DOI:
10.1016/j.geomphys.2017.07.002

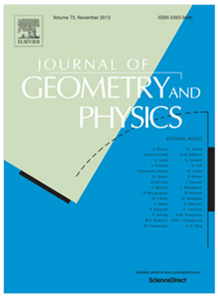
Document Version
Peer reviewed version

Citation for published version (APA):
Scott, S. (2018). log TQFT. *JOURNAL OF GEOMETRY AND PHYSICS*, *123*, 1-29.
https://doi.org/10.1016/j.geomphys.2017.07.002

Accepted Manuscript

log TQFT

Simon Scott



PII: S0393-0440(17)30171-7
DOI: http://dx.doi.org/10.1016/j.geomphys.2017.07.002
Reference: GEOPHY 3026

To appear in: *Journal of Geometry and Physics*

Received date : 16 June 2016
Revised date : 28 June 2017
Accepted date : 1 July 2017

Please cite this article as: S. Scott, log TQFT, *Journal of Geometry and Physics* (2017), http://dx.doi.org/10.1016/j.geomphys.2017.07.002

log TQFT

Simon Scott

Abstract

A class of logarithmic-functors is constructed which allow additive invariants on categories to be formulated as functors on A_∞ categories. Characters of such logarithmic representations provide a functorial structure for exotic torsions and include Chern characters, Reidemeister torsion and topological signatures. A number of examples are given.

The goal here is to put into place an algebraic theory, or rather a categorification, of logarithmic representations and their log-determinant characters.

Such characters provide a functorial setting for additive invariants arising as generalised Reidemeister torsions on bordism categories and may be viewed as semi-classical, positioned between genera (classical bordism invariants) and TQFTs (quantum bordism invariants); the former are homomorphisms

$$\mu : \Omega_* \to R$$

on the ring Ω_* of bordism classes of closed manifolds, such as the signature of a 4k dimensional manifold, while a TQFT (topological quantum field theory) of dimension n refers to a symmetric monoidal functor

$$Z : \mathbf{Bord}_n \to \mathsf{B}$$

from the bordism category $\mathbf{Bord}_n$, whose objects are smooth closed (n-1)-dimensional manifolds M and whose morphisms are n-dimensional bordisms, to a target symmetric monoidal category B.

The class of semi-classical bordism invariants considered here arise as characters of log-additive simplicial maps

$$\log : \mathcal{N}\mathbf{Bord}_n \to \mathcal{A} \tag{0.1}$$

from the nerve $\mathcal{N}\mathbf{Bord}_n$ of the bordism category to a simplicial set of rings $\mathcal{A}$. Such a map (0.1), called a *log-functor*, associates to each bordism $W \in \mathrm{mor}(M_0, M_1)$ between closed manifolds M_0 and M_1 a logarithm $\log_{M_0 \sqcup M_1}(W)$ in a ring $\mathsf{F}(M_0 \sqcup M_1) \in \mathcal{A}$ along with a hierarchy of compatible inclusions

$$
\begin{array}{ccccc}
 & & \mathsf{F}(M_0 \sqcup M_2) & & \\
 & & \downarrow & & \\
 & & \mathsf{F}(M_0 \sqcup M_1 \sqcup M_2) & & \\
 & \nearrow & & \nwarrow & \\
\mathsf{F}(M_0 \sqcup M_1) & & & & \mathsf{F}(M_1 \sqcup M_2)
\end{array}
\tag{0.2}
$$

such that when two bordisms $W \in \mathrm{mor}(M_0, M_1), W' \in \mathrm{mor}(M_1, M_2)$ are sewn together there is a log-additive identity in $\mathsf{F}(M_0 \sqcup M_1 \sqcup M_2)$

$$\log_{M_0 \sqcup M_2}(W \cup_{M_1} W') \;\approx\; \log_{M_0 \sqcup M_1}(W) + \log_{M_1 \sqcup M_2}(W'), \tag{0.3}$$

where $\approx$ indicates equality modulo finite sums of commutators. Neither commutators nor inclusion maps are seen by categorical trace maps $\tau_N : \mathsf{F}(N) \to R$ to a commutative ring R and so, irrespective of in which ring it may be convenient to view the logarithm of a bordism W, the resulting log-character $\tau(\log W) := \tau_{M_0 \sqcup M_1}(\log W) \in R$ is invariantly defined.

Characters of log-TQFTs capture a class of semi-local invariants that are of a somewhat more general nature than the local invariants that occur as genera but which, in view of the log-additive pasting property, are far simpler and more restricted (perhaps more delicate) than the globally determined invariants of a TQFT. Such trace-logs include instances of classical Whitehead and Reidemeister torsions and the topological signature σ and the (relative) Euler characteristic χ (note that σ is a genus while χ is not).

Log-Determinants of this type can arise formally in semi-classical expansions of Feymann path integrals, such as Reidemeister torsion $T_M(a)$ in the stationary phase expansion of Chern-Simons TQFT $Z_{\mathrm{cs}}(M) \sim \sum_a c(a)\sqrt{T_M(a)}$ over irreducible flat connections [20]. Indeed generally, path integral computations, both in theoretical particle physics and in TQFT in mathematics, proceed not through an absolute evaluation of the path integral but via computing appropriate asymptotic approximations. This suggests that the ontology proposed by TQFT would be enhanced by refinements which capture the functorial structure of such expansions. Though the specific focus here is on the first-order terms of a certain class of perturbation expansions determined by a generalised logarithmic property, the framework constructed provides a basis for encoding more general functorial asymptotics.

1 Logarithmic representations of monoids

A logarithmic representation of a monoid $\mathcal{Z}$ into a ring $\mathsf{B} = (\mathsf{B}, \cdot, +)$ means a homomorphism

$$\log : \mathcal{Z} \to \mathsf{B}/[\mathsf{B}, \mathsf{B}], \tag{1.1}$$

where

$$[\mathsf{B}, \mathsf{B}] = \{ \sum_{1 \leq j \leq n} [\beta_j, \beta_j'] \mid \beta_j, \beta_j' \in \mathsf{B}\} \tag{1.2}$$

is the subgroup of the abelian group $(\mathsf{B},+)$ consisting of finite sums of commutators $[\beta_j,\beta'_j] := \beta_j\cdot\beta'_j - \beta'_j\cdot\beta_j$ and $\mathsf{B}/[\mathsf{B},\mathsf{B}] := (\mathsf{B},+)/[\mathsf{B},\mathsf{B}]$ means the abelian quotient group. For $\mu,\nu\in\mathsf{B}$ we may use the notation

$$\mu \ \approx \ \nu \ \text{ if } \mu-\nu\in[\mathsf{B},\mathsf{B}], \quad \text{so } \mu=\nu \text{ in } \mathsf{B}/[\mathsf{B},\mathsf{B}]. \tag{1.3}$$

Thus, one has

$$\log(ba) = \log a + \log b \qquad \text{in } \mathsf{B}/[\mathsf{B},\mathsf{B}] \tag{1.4}$$

where $ba = b\circ a$ is the product in $\mathcal{Z}$, which the logarithm thus abelianises. In practise, logarithms usually arise in the following way:

Lemma 1.1 *A map $\ell og:\mathcal{Z}\to\mathsf{B}$ with*

$$\ell og(ba) = \ell og(b) + \ell og(a) + \sum_j [c_j, c'_j]$$

for some $c_j, c'_j\in\mathsf{B}$ defines a logarithm. If $0\to[\mathsf{B},\mathsf{B}]\to\mathsf{B}\to\mathsf{B}/[\mathsf{B},\mathsf{B}]\to 0$ is a split exact sequence of abelian groups then the converse holds.

Sums of logs are logs and so form an abelian group

$$\mathbb{L}\mathrm{og}(\mathcal{Z},\mathsf{B}) := \mathrm{Hom}(\mathcal{Z},\mathsf{B}/[\mathsf{B},\mathsf{B}]).$$

A trace on B with values in a commutative unital ring $(R,\cdot,+)$ is a homomorphism of abelian groups $\tau:(\mathsf{B},+)\to(R,+)$ which vanishes on commutators $\tau([b,b'])=0$, so $[\mathsf{B},\mathsf{B}]\subset\mathrm{Ker}\,(\tau)$. To give a trace τ is thus equivalent to giving a homomorphism of abelian groups

$$\widetilde{\tau}:\mathsf{B}/[\mathsf{B},\mathsf{B}]\to R.$$

Since sums of traces are traces, they likewise form an abelian group $\mathbb{T}\mathrm{race}(\mathsf{B},R)$.

A log-character (or logarithmic determinant or trace-log) on $\mathcal{Z}$ is an evaluation of the resulting canonical pairing

$$\mathbb{T}\mathrm{race}(\mathsf{B},R)\times\mathbb{L}\mathrm{og}(\mathcal{Z},\mathsf{B})\to\mathrm{Hom}(\mathcal{Z},(R,+)), \quad (\tau,\log)\mapsto\widetilde{\tau}\circ\log.$$

Such a character inherits the log-additivity property for $a,b\in\mathcal{Z}$

$$\widetilde{\tau}(\log ba) = \widetilde{\tau}(\log a) + \widetilde{\tau}(\log b) \quad \text{ in } R, \tag{1.5}$$

while composition with an exponential map $\varepsilon: R\to A^*$, $\varepsilon(x+y)=\varepsilon(x)\cdot\varepsilon(y)$, into the units of a commutative ring A associates a determinant $a\mapsto\det a := e\circ\widetilde{\tau}\circ\log(a)$ with the multiplicative property $\det ba = \det a\cdot\det b$ in A.

Example: Let $\mathcal{Z}=\mathrm{Fred}$ be the monoid of Fredholm operators on a Hilbert space, and $\mathsf{B}=\mathcal{F}$ the ideal of finite-rank operators. The map

$$\log:\mathrm{Fred}\to\mathcal{F}/[\mathcal{F},\mathcal{F}], \quad \log a := \pi([a,p]), \tag{1.6}$$

where $p\in\mathrm{Fred}$ is any parametrix for a and $\pi:\mathcal{F}\to\mathcal{F}/[\mathcal{F},\mathcal{F}]$ the quotient map, is a logarithm, the abstract Fredholm index of a, whilst its numeric log-character with respect to the canonical isomorphism $\mathcal{F}/[\mathcal{F},\mathcal{F}] \stackrel{\cong}{\to} \mathbb{C}, c \mapsto \widetilde{\mathrm{Tr}}\,(c)$, defined by the classical trace $\mathrm{Tr}\,:\mathcal{F} \to \mathbb{C}$ is the usual integer valued Fredholm index

$$\widetilde{\mathrm{Tr}}\,(\log a) = \mathrm{ind}\, a := \dim\ker(a) - \dim\mathrm{coker}(a)$$

and (1.5) is the classical additivity property of the index

$$\mathrm{ind}\, ba = \mathrm{ind}\, a + \mathrm{ind}\, b.$$

Somehwat more generally, on continuous families $\mathcal{Z} = \mathrm{Map}(M, \mathrm{Fred})$ of Fredholm operators, with continuous parametrix, parametrized by a manifold M, a log-character can be defined by sending $\mathsf{a} \in \mathrm{Map}(M, \mathrm{Fred})$ to its index bundle $\log \mathsf{a} := \mathrm{Ind}\, \mathsf{a} \in K_0(M)$. The top exterior power operation acts as an exponential map on the commutative ring $K_0(M)$ sending $\mathrm{Ind}\,\mathsf{a}$ to the isomorphism class of the determinant line bundle $\mathrm{Det}\,\mathsf{a}$ in the group $A \cong H^2(M, \mathbb{Z})$ of complex line bundles over M, with the log-additivity property $\mathrm{Ind}\,\mathsf{ba} = \mathrm{Ind}\,\mathsf{a} + \mathrm{Ind}\,\mathsf{b}$ in $K_0(M)$ exponentiating to the canonical multiplicativity property $\mathrm{Det}\,\mathsf{ba} = \mathrm{Det}\,\mathsf{a} \otimes \mathrm{Det}\,\mathsf{b}$ of the determinant line bundle in A. These facts persist to the case of families of Fredholm operators between non-isomorphic bundles, but need to be stated in terms of log-functors on categories, see §3 below.

A smooth version of this exists when a is defined by a geometric fibration of manifolds, mapping a to $\frac{1}{\mathrm{rk}(\mathrm{Ind}\,\mathsf{a})} F_{\mathsf{a}} \in \Omega^*(M, \mathrm{End}\,) = \mathsf{B}$ with F_{a} the curvature of a super connection on $\mathrm{Ind}\,\mathsf{a}$, the log-character defined by the super trace $\Omega^*(M, \mathrm{End}\,) \to \Omega^*(M)$ outputs the first Chern form of the determinant bundle and its exponential the Chern character form; all of which pushes down to cohomology.

Example: Let $\mathrm{Gl}(\Psi^{\infty}_{\mathrm{Ell}}(M, E, \mathbb{R}))$ be the group of elliptic invertible elements in the algebra $\Psi^{\infty}(M, E, \mathbb{R})$ of suspended ψdos introduced by Melrose [8] with its suspended-trace TR. Then

$$\log\,: \mathrm{Gl}_1(\Psi^{\infty}_{\mathrm{Ell}}(M, E, \mathbb{R})) \to \Psi^{\infty}(M, E, \mathbb{R}), \quad (\log\, A)(t) := A(t)^{-1}\,\dot{A}(t), \tag{1.7}$$

is a logarithm with character $\eta_{\mathrm{sus}}(A) := \mathsf{TR}(\log A)$ the *suspended eta invariant*.

Example: On $\mathrm{Gl}_n(\mathbb{C})$ (and likewise for the Fredholm determinant on Hilbert space) there is no globally defined logarithm exponentiating to the classical determinant $\det\,:\mathrm{Gl}_n(\mathbb{C}) \to \mathrm{Gl}_1(\mathbb{C})$. Rather, the logarithm is defined on the universal cover $\widehat{\mathrm{Gl}_n(\mathbb{C})}$ of based homotopy classes of paths $[[0,1], \mathrm{Gl}_n(\mathbb{C})]$ by

$$\widehat{\log}_n : \widehat{\mathrm{Gl}_n(\mathbb{C})} \to M_n(\mathbb{C}), \quad \widehat{\log}_n \beta := \int_0^1 \beta(t)^{-1}\,\dot{\beta}(t)\,dt, \tag{1.8}$$

on a smooth representative for $\beta = [t \mapsto \beta(t)]$, $\beta(1) = b$. $\widehat{\mathrm{Gl}_n(\mathbb{C})}$ is a principal $\pi_1(\mathrm{Gl}_n(\mathbb{C}))$ bundle, with fibre at $b \in \mathrm{Gl}_n(\mathbb{C})$ the paths from the identity ending at b, and with local sections defined by the complex powers: to $b \in \mathrm{Gl}_n(\mathbb{C})$ with spectral cut $\theta \in \mathbb{R}$ one associates (via functional calculus) the homotopy class of $\beta(t) := b^t_\theta$. Since the same θ works for elements of $\mathrm{Gl}_n(\mathbb{C})$ in a small neighbourhood of b, this extends to define the local section, while inserting into (1.8) gives the usual functional calculus *local* logarithm $\widehat{\log}_n([t \mapsto b^t_\theta]) = \log_\theta b$. For the affine representation $\pi_1(\mathrm{Gl}_n(\mathbb{C})) \to \mathbb{C}$ acting by $z \mapsto z + 2\pi i w(\gamma)$ with $w(\gamma)$ the winding number of a $\gamma \in \pi_1(\mathrm{Gl}_n(\mathbb{C}))$, the log-character $\mathrm{tr} \circ \widehat{\log}_n$ is a $\pi_1 := \pi_1(\mathrm{Gl}_n(\mathbb{C})) \cong \mathbb{Z}$ equivariant map $\widehat{\mathrm{Gl}_n(\mathbb{C})} \to \mathbb{C}$, or, equivalently, a section of the affine line bundle $\widehat{\mathrm{Gl}_n(\mathbb{C})} \times_{\pi_1} \mathbb{C}$ and $\exp(\mathrm{tr}\,\widehat{\log}_n \beta) = \det \beta(1) := \det b$. Equally, letting π_1 act on $M_n(\mathbb{C})$ by $a \mapsto a + (2\pi i w(\gamma) \oplus 0_{n-1})$, gives $\widehat{\log}_n$ as a section of the affine line bundle $\widehat{\mathrm{Gl}_n(\mathbb{C})} \times_{\pi_1} (M_n(\mathbb{C})/[M_n(\mathbb{C}), M_n(\mathbb{C})])$, isomorphic to $\widehat{\mathrm{Gl}_n(\mathbb{C})} \times_{\pi_1} \mathbb{C}$ by the classical trace isomorphism $M_n(\mathbb{C})/[M_n(\mathbb{C}), M_n(\mathbb{C})] \xrightarrow{\mathrm{tr}} \mathbb{C}$.

Thus, while the logarithm lives in an abstract complex line the log-determinant (and hence determinant) are honest complex numbers.

Example: Let $\Psi_{\mathrm{p.a.}}(M,\mathcal{E})$ be the monoid of classical ψdos on sections of a vector bundle $\mathcal{E} \to M$ admitting a principal angle $\theta \in \mathbb{R}$. As above, the functional calculus defines a logarithm, up to a choice of θ, $\Psi_{\mathrm{p.a.}}(M,\mathcal{E}) \to \Psi^{\leq 0}(M,\mathcal{E})$, $A \mapsto \log_\theta A$, to the algebra $\Psi^{\leq 0}$ of order zero ψdos — by ([17] §2.7.1.3) $\log_\theta A$ can be taken to be order zero, and is well defined because the dependence on the spectral cut lies in $[\Psi^{\mathbb{Z}}(M,\mathcal{E}), \Psi^{\mathbb{Z}}(M,\mathcal{E})]$. The character defined by the residue trace res : $\Psi^{\mathbb{Z}}(M,\mathcal{E}) \to \mathbb{C}$ gives a log determinant $\log\det_{\mathrm{res}}(A) := \mathrm{res}\,(\log A)$, the residue determinant. This along with the character given by the leading-symbol trace accounts for all log-determinants on classical ψdos [5].

Nevertheless, instead, if the classical trace is extended to a quasi-trace on $\Psi^{\leq 0}(M,\mathcal{E})$ by zeta-function regularisation, then the resulting quasi-character of $A \mapsto \log_\theta A$ is the spectral zeta determinant $\log\det_\zeta(A)$; the obstruction to the quasi-characacter being a homomorphism is the multiplicative anomaly.

Example: A log representation to a differential graded ring $\mathsf{B} = (\mathsf{B}, d)$ is a homomorphism $\log : \mathcal{Z} \to \mathsf{B}/([\mathsf{B},\mathsf{B}] + d\mathsf{B})$ with $[\mathsf{B},\mathsf{B}] + d\mathsf{B}$ the additive subgroup of sums of commutators and exact elements db some $b \in B$. Let M be a closed manifold and consider its odd topological K-theory $K_{-1}(M) := [M, \mathrm{Gl}_{\mathrm{Tr}}(H)]$ of homotopy classes of maps to the infinite general linear group $\mathrm{Gl}_{\mathrm{Tr}}(H)$ of invertible linear operators g with $g - I$ in the ideal $C_1(H)$ of trace class operators. Then

$$\ell og : K_{-1}(M) \to \Omega^*(M, \mathrm{End}\,(C_1(H))), \qquad \ell og(h) := \sum_k (-1)^{k-1} \frac{(k-1)!}{(2k-1)!} (h^{-1}dh)^{2k-1},$$

defines a logarithm in the abelianisation of the de Rham algebra $(\Omega^*(M, \mathrm{End}\,(C_1(H))), d)$ with character for the classical trace $\mathrm{Tr} : \Omega^*(M, \mathrm{End}\,(C_1(H))) \to H^*(M)$ equal to the odd Chern character.

On general categories matters are complicated by the fact that the respective logarithms of a pair of composable morphisms will, in general, take values in different rings, and so log-additivity (1.4) only becomes meaningful within the higher structure (0.2), (0.3), which we proceed to next.

2 Logarithmic representations of categories

All categories will be assumed to be small. Denote the set of morphisms in a category $\mathbf{C}$ between objects $x, y \in \mathrm{ob}(\mathbf{C})$ by $\mathrm{mor}_{\mathbf{C}}(x,y)$, or $\mathrm{mor}(x,y)$, and $\mathrm{end}(x) := \mathrm{mor}(x,x)$.

$\mathbf{C}$ is monoidal if it has a bifunctor $\otimes : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$ which is associative with identity object $1 = 1_{\mathbf{C}}$ up to coherent isomorphism. Any two coherence isomorphisms between associativity bracketings of an n-fold product $x_1 \otimes x_2 \otimes \cdots \otimes x_n$ for $x_j \in \mathrm{ob}(\mathbf{C})$ then coincide. To specify for each $\sigma \in S_n$ (symmetric group) a permutation isomorphism

$$\underbrace{x_1 \otimes \cdots \otimes x_n}_{:=x} \xrightarrow{s_\sigma(x)} \underbrace{x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)}}_{:=x_\sigma} \tag{2.1}$$

in $\mathrm{mor}_{\mathbf{C}}(x, x_\sigma)$ a braiding map $b_{w,y} : w \otimes y \to y \otimes w$ for each $w, y \in \mathrm{ob}(\mathbf{C})$ is assumed with $b_{y,w} = b_{w,y}^{-1}$, giving $\mathbf{C}$ the structure of a symmetric monoidal category: $\otimes$ is then commutative up to coherent isomorphism and (2.1) is uniquely defined for each associativity bracketing of x and x_σ. A functor $\mathsf{F} : \mathbf{C} \to \mathbf{A}$ out of a monoidal category $\mathbf{C}$ will be said to be strict if $\mathsf{F}(x_1 \otimes \cdots \otimes x_n)$ is independent of the choice of associativity bracketing of $x_1 \otimes \cdots \otimes x_n$ and if F maps the coherence isomorphisms to identity morphisms in $\mathbf{A}$. (The assumption that F is strict can be readily dropped provided one keeps track of the isomorphisms $\mathsf{F}((x \otimes y) \otimes z) \to \mathsf{F}(x \otimes (y \otimes z))$, and so on; for example, for a braided monoidal category).

Lemma 2.1 *For $x = x_1 \otimes \cdots \otimes x_n$ and $\sigma \in S_n$ one has a canonical isomorphism*

$$\mu_\sigma(x) := \mathsf{F}(s_\sigma(x)) : \mathsf{F}(x) \xrightarrow{\cong} \mathsf{F}(x_\sigma), \tag{2.2}$$

independent of a choice of associativity bracketing of x or x_σ, and satisfying

$$\mu_{\sigma' \circ \sigma}(x) = \mu_{\sigma'}(x_\sigma) \circ \mu_\sigma(x). \tag{2.3}$$

The *product functors* of a monoidal category $\mathbf{C}$ are (iterations of) the functors $\mathbf{C} \to \mathbf{C}$ obtained by holding fixed one of the inputs of the bifunctor $\otimes$: for $y \in \mathrm{ob}(\mathbf{C})$ the right-product functor $\mathsf{m}_{\otimes y} : \mathbf{C} \to \mathbf{C}$ takes $x \in \mathrm{ob}(\mathbf{C})$ to $x \otimes y \in \mathrm{ob}(\mathbf{C})$ and $\alpha \in \mathrm{mor}_{\mathbf{C}}(x, z)$ to $\alpha \otimes \iota \in \mathrm{mor}_{\mathbf{C}}(x \otimes y, z \otimes y)$, with ι the identity morphism, the left-product functor $\mathsf{m}_{w\otimes}(x) = w \otimes x$ is defined symmetrically. The product functors are not monoidal.

The following construction allows the classical additivity of logarithms to be promoted to a categorical additivity on composed morphisms.

Definition 2.2 *Let $\mathbf{C} = (\mathbf{C}, \otimes)$ be a symmetric monoidal category and let $\mathbf{C}^* = (\mathbf{C}^*, \otimes)$ be a groupoid whose objects are those of $\mathbf{C}$ and whose morphisms are a specified closed subclass of the isomorphisms of $\mathbf{C}$ (containing the coherence and permutation isomorphisms* (2.1)*).*

A monoidal product representation (MPR) of the reduced category $\mathbf{C}^*$ into an additive category $\mathbf{M}$ is a strict functor

$$\mathsf{F} : \mathbf{C}^* \to \mathbf{M} \tag{2.4}$$

along with for each $y \in \mathrm{ob}(\mathbf{C})$ a natural transformation of functors

$$\eta_{\otimes y} : \mathsf{F} \Rightarrow \mathsf{F}_{\otimes y} \tag{2.5}$$

from $\mathsf{F} : \mathbf{C}^* \to \mathbf{M}$ to $\mathsf{F}_{\otimes y} := \mathsf{F} \circ \mathsf{m}_{\otimes y} : \mathbf{C}^* \to \mathbf{M}$ compatible with $\otimes$ and the braiding. (The functor F is not assumed to be monoidal and in general will not be.)

Lemma 2.3 *If $\boldsymbol{S}$ is a symmetric monoidal category, monoidal product representations pull-back with respect to symmetric monoidal functors $\mathsf{J} : \boldsymbol{S}^* \to \boldsymbol{C}^*$.*

F is designed to represent the set of objects of $\mathbf{C}$ with its monoidal product, but not necessarily its morphisms. It is, however, sensitive to the permutation isomorphisms of Lemma 2.1, which intertwine with the covering maps $\eta_{\otimes y}$ as follows.

Lemma 2.4 *Let $y \in \mathrm{ob}(\boldsymbol{C})$. A monoidal product representation defines for each $x \in \mathrm{ob}(\boldsymbol{C})$ a morphism*

$$\eta_{\otimes y}(x) \in \mathrm{mor}_{\boldsymbol{M}}(\mathsf{F}(x), \mathsf{F}(x \otimes y)) \tag{2.6}$$

covering $\mathsf{m}_{\otimes y}$ such that for x, x_σ as in (2.1)

$$\eta_{\otimes y}(x_\sigma) \circ \mu_\sigma(x) = \mu_{\sigma\otimes 1}(x \otimes y) \circ \eta_{\otimes y}(x). \tag{2.7}$$

Proof: A natural transformation $\eta : \mathsf{G} \Rightarrow \mathsf{H}$ of functors $\mathsf{G}, \mathsf{H} : \mathbf{A} \to \mathbf{B}$ defines for $x \in \mathrm{ob}(\mathbf{A})$ a morphism $\eta(x) \in \mathrm{mor}_{\mathbf{B}}(\mathsf{G}(x), \mathsf{H}(x))$ with $\eta(z) \circ \mathsf{G}(\alpha) = \mathsf{H}(\alpha) \circ \eta(x)$ for $\alpha \in \mathrm{mor}_{\mathbf{A}}(x, z)$. Applied to $\mathsf{G} := \mathsf{F}$ and $\mathsf{H} := \mathsf{F}_{\otimes y}$, (2.5) gives $\eta_{\otimes y}(x) := \eta(x)$ in (2.6). For (2.7), take $z = x_\sigma$ and $\alpha = s_\sigma(x) \in \mathrm{mor}(x, x_\sigma)$, so $\eta(z) \circ \mathsf{G}(\alpha) = \eta_{\otimes y}(x_\sigma) \circ \mathsf{F}(s_\sigma(x)) = \eta_{\otimes y}(x_\sigma) \circ \mu_\sigma(x)$ while $\mathsf{H}(\alpha) \circ \eta(x) = \mathsf{F}_{\otimes y}(s_\sigma(x)) \circ \eta_{\otimes y}(x)$ and

$$\mathsf{F}_{\otimes y}(s_\sigma(x)) = \mathsf{F}(\mathsf{m}_{\otimes y}(s_\sigma(x))) = \mathsf{F}(s_\sigma(x) \otimes \iota_y) = \mathsf{F}(s_{\sigma\otimes 1}(x \otimes y)) = \mu_{\sigma\otimes 1}(x \otimes y).$$

□

In particular, since F is strict there is for each $x \in \mathrm{ob}(\mathbf{C})$ a canonical inclusion

$$\eta_x(1) : F(1) \hookrightarrow F(x). \tag{2.8}$$

Compatibility of the $\eta_{\otimes y}$ with $\otimes$ is the requirement $\eta_{\otimes(y\otimes z)} = \eta_{\otimes z} \circ \eta_{\otimes y}$, or, more fully,

$$\eta_{\otimes(y\otimes z)}(x) = \eta_{\otimes z}(x \otimes y) \circ \eta_{\otimes y}(x), \tag{2.9}$$

and compatibility with the braiding that

$$\eta_{\otimes(w\otimes z)}(x) = \mu_{1_x\otimes\sigma_{z,w}}(x \otimes z \otimes w)\eta_{\otimes(z\otimes w)}(x) \tag{2.10}$$

where $1_x \otimes \sigma_{z,w}$ is the permutation which fixes x and swaps w and z.

A monoidal product representation is *injective* if for each $x \in \mathrm{ob}(\mathbf{C})$ the morphisms $\eta_{\otimes y}(x)$ are left-invertible : there is a

$$\delta_{\otimes y}(x) \in \mathrm{mor}_{\mathbf{M}}(\mathsf{F}(x \otimes y), \mathsf{F}(x)) \tag{2.11}$$

with $\delta_{\otimes y}(x) \circ \eta_{\otimes y}(x) = i$, the identity morphism, and satisfying $\delta_{\otimes z} \circ \delta_{\otimes y} = \delta_{\otimes(z\otimes y)}$.

Somewhat more generally, it is useful to combine the above maps to define *insertion morphisms* for $x = x_0 \otimes \cdots \otimes x_n$ and $0 \le k \le n+1$ and $w \in \mathrm{ob}(\mathbf{C})$

$$\eta^k_w = \eta^k_w(x) : \mathsf{F}(x_0 \otimes \cdots \otimes x_n) \to \mathsf{F}(x_0 \otimes \cdots \otimes x_{k-1} \otimes w \otimes x_k \cdots \otimes x_n) \tag{2.12}$$

by

$$\eta^k_w(x) = \mu_{\sigma_{k,n+1}}(x \otimes w) \circ \eta_{\otimes w}(x), \tag{2.13}$$

where $\sigma_{k,n+1}$ is the permutation $(0, \ldots, n+1) \to (0, \ldots, k-1, n+1, k, \ldots, n)$. By *fiat*, $\eta_{\otimes y} := \eta^{n+1}_y(x)$ and $\eta_{y\otimes} := \eta^0_y(x)$. When it is clear what is meant, the superscript k and the domain specifier (x) may be omitted to write η_w.

For $\underline{w} = (w_1, \ldots, w_r) \in \mathrm{ob}(\Sigma(\mathbf{C}))$ the iterated insertion morphism

$$\eta_{\underline{w}} := \eta_{w_1}\eta_{w_2}\cdots\eta_{w_r} := \eta_{w_1} \circ \cdots \circ \eta_{w_r} : \mathsf{F}(x) \to \mathsf{F}(x_{\underline{w}}) \tag{2.14}$$

is unambiguously defined, independently of the ordering of the η_{w_j} (in the sense of Lemma 2.5); here, $x = x_0 \otimes \cdots \otimes x_n$ while $x_{\underline{w}}$ is the monoidal product of the x_i and w_l in a specified order. If the $\eta_{\otimes w}(x)$ are injective then so is (2.14): the *ejection morphism*

$$\delta^k_w = \delta^k_w(x) : \mathsf{F}(x_w) \to \mathsf{F}(x), \quad \delta^k_w(x) = \delta_{\otimes w}(x) \circ \mu_{\sigma^{-1}_{k,n+1}}(x_w), \tag{2.15}$$

for $x_w = x_0 \otimes \cdots \otimes x_{k-1} \otimes w \otimes x_{k+1} \otimes \cdots \otimes x_n$ and $0 \le k \le n$ and $w \in \mathrm{ob}(\mathbf{C})$ defines a left-inverse for η^k_w. The commutation properties are:

Lemma 2.5

$$\eta^l_z\eta^k_w = \eta^k_w\,\eta^{l-1}_z, \quad k < l, \tag{2.16}$$

$$\delta^l_w\delta^k_z = \delta^{k-1}_z\delta^l_w, \quad k < l, \tag{2.17}$$

$$\delta^l_w\eta^k_z = \begin{cases} \eta^{k-1}_z\delta^l_w & \text{if } k < l, \\ \eta^k_z\delta^{l-1}_w & \text{if } k > l, \\ 1 & \text{if } k = l \text{ and } w = z. \end{cases} \tag{2.18}$$

Proof: Here, $\eta^l_z\,\eta^k_w := \eta^l_z((x\otimes w)_{\sigma_{k,n+1}}) \circ \eta^k_w(x)$, where $x = x_1 \otimes \cdots \otimes x_n$, and so on. The case $\eta^{n+2}_z\,\eta^{n+1}_w = \eta^{n+1}_w\,\eta^{n+1}_z$ is

$$\eta_{\otimes z}(x \otimes w)\,\eta_{\otimes w}(x) = \mu_{1_x\otimes\sigma_{z,w}}(x \otimes z \otimes w)\,\eta_{\otimes w}(x \otimes z)\,\eta_{\otimes z}(x) \tag{2.19}$$

which is a restatement of the compatibility (2.9), (2.10). For the general case one has $\eta^l_z\,\eta^k_w := \mu_{\sigma_{l,m+2}}((x\otimes w)_{\sigma_{k,m+1}} \otimes z)\,\eta_{\otimes z}((x\otimes w)_{\sigma_{k,m+1}})\,\mu_{\sigma_{k,m+1}}(x\otimes w)\,\eta_{\otimes w}(x)$, by (2.13). From (2.7), $\eta_{\otimes z}(x\otimes w)\,\mu_{\sigma_{k,m+1}}(x\otimes w) = \mu_{\sigma_{k,m+1}\otimes 1_z}(x\otimes w\otimes z)\,\eta_{\otimes z}(x\otimes w)$, hence

$$\begin{aligned} \eta^l_z\,\eta^k_w &= \mu_{\sigma_{l,m+2}}((x\otimes w)_{\sigma_{k,m+1}} \otimes z)\,\mu_{\sigma_{k,m+1}\otimes 1_z}(x\otimes w\otimes z)\,\eta_{\otimes z}(x\otimes w)\,\eta_{\otimes w}(x) \\ &\overset{(2.19)}{=} \mu_{\sigma_{l,m+2}}((x\otimes w)_{\sigma_{k,m+1}} \otimes z)\,\mu_{\sigma_{k,m+1}\otimes 1_z}(x\otimes w\otimes z)\mu_{1_x\otimes\sigma_{z,w}}(x\otimes z\otimes w) \\ &\qquad \circ\,\eta_{\otimes w}(x\otimes z)\,\eta_{\otimes z}(x) \\ &\overset{(2.3)}{=} \mu_{\sigma_{l,m+2}\circ(\sigma_{k,m+1}\otimes 1_z)\circ(1_x\otimes\sigma_{z,w})}(x\otimes z\otimes w)\,\eta_{\otimes w}(x\otimes z)\,\eta_{\otimes z}(x). \end{aligned} \tag{2.20}$$

The elementary equality $\sigma_{l,m+2}\circ(\sigma_{k,m+1}\otimes 1_{m+2})\circ(1\otimes\sigma_{m+1,m+2}) = \sigma_{k,m+2}\circ(\sigma_{l-1,m+1}\otimes 1_{m+2})$ of permutations then yields (2.16). The other identities follow similarly.

□

The identities of Lemma 2.5 define a (parametrised weakly) simplicial set with p-simplices

$$\Delta_p = \{(\xi, x_0, \dots, x_{p-1}) \mid \xi \in F(x_0 \otimes \cdots \otimes x_{p-1}), x_j \in \mathrm{ob}(\mathbf{C})\} \subset \mathrm{ob}(\mathbf{M}) \times \mathrm{ob}(\mathcal{C}^p)$$

with face maps $d_k : \Delta_p \to \Delta_{p-1},\ (\xi, x_0, \dots, x_{p-1}) \mapsto (\delta^k_{x_k}(\xi), x_0, \dots, x_{k-1}, x_{k+1}, \dots, x_{p-1})$, and, for each $z \in \mathrm{ob}(\mathbf{C})$, degeneracy maps

$$s_k(z) : \Delta_p \to \Delta_{p+1},\ (\xi, x_0, \dots, x_{p-1}) \mapsto (\eta^k_z(\xi), x_0, \dots, x_{k-1}, z, x_k, \dots, x_{p-1}).$$

It is 'weakly' so insofar as the standard simplicial relation '$d_{j+1}s_j(z) = 1$' need not hold.

The morphisms $\delta^k_{\underline{w}}$ are not needed for the development of logarithms, but, when present, they enable more precision in the statement of some logarithm properties.

2.0.1 Example: *The free monoidal groupoid and additive categories*

One may assign a canonical MPR to any category $\mathbf{C}$ endowed with a functor $ß : \mathbf{C} \to \mathbf{A}$ to an additive category $\mathbf{A} = (\mathbf{A}, \oplus)$. First, associated to $\mathbf{C}$ one has an associated free monoidal groupoid $\Sigma(\mathbf{C})$ in which an object $\underline{x}$ of is an n-tuple $\underline{x} = (x_1, \dots, x_n) \in \mathrm{ob}(\mathbf{C}^n)$ of objects x_j of $\mathbf{C}$ and a morphism $\underline{\alpha} \in \mathrm{mor}_{\Sigma(\mathbf{C})}(\underline{x}, \underline{y})$ is an n-tuple $\underline{\alpha} = (\alpha_1, \dots, \alpha_n)$ of morphisms $\alpha_j \in \mathrm{mor}_{\mathbf{C}}(x_j, y_j)$. The monoidal structure $\otimes$ is on $\Sigma(\mathbf{C})$ is concatenation: for $\underline{x} = (x_1, \dots, x_n) \in \mathrm{ob}(\mathbf{C}^n)$ and $\underline{x}' = (x'_1, \dots, x'_m) \in \mathrm{ob}(\mathbf{C}^m)$, $\underline{x} \otimes \underline{x}' = (x_1, \dots, x_n, x'_1, \dots, x'_m) \in \mathrm{ob}(\mathbf{C}^{n+m})$ and likewise for morphisms (if $\mathbf{C}$ is symmetric monoidal it can be augmented by morphisms $\mathbf{C}^n \to \mathbf{C}^m$ for any n, m). Then there is the MPR $\mathsf{F}_ß : \Sigma(\mathbf{C}) \to \mathbf{A}$ which assigns to $\underline{x} \in \mathrm{ob}(\Sigma(\mathbf{C}))$ the ring

$$\mathsf{F}_ß(\underline{x}) := \mathrm{End}_{\mathbf{A}}(ß(x_1) \oplus ß(x_2) \oplus \cdots \oplus ß(x_n)) \tag{2.21}$$

whose elements are $n \times n$ matrices of morphisms of $\mathbf{A}$ with ij^{th} entry in $\mathrm{mor}_{\mathbf{A}}(ß(x_j), ß(x_i))$, and the sub representation which assigns the diagonal subring

$$\mathsf{F}_{ß,\mathsf{diag}}(\underline{x}) := \bigoplus_{j=1}^{n} \mathrm{mor}_{\mathbf{A}}(ß(x_j), ß(x_j)). \tag{2.22}$$

The covering maps $\eta_{\otimes y}$ are the canonical inclusions.

2.1 Tracial monoidal product representations

On a category of rings $\mathbf{R}$ one has the quotient functor $\Pi : \mathbf{R} \to \mathbf{R}/[\mathbf{R}, \mathbf{R}] \subset$ **Abelian**, to the category of abelian groups, already used for logarithms on monoids in §1, mapping $(R, \cdot, +) \in \mathrm{ob}(\mathbf{R}) \mapsto (R, +)/[R, R]$.

Definition 2.6 *A monoidal product representation F of a symmetric monoidal category $\boldsymbol{C}$ is said to be pretracial with respect to a background additive category $\boldsymbol{A}$ if the functor F ranges in the category of rings*

$$\mathsf{F} : \boldsymbol{C}^* \to \boldsymbol{Ring}$$

such that for each $x \in \mathrm{ob}(\boldsymbol{C})$

$$F(x) = \mathrm{end}_{\boldsymbol{A}}(\xi_x)$$

for some unique $\xi_x \in \mathrm{ob}(\boldsymbol{A})$*, and if the insertion morphisms (degeneracy maps)* $\eta_{\otimes y}(x)$ *of* (2.6) *are ring homomorphisms and the* $\mu_\sigma(x)$ *of* (2.2) *with* $x = x_1 \otimes \cdots \otimes x_n$ *are ring isomorphisms. We may indicate this by* $\mathsf{F} : \boldsymbol{C}^* \to \boldsymbol{Ring}_{\mathbf{Add}}$.

F *is said to be injective if the abelian group homomorphisms* $\delta_{\otimes y}(x)$ *of* (2.11) *preserve commutators:* $\delta_{\otimes y}(x)([\mathsf{F}(x \otimes y), \mathsf{F}(x \otimes y)]) \subset [\mathsf{F}(x), \mathsf{F}(x)]$.

Here, the ring product in $\mathrm{end}_{\mathbf{A}}(\xi_x)$ is defined by composition of morphisms and the abelian group product by the additive structure on $\mathbf{A}$.

Lemma 2.7 *Let* F *be pretracial and let* $\mathsf{F}(\boldsymbol{C}^*)$ *be the subcategory of* $\boldsymbol{Ring}_{\mathbf{Add}}$ *with objects* $\mathsf{F}(x)$ *for* $x \in \mathrm{ob}(\boldsymbol{C})$*. By composing with the quotient functor,* F *pushes-down to an induced monoidal product representation*

$$\mathsf{F}_\Pi : \boldsymbol{C}^* \to \mathsf{F}(\boldsymbol{C}^*)/[\mathsf{F}(\boldsymbol{C}^*), \mathsf{F}(\boldsymbol{C}^*)], \quad x \mapsto \mathsf{F}(x)/[\mathsf{F}(x), \mathsf{F}(x)]. \tag{2.23}$$

Proof: Since F is pretracial $\eta_{\underline{w}} : \mathsf{F}(x) \to \mathsf{F}(x_{\underline{w}})$ is a ring homomorphism, taking commutators to commutators. As such, it pushes-down to a homomorphism of abelian groups

$$\widetilde{\eta}_{\underline{w}} : \mathsf{F}(x)/[\mathsf{F}(x), \mathsf{F}(x)] \to \mathsf{F}(x_{\underline{w}})/[\mathsf{F}(x_{\underline{w}}), \mathsf{F}(x_{\underline{w}})], \quad \widetilde{\eta}_{\underline{w}}([\xi]) := \pi_x \circ \eta_{\underline{w}}(\xi), \tag{2.24}$$

with $\pi_x : \mathsf{F}(x) \to \mathsf{F}(x)/[\mathsf{F}(x), \mathsf{F}(x)]$ the quotient map, defining the insertion maps of a monoidal product representation. Since (2.16) persists to the quotient,

$$(\mathsf{F}(\mathbf{C}^*)/[\mathsf{F}(\mathbf{C}^*), \mathsf{F}(\mathbf{C}^*)], \widetilde{\eta}_z^j)$$

inherits the structure of a presimplicial set, while if F is injective then it inherits the structure of a simplicial set from $\mathsf{F}(\mathbf{C}^*)$. □

A monoidal category $\mathbf{E}$ has a trace τ if there exist objects $x \in \mathrm{ob}(\mathbf{E})$ with a non-empty closed subclass $\mathrm{end}_{\mathbf{E}}^\tau(x)$ of endomorphisms and a map

$$\tau_x : \mathrm{end}_{\mathbf{E}}^\tau(x) \to \mathrm{end}_{\mathbf{E}}(1)$$

with the trace property that for $\alpha \in \mathrm{mor}_{\mathbf{E}}(x, y)$ and $\beta \in \mathrm{mor}_{\mathbf{E}}(y, x)$ with $\beta \circ \alpha \in \mathrm{end}_{\mathbf{E}}^\tau(x)$ and $\alpha \circ \beta \in \mathrm{end}_{\mathbf{E}}^\tau(y)$ one has $\tau_x(b \circ \alpha) = \tau_y(a \circ \beta) \in \mathrm{end}_{\mathbf{E}}(1)$. An element $\delta \in \mathrm{end}_{\mathbf{E}}^\tau(x)$ is called τ-trace class and τ a categorical trace. For example, in $\mathbf{Bord}_n$ all bordisms are trace class for the trace sending $W \in \mathrm{end}(M)$ to the closed manifold formed by gluing the two boundary portions $\overline{M}$ and M of W via the diffeomorphism $\partial W \stackrel{\cong}{\to} \overline{M} \sqcup M$, see [11], [18]. On the other hand, for the classical trace Tr on the category of Hilbert spaces only preferred sub ideals of bounded operators are trace class. Nevertheless, the τ superscript in $\mathrm{end}_{\mathbf{E}}^\tau(x)$ will be omitted with the understanding that, where necessary, statements are meant for trace class morphisms.

Definition 2.8 *A pre-tracial monoidal product representation* $\mathsf{F} : \boldsymbol{C}^* \to \boldsymbol{Ring}_{\mathbf{Add}}$ *is said to be a tracial monoidal product representation of* $\boldsymbol{C}$ *if* $\boldsymbol{A}$ *has an* F*-compatible trace* τ*.*

F-compatible means that τ assigns to each $x \in \mathrm{ob}(\boldsymbol{C})$ a trace $\tau_x : \mathsf{F}(x) = \mathrm{end}_{\boldsymbol{A}}(\xi_x) \to \mathrm{end}_{\boldsymbol{A}}(1_{\boldsymbol{A}})$ satisfying the compatibility requirement that for all $x, y \in \mathrm{ob}(\boldsymbol{C})$

$$\tau_{x\otimes y} \circ \eta_{\otimes y}(x) = \tau_x \quad \textit{and} \quad \tau_{x_\sigma} \circ \mu_\sigma(x) = \tau_x. \tag{2.25}$$

Characters in a tracial monoidal product representation can be computed 'anywhere':

Lemma 2.9 *For a tracial monoidal product representation one has*

$$\tau_x = \tau_{x_{\underline{w}}} \circ \eta_{\underline{w}}. \tag{2.26}$$

Proof: Replacing $\tau_{x\otimes z}$ by $\tau_{x\otimes w\otimes z} \circ \eta_w$ defines another trace on on $\mathsf{F}(x \otimes z)$, but

$$\tau_{x\otimes w\otimes z} \circ \eta_w \overset{(2.10)}{=} \tau_{x\otimes w\otimes z} \circ \mu_\sigma(x \otimes z \otimes w) \circ \eta_{\otimes w}(x \otimes z) \overset{(2.25)}{=} \tau_{x\otimes z\otimes w} \circ \eta_{\otimes w}(x \otimes z) \overset{(2.25)}{=} \tau_{x\otimes z}.$$

Then (2.26) follows by iteration. □

Each of the above structures pushes-down to the quotient monoidal product representation F_Π (noted in (2.24) for the insertion maps) while for the trace τ one has for each object $x \in \mathrm{ob}(\mathbf{C})$ a commutative diagram

$$\begin{array}{ccc} \mathsf{F}(x) & \xrightarrow{\tau_x} & \mathrm{end}_{\mathbf{E}}(1) \\ \downarrow \pi_x & \overset{\tilde{\tau}_x}{\nearrow} & \\ \frac{\mathsf{F}(x)}{[\mathsf{F}(x),\mathsf{F}(x)]} & & \end{array} .$$

From this view point, π_x is a 'universal trace' on $\mathsf{F}(x)$ insofar as any trace factors uniquely through it: one has $\tau_x = \widetilde{\tau}_x \circ \pi_x$ and $\widetilde{\tau}_x = \widetilde{\tau}_{x_{\underline{w}}} \circ \widetilde{\eta}_{\underline{w}}$, with the second identity consequent on (2.26). Matters may be summarised as the commutativity of the diagram

$$\begin{array}{ccccc} \mathsf{F}(x) & & \xrightarrow{\eta_{\underline{w}}} & & \mathsf{F}(x_{\underline{w}}) \\ & \overset{\tau_x}{\searrow} & & \overset{\tau_{x_{\underline{w}}}}{\swarrow} & \\ \downarrow \pi_x & & \mathbb{C} & & \downarrow \pi_{x_{\underline{w}}} \\ & \overset{\tilde{\tau}_x}{\nearrow} & & \overset{\tilde{\tau}_{x_{\underline{w}}}}{\nwarrow} & \\ \frac{\mathsf{F}(x)}{[\mathsf{F}(x),\mathsf{F}(x)]} & & \xrightarrow{\tilde{\eta}_{\underline{w}}} & & \frac{\mathsf{F}(x_{\underline{w}})}{[\mathsf{F}(x_{\underline{w}}),\mathsf{F}(x_{\underline{w}})]} \end{array} . \tag{2.27}$$

In particular, (repeating (2.24)) $\pi_{x_{\underline{w}}} \circ \eta_{\underline{w}} = \widetilde{\eta}_{x_{\underline{w}}} \circ \pi_x$.

2.1.1 Example: *The free monoidal groupoid and additive categories*

Let $(\mathbf{A}, \tau)$ be an additive tracial category. To a category $\mathbf{C}$ endowed with a functor $ß : \mathbf{C} \to \mathbf{A}$ one has the MPR of Example 2.0.1 assigning to $\underline{x} \in \mathrm{ob}(\Sigma(\mathbf{C}))$ the ring $\mathsf{F}_ß(\underline{x}) := \mathrm{End}_{\mathbf{A}}(ß(x_1) \oplus ß(x_2) \oplus \cdots \oplus ß(x_n))$. This becomes a tracial MPR with the induced trace

$$\tau_{\underline{x}} : \mathsf{F}_ß(\underline{x}) \to \mathrm{end}_{\mathbf{A}}(1), \qquad \tau_{\underline{x}}((t_{ij}) = \sum_i \tau(t_{ii}).$$

2.1.2 Example: $\mathbf{Bord}_n$

There are the following two fundamental tracial MPRs on the bordism category which are often used:

Let $\mathsf{F} : \mathbf{Bord}_n^* \to \mathbf{Ring}_{\mathbf{Add}}$ be an unoriented pretracial monoidal product representation – *unoriented* is the assumption that $\mathsf{F}(M^{(-)}) = \mathsf{F}(M)$, where $M^{(-)}$ denotes M with one or more of its connected components with orientation reverse.

Let $\mathbf{Bord}_n^*$ be the subcategory of $\mathbf{Bord}_n$ whose morphisms are the coherence and permutation bordisms. Define a monoidal product representation $\mathsf{F}_{-\infty} : \mathbf{Bord}_n^* \to \mathbf{Alg}_{\mathsf{F}}$ by setting $\mathsf{F}_{-\infty}(M) := \Psi^{-\infty}(M) := \Psi^{-\infty}(M, \wedge T^*M)$ to be the algebra of smoothing operators on the de Rham complex $\Omega(M)$ with the coherence bordisms of the monoidal product $\sqcup$ mapped to the identity operator. An element $T \in \mathsf{F}_{-\infty}(M)$ is specified by a Schwartz kernel

$$k_{M} \in C^\infty(M \times M, ((\wedge T^*M)^* \otimes |\Lambda|_M^{\frac{1}{2}}) \boxtimes (\wedge T^*M \otimes |\Lambda|_M^{\frac{1}{2}})) \tag{2.28}$$

taking values in form valued half-densities

If M is disconnected and is written as a disjoint union $M = M_1 \sqcup \cdots \sqcup M_m$ of $M_j \in \mathrm{ob}(\mathbf{Bord}_n)$, then $\Omega(M) = \Omega(M_1) \oplus \cdots \oplus \Omega(M_m)$ with respect to which $T \in \mathsf{F}_{-\infty}(M)$ is an $n \times n$ block matrix $(T_{i,j})$ of smoothing operators $T_{i,j} \in \Psi^{-\infty}(M_j, M_i)$ specified by Schwartz kernels

$$k_{i,j} \in C^\infty(M_i \times M_j, ((\wedge T^*M_i)^* \otimes |\Lambda|_{M_i}^{\frac{1}{2}}) \boxtimes (\wedge T^*M_j \otimes |\Lambda|_{M_j}^{\frac{1}{2}})) \tag{2.29}$$

whose rows and columns are permuted by $\mu_\sigma(M)$ relative to a reordering σ of the M_j.

With $i : M := M_1 \sqcup \cdots \sqcup M_m \hookrightarrow M_N := M_1 \sqcup \cdots \sqcup N \sqcup \cdots \sqcup M_m$, the insertion maps are the canonical inclusions

$$\eta_N : \mathsf{F}_{-\infty}(M) \hookrightarrow \mathsf{F}_{-\infty}(M_N), \quad \eta_N(T) = i_N \circ T \circ i_N^*. \tag{2.30}$$

$\mathsf{F}_{-\infty}$ is pretracial, though not injective, and we may form the pushed-down insertion maps

$$\widetilde{\eta}_N = \widetilde{\eta}_N(M) : \frac{\mathsf{F}_{-\infty}(M)}{[\mathsf{F}_{-\infty}(M), \mathsf{F}_{-\infty}(M)]} \to \frac{\mathsf{F}_{-\infty}(M_N)}{[\mathsf{F}_{-\infty}(M_N), \mathsf{F}_{-\infty}(M_N)]}. \tag{2.31}$$

Lemma 2.10 *The linear map*

$$\mathrm{Tr}_{M} : \mathsf{F}_{-\infty}(M) \to \mathbb{C}, \quad \mathrm{Tr}_{M}(T) := \sum_{j=1}^m \mathrm{Tr}_{M_j}(T_{j,j}) := \sum_{j=1}^m \int_{M_j} \mathrm{tr}\,(k_{j,j}(m,m)), \tag{2.32}$$

is a trace and, up to a multiplication by a constant, is the unique trace on $\mathsf{F}_{-\infty}(\mathbf{Bord}_n^*)$. The quotients $\frac{\mathsf{F}_{-\infty}(M)}{[\mathsf{F}_{-\infty}(M), \mathsf{F}_{-\infty}(M)]}$ are complex lines and the trace defines and is defined by a linear isomorphism

$$\widetilde{\mathrm{Tr}}_{M} : \frac{\mathsf{F}_{-\infty}(M)}{[\mathsf{F}_{-\infty}(M), \mathsf{F}_{-\infty}(M)]} \stackrel{\cong}{\to} \mathbb{C} \tag{2.33}$$

with

$$\mathrm{Tr}_M = \widetilde{\mathrm{Tr}}_M \circ \pi_M. \tag{2.34}$$

One has

$$\mathrm{Tr}_M = \mathrm{Tr}_{M_N} \circ \eta_N \quad \text{on } \mathsf{F}_{-\infty}(M), \tag{2.35}$$

$$\widetilde{\mathrm{Tr}}_M = \widetilde{\mathrm{Tr}}_{M_N} \circ \widetilde{\eta}_N \quad \text{on } \mathsf{F}_{-\infty}(M)/[\mathsf{F}_{-\infty}(M), \mathsf{F}_{-\infty}(M)]. \tag{2.36}$$

We omit the straightforward proof.

The pushed-down insertion map $\widetilde{\eta}_N(M)$ in (2.31) is hence a linear isomorphism of complex lines, and fits into the commutative diagram (2.27) which, here, is

$$\begin{array}{ccccc}
\mathsf{F}_{-\infty}(M) & & \xrightarrow{\eta_N(M)} & & \mathsf{F}_{-\infty}(M_N) \\
 & \mathrm{Tr}_M \searrow & & \swarrow \mathrm{Tr}_{M_N} & \\
\downarrow \pi_M & & \mathbb{C} & & \downarrow \pi_{M_N} \\
 & \widetilde{\mathrm{Tr}}_M \nearrow & & \nwarrow \widetilde{\mathrm{Tr}}_{M_N} & \\
\frac{\mathsf{F}_{-\infty}(M)}{[\mathsf{F}_{-\infty}(M), \mathsf{F}_{-\infty}(M)]} & & \xrightarrow{\widetilde{\eta}_N(M) \cong} & & \frac{\mathsf{F}_{-\infty}(M_N)}{[\mathsf{F}_{-\infty}(M_N), \mathsf{F}_{-\infty}(M_N)]}
\end{array}, \tag{2.37}$$

and one has $\widetilde{\eta}_N(M) = \widetilde{\mathrm{Tr}}_{M_N}^{-1} \circ \widetilde{\mathrm{Tr}}_M$. Likewise, by the isomorphism (2.33), $\pi_M(A)$ may be characterised as *the abstract trace* of $A \in \mathsf{F}_{-\infty}(M)$, one has $\pi_M = \widetilde{\mathrm{Tr}}_M^{-1} \circ \mathrm{Tr}_M$.

The classical trace hence refines $\mathsf{F}_{-\infty}$ to a tracial monoidal product representation $(\mathsf{F}_{-\infty}, \mathrm{Tr})$. There is, on the other hand, the 'larger' monidal product representation

$$\mathsf{F}_{\mathbb{Z},-\infty} : \mathbf{Bord}_n^* \to \mathbf{Alg}_{\mathsf{F}}, \quad M \mapsto \mathsf{F}_{\mathbb{Z},-\infty}(M) \tag{2.38}$$

with $\mathsf{F}_{\mathbb{Z},-\infty}(M)$ the algebra of continuous operators on $\Omega(M)$ defined by Schwartz kernels which are smoothing off the 'matrix diagonal' and pseudodifferential along it: let $M_1, \ldots, M_m$ be the connected components of M and let $k_{i,j}$ be the restriction to $M_i \times M_j$ of the distributional kernel of $T \in \mathsf{F}_{\mathbb{Z},-\infty}(M)$. Then $k_{i,j}$ is required to be a smoothing kernel (2.29) if $i \neq j$, while if $i = j$ it may, more generally, be an integer order pseudodifferential operator (ψdo) kernel $k_{j,j} \in \mathcal{D}'(M_j \times M_j, ((\wedge T^* M_j)^* \otimes |\Lambda|^{\frac{1}{2}}_{M_j}) \boxtimes (\wedge T^* M_j \otimes |\Lambda|^{\frac{1}{2}}_{M_j}))$ in the space of conormal distributions on form-valued half-densities. Thus, there is an atlas of $M_j \times M_j$ in which $k_{j,j}$ can be written in each localisation as an oscillatory integral

$$k_{j,j}(x,y) = \int_{\mathbb{R}^n} e^{i\xi.(x-y)}\, \mathbf{b}^{[j]}(x,y,\xi)\; đ\xi\, |dx|^{\frac{1}{2}} |dy|^{\frac{1}{2}} \tag{2.39}$$

of a symbol (amplitude) $\mathsf{b}^{[j]}(x,y,\xi)$ of order $p_j \in \mathbb{Z} \cup \{-\infty\}$ (depending on the trivialisation). $\mathsf{F}_{\mathbb{Z},-\infty}(M)$ is filtered by the subspaces $\mathsf{F}_{p,-\infty}(M) = \Psi^{p,-\infty}(M)$ of operators with classical ψdos on the diagonal up to order $p \in \mathbb{Z}$. If $M = M_1 \sqcup \cdots \sqcup M_m$ then $\mathsf{F}_{\mathbb{Z},-\infty}(M)$ is identified with the matrix algebra $(T_{i,j})$ of operators $T_{i,j}$ with smoothing kernels off the matrix diagonal and with integer order ψdo oscillatory kernel (2.39) if $i = j$.

$\mathsf{F}_{\mathbb{Z},-\infty}$ is pretracial with quotient functor $\rho_M : \mathsf{F}_{\mathbb{Z},-\infty}(M) \to \mathsf{F}_{\mathbb{Z},-\infty}(M)/[\mathsf{F}_{\mathbb{Z},-\infty}(M), \mathsf{F}_{-\infty}(M)]$. It has a trace structure complementary to the classical trace and not quite unique:

Lemma 2.11 *Let M_j be the connected components of M. Then the linear space of traces on $\mathsf{F}_{\mathbb{Z},-\infty}(M)$ has (complex) dimension m: on $\mathsf{F}_{\mathbb{Z},-\infty}(M)$ each $\mathbf{c} = (c_1, \ldots, c_m) \in \mathbb{C}^m$ parametrises the linear sum of residue traces*

$$res_M^{\mathbf{c}}(B) = \sum_{j=1}^m c_j res_{M_j}(B_{jj}) := \sum_{j=1}^m c_j \int_{S^*M_j} b_{-n}^{[j]}(x,\eta)\, đ_S\eta\, |dx|. \tag{2.40}$$

Each such trace defines and is defined by a linear homomorphism

$$\widetilde{res}_M^{\mathbf{c}} : \frac{\mathsf{F}_{\mathbb{Z},-\infty}(M)}{[\mathsf{F}_{\mathbb{Z},-\infty}(M), \mathsf{F}_{\mathbb{Z},-\infty}(M)]} \stackrel{\cong}{\to} \mathbb{C} \quad \textit{with} \quad res_M^{\mathbf{c}} = \widetilde{res}_M^{\mathbf{c}} \circ \rho_M. \tag{2.41}$$

Each fixed choice of $\mathbf{c}$ (for all objects M) defines a tracial monoidal product representations $(\mathsf{F}_{\mathbb{Z},-\infty}, res^{\mathbf{c}})$.

These structures behave well with respect to diffeomorphisms:

Lemma 2.12 *Let $\mathsf{F} : \boldsymbol{Bord}_n^* \to \boldsymbol{Alg}_{\mathsf{F}}$, $M \mapsto (\mathsf{F}(M), \tau_M)$, be either one of the tracial monoidal product representations $(\mathsf{F}_{-\infty}, \mathrm{Tr})$ or $(\mathsf{F}_{\mathbb{Z},-\infty}, res^{\mathbf{c}})$. Let $M^{(-)}$ be M with one or more of its connected components with orientation reversed. Then $\mathsf{F}\big(M^{(-)}\big) = \mathsf{F}(M)$. A diffeomorphism $\phi : M \to N$ between $M, N \in \mathrm{ob}(\boldsymbol{Bord}_n)$ induces a canonical continuous isomorphism of algebras $\phi_\sharp : \mathsf{F}(M) \to \mathsf{F}(N)$, preserving the filtration by ΨDO order, and pushes-down to a continuous linear map*

$$\widetilde{\phi}_{M,N} : \mathsf{F}(M)/[\mathsf{F}(M), \mathsf{F}(M)] \to \mathsf{F}(N)/[\mathsf{F}(N), \mathsf{F}(N)].$$

Trace invariance: there is a commutative diagram

$$\begin{array}{ccccc}
\mathsf{F}(M) & & \stackrel{\phi_\sharp}{\longrightarrow} & & \mathsf{F}(N) \\
 & \stackrel{\tau_M}{\searrow} & & \stackrel{\tau_N}{\swarrow} & \\
\downarrow \pi_M & & \mathbb{C} & & \downarrow \pi_N \\
 & \stackrel{\tilde{\tau}_M}{\nearrow} & & \stackrel{\tilde{\tau}_N}{\nwarrow} & \\
\frac{\mathsf{F}(M)}{[\mathsf{F}(M),\mathsf{F}(M)]} & & \stackrel{\widetilde{\phi}_\sharp}{\longrightarrow} & & \frac{\mathsf{F}(N)}{[\mathsf{F}(N),\mathsf{F}(N)]}
\end{array} . \tag{2.42}$$

For $(\mathsf{F}_{-\infty}, \mathrm{Tr})$ the map $\widetilde{\phi}_\sharp$ is independent of the choice of ϕ: if M and N are diffeomorphic there is a canonical linear isomorphism of complex lines:

$$\vartheta_{M,N} : \frac{\mathsf{F}_{-\infty}(M)}{[\mathsf{F}_{-\infty}(M), \mathsf{F}_{-\infty}(M)]} \to \frac{\mathsf{F}_{-\infty}(N)}{[\mathsf{F}_{-\infty}(N), \mathsf{F}_{-\infty}(N)]}. \tag{2.43}$$

This is readily checked: the diffeomorphism ϕ induces a continuous linear pull-back isomorphism $\phi_* : \Omega(N) \stackrel{\cong}{\to} \Omega(M)$, with respect to which $\phi_\sharp(T) := \phi_*^{-1} \circ T \circ \phi_*$ is an algebra isomorphism defining an abelian group isomorphism $[\mathsf{F}(M), \mathsf{F}(M)] \stackrel{\cong}{\to} [\mathsf{F}(N), \mathsf{F}(N)]$. which with (2.12) gives (2.43). For the diagram, one uses the universality property of traces and Lidskii's theorem.

The classical and residue traces are, of course, complimentary but not unrelated. Using ζ-function regulation one extends (2.32) to a quasi-trace

$$\mathrm{Tr}^{\zeta}_{M} : \mathsf{F}_{\mathbb{Z},-\infty}(M) \to \mathbb{C}, \quad \mathrm{Tr}^{\zeta}_{M}(T) := \sum_{j=1}^{m} \mathrm{Tr}^{\zeta}_{M_j}(T_{j,j}) \tag{2.44}$$

with

$$\mathrm{Tr}^{\zeta}_{M_j}(T_{j,j}) := \mathrm{Tr}_{M_j}(T_{j,j} Q_j^{-z}) ||^{\mathrm{mer}}_{z=0} \tag{2.45}$$

relative to a choice of strongly elliptic regularising operators Q_j of strictly positive order on M_j, which for simplicity we shall take to be of Laplace-type (2nd order partial differential with leading symbol equal to that of a metric Laplacian), so that Q_j^{-z} is canonically defined. If $T \in \mathsf{F}_{\mathbb{Z},-\infty}(M)$ then $\mathrm{Tr}^{\zeta}_{M}(T)$ coincides with (2.32), but in general it does not vanish on commutators. Near $z = 0$ the meromorphically extended trace has the form

$$\mathrm{Tr}_{M_j}(T_{j,j} Q_j^{-z}) \;=\; \frac{2\mathrm{res}_{M_j}(T_{j,j})}{z} \;+\; \mathrm{Tr}^{\zeta}_{M_j}(T_{j,j})\, z^0 + \cdots$$

giving the relation between the three functionals $\mathrm{Tr}_M, \mathrm{Tr}^{\zeta}_M$ and res_{M_j}; the non-traciality of Tr^{ζ}_M means it does not push-down to the quotient $\mathsf{F}_{\mathbb{Z},-\infty}/[\mathsf{F}_{\mathbb{Z},-\infty}, \mathsf{F}_{\mathbb{Z},-\infty}]$, but nevertheless the numbers $\mathrm{Tr}^{\zeta}_M(T)$ generally provide an array of interesting invariants.

Finally, notice that $\mathrm{res}^{\mathbf{c}}_M$ for the general case $M = M_1 \sqcup \cdots \sqcup M_m$ is likewise a complex residue to a classical trace, but to the zeta extended *diagonal* classical trace on the diagonal subalgebra $\oplus_{j=1}^m \mathsf{F}_{\mathbb{Z},-\infty}(M_j)$ of $\mathsf{F}_{\mathbb{Z},-\infty}(M)$.

2.2 Logarithmic functors

The nerve $\mathcal{N}\mathbf{C}$ of a category $\mathbf{C}$ is the simplicial set whose p-simplices are diagrams

$$x_0 \stackrel{\alpha_0}{\to} x_1 \stackrel{\alpha_1}{\to} x_2 \to \cdots \to x_{p-1} \stackrel{\alpha_{p-1}}{\to} x_p \quad \in \mathcal{N}_p\mathbf{C} \tag{2.46}$$

of morphisms $\alpha_j \in \mathrm{mor}(x_j, x_{j+1})$. The j^{th} face map $d_j : \mathcal{N}_p\mathbf{C} \to \mathcal{N}_{p-1}\mathbf{C}$ of the simplex deletes x_j, replacing when $0 < j < p$

$$\cdots \to x_{j-1} \stackrel{\alpha_{j-1}}{\to} x_j \stackrel{\alpha_j}{\to} x_{j+1} \to \cdots \quad \text{by} \quad \cdots \to x_{j-1} \stackrel{\alpha_j \circ \alpha_{j-1}}{\longrightarrow} x_{j+1} \to \cdots$$

and the j^{th} degeneracy map $s_j : \mathcal{N}_p\mathbf{C} \to \mathcal{N}_{p+1}\mathbf{C}$ replaces

$$\cdots \to x_j \stackrel{\alpha_j}{\to} x_{j+1} \to \cdots \quad \text{by} \quad \cdots \to x_j \stackrel{\iota}{\to} x_j \stackrel{\alpha_j}{\to} x_{j+1} \to \cdots . \tag{2.47}$$

$\mathcal{N}\mathbf{C}$ carries more data than $\mathbf{C}$ — the objects and morphisms of $\mathbf{C}$ can be respectively identified with $\mathcal{N}_0\mathbf{C}$ and $\mathcal{N}_1\mathbf{C}$, while there is no right inverse to the composition face map $d_1 : \mathrm{mor}_{x_1}(x_0, x_2) \to \mathrm{mor}(x_0, x_2)$. The classifying space $B\mathbf{C}$ of $\mathbf{C}$ is the geometric realisation of $\mathcal{N}\mathbf{C}$.

Logarithms on a category $\mathbf{C}$ have to be differentiated between according to the substrata of marked morphisms in $\mathcal{N}_p\mathbf{C}$ on which they act. To this end, one has the stratum of $\underline{z} = (x_1, \ldots, x_{p-1})$-marked p-simplices (2.46) between $x, y \in \mathrm{ob}(\mathbf{C})$

$$\mathrm{mor}_{\underline{z}}(x, y) = \{x \overset{\alpha_0}{\to} x_1 \overset{\alpha_1}{\to} x_2 \to \cdots \to x_{p-1} \overset{\alpha_{p-1}}{\to} y\} \ \subset \ \mathcal{N}_p\mathbf{C}$$

$$:\cong \mathrm{mor}_{\mathbf{C}}(x, x_1) \times \mathrm{mor}_{\mathbf{C}}(x_1, x_2) \times \cdots \times \mathrm{mor}_{\mathbf{C}}(x_{p-1}, y).$$

If $\mathrm{mor}(x_j, x_{j+1}) = \emptyset$ some j then $\mathrm{mor}_{\underline{z}}(x, y) := \emptyset$, while $\mathrm{mor}_{\emptyset}(x, y) := \mathrm{mor}(x, y)$. One has the composition

$$\mathrm{mor}_{\underline{z}}(x, w) \times \mathrm{mor}_{\underline{z}'}(w, y) \overset{\circ}{\to} \mathrm{mor}_{\underline{z} \bullet w \bullet \underline{z}'}(x, y),$$

relative to concatenation $\bullet$, so $(x, z) \bullet y = (x, z, y)$ and so on, as a partially defined composition

$$\mathcal{N}_p\mathbf{C} \times \mathcal{N}_q\mathbf{C} \to \mathcal{N}_{p+q-1}\mathbf{C}$$

on compatible strata, while the face and degeneracy maps respectively restrict to simplicial maps

$$d_j : \mathrm{mor}_{\underline{z}}(x, y) \to \mathrm{mor}_{\delta_j(\underline{z})}(x, y), \qquad s_j : \mathrm{mor}_{\underline{z}}(x, y) \to \mathrm{mor}_{\sigma_j(\underline{z})}(x, y)$$

with $\delta_j : \mathbf{C}^p \to \mathbf{C}^{p-1}$ and $\sigma_j : \mathbf{C}^p \to \mathbf{C}^{p+1}$ defined in the evident way.

Recall that a simplicial map $f : X \to X'$ between simplicial sets $(X, d_j, s_j), (X', d'_j, s'_j)$ is given by maps $f_p : \Delta_p \to \Delta'_p$ between p-simplices which commute with the face and degeneracy maps, so that $f_{p-1} d_j = d'_j f_p$ and $f_p s_j = s'_j f_{p-1}$. Both these are implied by (but do not imply)

$$s'_j f_{p-1} d_j = f_p. \tag{2.48}$$

(2.48) is advantageous, here, insofar as it does not involve the boundary operators d'_j on X'. In the case where the range is only a presimplicial set (X', s'_j), so that $s'_l s'_k = s'_k s'_{l-1}$ for $k < l$, a map $f : (X, d_j, s_j) \to (X', s'_j)$ may be said to be *presimplicial* if (2.48) holds. (This applies equally when the domain is also only presimplicial (X, d_j).)

Definition 2.13 *Let $\boldsymbol{C} = (\boldsymbol{C}, \otimes)$ be a symmetric monoidal category and let*

$$\mathsf{F} : \boldsymbol{C}^* \to \mathbf{Ring}_{\mathbf{Add}}$$

be a (strict) pretracial monoidal product representation. Then a log-functor (or logarithmic-functor) on $\boldsymbol{C}$ taking values in F is a presimplicial log-additive map

$$\log : (\mathcal{N}\boldsymbol{C}, d_j, s_j) \to (\mathsf{F}(\boldsymbol{C}^*)/[\mathsf{F}(\boldsymbol{C}^*), \mathsf{F}(\boldsymbol{C}^*)], \widetilde{\eta}^j). \tag{2.49}$$

Such a structure is said to define a logarithmic representation of $\boldsymbol{C}$.

Unwrapping the definition, a log-functor comprises the following:

1. A (strict) pre-tracial monoidal product representation (on the set $\mathcal{N}_0\mathbf{C}$ of 0-simplices): $\mathsf{F}:\mathbf{C}^*\to\mathbf{Ring}_{\mathbf{Add}}$, and hence a quotient monoidal product representation

$$\mathbf{C}^*\to\mathsf{F}(\mathbf{C}^*)/[\mathsf{F}(\mathbf{C}^*),\mathsf{F}(\mathbf{C}^*)],\qquad z\in\mathrm{ob}(\mathbf{C})\mapsto\mathsf{F}(z)/[\mathsf{F}(z),\mathsf{F}(z)],$$

with insertion maps

$$\widetilde{\eta}_{\underline{w}}:\mathsf{F}(z)/[\mathsf{F}(z),\mathsf{F}(z)]\to\mathsf{F}(z_{\underline{w}})/[\mathsf{F}(z_{\underline{w}}),\mathsf{F}(z_{\underline{w}})].$$

2. A simplicial system of (strict) logarithm maps (on the set $\mathcal{N}_1\mathbf{C}$ of 1-simplices) assigning to $x,y\in\mathrm{ob}(\mathbf{C})$, with x,y not both the monoidal identity $1\in\mathrm{ob}(\mathbf{C})$, a map

$$\log_{x\otimes y}:\mathrm{mor}(x,y)\to\mathsf{F}(x\otimes y)/[\mathsf{F}(x\otimes y),\mathsf{F}(x\otimes y)],\tag{2.50}$$

$$\alpha\mapsto\log_{x\otimes y}\alpha=\log(x\stackrel{\alpha}{\to}y)$$

and, more generally, (on the set $\mathcal{N}_p\mathbf{C}$ of p-simplices) to each marking $\underline{z}=(z_1,\ldots,z_{p-1})$ a map

$$\log_{x\otimes\underline{z}\otimes y}:\mathrm{mor}_{\underline{z}}(x,y)\to\mathsf{F}(x\otimes\underline{z}\otimes y)/[\mathsf{F}(x\otimes\underline{z}\otimes y),\mathsf{F}(x\otimes\underline{z}\otimes y)]\tag{2.51}$$

where $x\otimes\underline{z}\otimes y:=x\otimes z_1\otimes\cdots\otimes z_{p-1}\otimes y\neq 1$,

$$\underline{\alpha}\mapsto\log_{x\otimes\underline{z}\otimes y}\underline{\alpha}:=\log_{x\otimes\underline{z}\otimes y}(x\stackrel{\alpha_0}{\to}z_1\stackrel{\alpha_1}{\to}z_2\to\cdots\to z_{p-1}\stackrel{\alpha_{p-1}}{\to}y),$$

such that for $x\stackrel{\alpha}{\to}z\stackrel{\beta}{\to}y\in\mathrm{mor}_z(x,y)$ associated to $\alpha\in\mathrm{mor}(x,z)$ and $\beta\in\mathrm{mor}(z,y)$ one has in

$$\mathsf{F}(x\otimes z\otimes y)/[\mathsf{F}(x\otimes z\otimes y),\mathsf{F}(x\otimes z\otimes y)]\tag{2.52}$$

the ($p=2$) log-additive property

$$\log_{x\otimes z\otimes y}(x\stackrel{\alpha}{\to}z\stackrel{\beta}{\to}y):=\widetilde{\eta}_{\otimes y}(\log_{x\otimes z}\alpha)+\widetilde{\eta}_{x\otimes}(\log_{z\otimes y}\beta),\tag{2.53}$$

or, equivalently,

$$\widetilde{\eta}_z(\log_{x\otimes y}(x\stackrel{\beta\circ\alpha}{\to}y))=\widetilde{\eta}_{\otimes y}(\log_{x\otimes z}\alpha)+\widetilde{\eta}_{x\otimes}(\log_{z\otimes y}\beta).\tag{2.54}$$

Notation: For brevity, in the left-hand side of (2.53) and (2.54) we write

$$\log_{x\otimes z\otimes y}\beta\alpha:=\log_{x\otimes z\otimes y}(x\stackrel{\alpha}{\to}z\stackrel{\beta}{\to}y),\qquad\log_{x\otimes y}\beta\alpha:=\log_{x\otimes y}(x\stackrel{\beta\circ\alpha}{\to}y).$$

In practise, (2.53) is generally obtained consequent on an equivalence

$$\log_{x\otimes z\otimes y}\beta\alpha=\eta_{\otimes y}(\log_{x\otimes z}\alpha)+\eta_{x\otimes}(\log_{z\otimes y}\beta)+\sum_{1\le j\le m}[\nu_j,\nu_j']$$

some $\nu_j, \nu'_j \in \mathsf{F}(x \otimes z \otimes y)$ and, likewise for (2.54). In this case, the presimpliciality of the log maps (2.50), (2.51) is for $p = 2$

$$\log_{x\otimes z\otimes y}(x \overset{\alpha}{\to} z \overset{\beta}{\to} y) - \eta_z \log_{x\otimes y}(x \overset{\beta\circ\alpha}{\to} y) \;\in\; [\mathsf{F}(x\otimes z\otimes y), \mathsf{F}(x\otimes z\otimes y)] \tag{2.55}$$

$$\log_{x\otimes x\otimes y}(x \overset{\alpha}{\to} x \overset{\beta}{\to} y) - \eta_{x\otimes} \log_{x\otimes y}(x \overset{\beta\circ\alpha}{\to} y) \;\in\; [\mathsf{F}(x\otimes x\otimes y), \mathsf{F}(x\otimes x\otimes y)] \tag{2.56}$$

$$\log_{x\otimes x\otimes y}(x \overset{\alpha}{\to} y \overset{\beta}{\to} y) - \eta_{\otimes y} \log_{x\otimes y}(x \overset{\beta\circ\alpha}{\to} y) \;\in\; [\mathsf{F}(x\otimes y\otimes y), \mathsf{F}(x\otimes y\otimes y)] \tag{2.57}$$

and, more generally, with $\underline{z} = (x_1, \ldots, x_{p-1})$, $\nu \in \mathrm{mor}_{\underline{z}}(x, y)$, $j \in \{1, \ldots, p-1\}$, that

$$\log_{\underline{z}} \nu - \eta_{x_j}(\log_{\delta_j(\underline{z})} d_j(\nu)) \;\in\; [\mathsf{F}(x\otimes \underline{z}\otimes y), \mathsf{F}(x\otimes \underline{z}\otimes y)] \tag{2.58}$$

plus the corresponding two end-point special cases ($x_0 = x, x_p = y$) generalising (2.56) and (2.57). These are the identities (2.48) for the presimplicial structures at hand.

Remark 2.14 [1] A log-functor is not in general a functor of categories, but is a functor of ∞-categories.

[2] Taking the geometric realization of (both sides of) (2.49) gives a 'logarithm' representation $|\log| : B\mathbf{C} \to |(\mathsf{F}(\mathbf{C}^*)/[\mathsf{F}(\mathbf{C}^*), \mathsf{F}(\mathbf{C}^*)]|$ of the (pre-) classifying space $B\mathbf{C}$ of the category $\mathbf{C}$.

The intertwining of the logarithm and the simplicial structures is clear when written as:

Lemma 2.15 *The log-additivity property* (2.54) *can be written*

$$\widetilde{\eta}_1 \log_{\delta_1(\underline{x})} \Big(d_1(x \overset{\alpha}{\to} z \overset{\beta}{\to} y)\Big) = \widetilde{\eta}_0 \log_{\delta_0(\underline{x})} \Big(d_0(x \overset{\alpha}{\to} z \overset{\beta}{\to} y)\Big) + \widetilde{\eta}_2 \log_{\delta_2(\underline{x})} \Big(d_2(x \overset{\alpha}{\to} z \overset{\beta}{\to} y)\Big).$$

where $\underline{x} = x \otimes y \otimes z$, $\eta_0 := \eta_{x\otimes}, \eta_1 := \eta_z, \eta_2 := \eta_{\otimes y}$, $x \overset{\alpha}{\to} z \overset{\beta}{\to} y \in \mathrm{mor}_z(x, y) \in \mathcal{N}_2\boldsymbol{C}$.

Here, the end-point face maps $d_0, d_p : \mathcal{N}_p\mathbf{C} \to \mathcal{N}_{p-1}\mathbf{C}$ are defined by deleting the 0^{th} or p^{th} morphism from a simplex; and the reason that (2.56), (2.57) are stated separately.

We note that a log-functor is effectively determined by its action on 1-simplices:

Lemma 2.16 *A simplicial system of logarithm maps* $\log_{x\otimes\underline{z}\otimes y}$ *is determined up to terms in* $[F, F]$ *by the log maps* $\log_{x\otimes y}$ *on* $\mathrm{mor}(x, y)$ *for each* $x, y \in \mathrm{ob}(\boldsymbol{C})$. *To define a compatible system of logarithm maps* $\log_{x\otimes\underline{z}\otimes y}$ *it is enough to define the* $\log_{x\otimes y}$ *on* $\mathrm{mor}(x, y)$ *satisfying* (2.54).

Proof: Compatibility (2.55) gives $\log_{x\otimes\underline{z}\otimes y} \delta = \widetilde{\eta}_{\underline{z}}(\log_{x\otimes y} \delta)$ in $\mathsf{F}(x\otimes\underline{z}\otimes y)/[\mathsf{F}(x\otimes\underline{z}\otimes y), \mathsf{F}(x\otimes\underline{z}\otimes y)]$ which is the first statement of the lemma. Given $\log_{x\otimes y}$, the second statement is that $\log_{x\otimes\underline{z}\otimes y} \delta := \widetilde{\eta}_{\underline{z}}(\log_{x\otimes y} \delta)$, defines by default a compatible system of logs (2.51).

□

Two p simplices which collapse to the same $(p - r)$ simplex have the same logarithm, and, likewise, inflating simplices does not change logarithms:

Lemma 2.17 *If* $d_1(x \overset{\alpha}{\to} z \overset{\beta}{\to} y) = d_1(x \overset{\alpha'}{\to} z \overset{\beta'}{\to} y)$ *(that is,* $\beta\alpha = \beta'\alpha'$*) in* $\mathrm{mor}(x,y)$ *then*

$$\log_{x\otimes z\otimes y} \beta\alpha = \log_{x\otimes z\otimes y} \beta'\alpha' \tag{2.59}$$

in $\mathsf{F}(x\otimes z\otimes y)/[\mathsf{F}(x\otimes z\otimes y), \mathsf{F}(x\otimes z\otimes y)]$*. More generally, if for* $\underline{z} = (x_1, \ldots, x_{p-1})$ *and* $\nu, \nu' \in \mathrm{mor}_{\underline{z}}(x,y)$ *and* $j = 1, \ldots, p-1$ *one has* $d_j(\nu) = d_j(\nu')$*, then*

$$\log_{\underline{z}} \nu = \log_{\underline{z}} \nu' \tag{2.60}$$

in $\mathsf{F}(x\otimes \underline{z}\otimes y)/[\mathsf{F}(x\otimes \underline{z}\otimes y), \mathsf{F}(x\otimes \underline{z}\otimes y)]$*. Iteratively, if* $d_k(d_j(\nu)) = d_k(d_j(\nu'))$ *then* (2.60) *continues to hold since*

$$\log_{\underline{z}} \nu = \widetilde{\eta}_{x_j}\widetilde{\eta}_{x_k} \log_{\delta_k(\delta_j(\underline{z}))} d_k(d_j(\nu)). \tag{2.61}$$

For $j < k$

$$\eta_{x_j}\eta_{x_k} \log_{\delta_k(\delta_j(\underline{z}))} d_k(d_j(\nu)) = \widetilde{\eta}_{x_{k+1}}\widetilde{\eta}_{x_j} \log_{\delta_j(\delta_{k-1}(\underline{z}))} d_j(d_{k-1}(\nu)). \tag{2.62}$$

Dually, for the degeneracy maps (2.48) *one has*

$$\log_{\sigma_j(\underline{z})} s_j(\nu) = \widetilde{\eta}^{j}_{x_j} \log_{\underline{z}} \nu \tag{2.63}$$

$$\log_{\sigma_k(\sigma_j(\underline{z}))} s_k(s_j(\nu)) = \widetilde{\eta}^{k}_{x_k}\eta^{j}_{x_j} \log_{\underline{z}} \nu \tag{2.64}$$

and a corresponding commutation formula to (2.62)*. For each of the above, the two end-point special cases corresponding to* (2.56) *and* (2.57) *also hold.*

Proof: By (2.55)

$$\log_{x\otimes z\otimes y}(x \overset{\alpha}{\to} z \overset{\beta}{\to} y) = \widetilde{\eta}_z \log_{x\otimes y}(x \overset{\beta\alpha}{\to} y) = \widetilde{\eta}_z \log_{x\otimes y}(x \overset{\beta'\alpha'}{\to} y) = \log_{x\otimes z\otimes y}(x \overset{\alpha'}{\to} z \overset{\beta'}{\to} y),$$

and in general $\log_{\underline{z}} \nu = \widetilde{\eta}_{x_j}(\log_{\delta_j(\underline{z})} d_j(\nu)) = \widetilde{\eta}_{x_j}(\log_{\delta_j(\underline{z})} d_j(\nu')) = \log_{\underline{z}} \nu$ by (2.58). The general version follows by iterating these equalities given that (2.61) holds, and that holds because the η_{x_l} are ring homomorphisms. (2.62) and its s_j counterpart are immediate from (2.5) and the simplicial identities $d^j d^k = d^k d^{j-1}$ and $s^j s^k = s^k s^{j+1}$ for $k < j$. The inflation formulae (2.63), (2.64) follow from (2.58) (resp. (2.62)) by replacing ν by $s_j(\nu)$ (resp. $s_k(s_j(\nu))$). The two end-point special cases of (2.60) hold from (2.56) and (2.57) by the same argument as the case $1 \le j \le p-1$, while for (2.63) this is shown in Proposition 2.18 (2.).

□

Log-functors transform naturally: if $\mathsf{J} : \mathbf{S} \to \mathbf{C}$ is a symmetric monoidal functor, then, since $\mathbf{C} \to \mathcal{N}\mathbf{C}$ is functorial, a logarithmic representation of $\mathbf{C}$ pulls-back to one of $\mathbf{S}$. Further basic properties of log-functors are listed in the following lemma:

Proposition 2.18 *1. Let* $p \in \mathrm{mor}_{\mathbf{C}}(x,x)$ *be a projection morphism:* $p \circ p = p$*. Then in* $\mathsf{F}(x\otimes x\otimes x)$

$$\eta_{x\otimes}(\log_{x\otimes x} p) \;\approx\; 0. \tag{2.65}$$

In particular, $\eta_{x\otimes}(\log_{x\otimes x}\iota) \approx 0$*, where* ι *is the identity morphism. If* F *is injective, in the sense of Definition 2.8, then in* $\mathsf{F}(x\otimes x)$

$$\log_{x\otimes x} p \;\approx\; 0. \tag{2.66}$$

2. For $\alpha \in \mathrm{mor}(x,y)$ *and identity morphisms* $\iota_x \in \mathrm{mor}(x,x)$*,* $\iota_y \in \mathrm{mor}(y,y)$

$$\log_{x\otimes y\otimes y}(\iota_y \circ \alpha) \;\approx\; \eta_{\otimes y}(\log_{x\otimes y}\alpha) \qquad \text{in } \mathsf{F}(x\otimes y\otimes y), \tag{2.67}$$

$$\log_{x\otimes x\otimes y}(\alpha \circ \iota_x) \;\approx\; \eta_{x\otimes}(\log_{x\otimes y}\alpha) \qquad \text{in } \mathsf{F}(x\otimes x\otimes y). \tag{2.68}$$

Notation: $\log_{x\otimes y\otimes y}(\iota_y\circ\alpha) := \log_{x\otimes y\otimes y}(x \xrightarrow{\alpha} y \xrightarrow{\iota_y} y)$.

3. For $\alpha,\beta \in \mathrm{mor}(x,x)$ *one has in* $\mathsf{F}(x\otimes x\otimes x)$

$$\eta_{\otimes x}\log_{x\otimes x}\beta\alpha \;\approx\; \eta_{\otimes x}\log_{x\otimes x}\alpha + \eta_{\otimes x}\log_{x\otimes x}\beta. \tag{2.69}$$

4. For $\alpha \in \mathrm{mor}(x,x)$ *and an isomorphism* $q \in \mathrm{mor}(w,x)$ *one has in* $\mathsf{F}(w\otimes x\otimes x\otimes w)$

$$\log_{w\otimes x\otimes x\otimes w}(q^{-1}\alpha q) \;\approx\; \eta_{w\otimes}\eta_{\otimes w}(\log_{x\otimes x}\alpha). \tag{2.70}$$

In the case $x = w$*, considering* $q^{-1}\alpha q \in \mathrm{mor}(x,x)$*, if* F *is injective then*

$$\log_{x\otimes x}(q^{-1}\alpha q) \;\approx\; \log_{x\otimes x}\alpha \tag{2.71}$$

in $\mathsf{F}(x\otimes x)$*. In either case, for a log-determinant structure one has in* $\mathrm{mor}_{\boldsymbol{A}}(1,1)$

$$\tau(\log q^{-1}\alpha q) = \tau(\log \alpha) \tag{2.72}$$

for any choice of representatives $\log_{x\otimes\underline{w}\otimes x} q^{-1}\alpha q$ *and* $\log_{x\otimes\underline{w}\otimes x}\alpha$ *of the logarithms.*

5. Let $\underline{w},\underline{w}' \in \mathrm{ob}(\Sigma(\boldsymbol{C}))$ *and let* $\alpha \in \mathrm{mor}_{\underline{w}}(x,z) \subset \mathcal{N}_p\boldsymbol{C}, \beta \in \mathrm{mor}_{\underline{w}'}(z,y) \subset \mathcal{N}_q\boldsymbol{C}$*. Then for a logarithmic representation one has in* $\mathsf{F}(x\otimes\underline{w}\otimes z\otimes\underline{w}'\otimes y)$

$$\log_{x\otimes\underline{w}\otimes z\otimes\underline{w}'\otimes y}(\beta\alpha) \;\approx\; \eta_{\underline{w}'\bullet y}(\log_{x\otimes\underline{w}\otimes z}\alpha) + \eta_{x\bullet\underline{w}}(\log_{z\otimes\underline{w}'\otimes y}\beta). \tag{2.73}$$

6. Let $\underline{w} = (w_1,\ldots,w_m) \in \mathrm{ob}(\Sigma(\boldsymbol{C}))$ *and let* $\alpha = \alpha_{m+1}\alpha_m\cdots\alpha_1 \in \mathrm{mor}_{\underline{w}}(x,y)$ *with* $\alpha_j : w_{j-1} \to w_j$ *and* $w_0 := x$*,* $w_{m+1} := y$*. Then*

$$\eta_{\underline{w}}\log_{x\otimes y}(\alpha_{m+1}\alpha_m\cdots\alpha_1) = \log_{x\otimes\underline{w}\otimes y}(\alpha_{m+1}\alpha_m\cdots\alpha_1) = \sum_{j=1}^{m+1}\eta_{j-1,j}\left(\log_{w_{j-1}\otimes w_j}\alpha_j\right)$$

in $\mathsf{F}(x\otimes\underline{w}\otimes y)$ *with* $\eta_{j-1,j} := \eta_{w_0}\circ\cdots\circ\eta_{w_{j-2}}\circ\eta_{w_{j+1}}\circ\cdots\circ\eta_{w_m}$*. In the case* $w_0 = w_1 = \cdots = w_{m+1} = x$ *and* F *is injective, this reduces in* $\mathsf{F}(x\otimes x)$ *to*

$$\log_{x\otimes x}(\alpha_{m+1}\alpha_m\cdots\alpha_1) \;\approx\; \sum_{j=1}^{m+1}\log_{x\otimes x}\alpha_j. \tag{2.74}$$

Proof: For 1. one has

$$\begin{aligned}
\log_{x\otimes x\otimes x}(x \xrightarrow{p} x \xrightarrow{p} x) &= \eta_{x\otimes}\log_{x\otimes x}(x \xrightarrow{p} x) + \eta_{\otimes x}\log_{x\otimes x}(x \xrightarrow{p} x)\\
&\overset{p\circ p = p}{=} \eta_{x\otimes}\log_{x\otimes x}(x \xrightarrow{p\circ p} x) + \eta_{\otimes x}\log_{x\otimes x}(x \xrightarrow{p\circ p} x)\\
&\overset{(2.56),(2.57)}{\approx} \log_{x\otimes x\otimes x}(x \xrightarrow{p} x \xrightarrow{p} x) + \log_{x\otimes x\otimes x}(x \xrightarrow{p} x \xrightarrow{p} x).
\end{aligned}$$

Hence $0 \approx \log_{x\otimes x\otimes x}(x \xrightarrow{p} x \xrightarrow{p} x) \overset{(2.56)}{\approx} \eta_{x\otimes}(\log_{x\otimes x} p \circ p) = \eta_{x\otimes}(\log_{x\otimes x} p)$. The other statements follow similarly.

□

If the pretracial monoidal product representation $\mathsf{F} : \mathbf{C}^* \to \mathbf{Ring_{Add}}$ is endowed with a trace τ then the τ-character of the log-functor defines a *log-determinant functor representation* of $\mathbf{C}$, mapping each $w \in \mathrm{ob}(\mathbf{C})$ to $\mathrm{end}_{\mathbf{A}}(1)$ and $\alpha \in \mathrm{mor}_{\underline{z}}(x, y)$ to the character

$$\widetilde{\tau}(\log\,\alpha) := \widetilde{\tau}_{x\otimes\underline{z}\otimes y}(\log_{x\otimes\underline{z}\otimes y}\alpha) \in \mathrm{end}_{\mathbf{A}}(1),$$

of $\log_{x\otimes\underline{z}\otimes y}\alpha \in \mathsf{F}(x\otimes\underline{z}\otimes y)/[\mathsf{F}(x\otimes\underline{z}\otimes y), \mathsf{F}(x\otimes\underline{z}\otimes y)]$. If $\varepsilon : \mathrm{mor}_{\mathbf{A}}(1,1) \to S$ is an exponential map (so $\varepsilon(\xi+\eta) = \varepsilon(\xi)\cdot\varepsilon(\eta)$) to a commutative ring S then one has a categorical determinant

$$\det{}_{\tau}\,\alpha := \varepsilon\left(\widetilde{\tau}_{x\otimes\underline{z}\otimes y}(\log_{x\otimes\underline{z}\otimes y}\alpha)\right) \in S. \tag{2.75}$$

We may write formally $\widetilde{\tau}(\log\,\alpha) = \log_{\varepsilon}\det{}_{\tau}\,\alpha$.

Lemma 2.19 *The character of $\alpha \in \mathrm{mor}_z(x, y) \in \mathcal{N}_p\boldsymbol{C}$ is invariantly defined: in $\mathrm{mor}_{\boldsymbol{A}}(1,1)$*

$$\widetilde{\tau}_{x\otimes z\otimes y}\left(\log_{x\otimes z\otimes y}\alpha\right) = \widetilde{\tau}_{x\otimes y}\left(\log_{x\otimes y}\alpha\right). \tag{2.76}$$

Likewise, for $\delta \in \mathrm{mor}(x, y)$ $\widetilde{\tau}_{x\otimes z\otimes y}\left(\eta_z(\log_{x\otimes y}\delta)\right) = \widetilde{\tau}_{x\otimes y}\left(\log_{x\otimes y}\delta\right)$, and more generally with $\underline{z} = (z_1, \ldots, z_r)$, $x = x_1 \otimes \cdots \otimes x_n$ one has

$$\widetilde{\tau}_{x_{\underline{z}}}\left(\log_{x_{\underline{z}}}\nu\right) = \widetilde{\tau}_x\left(\log_x \nu\right). \tag{2.77}$$

Proof: For $w \in \mathrm{ob}(\mathbf{C})$ one has $\log_{x_{\underline{w}}}(\nu) - \eta_{\underline{w}}(\log_{x_1\otimes\cdots\otimes x_n}\nu) \in [\mathsf{F}(x_{\underline{w}}), \mathsf{F}(x_{\underline{w}})]$ by (2.61) whilst $[\mathsf{F}(w), \mathsf{F}(w)] \subset \mathrm{Ker}\,(\tau_w)$. Hence (2.26) yields the conclusion.

□

Here, (2.76) is shorthand for $\widetilde{\tau}_{x\otimes z\otimes y}\left(\log_{x\otimes z\otimes y}(x \xrightarrow{\alpha} z \xrightarrow{\beta} y)\right) = \widetilde{\tau}_{x\otimes y}\left(\log_{x\otimes y}(x \xrightarrow{\beta\circ\alpha} y)\right)$, or $\widetilde{\tau}_{x\otimes z\otimes y}(\log_{x\otimes z\otimes y}\beta\alpha) = \widetilde{\tau}_{x\otimes y}(\log_{x\otimes y} d_1(\beta\alpha))$. By Lemma 2.19 *the* logarithmic character (2.2), of a morphism $\alpha \in \mathrm{mor}_{\mathbf{C}}(x, y)$ is independent of where it is computed, and likewise for (2.75).

Lemma 2.20 *For $\alpha \in \mathrm{mor}(x, z)$ and $\beta \in \mathrm{mor}(z, y)$*

$$\widetilde{\tau}\,(\log\,\beta\alpha) = \widetilde{\tau}\,(\log\,\alpha) + \widetilde{\tau}\,(\log\,\beta) \qquad \text{in } \mathrm{mor}_{\boldsymbol{A}}(1,1), \tag{2.78}$$

$$\det{}_{\tau}(\alpha\beta) = \det{}_{\tau}(\alpha)\cdot\det{}_{\tau}(\beta) \qquad \text{in } S, \tag{2.79}$$

The space $\mathbb{L}\mathrm{og}(\boldsymbol{C},\mathsf{F})$ of logarithms on $\boldsymbol{C}$ with respect to a fixed monoidal product representation F is an abelian group $\log_1, \log_2 \in \mathbb{L}\mathrm{og}(\boldsymbol{C},\mathsf{F}) \Rightarrow \log_1 + \log_2 \in \mathbb{L}\mathrm{og}(\boldsymbol{C},\mathsf{F})$ with respect to the additive structure of the category $\boldsymbol{A}$, as is the space $\mathbb{L}\mathrm{og}^{\chi}(\boldsymbol{C})$ of logarithmic characters $\tau(\log\alpha)$ independently of a particular F Likewise, the space $\mathbb{D}\mathrm{et}(\boldsymbol{C},S)$ of determinants is an abelian group with respect to the multiplication in the commutative ring $(S,\cdot)$.

If $\mathbf{C}$ is an additive category then $\tau \circ \log$ is a log-representation from the maximal sub groupoid of $\mathbf{C}$, whose morphisms are the isomorphisms of $\mathbf{C}$, to the isomorphism torsion group $K_1^{\mathrm{iso}}(\mathbf{C})$ of [13].

By statement 5 (and 6) of Proposition 2.18 it is enough to require log-additivity on 1-simplicies to infer it on p-simplices in $\mathcal{N}\mathbf{C}$. On the other hand, as far as computing log-determinant characters is concerned, log-additivity (2.54) can be formulated more generally as the existence of $\underline{w}_0, \underline{w}_1, \underline{w}_2 \in \mathrm{ob}(\mathbf{C})$ such that $\widetilde{\eta}_{\underline{w}_0}(\log_{x\otimes z}\alpha), \widetilde{\eta}_{\underline{w}_1}(\log_{z\otimes y}\beta), \widetilde{\eta}_{\underline{w}_2}(\log_{x\otimes y}(x \stackrel{\beta\circ\alpha}{\to} y))$ are all in the same $\mathsf{F}(v)$ with, in $\mathsf{F}(v)/[\mathsf{F}(v),\mathsf{F}(v)]$,

$$\widetilde{\eta}_{\underline{w}_1}(\log_{x\otimes y}(x \stackrel{\beta\circ\alpha}{\to} y)) = \widetilde{\eta}_{\underline{w}_2}(\log_{x\otimes z}\alpha) + \widetilde{\eta}_{\underline{w}_0}(\log_{z\otimes y}\beta). \tag{2.80}$$

Despite Lemma 2.16, it can be natural to define simplicial logarithms directly on strata $\mathrm{mor}_{\underline{z}}(x,y)$ in p-simplices with $p > 1$. In particular, this allows a log-functor to be extended to $\delta \in \mathrm{mor}_{\mathbf{C}}(1,1) = \mathrm{end}_{\mathbf{C}}(1)$ factorisable as $\delta = \beta\alpha$ for $\alpha \in \mathrm{mor}_{\mathbf{C}}(1,z)$ and $\beta \in \mathrm{mor}_{\mathbf{C}}(z,1)$ with $z \neq 1 \in \mathrm{ob}(\mathbf{C})$ (this is always the case on $\mathbf{Bord}_n$). Choosing such a factorisation, define

$$\log_z \delta := \log_z(1 \stackrel{\alpha}{\to} z \stackrel{\beta}{\to} 1) \ \in \mathsf{F}(z)/[\mathsf{F}(z),\mathsf{F}(z)]. \tag{2.81}$$

Here, we use $\log_z := \log_{1\otimes z\otimes 1}$ and $\mathsf{F}(1\otimes z\otimes 1) = \mathsf{F}(z)$, as F is exact and $\log$ is strict, which depends on δ and z but is independent of the particular choice of α,β. In the presence of a trace one then further has

$$\log_1 : \mathrm{end}_{\mathbf{C}}(1) \to \mathrm{end}_{\mathbf{A}}(1), \quad \log_1\delta := \widetilde{\tau}(\log_z(1 \stackrel{\alpha}{\to} z \stackrel{\beta}{\to} 1)), \tag{2.82}$$

independently of the particular choice of α,β and of z and by log-additivity

$$\log_1\delta := \widetilde{\tau}(\log_z\alpha) + \widetilde{\tau}(\log_z\beta) \tag{2.83}$$

as a particular case of the additivity of log-characters.

3 Examples

In this section we discuss a number of examples of logarithmic functors, with particular emphasis on the bordism category $\mathbf{Bord}_n$ associated to log-TQFTs. Such structures arise very naturally in topology and physics, and it remains to understand the universal structures that may reign over them. In this regard, on $\mathbf{Bord}_2$, for example, there is a classification of *unoriented* log-functors:

Theorem (Salvatori [9])

Let $F : \boldsymbol{Bord}_2^* \to \mathbf{Ring}$ *be an injective and unoriented monoidal product representation and let* $\log : \mathcal{N}\boldsymbol{Bord}_2 \to (F_\Pi(\boldsymbol{Bord}_2^*), \widetilde{\eta})$ *be an unoriented log-TQFT. Let* $\Sigma_{g,k}$ *denote an orientable, compact, and connected surface of genus* g*, whose boundary* $\partial\Sigma_{g,k}$ *has* k *connected components, i.e.* $\partial\Sigma_{g,k} \cong \sqcup_{j=1}^k S^1$*. Then,* $\forall g, k \in \mathbb{N}$*:*

$$\log_{\sqcup_{j=1}^k S^1} \overline{\Sigma_{g,k}} = \chi(\Sigma_{g,k}) \cdot \eta_{\sqcup_{j=1}^{k-1} S^1} \overline{D},$$

where D *is the disc,* $\chi(\Sigma_{g,k}) = \chi(\Sigma_g) - k$ *is the Euler characteristic of* $\Sigma_{g,k}$*, and* Σ_g *is the closed surface obtained from* $\Sigma_{g,k}$ *by gluing* k *discs along the boundary components.*

Remarks:

[1] More generally, handlebody methods for bordisms are expected to provide analogous results in higher dimensions.

[2] In a different view point, there is a conjectural *logarithmic cobordism hypothesis* for logarithmic n-functors on higher categories parallel to the cobordism hypothesis for TQFTs [6].

[3] See [9] for log functors ranging in cyclic cohomology $HC^k(\mathbf{B})$ (here we consider only $k = 0$).

Related methods for geometric fibrations have led to a cohomological characterization of the fibred bordism ring $\mathbf{Bord}_n(M/X)$ and ambient fibred bordism homology, as a precursor to fibred TQFT and log-TQFT [15].

3.1 Fredholm category

Let $\mathbf{C}_{\mathsf{Fred}}$ be the category whose objects are Hilbert spaces $H \in \mathrm{ob}(\mathbf{C}_{\mathsf{Fred}})$ and whose morphisms are Fredholm operators, with symmetric monoidal product defined by direct sum. Thus, $Z \in \mathrm{mor}(H, H')$ has a parametrix $Q \in \mathrm{mor}(H', H)$ so that

$$L_Z := QZ - I \in \mathcal{F}(H) \quad \text{and} \quad R_Z := ZQ - I' \in \mathcal{F}(H') \tag{3.1}$$

with $\mathcal{F}(H)$ the ideal of finite-rank operators. Define F by $H \to \mathcal{F}(H)$ with

$$\eta_K : \mathcal{F}(\underline{H}) \to \mathcal{F}(\underline{H}_K), \qquad \eta_K(Z) = i_K \circ Z \circ i_K^*,$$

where $i_K : \underline{H} := H_0 \oplus \cdots \oplus H_p \to \underline{H}_K := H_0 \oplus \cdots \oplus K \oplus \cdots \oplus H_p$ is the inclusion and $i_K^* : H_K \to H$ its adjoint (projection). Let $\widehat{A}$ be the inclusion of $A : H_i \to H_j$ in continuous linear operators on $H_1 \oplus H_2$: if $i = 1, j = 2$, then $\widehat{A} = \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix}$, and so on. Define $\log_{H \oplus H'} : \mathrm{mor}(H, H') \to \mathcal{F}_\Pi(H \oplus H') := \mathcal{F}(H \oplus H')/[\mathcal{F}(H \oplus H'), \mathcal{F}(H \oplus H')]$ by

$$\log_{H \oplus H'} Z = \pi_{H \oplus H'}([\widehat{Z}, \widehat{Q}] - J), \tag{3.2}$$

where $\pi_{H \oplus H'} : \mathcal{F}(H \oplus H') \to \mathcal{F}_\Pi(H \oplus H')$ is the quotient map and $J := -\widehat{I} + \widehat{I'} = -I \oplus I'$ the grading operator. Here, $\left[\widehat{Z}, \widehat{Q}\right]$ is not in $[\mathcal{F}(H \oplus H'), \mathcal{F}(H \oplus H')]$. But, recall, for continuous linear operators S, T on a Hilbert space V

$$ST \in \mathcal{F}(V) \text{ and } TS \in \mathcal{F}(V) \quad \Rightarrow \quad [S,T] \in [\mathcal{F}(V), \mathcal{F}(V)] \tag{3.3}$$

and the classical trace $\mathrm{Tr}_V : \mathcal{F}(V) \to \mathbb{C}$ defines a canonical isomorphism

$$\widetilde{\mathrm{Tr}}_V : \mathcal{F}(V)/[\mathcal{F}(V), \mathcal{F}(V)] \to \mathbb{C} \quad \text{with} \quad \widetilde{\mathrm{Tr}}_V \circ \pi_V = \mathrm{Tr}_V \tag{3.4}$$

as the canonical generator of the complex line $(\mathcal{F}(V)/[\mathcal{F}(V), \mathcal{F}(V)])^*$. Tr defines the unique trace on $\mathcal{F}$, equivalent to $A \in [\mathcal{F}(V), \mathcal{F}(V)] \iff \mathrm{Tr}(A) = 0$.

Lemma 3.1 *(3.2) is well-defined: $\left[\widehat{Z}, \widehat{Q}\right] - J$ is in $\mathcal{F}(H \oplus H')$ and (3.2) is independent of the choice of parametrix Q. The character is the Fredholm index*

$$\widetilde{\mathrm{Tr}}_{H \oplus H'}(\log_{H \oplus H'} Z) = \mathrm{ind}(Z) \in \mathbb{Z}. \tag{3.5}$$

Proof: $\left[\widehat{Z}, \widehat{Q}\right] = \left[\begin{pmatrix} 0 & 0 \\ Z & 0 \end{pmatrix}, \begin{pmatrix} 0 & Q \\ 0 & 0 \end{pmatrix}\right]$ and so $[\widehat{Z}, \widehat{Q}] - J = (I - QZ) \oplus (ZQ - I')$ is by (3.1) in $\mathcal{F}(H \oplus H')$. Q can be chosen with L_Z and R_Z projections onto the kernels of the operators Z and Z^*, respectively, giving, in view of (3.4), (3.5). If $P \in \mathrm{mor}(H', H)$ is a second parametrix then $\left(\left[\widehat{Z}, \widehat{Q}\right] - J\right) - \left(\left[\widehat{Z}, \widehat{P}\right] - J\right) = \left[\widehat{Z}, \widehat{Q} - \widehat{P}\right]$. But $\widehat{Z}(\widehat{Q} - \widehat{P}) = 0 \oplus Z(Q-P) \overset{(3.1)}{\in} \mathcal{F}(H \oplus H')$ and $(\widehat{Q} - \widehat{P})\widehat{Z} = (Q-P)Z \oplus 0 \in \mathcal{F}(H \oplus H')$ so $\left[\widehat{Z}, \widehat{Q} - \widehat{P}\right] \in [\mathcal{F}(H \oplus H'), \mathcal{F}(H \oplus H')]$ by (3.3). □

Let $\widetilde{\eta}_K : \mathcal{F}(\underline{H})/[\mathcal{F}(\underline{H}), \mathcal{F}(\underline{H})] \to \mathcal{F}(\underline{H}_K)/[\mathcal{F}(\underline{H}_K), \mathcal{F}(\underline{H}_K)]$ be the quotient linear isomorphism of complex lines induced from η_K. For the log-additivity property:

Lemma 3.2 *Let $Z \in \mathrm{mor}(H, H')$ and $Z' \in \mathrm{mor}(H', H'')$. Then*

$$\widetilde{\eta}_{H'}(\log_{H \oplus H''} Z'Z) = \widetilde{\eta}_{H''}(\log_{H \oplus H'} Z) + \widetilde{\eta}_H(\log_{H' \oplus H''} Z') \tag{3.6}$$

in $\mathcal{F}(H \oplus H' \oplus H'')/[\mathcal{F}(H \oplus H' \oplus H''), \mathcal{F}(H \oplus H' \oplus H'')]$.

Proof: Set $\log^Q Z := [\widehat{Z}, \widehat{Q}] - J \in \mathcal{F}(H \oplus H')$ and let $Q' \in \mathrm{mor}(H'', H')$ be a parametrix for Z'. Then (3.6) is equivalent to $\eta_{H'}(\log^{Q'Q} Z'Z) - \eta_{H''}(\log^Q Z) - \eta_H(\log^{Q'} Z') \in [\mathcal{F}, \mathcal{F}]$, with $\mathcal{F} = \mathcal{F}(H \oplus H' \oplus H'')$; changing the parametrices for Z, Z' or $Z'Z$ only produces a change in $[\mathcal{F}, \mathcal{F}]$ as accounted for in Lemma 3.1. One has

$$\eta_{H''}(\log^Q_{H \oplus H'} Z) = (I - QZ) \oplus (ZQ - I') \oplus 0,$$

$$\eta_H(\log^{Q'}_{H' \oplus H''} Z') = 0 \oplus (I' - Q'Z') \oplus (Z'Q' - I''),$$

$$\eta_{H'}(\log^{Q'Q}_{H \oplus H''} Z'Z) = (I - QQ'Z'Z) \oplus 0 \oplus (Z'ZQQ' - I'')$$

in $\mathcal{F}(H \oplus H' \oplus H'')$. The Fredholm property gives $ZQ = I' + R_Z, QZ = I + L_Z, Z'Q' = I'' + R_{Z'}, Q'Z' = I' + L_{Z'}$, for some $L_Z \in \mathcal{F}(H), R_Z, L_{Z'} \in \mathcal{F}(H), R_{Z'} \in \mathcal{F}(H'')$, and hence $Z'ZQQ' = I'' + R_{Z'} + Z'R_ZQ'$ and $QQ'Z'Z = I - L_Z - QL_{Z'}Z$. Thus

$$\eta_{H'}(\log^{Q'Q} Z'Z) - \eta_H(\log^{Q'} Z') - \eta_{H''}(\log^Q Z)$$

$$= \left[\begin{pmatrix} 0 & 0 & 0 \\ Z & 0 & 0 \\ 0 & Z' & 0 \end{pmatrix}, \begin{pmatrix} 0 & QL_{Z'} & 0 \\ 0 & 0 & R_Z Q' \\ 0 & 0 & 0 \end{pmatrix} \right] + \left[\begin{pmatrix} 0 & 0 & 0 \\ 0 & L_{Z'} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & R_Z & 0 \\ 0 & 0 & 0 \end{pmatrix} \right].$$

Each of these matrix products is in $\mathcal{F}$, so by (3.3) the above the commutators are in $[\mathcal{F}, \mathcal{F}]$. □

(2.25) holds so we have a tracial monoidal product representation and the log-character additivity formula (2.78) is (by (3.6), Lemma 3.1 and (3.4))

$$\mathrm{ind}\,(Z'Z) = \mathrm{ind}\,(Z) + \mathrm{ind}\,(Z'). \tag{3.7}$$

The logarithm (3.2) extends to p-simplices using Lemma 2.16, or, with $\underline{H} := (H_1, \ldots, H_{p-1})$, derectly on p-simplices $Z := H \stackrel{Z_0}{\to} H_1 \stackrel{Z_1}{\to} H_2 \to \cdots \to H_{p-1} \stackrel{Z_{p-1}}{\to} H_p \in \mathcal{N}_p\mathbf{C}_{\mathsf{Fred}}$ by

$$\log_{H \oplus \underline{H} \oplus H'} Z = \pi_{H \oplus \underline{H} \oplus H'}([\widehat{Z}, \widehat{Q}] - J_{m+1}), \tag{3.8}$$

with $Q_j : H_{j+1} \to H_j$ a parametrix to Z_j, $\widehat{Z}$ the $(m+2) \times (m+2)$ block matrix with $Z_1, \ldots, Z_m$ on the subdiagonal and zeroes elsewhere, $\widehat{Q}$ has Q_j on the upper-diagonal and zeroes elsewhere, and J_M with $-I$ in the $(1,1)$-position and I' in the $(m+2, m+2)$-position and zeroes elsewhere.

3.1.1 Families Index Formula

This case generalises to families of Fredholm operators, in which the logarithm becomes a suitable representative for the index bundle. Consider, for example, a geometric fibration $\pi : M \to X$ of spin manifolds with closed fibre of dimension n and the category $\mathbf{C}_\pi$ whose objects are Hermitian vector bundles $E \to M$ and whose morphisms $\mathsf{L} \in \mathrm{mor}_{\mathbf{C}_\pi}(E, E')$ are vertical (or fibrewise) smooth families of elliptic ψdos $\mathsf{L} \in \Psi^{\mathbb{Z}}_{M/X}(X, \pi_*(\mathrm{Hom}(E \otimes \wedge T(M/X), E' \otimes \wedge T(M/X)))$ with differential form valued coefficients, where $\pi_*(V) \to X$ is the bundle with fibre $C^\infty(\pi^{-1}(x), V_{|\pi^{-1}(b)})$ at $x \in X$. Give $\mathbf{C}_\pi$ the monoidal structure defined by direct sum and monoidal product representation sending an object E to the algebra of vertical integer order elliptic families $\mathcal{A}^{\mathbb{Z}}_{E \oplus E'}(M/X) := \Psi^{\mathbb{Z}}_{M/X}(X, \mathrm{End}\,(\pi_*(E \oplus E' \otimes \wedge T(M/X)))$ on $\pi_*(E)$ and morphisms to pull-backs. Assume the usual geometric data on the fibration in order to define a superconnection $\mathbb{A} : \Psi^{\mathbb{Z}}_{M/X}(X, \pi_*(E \otimes \wedge^k T(M/X))) \to \Psi^{\mathbb{Z}}_{M/X}(X, \pi_*(E \otimes \wedge^{k+1} T(M/X)))$ with 0-form componenet $\mathbb{A}_0 = \mathsf{L}$. Then a logarithmic functor

$$\log_{E \oplus E'} : \mathrm{mor}_{\mathbf{C}_\pi}(E, E') \to \mathcal{A}^{\mathbb{Z}}_{E \oplus E'}(M/X)/[\mathcal{A}^{\mathbb{Z}}_{E \oplus E'}(M/X), \mathcal{A}^{\mathbb{Z}}_{E \oplus E'}(M/X)],$$

is defined by

$$\log_{E \oplus E'} \mathsf{L} = \left(\frac{1}{2} \sum_{k \geq 0} \mathbb{A}^{2k} \right) \mathbb{L}\mathrm{og}_0 \mathbb{A}^2, \tag{3.9}$$

where $\mathbb{L}\mathrm{og}_0 \mathbb{A}^2$ is the zeta-function holomorphic functional calculus logarithm of the super-curvature $\mathbb{A}^2$. Integration over the fibre combined with the pointwise residue density defines the fibrewise residue trace

$$\mathrm{res}_{M/X} : \mathcal{A}^{\mathbb{Z}}(M/X) \to \Omega^*(X),$$

the essentially unique fibrewise trace. Applying this to define the character of the logarithm (3.9) gives the Atiyah-Singer family index density form

$$\operatorname{res}_{M/X}(\log_{E\oplus E'}\mathsf{L}) = \int_{M/X} \widehat{A}(M/X)\mathsf{ch}(E\oplus E') \quad \text{in } H^*(X),$$

whilst the character of the log-additivity property corresponds to the index bundle additivity $\operatorname{Ind}(L_1L_2) = \operatorname{Ind}(L_1) + \operatorname{Ind}(L_2)$ in $K_0(X)$.

Most details of the components of this construction, and proper definitions, can be found in [10] and in [17] (Ch. 5).

3.2 $\mathbf{Bord}_n$ and the topological signature

There are a number of bordism categories with natural logarithmic functors. Bordism classes will be denoted $\overline{W} \in \operatorname{mor}_{\mathbf{Bord_n}}(M_0, M_1)$, while $W = (W, \kappa_{\partial W}) \in \overline{W}$ will indicate a smooth representative of the class. Thus, W is an oriented smooth compact manifold of dimension $n+1$ whose boundary $\partial W \in \operatorname{ob}(\mathbf{Bord}_n)$ is endowed with an orientation preserving diffeomorphism $\kappa_{\partial W} : \partial W \to M_0^- \sqcup M_1$, the superscript indicating the reverse orientation on M_0. $\overline{W} = \overline{(W, \kappa_{\partial W})}$ denotes the equivalence class relative to oriented diffeomorphism; a bordism class $\overline{W} =$ depends on $\kappa_{\partial W}$ as well as on W.

Let $\mathsf{F} : \mathbf{Bord}_n^* \to \mathbf{Ring_{Add}}$ be an unoriented pretracial monoidal product representation §2.1.2. A log-TQFT on $\mathbf{Bord}_n$ relative to F is a log-additive presimplicial map

$$\log : \mathcal{N}\mathbf{Bord}_n \to \mathsf{F}(\mathbf{Bord}_n^*)/[\mathsf{F}(\mathbf{Bord}_n^*), \mathsf{F}(\mathbf{Bord}_n^*)],$$

defining for each p-simplex $M_0 \overset{\overline{W}_0}{\to} M_1 \overset{\overline{W}_1}{\to} M_2 \to \cdots \to M_{p-1} \overset{\overline{W}_{p-1}}{\to} M_p \in \mathcal{N}_p\mathbf{Bord}_n$ of bordisms between compact boundaryless manifolds M_j, a logarithm

$$\log_M(M_0 \overset{\overline{W}_0}{\to} M_1 \overset{\overline{W}_1}{\to} M_2 \to \cdots \to M_{p-1} \overset{\overline{W}_{p-1}}{\to} M_p) \;\in\; \mathsf{F}_\Pi(M) := \mathsf{F}(M)/[\mathsf{F}(M), \mathsf{F}(M)], \tag{3.10}$$

where $M = M_0 \sqcup M_1 \sqcup \cdots \sqcup M_p$, with

$$\log_{M_0\sqcup M_1\sqcup M_2}(M_0 \overset{\overline{W}_0}{\to} M_1 \overset{\overline{W}_1}{\to} M_2) = \widetilde{\eta}_{M_1}\log_{M_0\sqcup M_2}(M_0 \overset{\overline{W}_0\cup\overline{W}_1}{\longrightarrow} M_2), \tag{3.11}$$

where $\overline{W}_0 \cup \overline{W}_1$ is the composed bordism joined along M_1, and, on 1-simplices,

$$\widetilde{\eta}_{M_1}\log_{M_0\sqcup M_2}(\overline{W}_0\cup\overline{W}_1) = \widetilde{\eta}_{M_2}\log_{M_0\sqcup M_1}(\overline{W}_0) + \widetilde{\eta}_{M_0}\log_{M_1\sqcup M_2}(\overline{W}_1) \tag{3.12}$$

in $\mathsf{F}_\Pi(M_0 \sqcup M_1 \sqcup M_2)$. Though F is unoriented, the logarithms $\log_M(\overline{W})$ *will* in general depend on the orientation of the bordisms $\overline{W}$. The M_j need not be connected. On the other hand, writing $M_j = N_0 \sqcup \cdots \sqcup N_k$ is reflected functoriality in a canonical isomorphism $\mathsf{F}(M_j) \cong \mathsf{F}(N_0 \sqcup \cdots \sqcup N_k)$. A permutation of the ordering $N_{\sigma(0)} \sqcup \cdots \sqcup N_{\sigma(k)}$ yields (in accordance with (2.2)) a compatible canonical isomorphism $\mu_\sigma : \mathsf{F}(N_0 \sqcup \cdots \sqcup N_k) \overset{\cong}{\to} \mathsf{F}(N_{\sigma(0)} \sqcup \cdots \sqcup N_{\sigma(k)})$. In (3.10) there is no ambiguity because M is defined to be the given disjoint union in the order specified by the p-simplex.

The p-simplices of $\mathcal{N}\mathbf{Bord}_n$ may be viewed as bordisms which retain data of how they were formed by gluing other bordisms. Boundaryless bordisms $\overline{W} \in \mathrm{mor}_{\mathbf{Bord_n}}(\emptyset, \emptyset)$ need separate consideration: we are instructed by (2.81) to view $\overline{W}$ as a composed bordism $\emptyset \xrightarrow{\overline{W}_0} M \xrightarrow{\overline{W}_1} \emptyset$ relative to codimension 1 embedded submanifold $M \hookrightarrow W$ and set

$$\log_M \overline{W} := \log_M(\emptyset \xrightarrow{\overline{W}_0} M \xrightarrow{\overline{W}_1} \emptyset) \quad \in \mathsf{F}(M)/[\mathsf{F}(M), \mathsf{F}(M)].$$

Log-additivity then gives $\log_M \overline{W} = \log_M(\emptyset \overset{\overline{W}_0}{\to} M) + \log_M(M \overset{\overline{W}_1}{\to} \emptyset) \in \mathsf{F}(M)/[\mathsf{F}(M), \mathsf{F}(M)]$, and if tracial with character $\tau(\log \overline{W}) = \tau_M(\log \overline{W}_0) + \tau_M(\log \overline{W}_1) \in \mathrm{end}_{\mathbf{A}}(1)$ depending only on $\overline{W}$, not on its factorisation as $\overline{W}_0 \cup_M \overline{W}_1$.

Lemma 3.3 *Let $C_M \in \mathrm{mor}_{\mathbf{Bord_n}}(M, M)$ be the bordism class of $[0,1] \times M$. Then*

$$\widetilde{\eta}_M \log_{M \sqcup M}(C_M) = 0,$$

in $\mathsf{F}_\Pi(M \sqcup M \sqcup M)$ and $\log_{M \sqcup M}(C_M) = 0 \in \mathsf{F}_\Pi(M \sqcup M)$ if F is injective. For $\overline{W} \in$ $\mathrm{mor}(M_0, M_1)$

$$\widetilde{\eta}_{\sqcup N} \log_{M_0 \sqcup M_1}(M \overset{\overline{W}}{\to} N) = \log_{M_0 \sqcup M_1 \sqcup N}(M \overset{\overline{W}}{\to} N \overset{C_N}{\to} N) \tag{3.13}$$

in $\mathsf{F}_\Pi(M_0 \sqcup M_1 \sqcup N)$.

Proof: Restatements of Proposition 2.18 (1) and (2) to $\mathbf{Bord}_n$. □

A log-TQFT yields a 'scalar-valued' TQFT, in the following sense:

Lemma 3.4 *A logaritmic functor $\log : \mathcal{N}\boldsymbol{Bord}_n \to \boldsymbol{Ring}_{\mathbf{Add}}$ defined relative to a tracial MPR $\boldsymbol{Bord}_n^* \to (\mathsf{F}(\boldsymbol{Bord}_n^*), \tau)$ defines a monoid $(\mathrm{mor}_{\boldsymbol{A}}(1,1), +)$-valued symmetric monoidal functor $Z_{\log,\tau} : \boldsymbol{Bord}_n \to \mathrm{mor}_{\boldsymbol{A}}(1,1)$ by setting $Z_{\log,\tau}(M) = \mathrm{mor}_{\boldsymbol{A}}(1,1)$ and $Z_{\log,\tau}(\overline{W}) = \tau(\log \overline{W})$.*

Conversely, log-TQFTs can arise from TQFTs, for instance by pull-back. Specifically, the pull-back of the Fredholm category logarithm, of the previous example, by a TQFT $Z : \mathbf{Bord}_n \to \mathbf{Vect}$ yields a log-TQFT with

$$\log_{M_0 \sqcup M_1} \overline{W} := \log_{Z(M_0) \oplus Z(M_1)} Z(\overline{W}) \ \overset{(3.2)}{:=} \pi_{Z(M_0) \oplus Z(M_1)}([\widehat{Z(\overline{W})}, \widehat{Q}] - J),$$

in $\mathcal{F}_\Pi(Z(M_0) \oplus Z(M_1))$. Since the Hilbert spaces $Z(M_j)$ are finite-dimensional, its character is

$$\widetilde{\mathrm{tr}}\,(\log_{M_0 \sqcup M_1} \overline{W}) = \dim Z(M_1) - \dim Z(M_0).$$

For a surface Σ, for example, one has

$$\widetilde{\mathrm{tr}}\,(\log_{M_0 \sqcup M_1} \overline{\Sigma}) = \mu^{m_1} - \mu^{m_0}$$

with $\mu = \dim Z(S^1)$, $m_i = |\pi_0(M_i)|$.

This simple case is a particular instance of the rather more interesting log-TQFT associated to exotic Reidemeister torsion outlined in §3.4.

3.2.1 The topological signature

For a compact oriented manifold W of dimension $4k$ with boundary ∂W, the topological signature $\mathrm{sgn}(W)$ of W, is defined to be the signature of the quadratic form

$$\widehat{H}^{2k}(W) \times \widehat{H}^{2k}(W) \to \mathbb{R}, \quad (\xi, \xi') \mapsto < \xi \cup \xi', [W] >, \tag{3.14}$$

with $\widehat{H}^{2k}(W)$ the image of the inclusion $H^{2k}(W, \partial W) \to H^{2k}(W)$. This arises as a character of a logarithmic representation on bordisms as follows.

On a smooth representative $W \in \overline{W}$ of a bordism class $\overline{W} \in \mathrm{mor}_{\mathrm{Bord}_{4k}}(M_0, M_1)$, a choice of Riemannian metric g_W is made which in a collar neighbourhood U_j of each boundary component ∂W_j is a product metric $g_{U_j} = du_j^2 + g_{\partial W_j}$ with u_j a choice of normal coordinate in $(-1, 0]$ if ∂W_j is a component of M_0^- and in $[0, 1)$ if ∂W_j is a component of M_1; all logarithms will be independent of the choice of g_W and the choice of representative W. Associated to g_W is a Hodge star isomorphism $* : \Omega^p(W) \to \Omega^{4k-p}(W)$ and a signature operator

$$\eth^W = d + d^* : \Omega^+(W) \to \Omega^-(W)$$

between the eigenspaces $\Omega^\pm(W)$ of the involution $i^{p(p-1)}*$ on the de Rham complex.

Recall from [1], since W is isometric to a product near each boundary component ∂W_j the operator $\eth^W$ acts along tangential boundary directions by a self-adjoint signature operator B_j on the de Rham algebra $\Omega(\partial W_j)$, equal to $B_j^{2p} := (-1)^{k+p+1}(*d_j - d_j*)$ on $\Omega^{2p}(\partial W_j)$ and to $B_j^{2p-1} := (-1)^{k+p}(*d_j + d_j*)$ on $\Omega^{2p-1}(\partial W_j)$. Let $B_j^{ev} = \bigoplus_p B_j^{2p}$, $B_j^{odd} = \bigoplus_p B_j^{2p-1}$. Then B preserves form parity $B_j = B_j^{ev} \oplus B_j^{odd}$ relative to the de Rham algebra written as a direct sum of even and odd forms. The self-adjoint first-order elliptic operators B_j^{ev} and B_j^{odd} are spectrally identical, one has

$$h_j := \mathrm{Tr}\,(\Pi_0[B_j^{ev}]) = \mathrm{Tr}\,(\Pi_0[B_j^{odd}]) = \frac{1}{2}\,\mathrm{Tr}\,(\Pi_0[B_j]) \tag{3.15}$$

and $\eta_j := \eta(B_j^{ev}, 0) = \eta(B_j^{odd}, 0) = \frac{1}{2}\,\eta(B_j, 0)$, where $\Pi_0[S] \in \mathsf{F}_{-\infty}(\partial W_j)$ is the smoothing projection onto $\ker(S)$, and $\eta(S, 0)$ the η-invariant of an elliptic self-adjoint ψdo S. Let

$$\Pi_0^{ev} = \bigoplus_j \Pi_0[B_j^{ev}] \quad \in \mathsf{F}_{-\infty}(\partial W), \tag{3.16}$$

and likewise for Π_0^{odd}, and set $h := \mathrm{Tr}_{\,\partial W}(\Pi_0^{ev}) = \sum_j h_j$, $\eta := \eta(B^{ev}, 0) = \sum_j \eta_j$. The APS projection is the order zero ψdo projector

$$\Pi_\geq^{\partial W} = \bigoplus_{j=1}^r \Pi_\geq^{\partial W_j} \;\in \mathsf{F}_{\mathbb{Z}}(\partial W) := \bigoplus_{j=1}^r \Psi^{\mathbb{Z}}(\partial W_j, \wedge T^*\partial W_j) \tag{3.17}$$

where $\Pi_\geq^{\partial W_j}$ is the orthogonal projection onto the span of eigenforms of B_j with eigenvalue $\lambda \geq 0$. The Calderón projection, on the other hand, $C[\eth^W] \in \mathsf{F}_{\mathbb{Z}}(\partial W)$ is a projector onto the subspace $K(\eth^W) \subset \Omega(\partial W)$ of boundary sections which are restrictions to the boundary of interior solutions $\mathrm{Ker}\,(\eth^W) \subset \Omega(W)$; the Poisson operator $\mathcal{K}[\eth^W] : \Omega(\partial W) \to \Omega(W)$ associated to $\eth^W$ restricts in each Sobolev completion to a canonical isomorphism

$$K(\eth^W) \overset{\cong}{\to} \mathrm{Ker}\,(\eth^W) \quad \text{and then} \quad C[\eth^W] := \varrho\mathcal{K}[\eth^W], \tag{3.18}$$

where $\varrho : \Omega(W) \to \Omega(\partial W)$ is restriction to the boundary. See for instance §7 of [4].

Relative to an identification with its connected components $\partial W = \partial W_1 \sqcup \cdots \sqcup \partial W_n$ the projections may be written as $n \times n$ block matrices: $\Pi_{\geq}^{\partial W}$ is a diagonal direct sum of order zero ψdos whilst the Calderón projector $C[\eth^W]$ has order zero ψdos along the diagonal and has non-zero off-diagonal smoothing operator terms. The crucial analytic fact is:

Lemma 3.5

$$C[\eth^W] - \Pi_{\geq}^{\partial W} \ \in \mathsf{F}_{-\infty}(\partial W). \tag{3.19}$$

Proof: Since $\eth^W$ has the form $\sigma(du)(\partial_u + B_j)$ in a collar neighbourhood U_i of each connected component ∂W_i, the argument in [16] (Prop. 2.2), or the more general argument of [4] (Prop. 4.1), for the case for a single boundary readily adapts to the present multi-boundary context. □

The projection operators above are sensitive to orientation. For an oriented manifold N, let N^- denote the manifold with orientation reversed.

Lemma 3.6 $\Pi_{\geq}^{\partial W^-} = \Pi_{\leq}^{\partial W}$ *is the projection onto the span of eigenforms with eigenvalue* $\lambda \leq 0$. *Likewise,* $C[\eth^W]$ *and* $C[\eth^{W^-}]$ *are complementary projections modulo smoothing operators.*

Proof: Reversing the orientation on ∂W reverses the sign of the Riemannian volume form, and so the Hodge star $* \mapsto -*$. Thus $B_j^{2p} := (-1)^{k+p+1}(*d_j - d_j*)$ and $B_j^{2p-1} := (-1)^{k+p}(*d_j + d_j*)$ change sign, swapping negative and positive eigenvalues, which is the first assertion. Since $\partial(W^-) = (\partial W)^-$, the statement for the Calderón projection then follows from (3.19). □

A representative W for a bordism in $\mathrm{mor}_{\mathrm{Bord}_{4k}}(M_0, M_1)$ comes with an orientation preserving diffeomorphism $\kappa : \partial W \to M_0^- \sqcup M_1$. One has that $\kappa_\sharp(\Pi_{\geq}^{\partial W}),\ \kappa_\sharp(C[\eth^W]) \in \mathsf{F}_{\mathbb{Z}}(M_0 \sqcup M_1)$ are order zero ψdo projections, while

$$\kappa_\sharp(C[\eth^W]) - \kappa_\sharp(\Pi_{\geq}^{\partial W}) = \kappa_\sharp(C[\eth^W] - \Pi_{\geq}^{\partial W}) \ \in \mathsf{F}_{-\infty}(M_0 \sqcup M_1) \tag{3.20}$$

are smoothing operators. Also $\kappa_\sharp(\Pi_0^{ev}) \ \in \mathsf{F}_{-\infty}(M_0 \sqcup M_1)$. To define a logarithm

$$\log^{\mathrm{sgn}} : \mathcal{N}\mathbf{Bord}_{4k} \to \mathsf{F}_{-\infty}(\mathbf{Bord}_{4k}^*)/[\mathsf{F}_{-\infty}(\mathbf{Bord}_{4k}^*), \mathsf{F}_{-\infty}(\mathbf{Bord}_{4k}^*)]$$

it is enough to specify it on 1-simplices

$$\log^{\mathrm{sgn}}_{M_0 \sqcup M_1} \ : \ \mathrm{mor}_{\mathrm{Bord}_{4k}}(M_0, M_1) \to \mathsf{F}_{-\infty}(M_0 \sqcup M_1)/[\mathsf{F}_{-\infty}(M_0 \sqcup M_1), \mathsf{F}_{-\infty}(M_0 \sqcup M_1)].$$

Define

$$\log^{\mathrm{sgn}}_{M_0 \sqcup M_1}(\overline{W}) := \pi_{M_0 \sqcup M_1} \circ \kappa_\sharp \left(C[\eth^W] - \Pi_{\geq}^{\partial W} + \Pi_0^{ev} \right) \tag{3.21}$$

— equal to the sum of order zero ψdo projections in $\mathsf{F}^0_{\mathbb{Z}, -\infty}(M_0 \sqcup M_1)$ —

$$= \pi_{M_0 \sqcup M_1} \circ \kappa_\sharp(C[\eth^W]) - \pi_{M_0 \sqcup M_1} \circ \kappa_\sharp(\Pi_{\geq}^{\partial W}) + \pi_{M_0 \sqcup M_1} \circ \kappa_\sharp(\Pi_0^{ev}).$$

From (2.42) and (2.43)

$$\log^{\rm sgn}_{M_0 \sqcup M_1}(\overline{W}) = \vartheta_{\partial W,\, M_0 \sqcup M_1} \circ \pi_{\partial W}\left(C[\eth^W] - \Pi^{\partial W}_{\geq} + \Pi^{ev}_0\right). \tag{3.22}$$

Proposition 3.7 *The right-hand side of* (3.21) *depends only on the (oriented) bordism class* $\overline{W}$ *of* W *(independent of* g_W*) and has log-character*

$$\widetilde{\rm Tr}_{M_0 \sqcup M_1}(\log^{\rm sgn}_{M_0 \sqcup M_1}\overline{W}) = {\rm sgn}(W). \tag{3.23}$$

For use here and elsewhere, we note the following lemma:

Lemma 3.8 *Let* $H = H_+ \oplus H_-$ *be a Hilbert space polarised by infinite-dimensional subspaces* $H_\pm$*, and let* $\Pi_\pm$ *be the orthogonal projections with ranges* $H_\pm$*. Let* P_0, P_1 *be projections on* H *with* $P_j - \Pi_+$ *of trace-class* $(j = 0,1)$ *on* H*. Let* $W_j := {\rm ran}(P_j) \subset H$*, and let* ${\rm ind}_{W_0, W_1} a$ *denote the index of a Fredholm operator* $a : W_0 \to W_1$*. Then* $P_0 - P_1$ *is trace class on* H *and* $P_1 P_0 : W_0 \to W_1$ *is a Fredholm operator, and one has*

$${\rm ind}_{W_0, W_1}(P_1 P_0) = {\rm Tr}_H(P_0 - P_1). \tag{3.24}$$

Proof: Follows in a straightforward way using the methods of §7.1 of [12]. □

Proof of Proposition 3.7: Let $\eth^W_\geq$ be the APS boundary value problem [1]. So, $\eth^W_\geq = \eth^W$ with domain restricted to those sections $s \in \Omega^+(W)$ with $\Pi^{\partial W}_\geq(s_{|\partial W}) = 0$. Then, in the notation of Lemma 3.8,

$${\rm ind}\, \eth^W_\geq = {\rm ind}_{K(\eth^W_\geq),\, {\rm ran}(\Pi^{\partial W}_\geq)}\left(\Pi^{\partial W}_\geq \circ C(\eth^W_\geq)\right) \tag{3.25}$$

with $K(\eth^W_\geq)$ in (3.18) viewed as a closed subspace of the Hilbert space $H^{\partial W}$ of L^2 boundary sections polarised with $H^{\partial W}_+ = {\rm ran}(\Pi^{\partial W}_\geq)$, $H^{\partial W}_- = {\rm ran}(\Pi^{\partial W}_<)$ (the identity (3.25) for Dirac-type operators is well known, see for instance [3], [16]). With h and η defined following (3.16) and $L(w)$ the Hirzebruch L-polynomial in the Pontryagin forms, the APS signature theorem gives the first two equalities in

$$\begin{aligned}
{\rm sgn}(W) &\overset{[1],\ {\rm Thm}\ 4.14}{=} \int_W L(w) - \eta \\
&\overset{[1],\ {\rm eqn}\ 4.7}{=} {\rm ind}\,(\eth^W_\geq) + h \\
&\overset{(3.25)}{=} {\rm ind}_{K(\eth^W_\geq),\, {\rm ran}(\Pi^{\partial W}_\geq)}\left(\Pi^{\partial W}_\geq \circ C[\eth^W_\geq]\right) + {\rm Tr}_{\partial W}(\Pi^{ev}_0) \\
&\overset{(3.24)}{=} {\rm Tr}_{\partial W}(C[\eth^W] - \Pi^{\partial W}_\geq) + {\rm Tr}_{\partial W}(\Pi^{ev}_0) \\
&= {\rm Tr}_{\partial W}(C[\eth^W] - \Pi^{\partial W}_\geq + \Pi^{ev}_0) \\
&= {\rm Tr}_{M_0 \sqcup M_1}(\kappa_\sharp(C[\eth^W] - \Pi^{\partial W}_\geq + \Pi^{ev}_0)) \\
&\overset{(2.34)}{=} \widetilde{\rm Tr}_{M_0 \sqcup M_1}(\log^{\rm sgn}_{M_0 \sqcup M_1}\overline{W}).
\end{aligned}$$

The character $\widetilde{\rm Tr}_{M_0 \sqcup M_1}(\log^{\rm sgn}_{M_0 \sqcup M_1}\overline{W}) \in \mathbb{Z}$ is thus an oriented-homotopy invariant of W. Since $\widetilde{\rm Tr}_{M_0 \sqcup M_1} : \mathsf{F}_{-\infty}(M_0 \sqcup M_1)/[\mathsf{F}_{-\infty}(M_0 \sqcup M_1), \mathsf{F}_{-\infty}(M_0 \sqcup M_1)] \overset{\cong}{\to} \mathbb{C}$ is a linear isomorphism by Lemma 2.10, $\log^{\rm sgn}_{M_0\sqcup M_1}\overline{W}$ is hence a homotopy invariant of the manifold W; that is, with $\simeq_O$ indicating oriented homotopy equivalence,

$$W\simeq_O W' \Rightarrow {\rm sgn}W={\rm sgn}W' \Rightarrow \widetilde{\rm Tr}_{M_0\sqcup M_1}(\log^{\rm sgn}_{M_0\sqcup M_1}\overline{W}-\log^{\rm sgn}_{M_0\sqcup M_1}\overline{W}')=0$$

$$\Rightarrow \log^{\rm sgn}_{M_0\sqcup M_1}\overline{W}=\log^{\rm sgn}_{M_0\sqcup M_1}\overline{W}' \quad \text{in } \mathsf{F}_{-\infty}(M_0\sqcup M_1)/[\mathsf{F}_{-\infty}(M_0\sqcup M_1),\mathsf{F}_{-\infty}(M_0\sqcup M_1)].$$

In particular, the logarithm is an invariant of the bordism class of W in ${\rm mor}_{{\rm Bord}_{4k}}(M_0,M_1)$, and independent of any choice of Riemannian metric on W.

□

It is useful to note:

Lemma 3.9 $\log^{\rm sgn}_{M_0\sqcup M_1}(\overline{W})$ *in* (3.21)*, or* (3.22)*, is unchanged if* B^{ev} *is replaced by* B^{odd}

Proof: The difference is $\pi_{M_0\sqcup M_1}\circ\kappa_\sharp\left(\Pi_0^{ev}-\Pi_0^{odd}\right)$ which has character

$$\widetilde{\rm Tr}_{M_0\sqcup M_1}(\pi_{M_0\sqcup M_1}\circ\kappa_\sharp\left(\Pi_0^{ev}-\Pi_0^{odd}\right))={\rm Tr}_{M_0\sqcup M_1}(\Pi_0^{ev}-\Pi_0^{odd})$$

which by (3.15) is zero. Since $\widetilde{\rm Tr}_{M_0\sqcup M_1}$ is a isomorphism, the assertion follows. □

We may therefore better write

$$\begin{aligned}\log^{\rm sgn}_{M_0\sqcup M_1}(\overline{W}) &= \pi_{M_0\sqcup M_1}\circ\kappa_\sharp\left(C[\eth^W]-\Pi_\geq^{\partial W}+U^{\partial W}\right)\\ &= \vartheta_{\partial W,M_0\sqcup M_1}\circ\pi_{\partial W}\left(C[\eth^W]-\Pi_\geq^{\partial W}+U^{\partial W}\right)\end{aligned}$$

with $U^{\partial W}$ denoting either of the projections; this flexibility is important later.

The principal task at hand is to show log-additivity:

Theorem 3.10 *With respect to composition of bordisms*

$${\rm mor}_{{\rm Bord}_{4k}}(M_0,M_1)\times{\rm mor}_{{\rm Bord}_{4k}}(M_1,M_2)\to{\rm mor}_{{\rm Bord}_{4k}}(M_0,M_2),\quad(\overline{W}_0,\overline{W}_1)\mapsto\overline{W}_0\cup\overline{W}_1,$$

one has in $F_{-\infty}(M_0\sqcup M_1\sqcup M_2)/[F_{-\infty}(M_0\sqcup M_1\sqcup M_2),F_{-\infty}(M_0\sqcup M_1\sqcup M_2)]$

$$\widetilde{\eta}_{M_1}\log^{\rm sgn}_{M_0\sqcup M_2}(\overline{W}_0\cup\overline{W}_1) = \widetilde{\eta}_{M_2}\log^{\rm sgn}_{M_0\sqcup M_1}(\overline{W}_0) + \widetilde{\eta}_{M_0}\log^{\rm sgn}_{M_1\sqcup M_2}(\overline{W}_1). \tag{3.26}$$

Applying the trace $\widetilde{\rm Tr}_{M_0\sqcup M_1\sqcup M_2}$ to (3.26), one has from (3.23):

Corollary 3.11

$${\rm sgn}(W\cup_{M_1}W')={\rm sgn}(W)+{\rm sgn}(W'). \tag{3.27}$$

(3.27) was originally observed by Novikov (c1967) [1] and proved for closed $W\cup_{M_1}W'$ in [2].

[1]Contrasting with (Wall) non-additivity of the signature for higher codimension partitions [19].

Corollary 3.12 $\log^{\rm sgn}_{M_0\sqcup M_1}(\overline{W}_0)$ *is independent of the boundary diffeomorphism* κ*, and so depends only on the oriented diffeomorphism class of* W *(in fact, homotopy class).* $\log^{\rm sgn}_{M_0\sqcup M_2}(\overline{W}_0\cup\overline{W}_1)$ *is independent of the gluing diffeomorphism* ϕ *between the identified boundary components of* $W_0\in\overline{W}_0$ *and* $W_1\in\overline{W}_1$ *used to form* $\overline{W}_0\cup\overline{W}_1:=\overline{W_0\cup_\phi W_1}$*. The same statements hold for* ${\rm sgn}(W_0)$ *and* ${\rm sgn}(W_0\cup_\phi W_1)$*.*

The proof of Theorem 3.10 will occupy the remainder of this section.

Proposition 3.13 *The equality* (3.26) *holds if*

$$\widetilde{\eta}_{M_1\sqcup M_1}\log^{\rm sgn}_{M_0\sqcup M_2}(\overline{W}_0\cup\overline{W}_1) \;=\; \widetilde{\eta}_{M_1\sqcup M_2}\log^{\rm sgn}_{M_0\sqcup M_1}(\overline{W}_0) \;+\; \widetilde{\eta}_{M_0\sqcup M_1}\log^{\rm sgn}_{M_1\sqcup M_2}(\overline{W}_1) \quad (3.28)$$

holds in $\mathsf{F}_{-\infty}(M_0\sqcup M_1\sqcup M_1\sqcup M_2)/[\mathsf{F}_{-\infty}(M_0\sqcup M_1\sqcup M_1\sqcup M_2),\mathsf{F}_{-\infty}(M_0\sqcup M_1\sqcup M_1\sqcup M_2)]$.

Proof:

$$\widetilde{\rm Tr}_{M_0\sqcup M_1\sqcup M_1\sqcup M_2}(\widetilde{\eta}_{M_1\sqcup M_1}\log^{\rm sgn}_{M_0\sqcup M_2}(\overline{W}_0\cup\overline{W}_1)) \overset{(2.36)}{=} \widetilde{\rm Tr}_{M_0\sqcup M_2}(\log^{\rm sgn}_{M_0\sqcup M_2}(\overline{W}_0\cup\overline{W}_1))$$
$$\overset{(2.36)}{=} \widetilde{\rm Tr}_{M_0\sqcup M_1\sqcup M_2}(\widetilde{\eta}_{M_1}\log^{\rm sgn}_{M_0\sqcup M_2}(\overline{W}_0\cup\overline{W}_1)),$$

and, similarly,

$$\widetilde{\rm Tr}_{M_0\sqcup M_1\sqcup M_1\sqcup M_2}(\widetilde{\eta}_{M_1\sqcup M_2}\log^{\rm sgn}_{M_0\sqcup M_1}(\overline{W}_0)) = \widetilde{\rm Tr}_{M_0\sqcup M_1\sqcup M_2}(\widetilde{\eta}_{M_2}\log^{\rm sgn}_{M_0\sqcup M_1}(\overline{W}_0)),$$

$$\widetilde{\rm Tr}_{M_0\sqcup M_1\sqcup M_1\sqcup M_2}(\widetilde{\eta}_{M_0\sqcup M_1}\log^{\rm sgn}_{M_1\sqcup M_2}(\overline{W}_1)) = \widetilde{\rm Tr}_{M_0\sqcup M_1\sqcup M_2}(\widetilde{\eta}_{M_0}\log^{\rm sgn}_{M_1\sqcup M_2}(\overline{W}_1)).$$

Hence, if (3.28) holds, $\widetilde{\rm Tr}_{M_0\sqcup M_1\sqcup M_2}$ evaluated on

$$\widetilde{\eta}_{M_1}\log^{\rm sgn}_{M_0\sqcup M_2}(\overline{W}_0\cup\overline{W}_1) - \widetilde{\eta}_{M_2}\log^{\rm sgn}_{M_0\sqcup M_1}(\overline{W}_0) - \widetilde{\eta}_{M_0}\log^{\rm sgn}_{M_1\sqcup M_2}(\overline{W}_1)$$

is zero. Since $\widetilde{\rm Tr}_{M_0\sqcup M_1\sqcup M_2}$ is from (2.33) a linear isomorphism, (3.26) follows. □

Corollary 3.12 allows one to work with the geometric boundary of a representative W_0 of $\overline{W}\in{\rm mor}_{{\rm Bord}_{4k}}(M_0,M_1)$, rather than M_0,M_1. Thus, $\partial W_0 = X_0^-\sqcup X_1$ along with orientation preserving diffeomorphisms $\alpha_{\partial W_0}:X_0\to M_0$ and $\beta_{\partial W_0}:X_1\to M_1$. Likewise, $W_1\in\overline{W}_1\in{\rm mor}_{{\rm Bord}_{4k}}(M_1,M_2)$ has $\partial W_1 = Y_1^-\sqcup Y_2$ and oriented diffeomorphisms $\alpha_{\partial W_1}: Y_1\to M_1$ and $\beta_{\partial W_1}:Y_2\to M_2$. Let $\phi=\alpha_{\partial W_1}^{-1}\circ\beta_{\partial W_0}: X_1\overset{\cong}{\to} Y_1$. The space $W_0\cup_\phi W_1$ formed from W_0 and W_1 by identifying $x\in X_1$ with $\phi(x)\in Y_1$ has a smooth manifold structure compatible with those of W_0 and W_1 which is unique modulo oriented diffeomorphisms which fix $M_0,\phi(X_1)=Y_1$ and M_2. Then $\overline{W}_0\cup\overline{W}_1:=\overline{W_0\cup_\phi W_1}\in{\rm mor}_{{\rm Bord}_{4k}}(M_0,M_2)$ is the equivalence class of $W_0\cup_\phi W_1$ modulo such diffeomorphisms compatible with $\alpha_{\partial W_0}$ and $\beta_{\partial W_1}$. One has, further, the closed oriented hypersurface $N=\{[x]\mid x\in X_1\}\subset W_0\cup_\phi W_1$ with $[x]$ the equivalence class in the identification space $W_0\cup_\phi W_1$. We may choose a choose a Riemannian metric on $W_0\cup_\phi W_1$ which is isometric to a product in some collar neighbourhood $U\cong(-1,1)\times N$ of N in $W_0\cup_\phi W_1$, with N identified with $\{0\}\times N\subset U$. Define, then,

$$\log_{X_0\sqcup X_1}(\overline{W}_0) := \pi_{X_0\sqcup X_1}\left(C[\widetilde{\eth}^{W_0}] - \Pi_{\geq}^{X_0^-\sqcup X_1} + U^{X_0\sqcup X_1}\right),$$

$$\log_{Y_1 \sqcup Y_2}(\overline{W}_1) := \pi_{Y_1 \sqcup Y_2}\left(C[\eth^{W_1}] - \Pi_{\geq}^{Y_1^- \sqcup Y_2} + U^{Y_1 \sqcup Y_2}\right),$$

$$\log_{X_0 \sqcup Y_2}(\overline{W}_0 \cup \overline{W}_1) := \pi_{X_0 \sqcup Y_2}\left(C[\eth^{W_0 \cup_\phi W_1}] - \Pi_{\geq}^{X_0^- \sqcup Y_2} + U^{X_0 \sqcup Y_2}\right).$$

In terms other than $\Pi_{\geq}$ the orientation is not felt and so is not indicated.

Proposition 3.14 *The equality* (3.28) *holds if*

$$\widetilde{\eta}_{X_1 \sqcup Y_1}\log_{X_0 \sqcup Y_2}(\overline{W}_0 \cup \overline{W}_1) \;=\; \widetilde{\eta}_{Y_1 \sqcup Y_2}\log_{X_0 \sqcup X_1}(\overline{W}_0) \;+\; \widetilde{\eta}_{X_0 \sqcup X_1}\log_{Y_1 \sqcup Y_2}(\overline{W}_1) \qquad (3.29)$$

holds in $\mathsf{F}_{-\infty}(X_0 \sqcup X_1 \sqcup Y_1 \sqcup Y_2)/[\mathsf{F}_{-\infty}(X_0 \sqcup X_1 \sqcup Y_1 \sqcup Y_2), \mathsf{F}_{-\infty}(X_0 \sqcup X_1 \sqcup Y_1 \sqcup Y_2)]$.

Proof: Let $V_j, Z_j, M, N \in \mathrm{ob}(\mathbf{Bord}_n)$ with V_j and Z_j diffeomorphic and M and N diffeomorphic. Let $V := V_1 \sqcup \cdots \sqcup V_m$ and $Z := Z_1 \sqcup \cdots \sqcup Z_m$. By (2.43), there are then canonical identifications $\theta_{V,Z} : \mathsf{F}_\Pi(V) \to \mathsf{F}_\Pi(Z)$ and $\vartheta_{V_N,\, Z_M} : \mathsf{F}_\Pi(V_N) \to \mathsf{F}_\Pi(Z_M)$, where

$$V_N := V_1 \sqcup \cdots X_{k-1} \sqcup N \sqcup X_k \sqcup \cdots \sqcup V_m, \quad Z_M := Z_1 \sqcup \cdots Z_{k-1} \sqcup M \sqcup Z_k \sqcup \cdots \sqcup Z_m.$$

Moreover, the following diagram is easily seen to commute

$$\begin{array}{ccc} \mathsf{F}_\Pi(V_N) & \stackrel{\vartheta_{V_N,\, Z_M}}{\longrightarrow} & \mathsf{F}_\Pi(Z_M) \\ \uparrow \widetilde{\eta}^k_N & & \uparrow \widetilde{\eta}^k_M \\ \mathsf{F}_\Pi(V) & \stackrel{\vartheta_{V,\, Z}}{\longrightarrow} & \mathsf{F}_\Pi(Z) \end{array} \qquad (3.30)$$

Hence, taking $M := M_1 \sqcup M_2, N := Y_1 \sqcup Y_2, V := X_0 \sqcup X_1, Z := M_0 \sqcup M_1, V_M := X_0 \sqcup X_1 \sqcup Y_1 \sqcup Y_2, Z_M := M_0 \sqcup M_1 \sqcup M_1 \sqcup M_2$, one has

$$\begin{aligned} \widetilde{\eta}_{M_1 \sqcup M_2}\log^{\mathrm{sgn}}_{M_0 \sqcup M_1}(\overline{W}_0) &= \widetilde{\eta}_{M_1 \sqcup M_2} \circ \vartheta_{X_0 \sqcup X_1,\, M_0 \sqcup M_1}\log_{X_0 \sqcup X_1}(\overline{W}_0) \\ &= \vartheta_{X_0 \sqcup X_1 \sqcup Y_1 \sqcup Y_2,\, M_0 \sqcup M_1 \sqcup M_1 \sqcup M_2}\left(\widetilde{\eta}_{Y_1 \sqcup Y_2}\log_{X_0 \sqcup X_1}(\overline{W}_0)\right), \end{aligned}$$

$$\widetilde{\eta}_{M_0 \sqcup M_1}\log^{\mathrm{sgn}}_{M_1 \sqcup M_2}(\overline{W}_1) = \vartheta_{X_0 \sqcup X_1 \sqcup Y_1 \sqcup Y_2,\, M_0 \sqcup M_1 \sqcup M_1 \sqcup M_2}\left(\widetilde{\eta}_{X_0 \sqcup X_1}\log_{Y_1 \sqcup Y_2}(\overline{W}_1)\right)$$

$$\widetilde{\eta}_{M_1 \sqcup M_1}\log^{\mathrm{sgn}}_{M_0 \sqcup M_2}(\overline{W}_0 \cup \overline{W}_1) = \vartheta_{X_0 \sqcup X_1 \sqcup Y_1 \sqcup Y_2,\, M_0 \sqcup M_1 \sqcup M_1 \sqcup M_2}\left(\widetilde{\eta}_{X_1 \sqcup Y_1}\log_{X_0 \sqcup Y_2}(\overline{W}_0 \cup \overline{W}_1)\right).$$

Hence $(3.28) \;=\; \underbrace{\vartheta_{X_0 \sqcup X_1 \sqcup Y_1 \sqcup Y_2,\, M_0 \sqcup M_1 \sqcup M_1 \sqcup M_2}}_{\text{linear isomorphism}}((3.29))$. □

Proposition 3.15

The equality (3.29) *holds.*

Proof: It is convenient to take $W \in \overline{W}_0$ and $W' \in \overline{W}_1$ by cutting $W_0 \cup_\phi W_1 \in \overline{W}_0 \cup \overline{W}_1$ along the hypersurface N: let $W := (W_0 \cup_\phi W_1)\backslash(W_1\backslash N), W' := (W_0 \cup_\phi W_1)\backslash(W_0\backslash N)$. Set $X := X_0, Y := Y_2$. Then

$$\partial W = X^- \sqcup N, \quad \partial W' = N^- \sqcup Y, \quad X_1 = N = Y_1. \qquad (3.31)$$

From the sequences of inclusions $N \rightrightarrows W \sqcup W' \to W \cup_\phi W'$ one has the Mayer-Vietoris type sequence

$$0 \to \Omega^+(W \cup_\phi W') \to \Omega^+(W) \oplus \Omega^+(W') \to \Omega(N) \to 0$$

in which the first map is signed restriction of a form $\omega \mapsto (\omega_{|W}, -\omega_{|W'})$ ('restriction' meaning $\sigma_{|W_k} := \iota_k^*(\sigma)$ for the inclusions $\iota_k : W_k \hookrightarrow W$) and the second the sum of the boundary restrictions $(\sigma, \sigma') \mapsto \sigma_{|N} + \sigma'_{|N}$ defined using the collar neigbourhood of the hypersurface N in W; that is, they are sections of $(\wedge T^*W)_{|N}$, which includes the normal directions to N. It is a standard fact from [1] that the latter space is canoncially identified with the space of forms on N. We assume for now that at least one of W and W' has disconnected boundary. Then the non-exact sequence Mayer Vietoris sequence becomes exact on restriction to the kernels

$$0 \to \mathrm{Ker}\,(\eth^{W \cup_\phi W'}) \to \mathrm{Ker}\,(\eth^W) \oplus \mathrm{Ker}\,(\eth^{W'}) \to \Omega(N) \to 0, \tag{3.32}$$

by observing that $\mathrm{Ker}\,(\eth^{W \cup_\phi W'})$ to be the kernel of the map $\mathrm{Ker}\,(\eth^W) \oplus \mathrm{Ker}\,(\eth^{W'}) \to \Omega(N)$. For, in a collar $U = (-1,1) \times Z$, with Z a compact boundaryless manifold, the Riemannian metric can be chosen to be a product metric $g_{|U} = du^2 + g_Y$, and so that $\eth^U = (du\wedge + i_{du})\,(\partial_u + \eth^Y)$ relative to a (self-adjoint) signature-type operator $\eth^Y$ on Y. This implies any solution ψ to $\eth^U$ has the form $\psi(u,y) = \sum_k e^{-\lambda_k u}\psi_k(0)\phi_k(y)$ for a spectral resolution (λ_k, ϕ_k) of $\eth^Y$. The metric on $W \cup_N W'$ may be chosen to be a product in a tubular neighbourhood $(-1,1) \times N$ of the partitioning hypersurface N. Hence, matching of higher normal derivatives along N of elements of $\mathrm{Ker}\,(\eth^W)$ and $\mathrm{Ker}\,(\eth^{W'})$ follows from their zeroeth order matching pointwise along N (with a change of sign taking into account the sign of u in $(-1,1)$).

In view of the isomorphism (3.18), restricting solutions to the boundaries of the manifolds W and W' refines (3.32) to an exact sequence of maps on boundary sections

$$0 \to K(\eth^{W \cup_\phi W'}) \to K(\eth^W) \oplus K(\eth^{W'}) \to \Omega^*(N) \to 0. \tag{3.33}$$

Let H^N be the space of forms $\Omega(N)$, or in the following can be taken to be its L^2 completion, on N. The sequence (3.33) fits into a diagram

$$\begin{array}{ccccccccc} 0 & \to & K(\eth^{W \cup_\phi W'}) & \to & K(\eth^W) \oplus K(\eth^{W'}) & \to & H^N & \to & 0 \\ & & \downarrow G_0 & & \downarrow G_1 & & \downarrow {\scriptstyle id} & & \\ 0 & \to & \mathrm{ran}(\Pi_>^{\partial(W \cup_\phi W')} \oplus U^{\partial(W \cup_\phi W')}) & \to & \begin{array}{c}\mathrm{ran}(\Pi_>^{\partial W} \oplus U^{\partial W}) \\ \oplus \\ \mathrm{ran}(\Pi_>^{\partial W'} \oplus U^{\partial W'})\end{array} & \to & H^N & \to & 0 \end{array} \tag{3.34}$$

where in $\Psi^0(X \sqcup N \sqcup Y)$

$$\begin{aligned} G_0 &= (\Pi_>^{\partial(W \cup_\phi W')} \oplus U^{\partial(W \cup_\phi W')}) \circ C[\eth^{W_0 \cup_\phi W_1}], \\ G_1 &= (\Pi_>^{\partial W} \oplus U^{\partial W}) \circ C[\eth^W] \;\oplus\; (\Pi_>^{\partial W'} \oplus U^{\partial W'}) \circ C[\eth^{W'}], \\ &= \big((\Pi_>^{\partial W} \oplus U^{\partial W}) \oplus\; (\Pi_>^{\partial W'} \oplus U^{\partial W'})\big) \circ C[\eth^W] \oplus C[\eth^{W'}]. \end{aligned} \tag{3.35}$$

Next we show that the diagram has exact rows and is commutative up to adding a smoothing operator to the vertical Fredholm maps. We may write relative to (3.31) and using Lemma 3.6

$$\Pi_>^{\partial W} \oplus U^{\partial W} = \begin{pmatrix} \Pi_<^X \oplus U_-^X & 0 \\ 0 & \Pi_>^N \oplus U_+^N \end{pmatrix} \quad \in \ \Psi^0(X \sqcup N)$$

with $U_+^X = \Pi_0^{ev}(B_X)$ and $U_-^X = \Pi_0^{odd}(B_X)$, mindful of Lemma 3.9. While

$$\Pi_>^{\partial W'} \oplus U^{\partial W'} = \begin{pmatrix} \Pi_<^N \oplus U_-^N & 0 \\ 0 & \Pi_>^Y \oplus U_+^Y \end{pmatrix} \quad \in \ \Psi^0(N \sqcup Y),$$

$$\Pi_>^{\partial(W \cup_\phi W')} \oplus U^{\partial(W \cup_\phi W')} = \begin{pmatrix} \Pi_<^X \oplus U_-^X & 0 \\ 0 & \Pi_>^Y \oplus U_+^Y \end{pmatrix} \quad \in \ \Psi^0(X \sqcup Y).$$

These choices for the projections $U_\pm^Y$ provide a canonical identification

$$\mathrm{ran}(\Pi_>^{\partial(W \cup_\phi W')} \oplus U^{\partial(W \cup_\phi W')}) = \mathrm{ran}(\Pi_<^X \oplus U_-^X) \oplus \mathrm{ran}(\Pi_>^Y \oplus U_+^Y)$$

and, since $(\Pi_>^N \oplus U_+^N) \oplus (\Pi_<^N \oplus U_-^N) = id_N$, a canonical identification

$$\mathrm{ran}(\Pi_>^{\partial W} \oplus U^{\partial W}) \oplus \mathrm{ran}(\Pi_>^{\partial W'} \oplus U^{\partial W'}) = \mathrm{ran}(\Pi_<^X \oplus U_-^X) \oplus H_N \oplus \mathrm{ran}(\Pi_>^Y \oplus U_+^Y), \quad (3.36)$$

hence defining the maps in the lower exact sequence of the diagram.

The exactness of the top row has been accounted for above. As $K(\eth^{W \cup_\phi W'}) \subset H_X \oplus H_Y$, an element $\zeta \in K(\eth^{W \cup_\phi W'})$ may be written uniquely as $\zeta = (\xi_X, \eta_Y)$ with $\xi_X \in H_X$, $\eta_Y \in H_Y$. For convenience, and since it does not affect any previous construction, we also include the involution $(\alpha, \beta) \mapsto (\alpha, -\beta)$ on $K(\eth^{W'}) \subset H_N \oplus H_Y$, so that the inclusion

$$K(\eth^{W \cup_\phi W'}) \to K(\eth^W) \oplus K(\eth^{W'}) \quad \text{is} \ \ (\xi_X, \eta_Y) \mapsto (\xi_X, \nu_N) \oplus (-\nu_N, \eta_Y),$$

where $\nu_N = \nu_N(\xi_X, \eta_Y)$ is uniquely defined via unique continuation and the Poisson operator; (ξ_X, η_Y) corresponds uniquely via the Poisson operator to an element of $\mathrm{Ker}\,(\eth^{W \cup_\phi W'})$, then restrict to the hypersurfaces X, N and Y.

Now replace G_1 by $\mathcal{G}_1 = ((\Pi_<^X \oplus U_-^X) \oplus I_N) \circ C[\eth^W] + (I_N \oplus (\Pi_>^Y \oplus U_+^Y)) \circ C[\eth^{W'}]$ as a map

$$K(\eth^W) \oplus K(\eth^{W'}) \to \mathrm{ran}(\Pi_<^X \oplus U_-^X) \oplus H_N \oplus \mathrm{ran}(\Pi_>^Y \oplus U_+^Y),$$

where $C[\eth^W]$ and $(\Pi_<^X \oplus U_-^X) \oplus I_N$ mean $C[\eth^W] \oplus 0$ and $(\Pi_<^X \oplus U_-^X) \oplus I_N \oplus 0$, and so on.

Lemma 3.16 *With G_1 replaced by $\mathcal{G}_1$ the diagram* (3.34) *commutes.*

Proof: $\mathcal{G}_1$ evaluated on $(\xi_X, \lambda_N) \oplus (\mu_N, \eta_Y) \in K(\eth^W) \oplus K(\eth^{W'})$ is $\mathcal{G}_1((\xi_X, \lambda_N), (\mu_N, \eta_Y)) = ((\Pi_<^X \oplus U_-^X)\xi_X, \ \lambda_N + \mu_N, \ (\Pi_>^Y \oplus U_+^Y)\eta_Y)$. With G_1 replaced by $\mathcal{G}_1$: the left-hand square of (3.34) is

$$\begin{array}{ccc} (\xi_X, \eta_Y) & \to & ((\xi_X, \lambda), (-\lambda, \eta_Y)) \\ \downarrow & & \downarrow \\ ((\Pi_<^X \oplus U_-^X)\xi_X, (\Pi_>^Y \oplus U_+^Y)\eta_Y) & \to & ((\Pi_<^X \oplus U_-^X)\xi_X, \ 0, \ (\Pi_>^Y \oplus U_+^Y)\eta_Y) \end{array}$$

and the right-hand square is

$$\begin{array}{ccc} ((\xi_X,\lambda_N),(\mu_N,\eta_Y)) & \to & \lambda_N+\mu_N \\ \downarrow & & \downarrow \\ ((\Pi^X_< \oplus U^X_-)\xi_X,\ \lambda_N+\mu_N,\ (\Pi^Y_> \oplus U^Y_+)\eta_Y) & \to & \lambda_N+\mu_N. \end{array}$$

□

Lemma 3.17

$$G_1-\mathcal{G}_1 : K(\eth^W)\oplus K(\eth^{W'}) \to \operatorname{ran}(\Pi^X_< \oplus U^X_-)\oplus H_N \oplus \operatorname{ran}(\Pi^Y_> \oplus U^Y_+)$$

is the restriction of a smoothing operator $H_X \oplus H_N \oplus H_N \oplus H_Y \to H_X \oplus H_N \oplus H_Y$.

Proof: For $(\xi_X,\lambda_N)\oplus(\mu_N,\eta_Y)\in K(\eth^W)\oplus K(\eth^{W'})$

$G_1((\xi_X,\lambda_N),(\mu_N,\eta_Y)) := ((\Pi^X_<\oplus U^X_-)\xi_X,\ (\Pi^N_>\oplus U^N_+)\lambda_N+(\Pi^N_<\oplus U^N_-)\mu_N,\ (\Pi^Y_>\oplus U^Y_+)\eta_Y)$.

Hence $(G_1-\mathcal{G}_1)((\xi_X,\lambda_N),(\mu_N,\eta_Y)) = (0,\ (\Pi^N_<\oplus U^N_-)\lambda_N+(\Pi^N_<\oplus U^N_+)\mu_N,\ 0)$. Since $U^N_\pm$ is smoothing we may ignore this term, and it is enough to show that $(\xi_X,\lambda_N)\to(0,\ \Pi^N_<\lambda_N)$ and $(\mu_N,\eta_Y)\to(\Pi^N_<\mu_N,\ 0)$ are (restrictions of) smoothing operators. For this, on $(\xi_X,\lambda_N)\in K(\eth^W)=\operatorname{ran}(C[\eth^W](\xi_X,\lambda_N))$ one has $(\xi_X,\lambda_N)=C[\eth^W](\xi_X,\lambda_N)$. Writing $C[\eth^W]=\begin{pmatrix} C^{X,X} & C^{N,X} \\ C^{X,N} & C^{N,N}\end{pmatrix}$ as a 2x2 block matrix on $H_X\oplus H_N$, we see $C^{X,N}: H_X\to H_N$ and $C^{N,X}: H_N\to H_X$ are smoothing, in view of (3.19), this gives $\lambda_N = C^{X,N}\xi_X + C^{N,N}\lambda_N$ and that the first of the maps in question is the restriction of

$$\begin{pmatrix} 0 & 0 \\ \Pi^N_< C^{X,N} & \Pi^N_< C^{N,N}\end{pmatrix} \in \Psi^{\mathbb{Z}}(X\sqcup N).$$

Since $C^{X,N}$ is smoothing, we have only to show that $\Pi^N_< C^{N,N}\in\Psi^{-\infty}(N)$. But (3.19) states $\begin{pmatrix} C^{X,X} & C^{N,X} \\ C^{X,N} & C^{N,N}\end{pmatrix} - \begin{pmatrix} \Pi^X_< & 0 \\ 0 & \Pi^N_>\end{pmatrix}\in\Psi^{-\infty}(X\sqcup N)$ and, in particular, that $C^{N,N}-\Pi^N_>\in\Psi^{-\infty}(N)$. Hence, $\Pi^N_< C^{N,N}=\Pi^N_<(C^{N,N}-\Pi^N_>)$ is smoothing. □

Since G_1 is from (3.35) the direct sum of the operators $(\Pi^{\partial W}_>\oplus U^{\partial W})\circ C[\eth^W] : K(\eth^W)\to \operatorname{ran}(\Pi^{\partial W}_>\oplus U^{\partial W})$ and $(\Pi^{\partial W'}_>\oplus U^{\partial W'})\circ C[\eth^{W'}] : K(\eth^{W'})\to\operatorname{ran}(\Pi^{\partial W'}_>\oplus U^{\partial W'})$ and from (3.25) these are Fredholm, then G_1 is a Fredholm operator with index

$$\operatorname{ind}(G_1)=\operatorname{ind}\left((\Pi^{\partial W}_>\oplus U^{\partial W})\circ C[\eth^W]\right)\ +\ \operatorname{ind}\left((\Pi^{\partial W'}_>\oplus U^{\partial W'})\circ C[\eth^{W'}]\right).$$

By Lemma 3.17 $\operatorname{ind}(G_1) = \operatorname{ind}(\mathcal{G}_1)$. By Lemma 3.16 and Lemma 5 on p.202 of [7] $\operatorname{ind}(\mathcal{G}_1) = \operatorname{ind}(G_0)+\operatorname{ind}(id_{H_N}) = \operatorname{ind}(G_0)$. Hence $\operatorname{ind}(G_0) = \operatorname{ind}(G_1)$. That is, $\Pi^{\partial(W\cup_\phi W')}_>\oplus U^{\partial(W\cup_\phi W')})\circ C[\eth^{W_0\cup_\phi W_1}]$ has index equal to $\operatorname{ind}\left((\Pi^{\partial W}_>\oplus U^{\partial W})\circ C[\eth^W]\right)$ + $\operatorname{ind}\left((\Pi^{\partial W'}_>\oplus U^{\partial W'})\circ C[\eth^{W'}]\right)$. But

$$\begin{aligned}\operatorname{ind}\left((\Pi^{\partial W}_>\oplus U^{\partial W})\circ C[\eth^W]\right) &\overset{(3.24)}{=} \operatorname{Tr}_{X\sqcup N}\left(C[\eth^W]-\Pi^{\partial W}_>\oplus U^{\partial W}\right)\\ &\overset{(2.34)}{=} \widetilde{\operatorname{Tr}}_{X\sqcup N}\left(\pi_{X\sqcup N}\left(C[\eth^W]-\Pi^{\partial W}_>\oplus U^{\partial W}\right)\right)\\ &\overset{(2.36)}{=} \widetilde{\operatorname{Tr}}_{X\sqcup N\sqcup N\sqcup Y}\left(\widetilde{\eta}_{N\sqcup Y}\left(\pi_{X\sqcup N}\left(C[\eth^W]-\Pi^{\partial W}_>\oplus U^{\partial W}\right)\right)\right),\end{aligned}$$

$$\mathrm{ind}\left((\Pi_{>}^{\partial W'}\oplus U^{\partial W'})\circ C[\eth^{W'}]\right) \;=\; \widetilde{\mathrm{Tr}}_{X\sqcup N\sqcup N\sqcup Y}\left(\widetilde{\eta}_{X\sqcup N}\left(\pi_{X\sqcup N}\left(C[\eth^{W'}]-\Pi_{>}^{\partial W'}\oplus U^{\partial W'}\right)\right)\right),$$

$$\mathrm{ind}\left((\Pi_{>}^{\partial(W\cup_\phi W')}\oplus U^{\partial(W\cup_\phi W')})\circ C[\eth^{W_0\cup_\phi W_1}]\right)$$
$$=\widetilde{\mathrm{Tr}}_{X\sqcup N\sqcup N\sqcup Y}\left(\widetilde{\eta}_{N\sqcup N}\left(\pi_{X\sqcup Y}\left(C[\eth^{W_0\cup_\phi W_1}]-\Pi_{>}^{\partial(W\cup_\phi W')}\oplus U^{\partial(W\cup_\phi W')}\right)\right)\right).$$

The (reduced) trace $\widetilde{\mathrm{Tr}}_{X\sqcup N\sqcup N\sqcup Y}$ therefore vanishes on the element

$$\widetilde{\eta}_{N\sqcup N}\left(\pi_{X\sqcup Y}\left(C[\eth^{W_0\cup_\phi W_1}]-\Pi_{>}^{\partial(W\cup_\phi W')}\oplus U^{\partial(W\cup_\phi W')}\right)\right)$$
$$-\widetilde{\eta}_{X\sqcup N}\left(\pi_{X\sqcup N}\left(C[\eth^{W'}]-\Pi_{>}^{\partial W'}\oplus U^{\partial W'}\right)\right)-\widetilde{\eta}_{N\sqcup Y}\left(\pi_{X\sqcup N}\left(C[\eth^{W}]-\Pi_{>}^{\partial W}\oplus U^{\partial W}\right)\right)$$

in $\frac{\mathsf{F}_{-\infty}(X\sqcup N\sqcup N\sqcup Y)}{[\mathsf{F}_{-\infty}(X\sqcup N\sqcup N\sqcup Y),\mathsf{F}_{-\infty}(X\sqcup N\sqcup N\sqcup Y)]}$ By (2.33), this element is zero, which is (3.29). □

A closer look at the identity (3.26) reveals that it is equivalent to the Calderon projections fitting together with respect to gluing in the following way:

Corollary 3.18 *With $C(\eth^{W_1})^{\perp} := (I\oplus 0)-C(\eth^{W_1})\in\Psi^0(M_1\sqcup M_2)$., one has*

$$\eta_{M_1}C(\eth^{W_0\cup_{M_1}W_1})-\eta_{M_2}C(\eth^{W_0})-\eta_{M_0}C(\eth^{W_1})^{\perp}\in[\mathsf{F}_{-\infty}(M_0\sqcup M_1\sqcup M_2),\mathsf{F}_{-\infty}(M_0\sqcup M_1\sqcup M_2)].$$

3.3 Odd Chern character:

Consider a differential graded (dg-) category $\mathbf{A}$. Thus, each morphism space of $\mathbf{A}$ is a $\mathbb{Z}$-graded abelian group $\mathrm{mor}_{\mathbf{A}}(\xi,\eta)=\oplus_p\mathrm{mor}^p_{\mathbf{A}}(\xi,\eta)$ with a differential $d:\mathrm{mor}^p_{\mathbf{A}}(\xi,\eta)\to\mathrm{mor}^{p+1}_{\mathbf{A}}(\xi,\eta)$ satisfying $d\circ d=0$ and for the composition functor

$$\mathrm{mor}^p_{\mathbf{A}}(\xi,\eta)\times\mathrm{mor}^q_{\mathbf{A}}(\eta,\mu)\to\mathrm{mor}^{p+q}_{\mathbf{A}}(\xi,\mu),\qquad (f,g)\mapsto g\cdot f, \tag{3.37}$$

one has

$$d(g\cdot f)=dg\cdot f+(-1)^q g\cdot df\ \in\mathrm{mor}^{p+q+1}_{\mathbf{A}}(\xi,\mu). \tag{3.38}$$

Then, a *differential graded (dg-) log functor* is a differential graded (dg-) log-additive presimplicial map $\log:\mathcal{N}(\mathbf{C})\to\mathsf{F}(\mathbf{C}^*)$. This comprises, first, a monoidal product representation on F on $\mathbf{C}$ in which, writing $\mathsf{F}(x)=\oplus_p\mathsf{F}^p(x)$ (that is, $\mathsf{F}^p(x)=\mathrm{mor}^p_{\mathbf{A}}(\chi_x,\chi_x)$), with insertion maps $\eta^p_w:\mathsf{F}^p(x)\to\mathsf{F}^p(x_w)$ satisfying

$$d\circ\eta^p_{\underline{w}}=\eta^{p+1}_{\underline{w}}\circ d. \tag{3.39}$$

The superscript p may be omitted. $\mathsf{F}(x)$ is endowed with the graded commutator $[a,b]=a\circ b-(-1)^{|a|\,|b|}\,b\circ a$.

Secondly, setting $d\mathsf{F}(w)=\{df\mid f\in\mathsf{F}(w)=\mathrm{mor}_{\mathbf{A}}(\chi_w,\chi_w)\}$, a compatible system of *dg-logarithm maps.* Such a thing assigns to objects $x,y\in\mathrm{ob}(\mathbf{C})$ a map

$$\log_{x\otimes y}:\mathrm{mor}(x,y)\to\mathsf{F}(x\otimes y)/\left([\mathsf{F}(x\otimes y),\mathsf{F}(x\otimes y)]+d\mathsf{F}(x\otimes y)\right), \tag{3.40}$$

with range in the quotient of abelian groups $\mathsf{F}/\left([\mathsf{F},\mathsf{F}]+d\mathsf{F}\right)$, such that for $\beta\alpha \in \mathrm{mor}_z(x,y)$

$$\log_{x\otimes y}\beta\alpha = \log_{x\otimes z}\alpha + \log_{z\otimes y}\beta \tag{3.41}$$

in

$$\mathsf{F}(x\otimes z\otimes y)/\left([\mathsf{F}(x\otimes z\otimes y),\mathsf{F}(x\otimes z\otimes y)]+d\mathsf{F}(x\otimes z\otimes y)\right) \tag{3.42}$$

with respect to the covering inclusions of $\mathsf{F}(x\otimes y)$, $\mathsf{F}(x\otimes z)$ and $\mathsf{F}(z\otimes y)$ into $\mathsf{F}(x\otimes z\otimes y)$

Here, *compatibility* is the requirement that for $x,y,z\in \mathrm{ob}(\mathbf{C})$ and $\delta \in \mathrm{mor}_z(x,y)$

$$\log_{x\otimes z\otimes y}\delta - \eta_z(\log_{x\otimes y}\delta) \;\in\; [\mathsf{F}(x\otimes z\otimes y),\mathsf{F}(x\otimes z\otimes y)]+d\mathsf{F}(x\otimes z\otimes y). \tag{3.43}$$

(3.43) may be indicated by $\log_{x\otimes z\otimes y}\delta \;\approx_d\; \eta_z(\log_{x\otimes y}\delta)$ (whilst $\xi \;\approx\; \eta$ retains its meaning that $\xi-\eta\in[\mathsf{F},\mathsf{F}]$).

As with log-functors, (3.41) means $\widehat{\eta}_z(\log_{x\otimes y}\beta\alpha) = \widehat{\eta}_{\otimes y}(\log_{x\otimes z}\alpha)+\widehat{\eta}_{x\otimes}(\log_{z\otimes y}\beta)$ for the inclusions $\widehat{\eta}$ in the quotient (3.42). Such an equality is consequent on an equality for some $\mu_j,\mu'_j,\nu\in\mathsf{F}(x\otimes z\otimes y)$

$$\eta_z(\log_{x\otimes y}\beta\alpha) = \eta_{\otimes y}(\log_{x\otimes z}\alpha)+\eta_{x\otimes}(\log_{z\otimes y}\beta) + \sum_{1\le j\le n}[\mu_j,\mu'_j]+d\nu. \tag{3.44}$$

Theorem 3.19 *Let $\mathcal{C}$ be a symmetric monoidal groupoid. Let $\boldsymbol{A}=\boldsymbol{A}^{dg}$ be a differential graded k-category. Consider a monoidal functor*

$$\rho:(\mathcal{C},\otimes)\to(\mathrm{end}(\boldsymbol{A}),\oplus)$$

(with respect to the additive structure on $\boldsymbol{A}^{dg}$) of degree 0; so morphisms $\alpha\in\mathrm{mor}_{\boldsymbol{C}}(x,z)$ map to degree $p=0$ morphisms $\alpha_\rho:=\rho(\alpha)\in\mathrm{mor}^0_{\boldsymbol{A}}(\rho(x),\rho(z))$. Then for each $k\in\mathbb{N}$ there is a dg-log functor $\log_k\in\mathbb{L}\mathrm{og}(\boldsymbol{C},\boldsymbol{A}^{dg})$ defined on $\alpha\in\mathrm{mor}_{\boldsymbol{C}}(x,z)$ by

$$\log_k\alpha = (\alpha_\rho^{-1}d\alpha_\rho)^k \;\in \mathrm{end}^k_{\boldsymbol{A}}(\rho(x)) := \mathrm{mor}^k_{\boldsymbol{A}}(\rho(x),\rho(x)), \tag{3.45}$$

that is

$$\eta_z(\log_k\beta\alpha)\;\approx_d\;\eta_{\otimes y}(\log_k\alpha)+\eta_{x\otimes}(\log_k\beta) \qquad \text{in } \mathrm{end}^k_{\boldsymbol{A}}(\rho(x)\oplus\rho(z)). \tag{3.46}$$

$\log_k\alpha$ is trivial for $k\in 2\mathbb{N}$.

Proof. Since $\log_k\alpha=(\log_1\alpha)^k$ in $\mathrm{end}_{\mathbf{A}}(\rho(x))$ and $\log_1(\beta\alpha)=\log_1\alpha+\alpha_\rho^{-1}\log_1\beta\cdot\alpha_\rho$ by (3.38), then (3.46) is the equality $(\log_1\alpha+\alpha_\rho^{-1}\log_1\beta\cdot\alpha_\rho)^k\oplus 0 \;\approx_d\; \log_k\alpha\oplus\log_k\beta$, or

$$\begin{pmatrix}(\log_1\alpha+\alpha_\rho^{-1}\log_1\beta\cdot\alpha_\rho)^k & 0\\ 0 & 0\end{pmatrix}\approx_d\begin{pmatrix}\log_k a & 0\\ 0 & \log_k\beta\end{pmatrix}. \tag{3.47}$$

For this, note that in $\mathrm{end}^p_{\mathbf{A}}(\rho(x)\oplus\rho(z))$ for $b\in\mathrm{end}^p_{\mathbf{A}}(\rho(y))$

$$\begin{pmatrix}\alpha_\rho^{-1}\,b\,\alpha_\rho & 0\\ 0 & 0\end{pmatrix}=\begin{pmatrix}0 & 0\\ 0 & b\end{pmatrix}+\left[\begin{pmatrix}0 & \alpha_\rho^{-1}\,b\\ 0 & 0\end{pmatrix},\begin{pmatrix}0 & 0\\ \alpha_\rho & 0\end{pmatrix}\right], \tag{3.48}$$

that is,

$$\widehat{\eta}_{\otimes y}(\alpha_\rho^{-1}\,b\,\alpha_\rho)\;\approx\;\widehat{\eta}_{x\otimes}(b). \tag{3.49}$$

For $k=3$, for example, one has in $\mathrm{end}_{\mathbf{A}}(\rho(x))$

$$\begin{aligned}(\log_1\alpha+\alpha_\rho^{-1}\log_1\beta\cdot\alpha_\rho)^3 &= \log_3\alpha+(\alpha_\rho^{-1}\log_2\beta\cdot\alpha_\rho)\log_1\alpha+\ \log_2\alpha(\alpha_\rho^{-1}\log_1\beta\cdot\alpha_\rho)\\ &\quad+\log_1\alpha(\alpha_\rho^{-1}\log_1\beta\cdot\alpha_\rho)\log_1\alpha+(\alpha_\rho^{-1}\log_1\beta\cdot\alpha_\rho)\log_2\alpha\\ &\quad+(\alpha_\rho^{-1}\log_1\beta\cdot\alpha_\rho)\log_1\alpha(\alpha_\rho^{-1}\log_1\beta\cdot\alpha_\rho)+\alpha_\rho^{-1}\log_3\beta\cdot\alpha_\rho\end{aligned}$$

— cycling morphisms around modulo graded commutators —

$$\begin{aligned}&\approx\ \log_3\alpha+\alpha_\rho^{-1}\log_3\beta\cdot\alpha_\rho+\ 3\log_1\alpha(\alpha_\rho^{-1}\log_2\beta\cdot\alpha_\rho)+3\log_2\alpha(\alpha_\rho^{-1}\log_1\beta\cdot\alpha_\rho)\\ &=\log_3\alpha+\alpha_\rho^{-1}\log_3\beta\cdot\alpha_\rho+d(-3\log_1\alpha(\alpha_\rho^{-1}\log_1\beta\cdot\alpha_\rho)).\end{aligned}$$

Using (3.39), (3.48) / (3.49) this gives (3.47) for $k=3$. For the general case:

$$d\log_{2m-1}\alpha=-\log_{2m}\alpha, \tag{3.50}$$

$$d\,(\alpha_\rho^{-1}\log_{2m-1}\beta\cdot\alpha_\rho)\ \approx\ -\alpha_\rho^{-1}\log_{2m}\beta\cdot\alpha_\rho, \tag{3.51}$$

in $\mathrm{end}_{\mathbf{A}}(\rho(x))$. Hence $d\log_{2m}\alpha=0$, $d\,(\alpha_\rho^{-1}\log_{2m}\beta\cdot\alpha_\rho)\ \approx\ 0$, and so by (3.50) $\log_{2m}$ is trivial in $\mathsf{F}/([\mathsf{F},\mathsf{F}]+d\mathsf{F})$. For general k, as for $k=3$, it is enough to show

$$(\log_1\alpha+\alpha_\rho^{-1}\log_1\beta\cdot\alpha_\rho)^k-\log_k\alpha-\alpha_\rho^{-1}\log_k\beta\cdot\alpha_\rho\ \approx_d\ 0 \tag{3.52}$$

in $\mathrm{end}_{\mathbf{A}}(\rho(x))$. The left-hand side of (3.52) has the form

$$\sum\log_{m_1}\alpha\,(\alpha_\rho^{-1}\log_{n_1}\beta\cdot\alpha_\rho)\cdots\log_{m_l}\alpha\,(\alpha_\rho^{-1}\log_{n_l}\beta\cdot\alpha_\rho^{-1}) \tag{3.53}$$

summed over $m_j,n_j\in\mathbb{N}$ with $\sum_j m_j+n_j=k$ and $\sum_j m_j\neq0$, $\sum_j n_j\neq0$. Since $\log_m\alpha=\log_{m-1}\alpha\log_1\alpha=\log_1\alpha\log_{m-1}\alpha$, and for β, the degree one elements $\log_1\alpha$ or $\alpha_\rho^{-1}\log_1\beta\cdot\alpha_\rho$ may be cycled around with $\log_1\alpha\cdot\omega\ \approx\ (-1)^{k-1}\omega\cdot\log_1\alpha$ (and likewise for $\alpha_\rho^{-1}\log_1\beta\cdot\alpha_\rho$), and so (3.53) for $k=2p-1$ can be replaced by an $\approx$ sum

$$\sigma=\sum C_{m,n}\log_{m_1}\alpha\,(\alpha_\rho^{-1}\log_{n_1}\beta\cdot\alpha_\rho)\cdots\log_{m_r}\alpha\,(\alpha_\rho^{-1}\log_{n_r}\beta\cdot\alpha_\rho^{-1}),\qquad C_{m,n}\in\mathbb{N},$$

with all summands *inequivalent* up to $\approx$ and each summand containing an odd number of odd integers m_j,n_j and at least one even integer.

Strings of the form $\sigma_b=\log_{2m}\alpha\,(\alpha_\rho^{-1}\log_{n_1}\beta\cdot\alpha_\rho)\cdots\log_{m_r}\alpha\,(\alpha_\rho^{-1}\log_{n_r}\beta\cdot\alpha_\rho^{-1})$ (resp. $(\alpha_\rho^{-1}\log_{2n}\beta\cdot\alpha_\rho)\cdots\log_{m_r}\alpha)$ with all *odd* integer degrees m_j,n_j other than $m_1=2m$ (resp. $n_1=2n$) are present in

$$-\,d\left(\log_{2m-1}\alpha\,(\alpha_\rho^{-1}\log_{n_1}\beta\cdot\alpha_\rho)\cdots\log_{m_r}\alpha\,(\alpha_\rho^{-1}\log_{n_r}\beta\cdot\alpha_\rho^{-1})\right)$$

(resp. in $d(\alpha_\rho^{-1}\log_{2n-1}\beta\cdot\alpha_\rho)\cdots\log_{m_r}\alpha)$). so the sum of such strings is in $d\mathsf{F}(x)$. Any summand in σ is $\approx$ to a product of such strings and so occurs in an expansion of $d\tau_1\cdots d\tau_l=d(\tau_1 d\tau_2\cdots d\tau_l)$ for some τ_j. Repeating with the remaining summands in σ yields distinct summands from the previous ones, otherwise all would repeat including the initial one (assumed distinct), showing σ to be $\approx$ to an element of $d\mathsf{F}(x)$. □

When considered on a monoid $\log_k$ gives the odd-Chern characters of algebraic K_1 and K_{-1} topological K-theory. For a general groupoid one has the following.

If $\mathbf{C}$ is a topological category (one enriched over the category of topological spaces) the associated homotopy category $h\mathbf{C}$ has the same objects as $\mathbf{C}$ but $\mathrm{mor}_{h\mathbf{C}}(x,y) = [x,y]_{\mathbf{C}} := \pi_0(\mathrm{mor}_{\mathbf{C}}(x,y))$ is the set of homotopy classes of $\mathbf{C}$-morphisms $x \to y$. A dg-category $\mathbf{A}$ is tracial if endowed with a trace with $d\tau = \tau \circ d$; the trace is *closed* if $d\tau = 0$ for the differential d on $\mathrm{mor}_{\mathbf{A}}(1,1)$. $\mathbf{A}$ is a normed category if it is a topological category whose topologies are defined by norms.

Corollary 3.20 *If $\boldsymbol{C}$ is a topological category, and the dg-category $\boldsymbol{A}$ is normed, then*

$$\log_k \in \mathbb{L}\mathrm{og}(h\boldsymbol{C}, \mathsf{F}(\boldsymbol{C}^*)). \tag{3.54}$$

If $\boldsymbol{A} = (\boldsymbol{A}, \tau)$ is tracial with closed trace, then for $\alpha \in [x,y]_{\boldsymbol{C}}$

$$\tau(\log_k \alpha) \in H^{2k-1}(\mathrm{mor}_{\boldsymbol{A}}(1,1)), \tag{3.55}$$

where $H^*(K)$ is the cohomology of a chain complexK.

Proof: Consider a smooth (as $\mathbf{A}$ is normed) path $t \mapsto \alpha(t) \in \mathrm{mor}_{\mathbf{C}}(x,y)$, and hence a path of logarithms $t \mapsto \log_k \alpha_\rho := \log_k \rho(\alpha(t)) \in \mathsf{F}(x)$ for each $k \in \mathbb{N}$. Then in $\mathsf{F}(x)$ one has $\partial_t(\alpha_\rho^{-1} d\alpha_\rho) - d(\alpha_\rho^{-1}\dot{\alpha}_\rho) = [\alpha_\rho^{-1} d\alpha_\rho, \alpha_\rho^{-1}\dot{\alpha}_\rho]$, and, more generally,

$$\partial_t((\alpha_\rho^{-1} d\alpha_\rho)^{2k+1}) = \left(d(\alpha_\rho^{-1}\dot{\alpha}_\rho) + [\alpha_\rho^{-1} d\alpha_\rho, \alpha_\rho^{-1}\dot{\alpha}_\rho]\right)(\alpha_\rho^{-1} d\alpha_\rho)^{2k}$$
$$+(\alpha_\rho^{-1} d\alpha_\rho)\left(d(\alpha_\rho^{-1}\dot{\alpha}_\rho) + [\alpha_\rho^{-1} d\alpha_\rho, \alpha_\rho^{-1}\dot{\alpha}_\rho]\right)(\alpha_\rho^{-1} d\alpha_\rho)^{2k-1} + \dots$$

Modulo graded commutators, the factors $(\alpha_\rho^{-1} d\alpha_\rho)^r$ can be cycled around to see

$$\partial_t((\alpha_\rho^{-1} d\alpha_\rho)^{2k+1}) - (2k+1)\left(d(\alpha_\rho^{-1}\dot{\alpha}_\rho) + [\alpha_\rho^{-1} d\alpha_\rho, \alpha_\rho^{-1}\dot{\alpha}_\rho]\right)(\alpha_\rho^{-1} d\alpha_\rho)^{2k} \ \approx\ 0.$$

Since $\left(d(\alpha_\rho^{-1}\dot{\alpha}_\rho) + [\alpha_\rho^{-1} d\alpha_\rho, \alpha_\rho^{-1}\dot{\alpha}_\rho]\right)(\alpha_\rho^{-1} d\alpha_\rho)^{2k}$ equals $d\left(\alpha_\rho^{-1}\dot{\alpha}_t(d(-\alpha_\rho^{-1} d\alpha_\rho))^k\right)$ one has $\partial_t(\log_{2k-1}\alpha_\rho) \ \approx_d\ 0$, so $\log_{2k-1}\alpha_\rho$ is independent of the homotopy class of $\alpha(t)$. Since $d\tau(\log_{2k-1}\alpha) = \tau(d\log_{2k-1}\alpha) = -\tau((\alpha_\rho^{-1} d\alpha_\rho)^{2k})$ and $(\alpha_\rho^{-1} d\alpha_\rho)^{2k} = (\alpha_\rho^{-1} d\alpha_\rho)^{2k-1}\alpha_\rho^{-1} d\alpha_\rho \approx -\alpha_\rho^{-1} d\alpha_\rho(\alpha_\rho^{-1} d\alpha_\rho)^{2k-1} \ \approx\ -(\alpha_\rho^{-1} d\alpha_\rho)^{2k}$, then (3.55) follows. □

The cohomology classes so defined form the components of odd Chern character classes when constrtucted for the additive category$(\mathbf{C}, \oplus)$ whose objects are (complex) vector bundles $\mathcal{E} \to M$ with flat connection $\nabla^{\mathcal{E}}$, with $\mathrm{mor}_{\mathbf{C}}(\mathcal{E}, \mathcal{E}') = C^\infty(M, \mathrm{Gl}(\mathcal{E}, \mathcal{E}'))$ the space of bundle isomorphisms. Then $h\mathbf{C}$ has the same objects as $\mathbf{C}$ but

$$\mathrm{mor}_{h\mathbf{C}}(\mathcal{E}, \mathcal{E}') = [M, \mathrm{Gl}(\mathcal{E}, \mathcal{E}')]$$

is the set of homotopy classes of vector bundle isomorphisms $\mathcal{E} \to \mathcal{E}'$. Let $\mathbf{A}$ likewise have the same objects as $\mathbf{C}$ and with morphisms the chain complex of de Rham forms

$$\mathrm{mor}_{\mathbf{A}}(\mathcal{E}, \mathcal{E}') = \Omega(M, \mathrm{Hom}(\mathcal{E}, \mathcal{E}'))$$

with differential the flat connection $\nabla^{\mathcal{E},\mathcal{E}'}$ on $\mathrm{Hom}(\mathcal{E}, \mathcal{E}')$ induced $\mathcal{E}$ and $\mathcal{E}'$. In particular, with 1 the trivial line bundle $\mathrm{mor}_{\mathbf{A}}(1,1) = \Omega(M)$, so that $H^{2k-1}(\mathrm{mor}_{\mathbf{A}}(1,1)) = H^{2k-1}(M)$ is de Rham (singular) cohomology of M, while the unique trace tr_x on $\mathrm{End}(\mathcal{E}_x)$ defines the closed graded (super) trace $\mathrm{Tr}_s : \Omega(M, \mathrm{Hom}(\mathcal{E}, \mathcal{E}')) \to \Omega(M)$. $\mathbf{A}$ is enriched over the category $\mathbf{M}$ whose objects are de Rham complexes $\Omega(M,\mathcal{F})$ of forms for vector bundles $\mathcal{F}\to M$ and whose morphisms are pull-backs by bundle homomorphisms. Via the inclusion functor $\rho=\iota : C^\infty(M,\mathrm{Gl}(\mathcal{E},\mathcal{E}'))\to\Omega(M,\mathrm{Hom}(\mathcal{E},\mathcal{E}'))$ one then has logarithms

$$\log_{2k-1} : [M,\mathrm{Gl}(\mathcal{E},\mathcal{E}')]\to\Omega^{2k-1}(M,\mathrm{End}\,\mathcal{E}),\quad \log_{2k-1} g=(g^{-1}\nabla^{\mathcal{E},\mathcal{E}'}g)^{2k-1},$$

whose characters define cohomology classes $c_k(\mathcal{E},\mathcal{E}')=\mathrm{Tr}_s\left((g^{-1}\nabla^{\mathcal{E},\mathcal{E}'}g)^{2k-1}\right)\in H^{2k-1}(M)$.

3.4 Torsion:

We interpret results from [9] which identify generalised Euler numbers with exotic analytic torsion.

Consider the category h-$\mathbf{Bord}_n$ of h-(co)bordisms in which an object is a pair (M,ρ) with $M\in\mathrm{ob}(\mathbf{Bord}_n)$ a smooth closed manifold of dimension n augmented with an acyclic orthogonal finite dimension representation $\rho_M:\pi_1(M)\to O(m)$, or, equivalently, with a flat vector bundle $E_{\rho_M}\to M$ with vanishing de-Rham cohomology. A morphism $\overline{W}\in\mathrm{mor}_{h\text{-}\mathbf{Bord}_n}(M,M')$ is a bordism $\overline{W}\in\mathrm{mor}_{\mathbf{Bord}_n}(M,M')$ for which the inclusion maps $\iota_M : M\hookrightarrow W$ and $\iota_{M'} : M'\hookrightarrow W$ are homotopy equivalences and such that $j^*_{M,M'}(\rho_{M'})=\rho_M$, or, equivalently, that $j^*_{M,M'}(E_{\rho_{M'}})=E_{\rho_M}$ for the induced homotopy equivalence $j_{M,M'} : M\to M'$. h-$\mathbf{Bord}_n$ inherits from $\mathbf{Bord}_n$ its symmetric monoidal structure and h-$\mathbf{Bord}^*$ is the corresponding subcategory of $\mathbf{Bord}^*_n$.

Make use of a slightly modified version of Example 2.1.2 to define the pretracial monoidal product representation $\mathsf{F}_h :\ h\text{-}\mathbf{Bord}^*\to\mathbf{Alg}_{\mathsf{F}}$ which takes $M=M_1\sqcup\cdots\sqcup M_r$ to the matrix algebra $\mathsf{F}_h(M):=\mathsf{F}_{\mathbb{Z},-\infty}(M)$ of ψdos operators $(T_{i,j})$ acting on sections of $E_{\rho_M}\otimes\wedge^*(T^*M)$ with $T_{i,j}$ smoothing for $i\neq j$ and an integer order ψdo oscillatory kernel (2.39) for $i=j$.

For clarity we shall restrict attention to the case in which M is connected. We initially define a putative logarithm on *objects* (M,ρ) which depends on two choices: first, a Riemannian metric g_M on M associated to which in each form degree p there is a Hodge Laplacian $\Delta^E_p=(d^E_p)^*d^E_p+d^E_{p-1}(d^E_{p-1})^* : \Omega^p(M,E)\to\Omega^p(M,E)$ coupled to the flat connection on E, and, secondly, an $(n+1)$-tuple of complex numbers $\beta=(\beta_0,\ldots,\beta_n)\in\mathbb{C}^{n+1}$. Define, then,

$$L_{g_M,\beta}(M,\rho)=\pi_h\left(\frac{1}{2}\bigoplus_{p\geq 0}(-1)^p\beta_p\log\Delta^E_p\right)\quad\in\mathsf{F}_h(M)/[\mathsf{F}_h(M),\mathsf{F}_h(M)]$$

with $\pi_h:\mathsf{F}_h(M)\to\mathsf{F}_h(M)/[\mathsf{F}_h(M),\mathsf{F}_h(M)]$ the quotient map. The following is obtained by extending methods of Ray and Singer [14]:

Theorem 3.21 [9] *The element $L_{g_M,\beta}(M,\rho)$ is independent of the choice of Riemannian metric g_M if and only if $\beta_p=1$ for each p or if $\beta_p=p$ for each p.*

It follows that only in these two cases, which we denote by $L_1(M,\rho)$ and $L_p(M,\rho)$ respectively, there are associated logarithms $\log_1$ and $\log_p$ on $\mathrm{mor}_{h\text{-}\mathbf{Bord}_n}(M,M')$ defined by

$$\log_1(\overline{W}) = \log_1(M,\rho) - \log_1(M',\rho'), \quad \log_p(\overline{W}) = \log_p(M,\rho)) - \log_p(M',\rho').$$

There are, then, the corresponding characters of these logarithmic-representations, or, more precisely, there is one character and one quasi-character for each of $\log_1$ and $\log_p$. The quasi-character is the evaluation using the zeta quasi-trace MPR with $Q = \oplus_p \Delta_p^E$ in (2.45): for this, one has [9]

$$\mathrm{Tr}^{\zeta}_M(\log_1(M,\rho)) := 0. \tag{3.56}$$

and from [14]

$$\mathrm{Tr}^{\zeta}_M(\log_p(M,\rho)) := \begin{cases} 0 & \text{if } \dim M \text{ even}, \\ \tau(M,E_\rho) & \text{if } \dim M \text{ odd}, \end{cases} \tag{3.57}$$

where $\tau(M,E_\rho)$ is analytic torsion, equal by the Cheeger-Muller theorem to simplicial Reidemester torsion. The character proper, on the other hand, is the evaluation using the residue trace MPR of (2.40): for this, one has

Theorem 3.22 *[9]*

$$\mathit{res}_M(\log_1(M,\rho)) := \begin{cases} \chi(M,E) & \textit{if } \dim M \text{ even}, \\ 0 & \textit{if } \dim M \text{ odd}, \end{cases} \tag{3.58}$$

$$\mathit{res}_M(\log_p(M,\rho)) := \begin{cases} \chi_p(M,E) & \textit{if } \dim M \text{ even}, \\ 0 & \textit{if } \dim M \text{ odd}, \end{cases} \tag{3.59}$$

with $\chi(M,E) = \sum_{j\geq 0}(-1)^p \dim H^p(M,E_\rho)$ *and* $\chi({}_pM,E) = \sum_{j\geq 0}(-1)^p p \dim H^p(M,E_\rho)$.

It follows that the quasi-characters of the logarithmic functors are

$$\mathrm{Tr}^{\zeta}_M(\log_1(\overline{W})) = 0 \tag{3.60}$$

and

$$\mathrm{Tr}^{\zeta}_M(\log_p(\overline{W})) = \tau(M,E_\rho) - \tau(M',E'_{\rho'}) \tag{3.61}$$

and hence that the latter computes the Whitehead torsion $\tau^{\mathrm{Wh}}(W)$ of an even-dimensional h-cobordism

$$\mathrm{Tr}^{\zeta}_M(\log_p(\overline{W})) = \log\det{}_\rho(\tau^{\mathrm{Wh}}(W)), \tag{3.62}$$

where $\det_\rho : \mathrm{Wh}(\pi_1(M')) \to \mathbb{R}_+$ is the induced determinant on the Whitehead group - this uses a standard identification of the difference of Reidemeister torsions in (3.61) with the the right-hand side of (3.62). On the other hand, in view of (3.58), (3.59) and homotopy equivalence of M and M', the residue characters both vanish

$$\mathrm{res}_M(\log_1(\overline{W})) = 0, \quad \mathrm{res}_M(\log_p(\overline{W})) = 0. \tag{3.63}$$

Nevertheless, since Theorem 3.21 applies to any closed manifold M, if the logarithms $\log_1$ and $\log_p$ are considered on $\mathbf{Bord}_n$ and taking $\mathrm{res}^{\mathbf{c}}_M$ from (2.40) for the general case $M = M_1 \sqcup \cdots \sqcup M_m$, then the characters are non-trivial in general with

$$\mathrm{res}^{\mathbf{c}}_M(\log_p(\overline{W})) = \sum_{i=1}^m c_i\,\left(\chi_p(M_i,E_i) - \chi_p(M'_i,E'_i)\right), \tag{3.64}$$

and similarly for $\log_1(\overline{W})$. Though not entirely satisfatory in view of independence of the choice of bordism, this can be written in terms of $\chi(W)$ and $\chi(M')$, in analogy to (3.62).

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