are also required for the application of local limit theorems such as in $[8, 9]$.

In this paper, motivated by the results of Spitzer [1] for genuinely $d$-dimensional random walks and the approach of Becker and König [10], we will study the asymptotic behaviour $\mathbb{E}$-asymmetrically distributed, and genuinely $d$-dimensional with $d \leq 3$. If

$$\lim_{n \to \infty} \frac{\text{var}(L_n(\alpha))}{\text{var}(L^{SRW}_n(\alpha))} > 0,$$

then $\mathbb{E}|X|^2 < \infty$ and $\mathbb{E}X = 0$.

As it follows from Theorem 3 given below for $d = 2, 3$ and from Theorem 5.2.3 in Chen [12] for $d = 1$, if $\mathbb{E}X = 0$ and $0 < \mathbb{E}|X|^2 < \infty$, then $\lim_{n \to \infty} \var(L_n(\alpha))/\var(L^{SRW}_n(\alpha)) > 0$.

For any genuinely $d$-dimensional random walk with finite second moments and zero mean, the asymptotic behaviour of $\var(L_n(\alpha))$ is similar to that of the $d$-dimensional simple symmetric random walk. Also, as it follows from our general bounds (see Proposition 4 and Corollary 7) that the asymptotic results for the genuinely $d$-dimensional random walk can be reproduced by those of the symmetric one-dimensional random walk with appropriately chosen heavy tails, as was indicated by Kesten and Spitzer [2]. The proofs are based on adapting the Tauberian approach developed in [13].

### Theorem 1

Let $X, X_1, X_2, \ldots$ be independent, identically distributed, and genuinely $d$-dimensional $\mathbb{Z}^d$-valued random variables, for any $d \geq 1$. Then, there exist positive constants $C_{a, X} > \epsilon_{a, X} > 0$, depending on $\alpha$ and the distribution of $X$, such that for all $n$ large enough

$$\text{var}(L_n(\alpha)) \leq \epsilon_{a, X} \text{var}(L^{SRW}_n(\alpha, d)) \leq C_{a, X} \var(v_{d, a}(n)).

$$

The result was motivated by [1, 10] and improves related results of Becker and König for $d = 3$ and $d = 4$. Several cases treated in [3, 4, 10–13] can then be obtained as particular cases.

Moreover, we also show the surprising converse. More precisely, we show that the right asymptotic behaviour of $\var(L_n^2)$ implies that the jumps must have zero mean and finite second moment.

### Theorem 2

Let $X, X_1, X_2, \ldots$ be independent, identically distributed, and genuinely $d$-dimensional with $d \leq 3$. If

$$\liminf_{n \to \infty} \frac{\text{var}(L_n(\alpha))}{\text{var}(L^{SRW}_n(\alpha))} > 0,$$

then $\mathbb{E}|X|^2 < \infty$ and $\mathbb{E}X = 0$.

2. Proofs

#### 2.1. General Bounds

We first develop a technique to treat random walks with independent but not necessarily identically distributed increments.
Proposition 4 (general upper bound). Assume that \( X_1, X_2, \ldots \) are independent \( \mathbb{Z}^d \)-valued random variables and let \( S_{u,v} := X_u + \cdots + X_{u+v} \). Suppose further that for all \( n \in \mathbb{N} \) and integers \( a, b, v \geq 0 \), with \( a + u \leq b \) and any \( x \in \mathbb{Z}^d \), one has
\[
\mathbb{P}(S_{a,u} \pm S_{b,v} = x) \leq \Phi(u + v), \quad (A)
\]
\[
\mathbb{P}(S_{a,u} = 0) - \mathbb{P}(S_{a,u} + S_{b,v} = 0) \leq \Psi(u, v), \quad (B)
\]
where \( \Phi(u) \) is nonincreasing and \( \Psi(u, v) \) is nonincreasing in \( u \) and is nondecreasing and subadditive in \( v \) in the sense that \( \Psi(u, v + w) \leq A \Psi(u, v + \Psi(u, w)) \), for some constant \( A \) independent of \( u, v \), and \( w \). Then, for some constant \( K = cA(1 + A)^{\alpha-2} \) depending only on \( \alpha \)
\[
\var(L_n(\alpha)) \leq Kn\left(\sum_{i=0}^{n-1} \Phi(i)\right)^{2\alpha-4} \cdot \sum_{i,j=0}^{n-1} \Phi(j \lor i) \cdot \Phi(i + k, j).
\]

Proof of Proposition 4. We first write out the variance as a sum
\[
\var(L_n(\alpha)) = (\alpha!)^2 \sum_{k_1 \leq \cdots \leq k_n} \mathbb{P}[S_{k_1} = \cdots = S_{k_n}, S_i = \cdots = S_{i_n}].
\]

\[
I_n := \sum_{k_1 \leq \cdots \leq k_n} \mathbb{P}[S_{k_1} = \cdots = S_{k_n}, S_i = \cdots = S_{i_n}]
\]
\[
= \sum_{x, y \in \mathbb{Z}^d} \sum_{p_1 \geq \cdots \geq p_{2\alpha}} \sum_{e \in \{0,1\}^{2\alpha}} \mathbb{P}(S_{p_1} = x, S_{p_2} = x + e_1, \ldots, S_{p_{2\alpha}} = x + e_{2\alpha})
\]
\[
\leq \sum_{x, y \in \mathbb{Z}^d} \sum_{m_0, m_1, \ldots, m_{2\alpha-1}} \sum_{e \in \{0,1\}^{2\alpha}} \mathbb{P}(S_{m_0} = x) \mathbb{P}(S_{m_1, m_1} = \delta_1 y) \cdots \mathbb{P}(S_{m_{2\alpha-2}, m_{2\alpha-1}} = \delta_{2\alpha-1} y)
\]
\[
= \sum_{x, y \in \mathbb{Z}^d} \sum_{m_0, m_1, \ldots, m_{2\alpha-1}} \sum_{e \in \{0,1\}^{2\alpha}} \mathbb{P}(S_{m_0, m_1} = \delta_1 y) \cdots \mathbb{P}(S_{m_{2\alpha-2}, m_{2\alpha-1}} = \delta_{2\alpha-1} y).
\]

Summing over the free index \( m_0 \), it is clear that
\[
I_n \leq (n + 1) \cdot \sum_{m_0, m_1, \ldots, m_{2\alpha-1}} \mathbb{P}(S_{m_0, m_1} = \delta_1 y).
\]

(12)

For any \( \delta = (\delta_1, \ldots, \delta_{2\alpha-1}) \) with \( v(\delta) = v \), exactly \( u = 2\alpha - 1 - v \) elements are equal to 0, and therefore by Assumption (A) with \( x = 0 \) we have
\[
I_n \leq C(n + 1) \cdot \sum_{i=0}^{\alpha} \left( \sum_{j=0}^{u} \Phi(i) \right)^{2\alpha - 1 - v} \cdot \sum_{j_1, \ldots, j_v} \mathbb{P}(S_{w, j_1} = \delta_1 y) \cdots \mathbb{P}(S_{w, j_v} = \delta_v y).
\]

(13)

Letting \( (S_{n, h \in \mathbb{N}}^n) \) denote an independent copy of the random walk \( (S_n)_{n \in \mathbb{N}} \) and assuming without loss of generality that \( j_1 \leq \cdots \leq j_v \), we have that for any \( \delta \in \{-1, +1\}^v \)
\[
\sum_{y \in \mathbb{Z}^d} \mathbb{P}(S_{w, j_1} = \delta_1 y) \leq \left( \sum_{y \in \mathbb{Z}^d} \mathbb{P}(S_{w, i} = y) \right)^{v-1} \sup_{w, i} \mathbb{P}(S_{w, j_1} = \delta_1 y) \cdots \mathbb{P}(S_{w, j_v} = \delta_v y).
\]

(14)
Let $G_n := \sum_{i=0}^{n} \phi(i)$. Since $\phi$ is nonincreasing we have that

$$
\Delta_{n, v} := \sum_{0 \leq i, j < -s, j \leq n} \prod_{t=2}^{v} \phi(j_t \lor j_t')
\leq \sum_{i=0}^{n} \phi(j_i) \sum_{0 \leq i, j < -s, j \leq n} \prod_{t=2}^{v} \phi(j_t \lor j_t')
= G_n \Delta_{n, v, 1},
$$

and iterating this procedure, for $v \geq 3$, we have that $\Delta_{n, v} \leq \Delta_{n, \delta_{n, v^{-3}}}$. Combining the two bounds and summing over $v = 3, \ldots, 2\alpha - 1$, we have that

$$
I_n \leq \sum_{v=3}^{2\alpha-1} c(\alpha) n^{-2\alpha-1-v} \Delta_{n, v} \leq c(\alpha) n^{2\alpha-1-v-3} \Delta_{n, 3}
$$

$$
= c(\alpha) n^{2\alpha-4} \Delta_{n, 3},
$$

(16)

where $c(\alpha)$ is a constant depending only on $\alpha$.

**Terms with $v = 2$.** Next we consider the sum $J_n$ over the terms with $v = 2$, which occurs when, for some $j$, the indices $l_1, \ldots, l_\alpha$ all lie in $[k_p, k_{p+1}]$. Then it is easy to see that this sum $J_n$ is bounded above by

$$
J_n \leq C n \sup_{w_{u, -1} \leq m_{u, -3} \leq w_{\alpha, m_{\alpha}}} \sum_{m_{\alpha} = 0}^{n} \prod_{t=1}^{\alpha-1} \mathbb{P} (S_{w_{u, m_t} = 0}) \cdot \mathbb{P} (S_{w_{u, m_t} = 0} \cdot \mathbb{P} (S_{w_{u, m_t} + \cdots + S_{w_{u, m_t} = 0}) \right) \leq C n \Delta_{n, 2}^{-2}
$$

$$
\cdot \sup_{w_{u, -1} \leq m_{u, -3} \leq w_{\alpha, m_{\alpha}}} \sum_{m_{\alpha} = 0}^{n} \prod_{t=1}^{\alpha-1} \mathbb{P} (S_{w_{u, m_t} = 0}) \cdot \mathbb{P} (S_{w_{u, m_t} + \cdots + S_{w_{u, m_t} = 0}) \right)
$$

$$
\leq C n \Delta_{n, 2}^{-2} \sum_{m_{\alpha} = 0}^{n} \prod_{t=1}^{\alpha-1} \mathbb{P} (S_{w_{u, m_t} = 0}) \cdot \mathbb{P} (S_{w_{u, m_t} + \cdots + S_{w_{u, m_t} = 0}) \right)
$$

$$
= C n \Delta_{n, 3}^{-2} A_{\alpha} \left(1 + A_{\alpha}\right)^{\alpha-2} \cdot \left(\sum_{m_{\alpha} = 0}^{n} \prod_{t=1}^{\alpha-1} \mathbb{P} (S_{w_{u, m_t} = 0}) \cdot \mathbb{P} (S_{w_{u, m_t} + \cdots + S_{w_{u, m_t} = 0}) \right)
$$

$$
m_1 \leq C n \Delta_{n, 3}^{-2} A_{\alpha} \left(1 + A_{\alpha}\right)^{\alpha-2} \sum_{i, j=0}^{n} \phi(j) \psi(i)
$$

$$
+ k, j, i, j, k = 0
\right)
$$

\[ \Box \]

The following corollary provides explicit bounds in the cases that are usually considered in the literature.

**Corollary 5.** Assume that the conditions of Proposition 4 are satisfied with $\phi(m) = T/m$ and $\psi(m, k) = Tm^{-k}(k \land m)$. Then,

$$
\text{var} (L_n (\alpha)) \leq C n \left(\sum_{i=0}^{n} \phi\left(\left[\frac{i}{2}\right]\right) \sum_{j=0}^{n} \psi\left(\left[\frac{j}{2}\right]\right) \sum_{k=0}^{n} \psi\left(\left[\frac{k}{2}\right]\right) \right).
$$

(20)

It is straightforward to see that Corollary 5 includes random walks with mean zero and finite second moment; for example, $d = 2$ corresponds to $r = 1$ and $d = 3$ to $r = 3/2$. Therefore several relevant results in [3, 7–13] are obtained as a special case of Corollary 5 and extended to the case of independent but not necessarily identically distributed variables, for example, by applying the local limit theorem, as conducted in [8].

Also when the random walk increment $X$ is in the domain of attraction of the one-dimensional symmetric Cauchy law [13, 14] or in the case of planar random walk with second moments [3, 7–9, 11], it is well known that the conditions of Proposition 4 are satisfied with $\phi(m) = T/m$ and $\psi(m, k) = Tm^{-k}(k \land m)$.

However, we can do better for symmetric variables and show that condition (A) implies (B), which together with the comparison technique motivates the following results. For a real number $x$, we write $[x]$ for the integer part of $x$.

**Proposition 6 (bounds via comparison with characteristic function of symmetric random variables).** Let $X_1, X_2, \ldots$, be independent $\mathbb{Z}$-valued random variables and let $f_i(t) := \text{E} \exp(\text{i} t X_i)$. Assume that there exist a measurable function $f : \Sigma \rightarrow [0, 1]$ and a positive nonincreasing sequence $(\phi(m))_{m \in \mathbb{N}_0}$, such that

$$
|1 - f_i(t)| \leq T f(t),
$$

$$
|f_i (\pm t)| \leq f(t),
$$

(19)

$$
\int_{\Sigma} f(t)^m dt \leq \phi(m),
$$

for all integers $i, m \geq 0$, all $t \in \Gamma$, and some positive constant $T$. Then there exists another positive constant $K = C(\alpha, d, T)$ such that

$$
\text{var} (L_n (\alpha)) \leq Kn \left(\sum_{i=0}^{n} \phi\left(\left[\frac{i}{2}\right]\right) \sum_{j=0}^{n} \psi\left(\left[\frac{j}{2}\right]\right) \sum_{k=0}^{n} \psi\left(\left[\frac{k}{2}\right]\right) \right).
$$

(20)
Proof of Proposition 6. Using the notation of Proposition 4, for positive integers \(a, u, b,\) and \(v,\) with \(a + u \leq b,\) \(e_j = \pm 1,\) and any \(x \in \mathbb{R}^d\)

\[
P(S_{a,u} + e \cdot S_{b,v} = x) \leq \frac{1}{(2\pi)^{d/2}} \int_{f \in [a, a+u]\cup[b, b+v]} \prod_{j \in [a, a+u]\cup[b, b+v]} |f_j(\epsilon_j t)| dt
\]

\[
\leq \frac{1}{(2\pi)^d} \int_0^\infty f(t)^{u+v} dt = \frac{1}{(2\pi)^d} \phi(u + v).
\]

To find \(\psi(u, v),\) notice that since \(f(t) \geq 0,\)

\[
\phi(m) \geq m \int f(t)^m (1 - f(t)) dt
\]

\[
= m \sum_{j=0}^{m-1} \int f(t)^{m+j} (1 - f(t)) dt
\]

\[
\geq m \int f(t)^m (1 - f(t)) dt = mQ(2m)
\]

whence \(Q(m) \leq 2\phi([m/2])/m.\) Therefore,

\[
\mathbb{P}(S_{a,u} = 0) - \mathbb{P}(S_{a,u} + S_{b,1} = 0)
\]

\[
= \frac{1}{(2\pi)^d} \int_0^\infty \prod_{j \in [a, a+u]} f_j(\epsilon_j t) (1 - f_{b+1}(t)) dt
\]

\[
\leq CT \int_0^\infty f(t)^m |1 - f(t)| dt \leq \frac{CT\phi([u/2])}{u}.
\]

A telescoping argument implies that

\[
\mathbb{P}(S_{a,u} = 0) - \mathbb{P}(S_{a,u} + S_{b,v} = 0) \leq CT\phi \left( \frac{u}{2} \right) \frac{v}{u}.
\]

On the other hand for \(u \leq v\) we can obtain a tighter bound through

\[
\mathbb{P}(S_{a,u} = 0) - \mathbb{P}(S_{a,u} + S_{b,v} = 0) \leq \mathbb{P}(S_{a,u} = 0)
\]

\[
\leq \phi(u).
\]

Combining the two bounds above it follows that (B) is satisfied with \(\psi(u, v) = \phi([u/2]) \min(u, v)/u.\) Thus all conditions of Proposition 4 are satisfied and the result follows.

The following corollary allows for the case where \(\phi(m)\) is regularly varying.

Corollary 7. Assume that the conditions of Proposition 6 are satisfied with \(\phi(m) = h(m)m^{-r},\) \(r \geq 1,\) where \(h(\cdot)\) is slowly varying at \(\infty.\) Then,

\[
\var(L_n(\alpha)) \leq K\Delta_n(\alpha, \phi)
\]

\[
\leq c_4T^{2r-2} \left\{ \begin{array}{ll}
\sum_{k=1}^n h(k)^2/k, & \text{for } r = \frac{3}{2}, \\
\sum_{k=1}^n h(k)^2/k, & \text{for } r > \frac{3}{2}.
\end{array} \right.
\]

Several results in [3, 7–13] are obtained as a special case of Corollary 7 and can be extended to dependent variables, for example, a random walk driven by a hidden Markov chain. In addition, following [2], we can construct a one-dimensional symmetric random walk with characteristic function \(f(t) = 1 - c|t|^{1/2} + o(|t|^{1/2}),\) where \(r = 2/d\) for \(d = 2, 3\) and \(r = 1/2\) for \(d \geq 4,\) whose asymptotic behavior is similar to that of genuinely \(d\)-dimensional random walk.

The following example of genuinely 2-dimensional recurrent walk with infinite variance was motivated by Spitzer [1, pp. 87].

Example 8. Let \(X_1, X_2, \ldots\) be independent, identically distributed, \(Z^2\)-valued random variables, such that \(P[X_1 = k] = c/(k^2 \log(k)^\beta),\) for \(k \geq 4\) and \(\beta \in (0, 1].\) Let \(S_n\) be the corresponding random walk in \(Z^2.\) Then we have

\[
\var(L_n(\alpha)) \leq cN^2 \sum \log n^\beta, \log \log n^{2r-4} \log n^{(1-\beta)}
\]

\[
\leq c_4T^{2r-2} \left\{ \begin{array}{ll}
\sum_{k=1}^n h(k)^2/k, & \text{for } r = \frac{3}{2}, \\
\sum_{k=1}^n h(k)^2/k, & \text{for } r > \frac{3}{2}.
\end{array} \right.
\]

for \(n \geq 10.\) Under these assumptions we have that \(P(S_n = 0) \leq c/n \log n^{-1/\beta},\) which is in the critical range, where the random walk is recurrent, without second moment. To see why, we note that by a lengthy but straightforward calculation it can be shown that the characteristic function of \(X\) satisfies (19) with

\[
\phi(n) = \frac{c}{n \log (e \vee n)^{1-\beta}}.
\]

\[
f(t) = \exp \left[ -A|t|^2 h (|t|^2) \right],
\]

where \(h(r) = \left[ 1 + \log \left( \frac{1}{r} \right) \right]^{1-\beta}.
\]

The sequence \(\phi(m)\) is identified via Fourier inversion, polar coordinates, and a Laplace argument,

\[
\int_0^\infty f(t)^n dt \leq c \int_0^\infty \exp \left( -nr \left( 1 + \log \left( \frac{1}{r} \right) \right) \right)
\]

\[
+ O(e^{-n}) \frac{e^{c}}{n \log (e \vee n)^{1-\beta}} = \phi(n).
\]

2.2. Bounds for Identically Distributed Variables

Proposition 9 (general upper bound for i.i.d.). Let \(X, X_1, X_2, \ldots\) be independent, identically distributed,
$\mathbb{Z}^d$-valued random variables. Suppose that for any $x \in \mathbb{Z}^d$ and all positive integers $a, u, b,$ and $v$, with $a + u \leq b$, it holds that
\[
\mathbb{P}(S_{a,u} \pm S_{b,v} = x) \leq \phi(u + v),
\]
where $\phi(m)_{m \in \mathbb{N}_0}$ is a nonincreasing sequence. Then for some constant $K = c(\alpha)$ we have that
\[
\var(L_n(\alpha)) \leq Kn \left( \sum_{i=0}^{n-1} \phi(i) \right)^{2\alpha - 4} \sum_{j=0}^{\alpha} j \phi(j) \sum_{k=j}^{n\alpha + 1} \phi \left( \left[ \frac{k}{\alpha} \right] \right).
\]

Proof of Proposition 9. By inspecting the proof of Proposition 6, we notice that we only need to bound the term $J_n$. Consider typical ordering
\[
0 \leq i_1 \leq \cdots \leq i_k \leq j_1 \leq \cdots \leq j_n \leq i_{k+1} \leq \cdots \leq i_n
\]
and let us change variables to $(m_0, \ldots, m_{2\alpha})$ such that $m_0 + \cdots + m_{2\alpha} = n$. Then the contribution to $J_n$ is given by
\[
\sum_{m_0, \ldots, m_{2\alpha}} \prod_{j \leq k \leq \alpha} \mathbb{P}(S_{m_j} = 0)
\cdot \left[ \mathbb{P}(S_{m_k + m_{k+1}} = 0) - \mathbb{P}(S_{m_k + m_{k+1}} = 0) \right].
\]
We keep $m_j$ fixed for $j \neq \alpha, k + \alpha$ and we sum over $m = m_k + m_{k+1}$ from 0 to some $M = M(n, (m_j)_{j \leq k \leq \alpha})$. Then for given $m_{k+1}, \ldots, m_{k+\alpha-1}$, the term in the sum is
\[
\sum_{m=0}^{M} (m+1) \left[ \mathbb{P}(S_m = 0) - \mathbb{P}(S_{m+q} = 0) \right],
\]
where $q = m_{k+1} + \cdots + m_{k+\alpha-1}$. Since $M \leq n - q$, it is an easy exercise to show that this sum is bounded above by
\[
\sum_{m=0}^{M} (m+1) \left[ \mathbb{P}(S_m = 0) - \mathbb{P}(S_{m+q} = 0) \right] \leq \sum_{m=0}^{n-q} (m+1) \mathbb{P}(S_m = 0) + q^\alpha (n-q) \geq \sum_{m=0}^{n-q} \mathbb{P}(S_m = 0) \leq \sum_{m=0}^{m^*} (m+1) \mathbb{P}(S_m = 0)
\]
\[+ \alpha m^* \sum_{m=m^*}^{n} \mathbb{P}(S_m = 0),
\]
where $m^* = \max\{m_{k+1}, \ldots, m_{k+\alpha-1}\}$. The result follows by summing over all indices apart from $m^*$ and changing the order of summation.

2.3. Proofs of Main Results

Proof of Theorem 1. We apply a comparison argument found to be useful in many areas (e.g., Montgomery-Smith and Pruss [15], and Lefèvre and Utev [16]). More specifically we bound the quantity $\var(L_n)$ by the corresponding quantity for the symmetrised random walk.

Following Spitzer’s argument we notice that with $f(t) = \mathbb{E}[\exp(it \cdot X_1)]$
\[
\mathbb{P}(S_{a,u} + eS_{b,v} = x) \leq c \int \left[ |f(t)|^u |f(-t)|^v \right] dt
\]
\[= c \int \left[ |f(t)|^{u/2} |f(-t)|^{v/2} \right] dt.
\]
Since $|f(t)|^2$ is the characteristic function of a symmetric random variable in $\mathbb{Z}^d$, for some positive $\lambda$, we have $1 - |f(t)|^2 \geq \lambda t^2$, and, hence,
\[
\mathbb{P}(S_{a,u} + eS_{b,v} = x) \leq c \int \exp \left[ -\frac{\lambda (u + v)}{2} |t|^2 \right] dt
\]
\[\leq c (u + v)^{-d/2}.
\]
The result follows from Proposition 9 applied with $\phi(m) = m^{-d/2}$.

The proof of Theorem 2 will be based on the following lemma.

Lemma 10. Assume $X, X_1, X_2, \ldots$ are independent, identically distributed, genuinely $d$-dimensional random variables such that $\mathbb{E}[X^2] = \infty$. Then there exists a monotone, slowly varying sequence $(h_n)_{n \in \mathbb{N}_0}$, such that $h_n \to 0$ as $n \to \infty$ and
\[
\sup_{x \in \mathbb{Z}^d} \mathbb{P}(S_n = x) \leq c_d \int \left[ \mathbb{E}[e^{tX}]^n \right] dt \leq h_n n^{-d/2}.
\]

Proof of Lemma 10. Without loss of generality we assume that $X$ is symmetric. Let $\sigma_{\alpha} := \mathbb{E}[e \cdot X^\alpha \mathbb{I}(|X| \leq L)]$. Following Spitzer, since $X$ is genuinely $d$-dimensional, we may assume that there exist positive constants $c, W$, such that for any unit vector $|e| = 1$ we have that $\sigma_{\alpha} \geq c$ and $1 - f(t) \geq c |t|^2$ for all $t \in \Gamma$. Let $\lambda_d$ be the $d$-dimensional Lebesgue measure on $\mathbb{R}^d$ and $\mu_d$ the Lebesgue-Haar measure on $S^{d-1} := \{ e \in \Gamma : |e| = 1 \}$. Notice that since $\mathbb{E}[X^2] = \infty$, for any $K$, we have $\mu_d \{ e : \sigma_{\alpha} e \leq K \} = 0$.

Fix a small positive $x$ such that $\sqrt{c/x} \geq 2W$, and for any $\epsilon > 0$ let $K = K(\epsilon) = e^{-d/2}$. Then there exists $L = L(\epsilon) > 0$ small enough so that $\mu_d \{ e : \sigma_{\alpha} e \leq K \} \leq e^{d/2}$. We partition $S^{d-1}$ in two sets
\[
A_{L,K} = \{ e \in S_d : \sigma_{\alpha} \geq K \},
\]
\[
\overline{A}_{L,K} = \{ e \in S_d : \sigma_{\alpha} < K \},
\]
so that, for any direction $e$ in $\overline{A}_{L,K}$,
\[
\{ z \in \mathbb{R} : 1 - f(ze) \leq x \} \subseteq \{ z : c z^2 \leq x \}
\]
\[\subseteq \left\{ z : |z| \leq \sqrt{\frac{1}{c}} \right\}.
\]
Hence, using $d$-dimensional spherical coordinates,
\[
\lambda_d \{(z,e) \in \mathbb{R} \times A_{L,K} : 1 - f(ez) \leq x\} \\
\leq \mu_d \left[ A_{L,K} \right] \left( \frac{x}{c} \right)^{d/2} \left( \frac{1}{d} \right) \leq e^{d/2} \left( \frac{x}{c} \right)^{d/2} \left( \frac{1}{d} \right).
\]  
(41)

On the other hand, for any $t$,
\[
1 - f(t) = 2 \sum_{k \in \mathbb{Z}^d} \sin \left( \frac{|t \cdot k|}{2} \right) P(X = k) \\
\geq \left( \frac{1}{4} \right) E \left[ (t \cdot X)^2 I \left( |t \cdot X| \leq \frac{1}{2} \right) \right] \\
= \left( \frac{|t|^2}{4} \right) \sigma_{t/|t|,|t|/2|t|}.
\]

Now, assume that $\sqrt{c/\gamma} \geq 2L$. Then for any direction $e \in A_{L,K}$, by choice of $x$ and since $\sigma_{x,L}$ is increasing in $L$, for $cz^2 \leq 1 - f(ez) \leq x$ or $|z| \leq \sqrt{x/c}$, it must be the case that
\[
x \geq 1 - f(ez) \geq \left( \frac{z^2}{4} \right) \sigma_{e,1/2e} \geq \left( \frac{z^2}{4} \right) \sigma_{e,L} \\
\geq \left( \frac{z^2}{4} \right) K,
\]

implying that, on the set $A_{L,K}$, it must be that $|z| \leq 2\sqrt{x/K}$. Changing to $d$-dimensional polar coordinates, we find that
\[
\lambda_d \{(z,e) \in \mathbb{R} \times A_{L,K} : 1 - f(ez) \leq x\} \\
\leq \int_{e \in A_{L,K}} \int_0^{\sqrt{x/K}} r^{d-1} dr \leq C_d e^{d/2} x^{d/2}.
\]  
(44)

Overall, for $x \leq c/4L^2$, $\lambda_d \{|t : 1 - f(t) \leq x\} \leq c_d(xe)^{d/2}$, and hence $\{t \in \Gamma : 1 - f(t) \leq x\}$ has Lebesgue measure $o(x^{d/2})$.

Let $F(x)$ be the cumulative distribution function of the random variable $\log(1/f(t))$ defined on the probability space $\Gamma$ with normalised Lebesgue measure. Then $F$ is continuous at $x = 0$ and supported on $\mathbb{R}^+$. Moreover, we have that $F(x) = o(x^{d/2})$ as $x \downarrow 0$. Therefore, for some positive sequence $(e_n)_{n \in \mathbb{N}}$ with $e_n \to 0$, we have that
\[
\frac{1}{(2\pi)^d} \int_\Gamma f(t)^n dt = \int_0^{\infty} e^{-nx} dF(x) \\
= n \int_0^{\infty} e^{-nx} F(x) dx \leq n^{-d/2} e_n.
\]  
(45)

It remains to show that there exists a positive, monotone, slowly varying sequence $(h_n)_{n \in \mathbb{N}}$, such that $e_n \leq h(n) \to 0$ as $n \to \infty$. Let $\tilde{\delta}_n = \sup_{j \geq n} e_j$ and $\delta_0 = 0$ and for $n \geq 1$ define $a_n$ recursively by $a_n = \min\{2a_{n-1}, 1/\delta_n\}$, for $2^{-n} < \tilde{\delta}_n < 2^{-n}$, so that $a_n \to 0$ is monotone, $a_{n+1} \leq 2a_{n-1}$ implying that $a_n \leq 4\tilde{\delta}_n$, and $1/a_n \geq \delta_n \geq e_n$. Finally, take $h_n = 1/\max(a_0, \log a_n)$.

**Proof of Theorem 2.** Assume that $\mathbb{E}|X|^2 = \infty$ and $d = 2$ or $d = 3$. Then, by Lemma 10 there exists a slowly varying sequence $h_n \to 0$ as $n \to \infty$ such that $\int_1^\infty |\exp(\cdot X)|^d dt \leq h_n n^{-d/2}$.

Applying Corollary 7 with $r = 1$ and $r = 3/2$ we, respectively, find that
\[
\var_{\{L_n(\alpha)\}} \leq \begin{cases} 
Kn \left( \sum_{k=1}^{n} \frac{h(k)}{k} \right)^{2a-4} = o\left( n^4 (\log n)^{2a-4} \right), & \text{for } d = 2, \\
Kn \left( \sum_{k=1}^{n} \frac{h(k)}{k} \right)^{2a-4} = o(n \ln n), & \text{for } d = 3.
\end{cases}
\]

Finally assume that $\mathbb{E}|X|^2 < \infty$ and $\mathbb{E}[X] = \mu \neq 0$. Then $P(S_n = 0) = P(S_n^* = -nu)$ whence it follows that $P(S_n = 0) = o(n^{d/2})$ (see, e.g., [17, Theorem 2.3.10]). Then inspecting the proof of Proposition 4, one can readily obtain the desired bound for the $I_n$ term, while with slight modification the bound for the $I_n$ term also follows.

Note that for $d = 1$ the situation is much simpler since then $\var(L_n^{SRW}(\alpha)) \sim C[\mathbb{E}L_n^{SRW}(\alpha, d)]^3$ and if $\mathbb{E}|X|^2 = \infty$ or $\mathbb{E}[X] \neq 0$, $\mathbb{E}L_n^{SRW}(\alpha, d) = o(n^{1+\alpha/2})$.

**Proof of Theorem 3.** We first give the proof for the case $d = 1$. As in the proof of Proposition 4 we begin from expression (10) and define the sequences $p_i$ and $\delta_i$ for $i = 1, \ldots, 2n - 1$, and the quantity $\nu(\delta) = \sum_{i=1}^{2n-1} |\delta_i|$. Recall that $\nu(\delta)$ measures the interlacement of the two sequences $k_1, \ldots, k_n$ and $l_1, \ldots, l_n$. For example, $\nu(\delta) = 1$ occurs when either $k_\alpha \leq l_1$ or $l_\alpha \leq k_1$, in which case the contribution vanishes by the Markov property. On the other hand $\nu(\delta) = 2$ when, for example, $l_1, \ldots, l_{\alpha-1} \in [k_\alpha, k_{\alpha+1}]$ for some $i$. Finally $\nu(\delta) = 3$ occurs when, for example,
\[
k_1 \leq \cdots \leq k_\alpha \leq l_1 \leq \cdots \leq l_{\alpha-1} \leq k_{\alpha+1} \leq \cdots \leq k_n \leq l_n
\]  
(47)

From the proof of Proposition 4, and using the bound $P(S_n = 0) \leq c/n$, the terms of the sum are bounded above by $n^2 \log(n)^{2a-1-\nu(\delta)}$, and thus the leading term appears when either $\nu(\delta) = 2, 3$, with other terms giving strictly lower order. We will therefore analyze these two situations in detail in order to derive the exact asymptotic constants. When $\nu = 3$, the two terms in the difference individually give the correct order and will be treated by the classical Tauberian theory. However for $\nu = 2$, the two terms only give the correct order when considered together. This however forbids the use of Karamata’s Tauberian theorem since the monotonicity restriction would require roughly that $X_i$ is symmetric. Thus the complex Tauberian approach, as developed in [13], is required to justify the answer.

**Case 1 ($\nu(\delta) = 3$).** Assume that part of the sequence $\{l_1, \ldots, l_n\}$ lies between $k_\alpha$ and $k_{\alpha+1}$ and the rest between $k_\alpha$ and $k_{\alpha+1}$. Then using the change of variables
\[ i_1 = m_0, \]
\[ i_2 = m_0 + m_1, \]
\[ \vdots \]
\[ i_r = m_0 + \cdots + m_{r-1} \]
\[ j_1 = m_0 + \cdots + m_r, \]
\[ j_2 = m_0 + \cdots + m_{r+1}, \]
\[ \vdots \]
\[ j_s = m_0 + \cdots + m_{r+s-1}, \]
\[ i_{r+1} = m_0 + \cdots + m_{r+s}, \]
\[ i_{r+2} = m_0 + \cdots + m_{r+s+1}, \]
\[ \vdots \]
\[ i_a = m_0 + \cdots + m_{a+s-1}, \]
\[ j_{s+1} = m_0 + \cdots + m_{a+s}, \]
\[ j_{s+2} = m_0 + \cdots + m_{a+s+1}, \]
\[ \vdots \]
\[ j_a = m_{2a-1}, \]
\[ n = m_0 + \cdots + m_{2a}, \]

we rewrite the positive term in (10) as

\[ a(n) = \sum \mathbb{P}[S(i_1) = \cdots = S(i_a); S(j_1) = \cdots = S(j_a)] \]
\[ = \sum_{m_0, \ldots, m_{2a-1}} \prod_{j=r+1}^{a+s+1} \mathbb{P}(S_{m_j} = 0) \cdot \mathbb{P}(S_{m} + S'_{m} = S_{m_{r+s}} + S'_{m_{r+s}} = 0). \]

Notice that from [13] we have that \( \sum_{n \geq 0} \lambda^n P(S_n = 0) \sim \log(1/(1-\lambda))/\pi \nu. \) Let

\[ a(\lambda) = (1-\lambda)^{-3} \left[ -\log (1-\lambda) \right]^{2a-4}, \]
\[ c_\nu = \left( \pi \nu \right)^{-2a+4}. \]

Then, by direct calculations and Fourier inversion formula

\[ \sum_{n \geq 0} \lambda^n a(n) = c_\nu (1-\lambda) a(\lambda) \]
\[ \cdot \sum_{x, \lambda z, k_{x}} \lambda^{k_1 + k_2} \mathbb{P}(S_{k_1} = x) \mathbb{P}(S_{k_2} = -x) \]
\[ \cdot \mathbb{P}\left( S_{k_1} = x \right) = c_\nu (1-\lambda) a(\lambda) \frac{1}{(2\pi)^2} \]
\[ \cdot \int \int \frac{dt \, ds}{(1-\lambda f(t))(1-\lambda f(s))(1-\lambda f(t+s))} \]
\[ \sim c_\nu (1-\lambda) a(\lambda) \frac{1}{(2\pi)^2} \frac{1}{1-\lambda} \]
\[ \cdot \int \int \frac{dx \, dy}{(1+|x|)(1+|y|)(1+|x+y|)} \sim \left( \frac{1}{4\nu^2} \right) \]
\[ \cdot c_\nu a(\lambda). \]

Next we consider the negative term in (10)

\[ b(n) = \sum \mathbb{P}\left[ S_{m_1} = \cdots = S_{m_{r+s}} = S_{m_{r}} + \cdots = S_{m_{a+s-1}} = 0 \right] \]
\[ + S_{m_{r+s}} = S_{m_{r+s+1}} = \cdots = S_{m_{a+s}} = 0 \right] \mathbb{P}\left[ S_{m_{r}} = \cdots = S_{m_{a+s}} = 0 \right]. \]

By direct calculations and (6),

\[ \sum_{n \geq 0} \lambda^n b(n) = \left( \frac{1}{\pi \nu} \log \left( \frac{1}{1-\lambda} \right) \right)^{a+s+r-2} (1-\lambda)^{-2} \]
\[ \cdot \sum_{m_0, m_{r}, m_{a+s}} \lambda^{m_r + m_{a+s}} \]
\[ \cdot \prod_{t=r+1}^{a+s+1} \mathbb{P}(S_{m_t} = 0) \cdot \mathbb{P}(S_{m_r + \cdots + m_{a+s}} = 0) \]
\[ \cdot \mathbb{P}(S_{m_r + \cdots + m_{a+s}} = 0), \]

and using Fourier inversion and (6) the internal sum behaves as

\[ (2\pi)^{-a+s+r} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \left[ \prod_{j=r+1}^{a+s+1} \frac{1}{1-\lambda \phi(t_j)} \left( 1-\lambda \phi(t_j) \right)^{-1} \left( 1-\lambda \phi(t_j) \right)^{-1} \right] \]
\[ \cdot \frac{d\gamma}{d\nu} \sim \left( \pi \nu \right)^{-a+s+r} (1-\lambda)^{-1} \]
\[ \cdot \log \left( \frac{1}{1-\lambda} \right)^{a+s+r-2} \frac{\pi^2}{6}. \]
Then, we have $\sum_n \lambda^n b(n) \sim (\pi^2/6(\pi\gamma)^2)\alpha(\lambda)$, whence the Tauberian theorem implies that $a(n) - b(n) \sim n^2 \log(n)^{2\alpha-4}/24\pi^2\gamma^2\alpha^2$. Most importantly we see that the lengths and locations of the chains, $r$ and $s$, do not affect the asymptotic behaviour. Noting that if $1 \leq r, s \leq \alpha - 1$, we can partition $2\pi = r + s + (\alpha - r) + (\alpha - s)$ in $(\alpha - 1)^2$ ways, and thus overall the total contribution from terms with $v = 3$ is

$$\int \frac{(\pi^2/6(\pi\gamma)^2)\alpha(\lambda)}{r!s!(\alpha-r)!\alpha-s!} d\lambda \sim \pi \frac{(\alpha - 1)^2}{12\pi^2\gamma^2\alpha^2} n^2 \log(n)^{2\alpha-4}. \quad (55)$$

Case 2 ($\nu(\delta) = 2$). The typical term $c(n)$ was introduced in (33) in the proof of Proposition 9. Now we let $\lambda \in C$, with $|\lambda| < 1$. By lengthy but direct calculations we can derive an expression of the form

$$\sum_n \lambda^n c(n) = \frac{\alpha - 1}{\gamma^2} \log(\lambda) + o(a(\lambda)), \quad \lambda \rightarrow 1. \quad (56)$$

The approach developed in [13] can then be used to bound the error terms and show that $c(n) \sim [(\alpha - 1)/\gamma^2] n^2 \log(n)^{2\alpha-4}$. Finally taking into account the fact that $I_1, \ldots, I_{\alpha}$ can be in any of the $\alpha - 1$ intervals $[k_i, k_{i+1}]$, for $i = 1, \ldots, \alpha - 1$, the result follows the overall contribution of terms with $v(\delta) = 2$

$$\int (\pi^2/6(\pi\gamma)^2)\alpha(\lambda) d\lambda \sim \pi \frac{(\alpha - 1)^2}{2\gamma^2} n^2 \log(n)^{2\alpha-4}. \quad (57)$$

The case for $d = 2$ is very similar, so we move on to the case $d = 3$.

Case 3 ($\nu(\delta) = 3$ and $\alpha = 2$). Using the same notation as before, we have three terms to consider $a(n), b(n)$, and $c(n)$. We first consider $c(n)$. Letting $K = e/\sqrt{1 - \lambda}$ and using the usual power series construction and spherical coordinates

$$\sum_n \lambda^n c(n) = (1 - \lambda)^{-1} (2\pi)^{-6}$$

$$I_1(\lambda) \sim |\Sigma|^{-1} \int_{r,s=0}^{2\pi} \int_{\theta_1, \theta_2=0}^{\pi} \sin(\theta_1) \sin(\theta_2) d\theta_1 d\theta_2 \frac{r^2 s^2 \sin(\theta_1) \sin(\theta_2) d\theta_1 d\theta_2 d\phi_1 d\phi_2 dr ds}{(1 - \lambda + \lambda r^2)(1 - \lambda + \lambda s^2)[1 - \lambda + \lambda (r^2 + s^2 + 2Ars)]}$$

$$= |\Sigma|^{-1} \int_{\theta_1, \theta_2=0}^{\pi} \frac{2\pi}{\sin(\theta_1) \sin(\theta_2)} d\theta_1 d\theta_2 \frac{r^2 s^2 \sin(\theta_1) \sin(\theta_2) d\theta_1 d\theta_2 d\phi_1 d\phi_2 dr ds}{(1 + r^2)(1 + s^2)[1 + r^2 + s^2 + 2Ars]}$$

$$\sim |\Sigma|^{-1} \log(K) \int_{\theta_1, \theta_2=0}^{\pi} \frac{2\pi}{\sin(\theta_1) \sin(\theta_2)} d\theta_1 d\theta_2 \frac{\arccos(A(\theta_1, \theta_2, \phi_1, \phi_2)) d\phi_1 d\phi_2 d\theta_1 d\theta_2}{\sqrt{1 - A(\theta_1, \theta_2, \phi_1, \phi_2)^2}}$$

The other integral is slightly easier

$$I_2(\lambda) \sim |\Sigma|^{-1} \frac{\pi}{2} \log K$$

$$\int_{\theta_1, \theta_2=0}^{\pi} \int_{\phi_1, \phi_2=0}^{2\pi} \sin(\theta_1) \sin(\theta_2) d\phi_1 d\phi_2 d\theta_1 d\theta_2, \quad (62)$$

and thus overall we must have that

$$I_1 - I_2(\lambda) \sim \frac{1}{2} (2\pi)^{-6} |\Sigma|^{-1} (1 - \lambda)^{-2} \log \left( \frac{1}{1 - \lambda} \right)$$

$$\int_{\theta_1, \theta_2=0}^{\pi} \int_{\phi_1, \phi_2=0}^{2\pi} \frac{\arccos(A)}{\sqrt{1 - A^2}} \frac{\pi}{2} \sin(\theta_1)$$
\begin{align*}
&\cdot \sin(\theta_2) \, d\phi_1 \, d\phi_2 \, d\theta_1 \, d\theta_2 = \kappa_2 (1 \\
&- \lambda)^{-2} \log \left( \frac{1}{1-\lambda} \right),
\end{align*}

whence it follows that \( \text{var}(L_n(2)) \sim (\kappa_1 + \kappa_2) n \log n. \)

To prove the last claim let \( S'_n = X'_1 + \cdots + X'_n \) be another random walk, independent of \( S_n \), such that its characteristic function \( f'(t) = \mathbb{E}[\exp(itX')] \) also satisfies the expansion (6).

Then using [13, Lemma 3.1], one can adapt the proof of [13, Theorem 2.1] to show that \( L'_n(\alpha) = L_n(\alpha) + o(L_n(\alpha)). \)

\[ \square \]

**Competing Interests**

The authors declare that they have no competing interests.

**References**


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