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From non-ergodic eigenvectors to local resolvent statistics and back: a random matrix perspective

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Abstract – We study the statistics of the local resolvent and non-ergodic properties of eigenvectors for a generalised Rosenzweig-Porter $N \times N$ random matrix model, undergoing two transitions separated by a delocalised non-ergodic phase. Interpreting the model as the combination of on-site random energies $\{a_i\}$ and a structurally disordered hopping, we found that each eigenstate is delocalised over $N^{2-\gamma}$ sites close in energy $|a_j - a_i| \leq N^{1-\gamma}$ in agreement with Kravtsov *et al.* (*New. J. Phys.*, **17** (2015) 122002). Our other main result, obtained combining a recurrence relation for the resolvent matrix with insights from Dyson's Brownian motion, is to show that the properties of the non-ergodic delocalised phase can be probed studying the statistics of the local resolvent in a non-standard scaling limit.

The theoretical study of the non-equilibrium dynamics of isolated quantum systems has attracted considerable interest in recent years, partly due to advances in experiments on trapped ultra-cold atomic gases [1]. One of the most fundamental questions that arose is about the applicability of statistical mechanics to quantum systems in presence of interactions and disorder, and the related Many-body localisation (MBL) transition [2]. A system is in a MBL phase if taking interactions into account the many-body eigenstates are localised in Fock space. The Fock space can be seen as a lattice with connectivity determined by two-body interactions. Its structure is that of a very high dimensional lattice where loops are scarce, therefore reminiscent of the Bethe lattice and random regular graphs (RRG). Starting from the pioneering work [3], Anderson localization on such lattices has been considered by many as a simplified case to study questions related to the MBL transition. It attracted a lot of attention recently [4–6] because it could provide a test ground to analyse the delocalised non-ergodic or “bad metal” regime, which was predicted as an intermediate phase separating the fully delocalised and the MBL phases [2, 3]. In

this regime, eigenstates would be delocalised over a large number of configurations, but which only cover a very tiny fraction, vanishing for large system size, of the entire Fock space. Although the existence of the MBL transition is now well established (at least for one dimensional systems) [7], the understanding of the delocalised non-ergodic phase is far from being completed. Some numerical results seem to indicate its presence in many-body systems [8, 9] whereas its existence on Bethe lattices is under intense scrutiny and debated [4, 6, 10–12]. It is not clear at this stage whether the sub-diffusive behaviour found before the MBL transition [13–17] is somehow related to it.

Given this state of the art, it is therefore useful to study simpler models that could provide a playground to explore its nature and sharpen the questions about it. With this aim, the authors of Ref. [18] proposed a random matrix model, the generalised Rosenzweig-Porter (GRP) model, as a relative of the RRG with random on-site energy. This was motivated by the known relation between the RRG and Gaussian invariant ensembles of random matrix theory [19]. They showed that the GRP model indeed undergoes two transitions: a localisation transition and a separate ergodic transition, with an intermediate delocalised non-ergodic phase separating the two.

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In this work we also focus on this model. Our aim is to further characterise the intermediate phase of the GRP model. We do so by applying a technique based on a recurrence relation for the resolvent matrix, and the Dyson Brownian motion. Our main results consist in linking the statistics of the local resolvent to the properties of the mixed phase, and in combining these insights with the Brownian motion analysis to derive the scaling of the eigenstates. Besides the interest in the MBL context, our results are also relevant in other physical situations where quasi-delocalised states emerge, such as jamming [20] and random matrix theory [21,22].

The GRP random matrix model. – Following [18], we consider a generalisation of the Rosenzweig-Porter model [23], with the Hamiltonian given by a $N \times N$ Hermitian matrix

$$H = A + \frac{\mu}{N^{\gamma/2}} V, \quad (1)$$

where A is diagonal with real entries a_i , independently drawn from a probability density $p_A(a_i)$, while V belongs to the Gaussian unitary ensemble (GUE) with variance 1. Drawing analogies with the RRG, the GUE matrix V corresponds to the structural geometrical disorder, while A to the on-site disorder.¹ The parameter γ controls the relative magnitude of the two terms: it is a proxy for the strength of the on-site disorder.

For $\gamma > 2$, standard second order perturbation theory shows that the GUE term is a small regular perturbation (the perturbation of the eigenvalues is much smaller than their typical level spacing). As a consequence, the Hamiltonian is close to A and hence eigenstates are completely localised. Similarly, for $\gamma < 1$ the first term is a small regular perturbation, hence the rotationally invariant V term dominates, and the eigenstates are uniformly distributed on the unitary sphere, as for the GUE. The value $\gamma = 1$ was indeed shown to play a special role for the density of states [24–28], which is given for $\gamma \leq 1$ by the Wigner semicircle distribution, and for $\gamma > 1$ by p_A . The value $\gamma = 2$ instead governs the level statistics on the scale of the typical level spacing. Computing the spectral form factor, the unfolded two-point correlation function was shown to be universal, *i.e.* it does not depend on the specific form of p_A [18,25]. It has the Wigner-Dyson form for $\gamma < 2$, and Poisson for $\gamma > 2$. These results confirm that for $\gamma < 1$ and $\gamma > 2$ the system is respectively fully delocalised and fully localised. The regime $\gamma \in (1,2)$ instead is special: the density of states is given by p_A and not by Wigner semi-circle but nevertheless the nearest neighbours level statistics has the Wigner-Dyson form. As shown in [18,29] and discussed later on, this regime provides a simple example of a delocalised non-ergodic phase.

¹The analogy with the RRG would suggest to choose V real symmetric (GOE). We consider the unitary model because most of the literature focuses on it. Our conclusions apply to both versions of the model.

The delocalised non-ergodic phase. – The authors of Ref. [18] characterised the eigenstates for $\gamma \in (1,2)$, finding the support set [5] to be a fractal over $N^{D_1} = N^{2-\gamma}$ sites. For large N , the eigenstates are supported over a large number of sites, so they are delocalised – but only over a fraction $\propto N^{1-\gamma}$ of all sites, which tends to zero in the thermodynamic limit.

To study the spectral statistics we focus on the resolvent matrix

$$\mathbf{G}(z) = (z - H)^{-1}, \quad (2)$$

a standard tool of random matrix theory. It is a random complex function, which evaluated at $z = \lambda - i\eta$ carries information about spectral quantities at energy λ , on a scale η . The (global) resolvent is $G(z) = \text{Tr } \mathbf{G}(z)/N$, while the diagonal elements of \mathbf{G} are known as the local resolvent. The behaviour of $G(z)$ is completely featureless: in the large N limit it converges to a non-fluctuating value of order one as long as $\eta > 1/N$, as can be checked by using the spectral representation of $G(z)$. The statistics of the local resolvent can be instead used as a tool to probe localisation transitions. In general one focuses on its imaginary part for $\eta \rightarrow 0$ *after* the $N \rightarrow \infty$ limit is taken: in the localised phase the imaginary part vanishes whereas it remains finite in the delocalised phase, see, *e.g.*, [30]. As we shall show below, in order to probe the non-ergodic delocalised phase one instead needs to consider a different scaling limit and study how the statistics of the local resolvent evolves when η goes to zero as $1/N^\delta$ for $\delta < 1$.

In the next section we derive the probability distribution for the local resolvent in the delocalised non-ergodic phase. We then combine this with results from the Dyson Brownian motion analysis to get a complete picture of the non-ergodic delocalised phase.

Local resolvent statistics and non-ergodic delocalised phase. – Using the block matrix inversion formula, it is possible to derive the equality in distribution between random variables

$$G_{00}^{(N+1)}(z)^{-1} \stackrel{d}{=} z - H_{00} - \frac{\mu^2}{N^\gamma} \sum_{ij} G_{ij}^{(N)}(z) V_{0i} V_{j0}, \quad (3)$$

relating the probability distribution of the $N+1$ -dimensional local resolvent to those of the N -dimensional local resolvent and uncorrelated matrix elements.

We now argue that the sum in the RHS of (3) is self-averaging with respect to the V_{0i} s and the V_{ij} s in the large N limit. Defining the random variables $X = 1/N \sum_i G_{ii}(z) |V_{0i}|^2$ and $Y = 2/N \sum_{i < j} G_{ij}(z) V_{0i} V_{j0}$, eq. (3) can be written as

$$G_{00}^{(N+1)}(z)^{-1} \stackrel{d}{=} z - H_{00} - \frac{\mu^2}{N^{\gamma-1}} (X + Y). \quad (4)$$

The expected value $\langle G_{ii}(z) \rangle$ is of order one and does not depend on i , hence $\langle X \rangle = \langle G_{ii}(z) \rangle = G(z)$. The variance of X reads $\frac{1}{N^2} \sum_{ij} \langle G_{ii}(z) G_{jj}(z) |V_{0i}|^2 |V_{0j}|^2 \rangle_c$ and can be broken up into two terms, corresponding to $i \neq j$ and

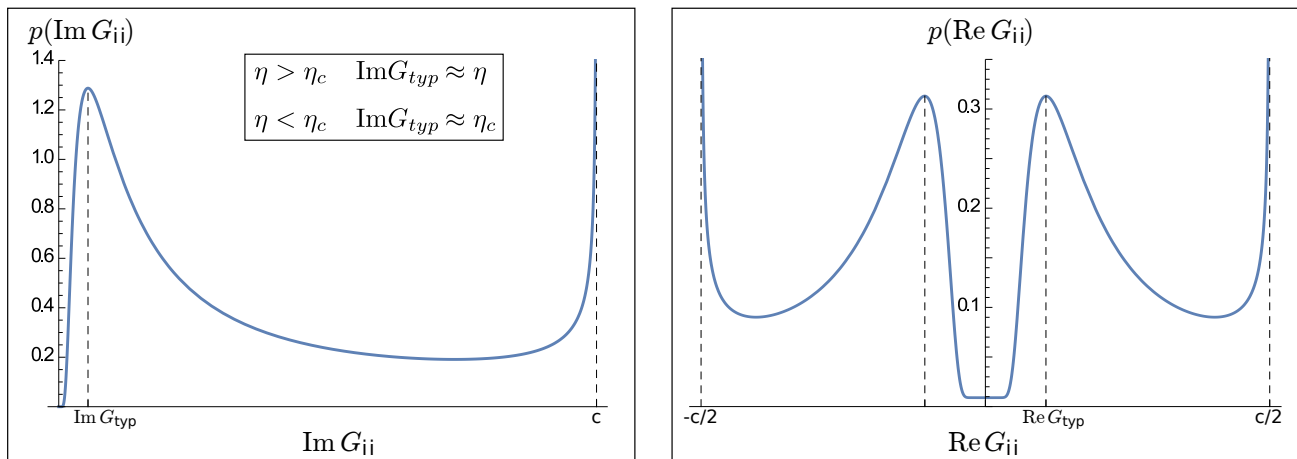


Fig. 1: Probability distributions for the imaginary (left) and real (right) parts of the local resolvent $G_{ii}(z)$. The values $\lambda = 0$, $\eta = 0.4$ (left), $\eta = 0.15$ (right) and a standard normal distribution for p_A are used for the plots. For small $\eta > \eta_c$, $\text{Im } G_{typ} \propto \eta$, for $\eta < \eta_c$, $\text{Im } G_{typ} \propto \eta_c$ while $\text{Re } G_{typ}$ is of order one in both cases. The cutoffs are controlled by $c = 1/\eta$ or $c = 1/\eta_c$ respectively.

$i = j$ respectively. By perturbative arguments [27], it can be shown that correlations between different components of the local resolvent matrix go to zero for $N \rightarrow \infty$ thus implying that the former contribution to the sum vanishes. The latter can be similarly shown to be negligible using perturbation theory in V for $\eta = 1/N^\delta$ with $\delta < 1$. As a consequence X becomes a non-fluctuating quantity, equal to $G(z)$, in the large N limit. Instead Y can be neglected since its average is zero and arguments analogous to the ones above imply that its variance vanishes. Therefore we obtain the relation

$$G_{00}(z)^{-1} \stackrel{d}{=} z - a - \frac{\mu^2}{N^{\gamma-1}} G(z) - \frac{\mu}{N^{\gamma/2}} V_{00}, \quad (5)$$

where all the randomness in the RHS comes from $a \sim p_A(a)$ and $V_{00} \sim \mathcal{N}(0, 1)$. Since all diagonal elements of \mathbf{G} are statistically equivalent, eq. (3) establishes the distribution of G_{ii}^{-1} for every i . Looking at eq. (5) one immediately realises that the values $\gamma = 1$ and $\gamma = 2$ play a special role. In our region of interest $\gamma \in (1, 2)$, the last term can be neglected.

By taking the average of the local resolvent and using eq. (5), one finds that up to corrections small in N , the global resolvent is that of A ,

$$G(z) = \bar{G}(z) = \bar{G}_A(z) = \mathcal{P} \int \frac{p_A(a)}{z - a} da, \quad (6)$$

where \mathcal{P} indicates the Cauchy principal value of the integral. Now that $G(z)$ is known, and is determined from p_A , the distribution of $G_{ii}(z)$ can be obtained.

We will study separately the real and imaginary parts of $G_{ii}(z)$ at $z = \lambda - i\eta$, taking the large N limit with η either fixed, or scaling as $\eta \approx N^{-\delta}$ ($\delta < 1$). Up to terms that can be neglected in both cases, we derive from (6)

the expressions

$$\text{Re } G_{ii}(z) = \frac{\lambda - a_i}{(\lambda - a_i)^2 + \left(\eta + \frac{\mu^2}{N^{\gamma-1}} \text{Im } \bar{G}\right)^2}, \quad (7)$$

$$\text{Im } G_{ii}(z) = \frac{\eta + \frac{\mu^2}{N^{\gamma-1}} \text{Im } \bar{G}(z)}{(\lambda - a_i)^2 + \left(\eta + \frac{\mu^2}{N^{\gamma-1}} \text{Im } \bar{G}\right)^2}. \quad (8)$$

Let us first focus on the usual scaling limit that corresponds to the large N limit with η small but fixed. The second term in the numerator of (8) then is subleading. Neglecting it, we obtain the distribution

$$p_{\text{Im } G}(x) = [\varphi_+(x) + \varphi_-(x)] \frac{\sqrt{\eta}}{2x^{3/2}\sqrt{1-\eta x}}, \quad (9)$$

where $\varphi_\pm(x) = p_A\left(\lambda \pm \sqrt{\eta(1/x - \eta)}\right)$. The distribution displays some interesting features that do not depend on the specific form of p_A . Its typical form is plotted in Fig. 1. It has a peak of height $\mathcal{O}(1/\eta)$ at $\text{Im } G_{ii} \approx \eta$; for large $\text{Im } G_{ii} \ll \eta^{-1}$ it has a power law decay

$$p_{\text{Im } G}(x) \propto p_A(\lambda) \eta^{1/2} x^{-3/2} \quad (10)$$

with a cutoff at $x = \text{Im } G_{ii} = \eta^{-1}$, where it diverges as $(\eta^{-1} - x)^{-1/2}$. Note that the $\eta \rightarrow 0^+$ limit of the distribution is singular and must be taken after the integration when computing expectation values. These features are typical of localised phases, see, *e.g.*, [30, 31]. In order to unveil that for $\gamma \in (1, 2)$ the system is instead delocalised but non-ergodic, one has to study the statistics of $\text{Im } G_{ii}(\lambda - i\eta)$ with the scaling $\eta \approx N^{-\delta}$.

Looking at eq. (8), we foresee three possible behaviours, with a critical value $\eta_c = N^{1-\gamma}$ discriminating between them.

- If $\eta \gg \eta_c$ ($\delta > \gamma - 1$) then η dominates the numerator of (8) and the previous discussion still holds.
- If $\eta \ll \eta_c$ ($\delta < \gamma - 1$) then the $\text{Im} \bar{G}$ term dominates. For finite large N the previous arguments still work, but with $\mu^2 N^{1-\gamma} \text{Im} \bar{G}$ replacing η .
- In the critical case $\eta = \eta_c$, the two terms are of the same order and they both contribute to the quantity setting the scales for the probability distribution.

A similar treatment yields the statistics of $\text{Re} G_{ii}(z)$. The result is qualitatively similar to the imaginary part, but more involved and perhaps less instructive. For the real part, the limit $\eta \rightarrow 0^+$ is not singular and its typical value is of order one. The typical plot for $p_{\text{Re}G}(x)$ is shown in Fig. 1 (right).

In summary, we find that the usual scaling ($N \rightarrow \infty$ first and $\eta \rightarrow 0$ later) is blind to the existence of the non-ergodic delocalised phase, whose existence can be instead revealed focusing on $\eta = 1/N^\delta$. For any $\delta < 1$, in a standard localised phase the typical value of $\text{Im} G_{ii}(x)$ is always of the order of η , whereas in a standard delocalised phase the typical value of $\text{Im} G_{ii}(x)$ tends to a finite value. The behaviour in the non-ergodic delocalised phase is intermediate between these two cases: the typical value of $\text{Im} G_{ii}(x)$ decreases with η , as it would happen in a localised phase, but only until the value η_c is reached. For $\eta \ll \eta_c$ it remains of the order of η_c , as it would happen in a delocalised phase (with the important difference that η_c is not of order one but vanishes as $N^{1-\gamma}$). In the next section we relate this result to the scaling of the eigenstate components using the Dyson Brownian motion technique.

Dyson Brownian motion. – The Dyson Brownian motion (DBM) is a matrix-valued stochastic process in which each element of the matrix undergoes an independent Brownian motion,

$$dM_{ij}(t) = \sqrt{\frac{(1 + \delta_{ij})\sigma^2}{2}} dW_{ij}, \quad (11)$$

where W_{ij} are independent standard complex Wiener processes, with $W_{ij} = W_{ji}^*$. With initial conditions $M(0) = 0$, $M(t)$ is a random GUE matrix with variance $\sigma^2 t$. This technique was in fact introduced by Dyson to study spectral properties of the Gaussian invariant ensembles [32].

If we consider instead initial conditions $M(0) = A$ and set $\sigma^2 = N^{-\gamma}$, at $t = \mu^2$ we obtain $M(\mu^2) = H$, the Hamiltonian (1) of the generalised Rosenzweig-Porter model. Using perturbation theory on a discretised version of (11) (or equivalently Itô calculus), stochastic differential equations can be derived, describing the evolution of the eigenvalues and eigenvectors under the Brownian evolution. Denoting the i -th eigenvalue (sorted in increasing order) by λ_i and the corresponding eigenvector $\psi^{(i)} = (\psi_1^{(i)}, \dots, \psi_N^{(i)})$, we obtain the stochastic differential equations [33, 34]

$$d\lambda_i = \frac{1}{N^\gamma} \sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j} dt + \frac{1}{N^{\gamma/2}} db_i, \quad (12)$$

$$d\psi^{(i)} = - \left[\sum_{j \neq i} \frac{1}{(\lambda_i - \lambda_j)^2} \right] \frac{\psi^{(i)}}{2N^\gamma} dt + \frac{1}{N^{\gamma/2}} \sum_{j \neq i} \frac{\psi^{(j)} db_{ij}}{\lambda_i - \lambda_j}, \quad (13)$$

with initial conditions $\lambda_i(0) = a_i$, $\psi_j^{(i)}(0) = \delta_{ij}$. The noise terms are real (b_i) and complex ($b_{ij} = b_{ji}^*$) standard Wiener processes.

It is also useful to establish the equation verified by the following quantities [33, 34]

$$u_{i|j} = [|\psi_j^{(i)}|^2], \quad (14)$$

where $[\dots]$ indicates the average over the eigenvector noise b_{ij} . Since the evolution of the eigenvalues is decoupled from that of the eigenvectors, the average $[\dots]$ does not affect the eigenvalues. Using Itô's calculus one obtains for a fixed realisation of the eigenvalues the evolution equation [34]

$$\partial_t u_{i|j}(t) = N^{-\gamma} \sum_{k \neq i} \frac{u_{k|j}(t) - u_{i|j}(t)}{(\lambda_k - \lambda_i)^2}, \quad (15)$$

with initial conditions $u_{i|j} = \delta_{ij}$.

Resolvent and DBM. – We first show that the DBM provides an alternative way to find the previous results on the resolvent for $\gamma \in (1, 2)$. To do so, we first consider the (non-averaged) eigenvalue density

$$\rho(\lambda, t) = \frac{1}{N} \sum_i \delta(\lambda - \lambda_i(t)), \quad (16)$$

and its Stieltjes transform, the resolvent

$$G(z, t) = \int \frac{\rho(\lambda, t)}{z - \lambda} d\lambda = \frac{1}{N} \text{Tr} \mathbf{G}(z, t). \quad (17)$$

Dean's equation [35] provides a way to derive from the SDE for the eigenvalues (12) a closed stochastic evolution equation for $\rho(\lambda, t)$. A Stieltjes transformation then gives a closed equation for $G(z, t)$, which is a stochastic (complex) Burgers' equation

$$\partial_t G(z, t) = -\frac{1}{N^{\gamma-1}} G(z, t) \partial_z G(z, t) + \frac{1}{N^{\gamma/2}} \bar{\eta}(z, t), \quad (18)$$

where $\bar{\eta}$ is an order one, Gaussian noise with

$$\langle \bar{\eta}(z, t) \bar{\eta}(z', t') \rangle = -\delta(t - t') \partial_z \partial_{z'} \frac{\langle G(z, t) - G(z', t') \rangle}{z - z'}. \quad (19)$$

Note that G can be written explicitly as a function of $\{\lambda_i\}$, so its evolution equation can be derived directly from eq. (12) using Itô's lemma [34]. The result is again a stochastic Burgers' equation, however the noise term appears in a less appealing form. While still quite complicated, the form (18) clarifies what the order in N of each

term is, and the transitions at $\gamma = 1$ and $\gamma = 2$ appear naturally in it.

In the intermediate phase the leading term is 0, *i.e.* $G(z, t) \approx G(z, 0) = G_A(z)$ at all t , as found previously. The first correction gives a deterministic inviscid Burgers' equation, well known in random matrix theory² [36, 37].

We can then focus on the local resolvent, averaged over the off-diagonal noise W_{ij} only [34]:

$$U_j(z, t) = [G_{jj}(z, t)] = \sum_i \frac{u_{i|j}}{z - \lambda_i(t)}. \quad (20)$$

The evolution equation for U_j is derived from equations (12,15) again using Itô's lemma. The resulting stochastic equation has a structure similar to (18). Keeping only the leading term in N for $\gamma \in (1, 2)$, we obtain the evolution equation

$$\partial_t U_j(z, t) = -\frac{1}{N^{\gamma-1}} G_A(z) \partial_z U_j(z, t). \quad (21)$$

The evolution is deterministic, and the randomness in U_j comes only from the initial condition $U_j(z, 0) = 1/(z - a_j)$. The solution to (21) is

$$U_j(z, t)^{-1} = z - a_j - \frac{t}{N^{\gamma-1}} G_A(z). \quad (22)$$

Comparing with eqs. (7,8), the result evaluated at $t = \mu^2$ coincides with what we obtained for the local resolvent G_{ii} from the recurrence equation (3). With that technique it is not necessary to take the [...] -expected value, which means that G_{ii} is self-averaging for large N with respect to the W_{ij} s.

Eigenvectors delocalisation and statistics of the local resolvent. – In order to understand the amount of delocalisation of the eigenvectors we focus on the solution of (15) at $t = \mu^2$. To extract information on the region of the Hilbert space over which the eigenstates are delocalised, we consider the following ansatz, inspired by the results on the statistics of the local resolvent and of Ref. [18]. Assume that $u_{i|j}$ is of order $N^{-\alpha}$ for $|i-j| \approx N^\alpha$ and much smaller on the remaining $\approx N - N^\alpha$ sites. Then the sum in (15) has N^α contributions, each of which is of order $N^{-2\alpha+2}$ because $\lambda_i - \lambda_j \approx N^{\alpha-1}$. Hence

$$\partial_t u_{i|j} \approx \frac{1}{N^\gamma} N^\alpha \cdot N^{-\alpha} N^{2-2\alpha} \stackrel{!}{\approx} N^{-\alpha}, \quad (23)$$

showing that the ansatz is consistent only if $\alpha = 2 - \gamma$ and thus establishing that the eigenvectors are delocalised but only on $N^{2-\gamma}$ sites. This result supports the picture that going from the localised to the intermediate phase the states spread from a single site to $\approx N^{2-\gamma}$ states closest in energy, and is compatible with the result of Ref. [18] for the fractal dimension of the eigenstates. Correspondingly,

²The treatment of the Dyson Brownian motion is normally applied to models corresponding to $\gamma = 1$ [34, 36, 37]. The standard Burgers' equation is recovered with the rescaling $t \rightarrow N^{\gamma-1}t$.

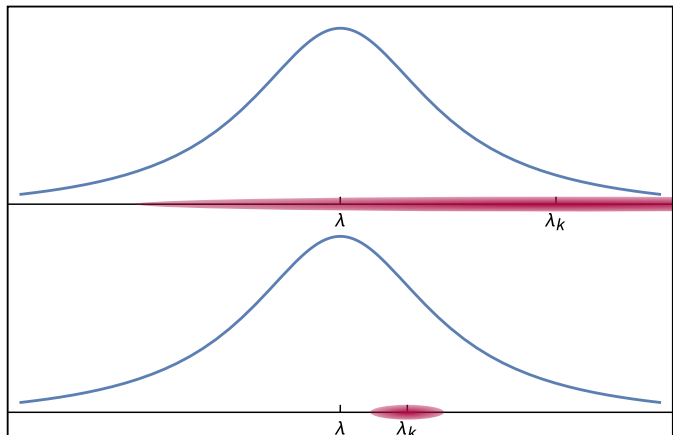


Fig. 2: Representation of the packet picture for the contributions to $\text{Im } G_{ii}$, see eq. (24). The shaded area represents the packet associated to the level λ_k , of width $\approx \eta_c = N^{1-\gamma}$. The width of the Lorentzian is η . If $\eta > \eta_c$ (lower panel) all the levels in the packet contribute approximately the same. If $\eta < \eta_c$ (upper panel) this is no longer the case.

the eigenvalues are correlated if their distance is of order $N^{1-\gamma}$ or less but become uncorrelated on larger scales. Hence one expects Poisson-like statistics on larger scales in agreement with the results found in Ref. [18].

We now show that these insights provide a natural explanation of the statistics of the local resolvent found previously. The imaginary part of the local resolvent is given by the sum

$$\text{Im } G_{ii}(\lambda - i\eta) = \sum_k \frac{|\psi_k^{(i)}|^2 \eta}{(\lambda - \lambda_k)^2 + \eta^2}. \quad (24)$$

The k -th level's contribution to the sum is given by two factors: the eigenvector weight $|\psi_k^{(i)}|^2$, and a weight depending on λ_k as a Lorentzian centred in λ . In the completely localised phase ($\gamma > 2$) $|\psi_k^{(i)}|^2 = \delta_{ki}$, so only λ_i contributes. For large N , $\lambda_i \approx a_i$ is drawn randomly from p_A . If λ is in the bulk of the spectrum, typically $|\lambda - \lambda_i| \approx 1$, giving a peak of the distribution at $\text{Im } G_{ii} \approx \eta$. If instead $|\lambda - \lambda_i| \lesssim \eta$, which happens with probability $\propto p_A(\lambda)\eta$, then $\text{Im } G_{ii} \lesssim 1/\eta$. These are the rare fluctuations that populate the tail of the distribution close to the cutoff at $1/\eta$. In between there is the regime $1 \ll |\lambda - \lambda_i| \ll \sqrt{\eta}$, where the density of states is approximately constant, corresponding to the power law $\propto p_A(\lambda) \text{Im } G^{-3/2}$.

We now turn to our region of interest: the delocalised non-ergodic phase $1 < \gamma < 2$. From the the previous discussion on eigenvector delocalisation, we know that the leading contribution comes from $\approx N^{2-\gamma}$ levels, which come in “packets” (or mini-bands) of eigenvalues of width $\eta_c = N^{1-\gamma}$, centred around λ_i . For large enough N and $\eta_c \ll \eta \ll 1$, the width of the packet is much smaller than that of the Lorentzian, so the packet behaves coherently, with all the eigenvalues in the packet being close to λ if

and only if λ_k is, and so on (see Fig. 2 - lower panel). The total contribution from the packet is identical to the single-level contribution in the localised phase. In particular, the typical value of the local resolvent is of the order of η . This explains the result we found for $\eta_c \ll \eta \ll 1$, see eq. (9).

The picture changes when the width of the Lorentzian becomes smaller than that of the eigenvalue packet. In this case, the probability that the Lorentzian overlaps with the packet is of order η_c . When this happens, the sum over k in (24) is of the order of $1/\eta_c$ independently of the value of η as long as $\eta > 1/N$. States which are $O(1)$ away from the centre of the Lorentzian are only important in determining the typical value of $\text{Im } G_{ii}$: their weights are of order $N^{-\gamma}$ [18, 33] and hence their overall contribution leads to the result $\text{Im } G_{typ} \sim N^{1-\gamma} = \eta_c$ found previously.

Conclusion. – We investigated the localisation properties of the generalised Rosenzweig-Porter model, using a recurrence relation for the local resolvent and the Dyson Brownian motion. Our main focus was the non-ergodic delocalised phase unveiled in Ref. [18], and of which we confirmed the existence using complementary techniques. Interpreting the model as the combination of on-site random energies a_i and a structurally disordered hopping, we found that each eigenstate is delocalised over $N^{2-\gamma}$ sites close in energy $|a_j - a_i| \leq N^{1-\gamma}$, in agreement with the fractal properties found in Ref. [18].

The other main result of our work is the characterisation of the statistics of the local resolvent in the non-ergodic delocalised phase. In particular, we showed that its existence can be revealed studying a non-standard scaling limit in which the small additional imaginary part η vanishes as $1/N^\delta$. The value η_c at which the statistics displays a cross-over from a behaviour characteristic of standard localised phases to a behaviour similar to the one of standard delocalised phases is equal to the typical level spacing, $1/N$, times the number of sites, $N^{2-\gamma}$, over which the eigenvectors are delocalised. Thus, from the local resolvent statistics one has a direct access to the non-ergodic properties of the delocalised phase. After the completion of this work, we became aware of [10] in which the statistics of the local resolvent in a non-standard scaling limit is also proposed and used to probe the existence of a delocalised non-ergodic phase. However, the type of cross-over and of non-ergodic delocalised phase are different from the ones studied in this work. In the case studied in [10], the delocalised non-ergodic phase should have a typical imaginary part of the local resolvent that does not vanish in the large N limit, moreover below a cross-over scale η_c the local resolvent, i.e. the local density of states, should cease to be a smooth function. This is a distinct cross-over from the one found in our work, signalling that the two non-ergodic delocalised phases are different.

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REFERENCES

- [1] J. Eisert, M. Friesdorf, and C. Gogolin, *Nature Physics* **11**, 124 (2015).
- [2] D. Basko, I. Aleiner, and B. Altshuler, *Annals of Physics* **321**, 1126 (2006).
- [3] B. L. Altshuler, Y. Gefen, A. Kamenev, and L. S. Levitov, *Phys. Rev. Lett.* **78**, 2803 (1997).
- [4] G. Biroli, A. C. Ribeiro-Teixeira, and M. Tarzia, arXiv:1211.7334.
- [5] A. De Luca, A. Scardicchio, V. E. Kravtsov, and B. L. Altshuler, arXiv:1401.0019.
- [6] A. De Luca, B. L. Altshuler, V. E. Kravtsov, and A. Scardicchio, *Phys. Rev. Lett.* **113**, 046806 (2014).
- [7] R. Nandkishore and D. A. Huse, *Annual Review of Condensed Matter Physics* **6**, 15 (2015).
- [8] M. Pino, L. B. Ioffe, and B. L. Altshuler, *Proc. Natl. Acad. Sci. U.S.A.* **113**, 536 (2016).
- [9] J. Goold *et al.*, *Phys. Rev. B* **92**, 180202 (2015).
- [10] B. Altshuler, E. Cuevas, L.B.Ioffe, and V.E.Kravtsov, arXiv:1605.02295.
- [11] K. Tikhonov, A. Mirlin, and M. Skvortsov, arXiv:1604.05353.
- [12] K. Tikhonov and A. Mirlin, arXiv:1608.00331.
- [13] Y. Bar Lev, G. Cohen, and D. R. Reichman, *Phys. Rev. Lett.* **114**, 100601 (2015).
- [14] Y. Bar Lev and D. R. Reichman, *EPL* **113**, 46001 (2016).
- [15] R. Vosk, D. A. Huse, and E. Altman, *Phys. Rev. X* **5**, 031032 (2015).
- [16] A. C. Potter, R. Vasseur, and S. A. Parameswaran, *Phys. Rev. X* **5**, 031033 (2015).
- [17] J.-y. Choi *et al.*, *Science* **352**, 1547 (2016).
- [18] V. E. Kravtsov, I. M. Khaymovich, E. Cuevas, and M. Amini, *New Journal of Physics* **17**, 122002 (2015).
- [19] I. Oren and U. Smilansky, *Journal of Physics A* **43**, 225205 (2010).
- [20] M. L. Manning and A. J. Liu, *EPL* **109**, 36002 (2015).
- [21] X. Cao, A. Rosso, J.-P. Bouchaud, and P. L. Doussal, arXiv:1607.04173.
- [22] P. Bourgade, private communication.
- [23] N. Rosenzweig and C. E. Porter, *Phys. Rev.* **120**, 1698 (1960).
- [24] E. Brézin and S. Hikami, *Nucl. Phys. B* **479**, 697 (1996).
- [25] H. Kunz and B. Shapiro, *Phys. Rev. E* **58**, 400 (1998).
- [26] A. Pandey, *Chaos, Solitons & Fractals* **5**, 1275 (1995).
- [27] A. Altland, M. Janssen, and B. Shapiro, *Phys. Rev. E* **56**, 1471 (1997).
- [28] T. Guhr, *Annals of Physics* **250**, 145 (1996).
- [29] P. Shukla, *New Journal of Physics* **18**, 021004 (2016).

- [30] E. Tarquini, G. Biroli, and M. Tarzia, *Phys. Rev. Lett.* **116**, 010601 (2016).
- [31] A. D. Mirlin and Y. V. Fyodorov, *Phys. Rev. Lett.* **72**, 526 (1994).
- [32] F. J. Dyson, *Journal of Mathematical Physics* **3** (1962).
- [33] P. Bourgade and H.-T. Yau, arXiv:1312.1301.
- [34] R. Allez, J. Bun, and J.-P. Bouchaud, arXiv:1412.7108.
- [35] D. S. Dean, *Journal of Physics A* **29**, L613 (1996).
- [36] L. C. G. Rogers and Z. Shi, *Probability Theory and Related Fields* **95**, 555 (1993).
- [37] J.-P. Blaizot and M. A. Nowak, *Phys. Rev. E* **82**, 051115 (2010).