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# Design of Fuzzy Functional Observer-Controller via Higher Order Derivatives of Lyapunov Function for Nonlinear Systems

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**Abstract**—In this paper, we investigate the stability of Takagi–Sugeno fuzzy-model-based (FMB) functional observer-control system. When system states are not measurable for state-feedback control, a fuzzy functional observer is designed to directly estimate the control input instead of the system states. Although the fuzzy functional observer can reduce the order of the observer, it leads to a number of observer gains to be determined. Therefore, a new form of fuzzy functional observer is proposed to facilitate the stability analysis such that the observer gains can be numerically obtained and the stability can be guaranteed simultaneously. The proposed form is also in favor of applying separation principle to separately design the fuzzy controller and the fuzzy functional observer. To design the fuzzy controller with the consideration of system stability, higher order derivatives of Lyapunov function (HODLF) are employed to reduce the conservativeness of stability conditions. The HODLF generalizes the commonly used first-order derivative. By exploiting the properties of membership functions and the dynamics of the FMB control system, convex and relaxed stability conditions can be derived. Simulation examples are provided to show the relaxation of the proposed stability conditions and the feasibility of designed fuzzy functional observer-controller.

**Index Terms**—Fuzzy functional observer-controller, higher order derivatives of Lyapunov function (HODLF), nonlinear system, stability analysis, Takagi–Sugeno (T–S) fuzzy model.

## I. INTRODUCTION

**S**TABILITY of nonlinear systems is complex and difficult to be systematically analyzed. Fuzzy-model-based (FMB) control scheme [1] has been proposed as an efficient approach to conduct stability analysis and control synthesis for nonlinear systems. The nonlinear systems can be separated to several linear subsystems which are smoothly combined by membership functions. In this way, linear control techniques such as state-feedback control can be applied and extended to

fuzzy state-feedback controller for nonlinear systems. To begin with, Takagi–Sugeno (T–S) fuzzy model [2] or polynomial fuzzy model [3] are established via the sector nonlinearity technique [1] (or other modeling methods) to describe the nonlinear systems. The parallel distributed compensation (PDC) [4] is then exploited to design the fuzzy controller. Based on the framework of FMB control system, the Lyapunov stability theory [4] is employed to carry out the stability analysis. The stability conditions are in terms of linear matrix inequalities (LMIs) [1], [5] or sum of squares (SOSs) [6]. By numerically solving the stability conditions via convex programming techniques, if a feasible solution exists, the stability of the closed-loop nonlinear system can be guaranteed and the feedback gains in the fuzzy controller can be obtained simultaneously.

In the development of FMB control scheme, the conservativeness of stability conditions is a critical problem which attracts researchers' attention. When solving the stability conditions, the conservativeness results in infeasible solutions, which means the feedback gains cannot be obtained. It restricts the applicability of FMB control scheme. There are several sources of conservativeness, one of which is the double fuzzy summation. Pólya's theory [7] was applied to investigate higher dimensions of fuzzy summation, which offers progressively necessary and sufficient conditions. The application of this theory also generalizes some earlier works [8]. Another source is the membership-function-independent stability conditions, which means the stability conditions do not depend on the membership functions under consideration. Therefore, the membership-function-dependent approach is exploited to make the stability conditions considering the specific membership functions, which can reduce the conservativeness. This type of approaches includes polynomial constraints [9], symbolic variables [10], and approximated membership functions [11].

Apart from the above two sources, the form of Lyapunov function affects the conservativeness meanwhile. The quadratic Lyapunov function and its first-order derivative are commonly investigated in the stability analysis [4]. To relax the stability conditions, more general types of Lyapunov function candidates have been employed such as piecewise linear Lyapunov function [12], [13], switching Lyapunov function [14], fuzzy Lyapunov function [15]–[17], and polynomial Lyapunov function [14]. Furthermore, instead of using the first-order derivative, higher order derivatives of

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Lyapunov function (HODLF) have been considered to relax the stability conditions. The HODLF was proposed in [18], and later generalized by [19]. One of the advantages in [19] is that the stability conditions are convex which can be numerically solved by convex programming techniques. However, only specific types of nonlinear systems were studied such as polynomial systems. Consequently, the HODLF should be combined with FMB control scheme such that general nonlinear systems can be dealt with. In discrete-time FMB control system, the nonmonotonic Lyapunov function [20]–[22] and the multistep Lyapunov function were investigated [23]–[26]. Similar to HODLF, they involve the difference of Lyapunov function in more steps instead of only one step. To the best of our knowledge, the HODLF has not been applied to continuous-time FMB control system. In continuous-time FMB control system, the HODLF is difficult to be exploited to relax the stability conditions due to the existence of the derivative of membership functions. The combination of HODLF and continuous-time FMB control system is important since it improves the applicability of both HODLF and FMB control scheme, which is a worthwhile investigation.

With respect to other development of FMB control scheme, it has been extended by considering various control problems [27]–[30], which also enhance the applicability of FMB control scheme since these control problems exist in real applications. The fuzzy observer [5], [31]–[38] has been investigated to estimate the system states when they are not measurable. In the case that the premise variables of membership functions are measurable, the separation principle [39] can be applied to design the fuzzy observer and fuzzy controller separately.

While the fuzzy observer is widely studied, the fuzzy functional observer receives relatively less attention. Since the ultimate goal of estimating the system states is for state-feedback control, it is more straightforward to estimate the control input instead of the system states. Moreover, the order of the functional observer is lower than the traditional observer, which reduces the complexity of the observer. In [40], the fuzzy functional observer was proposed. Although the separation principle can be exploited to separately design the fuzzy controller and fuzzy observer, a number of observer gains have to be manually designed. To ease the design procedure, the technique for linear functional observer [41] was employed to design the fuzzy functional observer in [42]. Nevertheless, the stability of the FMB functional observer-control system has to be checked after designing the feedback gains due to the nonconvex stability conditions. These limitations motivate us to explore a one-step design and extend the functional observer to nonlinear systems under the FMB control paradigm, which means the stability can be guaranteed while the feedback gains are acquired.

In this paper, we aim to enhance the applicability of FMB control scheme by relaxing the stability conditions and considering unmeasurable system states for feedback control. The HODLF in [19] is exploited to achieve the relaxation of stability analysis when designing the fuzzy controller. To tackle the difficulty of the derivative of membership functions and obtaining convex stability conditions, the technique used

in [15] is employed and improved in this paper. First, more properties of membership functions and the dynamics of FMB control system are utilized to derive convex conditions due to the occurrence of higher order terms. Second, the lower bound of the derivative of membership functions is allowed to be different from the upper bound, which leads to more relaxed conditions. Compared with existing work in discrete time [20]–[26], this is the first attempt to consider HODLF in continuous-time FMB control systems. Also, it can be demonstrated from the simulation that the proposed stability conditions from HODLF are more relaxed than those from the fuzzy Lyapunov function in [15] by comparing the stabilization region. Note that the boundary requirement of the derivative of membership functions may not be met in some cases [17]. More advanced techniques such as [16] and [17] may be applied in the future to meet the boundary requirement or to provide more relaxed conditions. Other than the relaxation of stability analysis, we design the fuzzy functional observer to estimate the control input due to the unmeasurable system states. We extend the technique for linear functional observer [41] to design the fuzzy functional observer. To facilitate the analysis, we propose a new form of fuzzy functional observer. Based on the proposed form, the separation principle [39] can be applied to design the fuzzy functional observer separately from the fuzzy controller. In addition, convex stability conditions can be derived. Compared with existing fuzzy functional observers [40], [42], the proposed fuzzy functional observer can be designed by numerically solving the stability conditions and the stability of FMB observer-control system is guaranteed simultaneously.

This paper is organized as follows. The notations, formulation of T–S fuzzy model and controller, and useful lemmas are presented in Section II. Stability analysis of FMB functional observer-control system is conducted via HODLF in Section III. Simulation examples are given in Section IV to demonstrate the proposed design procedure. Finally, the conclusion is drawn in Section V.

## II. PRELIMINARY

### A. Notation

The following notation is employed throughout this paper. The expressions of  $\mathbf{M} > 0$ ,  $\mathbf{M} \geq 0$ ,  $\mathbf{M} < 0$ , and  $\mathbf{M} \leq 0$  denote the positive, semi-positive, negative, and semi-negative definite matrices  $\mathbf{M}$ , respectively. The symbol “\*” in a matrix represents the transposed element in the corresponding position. The symbol “diag{...}” stands for a block-diagonal matrix. The superscript “ $-T$ ” represents the inverse of the transpose. The superscript “+” stands for the Moore–Penrose generalized inverse.

### B. T–S Fuzzy Model

The  $i$ th rule of the T–S fuzzy model [2], [43] representing a nonlinear plant is given as follows:

$$\text{Rule } i : \text{ IF } f_1(\mathbf{x}(t)) \text{ is } M_1^i \text{ AND } \dots \text{ AND } f_\psi(\mathbf{x}(t)) \text{ is } M_\psi^i \\ \text{ THEN } \dot{\mathbf{x}}(t) = \mathbf{A}_i \mathbf{x}(t) + \mathbf{B}_i \mathbf{u}(t)$$

where  $\mathbf{x}(t) = [x_1(t), x_2(t), \dots, x_n(t)]^T$  is the state vector, and  $n$  is the dimension of the nonlinear system;  $f_\eta(\mathbf{x}(t))$  is the

premise variable corresponding to its fuzzy term  $M_\eta^i$  in rule  $i$ ,  $\eta = 1, 2, \dots, \Psi$ , and  $\Psi$  is a positive integer;  $\mathbf{A}_i \in \mathfrak{R}^{n \times n}$  and  $\mathbf{B}_i \in \mathfrak{R}^{n \times m}$  are the known system and input matrices, respectively;  $\mathbf{u}(t) \in \mathfrak{R}^m$  is the control input vector. The dynamics of the nonlinear system is described by the following T-S fuzzy model:

$$\dot{\mathbf{x}}(t) = \sum_{i=1}^p w_i(\mathbf{x}(t))(\mathbf{A}_i \mathbf{x}(t) + \mathbf{B}_i \mathbf{u}(t)) \quad (1)$$

where  $p$  is the number of fuzzy rules;  $w_i(\mathbf{x}(t))$  is the normalized grade of membership,  $w_i(\mathbf{x}(t)) = ((\prod_{\eta=1}^{\Psi} \mu_{M_\eta^i}(f_\eta(\mathbf{x}(t)))) / (\sum_{k=1}^p \prod_{\eta=1}^{\Psi} \mu_{M_\eta^k}(f_\eta(\mathbf{x}(t)))))$ ,  $w_i(\mathbf{x}(t)) \geq 0$ ,  $i = 1, 2, \dots, p$ , and  $\sum_{i=1}^p w_i(\mathbf{x}(t)) = 1$ ;  $\mu_{M_\eta^i}(f_\eta(\mathbf{x}(t)))$ ,  $\eta = 1, 2, \dots, \Psi$ , are the grades of membership corresponding to the fuzzy term  $M_\eta^i$ .

### C. T-S Fuzzy Controller

For brevity, the time  $t$  associated with variables is dropped from now for the case without ambiguity. Using the PDC approach [4], the  $j$ th rule of the fuzzy controller is described as follows:

$$\begin{aligned} \text{Rule } j: & \text{ IF } f_1(\mathbf{x}) \text{ is } M_1^j \text{ AND } \dots \text{ AND } f_\Psi(\mathbf{x}) \text{ is } M_\Psi^j \\ & \text{ THEN } \mathbf{u} = \mathbf{G}_j \mathbf{x} \end{aligned}$$

where  $\mathbf{G}_j \in \mathfrak{R}^{m \times n}$  is the controller gain. The fuzzy controller, which is to control the nonlinear system, is given by

$$\mathbf{u} = \sum_{j=1}^p w_j(\mathbf{x}) \mathbf{G}_j \mathbf{x}. \quad (2)$$

### D. Useful Lemmas

The following lemmas are employed in the later analysis.

*Lemma 1 (HODLF):* The nonlinear system  $\dot{\mathbf{x}} = f(\mathbf{x})$  ( $f: \mathfrak{R}^n \rightarrow \mathfrak{R}^n$  has an equilibrium point at the origin) is guaranteed to be asymptotically stable if there exist Lyapunov functions  $V_1(\mathbf{x})$  and  $V_2(\mathbf{x})$  such that the following conditions are satisfied [19]:

$$W(0) = \dot{V}_2(0) + V_1(0) = 0 \quad (3)$$

$$W(\mathbf{x}) = \dot{V}_2(\mathbf{x}) + V_1(\mathbf{x}) > 0 \quad \forall \mathbf{x} \neq \mathbf{0} \quad (4)$$

$$\dot{W}(\mathbf{x}) = \ddot{V}_2(\mathbf{x}) + \dot{V}_1(\mathbf{x}) < 0 \quad \forall \mathbf{x} \neq \mathbf{0}. \quad (5)$$

*Lemma 2:* With matrices  $\mathbf{X}$  and  $\mathbf{Y}$  of appropriate dimensions and scalar  $\beta > 0$ , the following inequality holds [44]:

$$\mathbf{X}^T \mathbf{Y} + \mathbf{Y}^T \mathbf{X} \leq \beta \mathbf{X}^T \mathbf{X} + \frac{1}{\beta} \mathbf{Y}^T \mathbf{Y}.$$

## III. STABILITY ANALYSIS

In this section, we conduct the stability analysis for T-S FMB functional observer-control systems. A new form of the fuzzy functional observer will be proposed to make the augmented system in triangular form such that the separation principle can be applied. Since the fuzzy controller and fuzzy functional observer will be separately designed according to separation principle, we first design the fuzzy controller. The stability conditions are derived via HODLF. After that,

we design the fuzzy functional observer, where it will be demonstrated that the separation principle can be applied.

### A. Design of T-S Fuzzy Controller via HODLF

For brevity, the membership function  $w_i(\mathbf{x})$  is denoted as  $w_i$ . The FMB control system consisting of the T-S fuzzy model (1) and the fuzzy controller (2) is formulated as follows:

$$\dot{\mathbf{x}} = \sum_{i=1}^p \sum_{j=1}^p h_{ij} (\mathbf{A}_i + \mathbf{B}_i \mathbf{G}_j) \mathbf{x} \quad (6)$$

where  $h_{ij} \equiv w_i w_j$ .

The control objective is to make the T-S FMB control system (6) asymptotically stable, i.e.,  $\mathbf{x} \rightarrow \mathbf{0}$  as time  $t \rightarrow \infty$ , by determining the feedback gain  $\mathbf{G}_j$ .

*Theorem 1:* The FMB control system (6) with differential membership functions is guaranteed to be asymptotically stable if there exist an invertible matrix  $\mathbf{X} \in \mathfrak{R}^{n \times n}$ , matrices  $\tilde{\mathbf{P}}_{1i} = \tilde{\mathbf{P}}_{1i}^T \in \mathfrak{R}^{n \times n}$ ,  $\tilde{\mathbf{P}}_2 = \tilde{\mathbf{P}}_2^T \in \mathfrak{R}^{n \times n}$ ,  $\tilde{\mathbf{Y}}_1 = \tilde{\mathbf{Y}}_1^T \in \mathfrak{R}^{n \times n}$ ,  $\tilde{\mathbf{Y}}_2 \in \mathfrak{R}^{n \times n}$ ,  $\tilde{\mathbf{S}}_i = \tilde{\mathbf{S}}_i^T \in \mathfrak{R}^{n \times n}$ ,  $\mathbf{N}_j \in \mathfrak{R}^{m \times n}$ , and predefined scalars  $\beta_{ij} > 0$ ,  $\mu_1, \mu_2, \dots, \mu_6$ ,  $i, j = 1, 2, \dots, p$  such that the following LMI-based conditions are satisfied:

$$\tilde{\Theta}_{ij} + \tilde{\Theta}_{ji} > 0 \quad \forall i \leq j \quad (7)$$

$$\tilde{\mathbf{P}}_{1i} - \tilde{\mathbf{Y}}_1 \leq \tilde{\mathbf{S}}_i \quad \forall i \quad (8)$$

$$\tilde{\mathbf{S}}_i \geq 0 \quad \forall i \quad (9)$$

$$\Psi_{ij} + \Psi_{ji} < 0 \quad \forall i \leq j \quad (10)$$

where

$$\begin{aligned} \tilde{\Theta}_{ij} &= \begin{bmatrix} \tilde{\Theta}_{ij}^{(11)} & * \\ \tilde{\Theta}_{ij}^{(21)} & \tilde{\Theta}_{ij}^{(22)} \end{bmatrix} \\ \tilde{\Theta}_{ij}^{(11)} &= \tilde{\mathbf{P}}_{1i} + \mu_1 (\mathbf{A}_i \mathbf{X}^T + \mathbf{B}_i \mathbf{N}_j) + \mu_1 (\mathbf{A}_i \mathbf{X}^T + \mathbf{B}_i \mathbf{N}_j)^T \\ \tilde{\Theta}_{ij}^{(21)} &= \tilde{\mathbf{P}}_2 - \mu_1 \mathbf{X} + \mu_2 (\mathbf{A}_i \mathbf{X}^T + \mathbf{B}_i \mathbf{N}_j) \\ \tilde{\Theta}_{ij}^{(22)} &= -\mu_2 (\mathbf{X} + \mathbf{X}^T) \\ \Psi_{ij} &= \begin{bmatrix} \tilde{\tilde{\Theta}}_{ij} & * & * & * & \dots & * \\ \tilde{\mathbf{Y}} & \Psi^{(22)} & * & * & \dots & * \\ \tilde{\Omega}_{11} & \mathbf{0} & -\beta_{11} \mathbf{I} & * & \dots & * \\ \tilde{\Omega}_{12} & \mathbf{0} & \mathbf{0} & -\beta_{12} \mathbf{I} & \dots & * \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \tilde{\Omega}_{pp} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & -\beta_{pp} \mathbf{I} \end{bmatrix} \\ \Psi^{(22)} &= -\frac{1}{\sum_{r=1}^p \sum_{s=1}^p \beta_{rs}} \mathbf{I} \\ \tilde{\tilde{\Theta}}_{ij} &= \begin{bmatrix} \tilde{\tilde{\Theta}}_{ij}^{(11)} & * & * \\ \tilde{\tilde{\Theta}}_{ij}^{(21)} & \tilde{\tilde{\Theta}}_{ij}^{(22)} & * \\ \tilde{\tilde{\Theta}}_{ij}^{(31)} & \tilde{\tilde{\Theta}}_{ij}^{(32)} & \tilde{\tilde{\Theta}}_{ij}^{(33)} \end{bmatrix} \\ \tilde{\tilde{\Theta}}_{ij}^{(11)} &= \sum_{r=1}^p (\bar{\phi}_r - \underline{\phi}_r) \tilde{\mathbf{S}}_r + \sum_{r=1}^p \underline{\phi}_r (\tilde{\mathbf{P}}_{1r} - \tilde{\mathbf{Y}}_1) \\ &\quad + (\mathbf{A}_i \mathbf{X}^T + \mathbf{B}_i \mathbf{N}_j) + (\mathbf{A}_i \mathbf{X}^T + \mathbf{B}_i \mathbf{N}_j)^T \\ \tilde{\tilde{\Theta}}_{ij}^{(21)} &= \tilde{\mathbf{P}}_{1i} - \mathbf{X} + \mu_3 (\mathbf{A}_i \mathbf{X}^T + \mathbf{B}_i \mathbf{N}_j) \\ &\quad + \mu_4 (\mathbf{A}_i \mathbf{X}^T + \mathbf{B}_i \mathbf{N}_j)^T \end{aligned}$$

$$\begin{aligned}
 \tilde{\Xi}_{ij}^{(22)} &= 2\tilde{\mathbf{P}}_2 - \mu_3(\mathbf{X} + \mathbf{X}^T) + \mu_5(\mathbf{A}_i\mathbf{X}^T + \mathbf{B}_i\mathbf{N}_j) \\
 &\quad + \mu_5(\mathbf{A}_i\mathbf{X}^T + \mathbf{B}_i\mathbf{N}_j)^T \\
 \tilde{\Xi}^{(31)} &= \tilde{\mathbf{P}}_2 - \mu_4\mathbf{X} \\
 \tilde{\Xi}_{ij}^{(32)} &= \mu_6(\mathbf{A}_i\mathbf{X}^T + \mathbf{B}_i\mathbf{N}_j) - \mu_5\mathbf{X} \\
 \tilde{\Xi}^{(33)} &= -\mu_6(\mathbf{X} + \mathbf{X}^T) \\
 \tilde{\Upsilon} &= [\mu_4\mathbf{I} \quad \mu_5\mathbf{I} \quad \mu_6\mathbf{I}] \\
 \tilde{\Omega}_{ij} &= \left[ \rho_{ij}(\mathbf{A}_i\mathbf{X}^T + \mathbf{B}_i\mathbf{N}_j - \tilde{\mathbf{Y}}_2) \quad \mathbf{0} \quad \mathbf{0} \right]
 \end{aligned}$$

$\underline{\phi}_i$  and  $\bar{\phi}_i$  are the lower and upper bounds of  $\dot{w}_i$ , respectively;  $\rho_{ij}$  is the upper bound of  $|\dot{h}_{ij}|$ ; and the controller gains are obtained by  $\mathbf{G}_j = \mathbf{N}_j\mathbf{X}^{-T} \forall j$ .

*Proof:* To ensure the stability of (6), we employ the HODLF (Lemma 1). Choosing a fuzzy Lyapunov function candidate  $V_1(\mathbf{x}) = \mathbf{x}^T (\sum_{i=1}^p w_i \mathbf{P}_{1i}) \mathbf{x}$  and a quadratic Lyapunov function candidate  $V_2(\mathbf{x}) = \mathbf{x}^T \mathbf{P}_2 \mathbf{x}$  where  $\mathbf{P}_{1i} \in \mathfrak{N}^{n \times n} \forall i$  and  $\mathbf{P}_2 \in \mathfrak{N}^{n \times n}$  are symmetric matrices, (3) in Lemma 1 is satisfied.

*Remark 1:* In general, the Lyapunov functions  $V_1(\mathbf{x})$  and  $V_2(\mathbf{x})$  can be chosen as any candidates by users. In this paper, we aim to compare the HODLF with existing fuzzy Lyapunov function. Consequently, we choose  $V_1(\mathbf{x})$  as a fuzzy Lyapunov function candidate. It can be seen that the additional matrix  $\mathbf{P}_2$  may lead  $W(\mathbf{x})$  to provide more relaxed stability conditions than only employing fuzzy Lyapunov function  $V_1(\mathbf{x})$ . Note that the HODLF is not strictly relaxed than the compared one due to the introduction of conservativeness in the analysis.

To satisfy conditions (4) and (5) and facilitate stability analysis, the following properties are exploited [15]:

$$\begin{aligned}
 \Gamma_1 &= 2(\mathbf{x}^T \mu_{k_1} \mathbf{M} + \dot{\mathbf{x}}^T \mu_{k_2} \mathbf{M}) \\
 &\quad \times \left( \sum_{i=1}^p \sum_{j=1}^p h_{ij}(\mathbf{A}_i + \mathbf{B}_i \mathbf{G}_j) \mathbf{x} - \dot{\mathbf{x}} \right) = 0 \quad (11)
 \end{aligned}$$

$$\begin{aligned}
 \Gamma_2 &= 2(\mathbf{x}^T \mu_{k_3} \mathbf{M} + \dot{\mathbf{x}}^T \mu_{k_4} \mathbf{M} + \ddot{\mathbf{x}}^T \mu_{k_5} \mathbf{M}) \\
 &\quad \times \left( \sum_{r=1}^p \sum_{s=1}^p \dot{h}_{rs}(\mathbf{A}_r + \mathbf{B}_r \mathbf{G}_s) \mathbf{x} \right. \\
 &\quad \left. + \sum_{i=1}^p \sum_{j=1}^p h_{ij}(\mathbf{A}_i + \mathbf{B}_i \mathbf{G}_j) \dot{\mathbf{x}} - \ddot{\mathbf{x}} \right) = 0 \quad (12)
 \end{aligned}$$

$$\Gamma_3 = \sum_{r=1}^p \dot{w}_r \mathbf{Y}_1 = \mathbf{0} \quad (13)$$

$$\Gamma_4 = \sum_{r=1}^p \sum_{s=1}^p \dot{h}_{rs} \mathbf{Y}_2 = \mathbf{0} \quad (14)$$

where  $\mathbf{M} \in \mathfrak{N}^{n \times n}$  is an invertible matrix;  $\mu_{k_l} \forall k_l$  are arbitrary scalars;  $\mathbf{Y}_1 \in \mathfrak{N}^{n \times n}$  is a symmetric matrix; and  $\mathbf{Y}_2 \in \mathfrak{N}^{n \times n}$  is an arbitrary matrix.

*Remark 2:* In [15], only properties (11) and (13) are used in the analysis. In this paper, however, the terms  $\ddot{\mathbf{x}}$  and  $\dot{h}_{rs}$  appear in the analysis resulted from applying HODLF. Therefore, properties (12) and (14) are added to handle this more complex situation.

Defining the augmented vector  $\mathbf{z}_1 = [\mathbf{x}^T \quad \dot{\mathbf{x}}^T]^T$  and using property (11) with  $k_1 = 1$  and  $k_2 = 2$ , we have

$$W(\mathbf{x}) = 2\dot{\mathbf{x}}^T \mathbf{P}_2 \mathbf{x} + \mathbf{x}^T \sum_{i=1}^p w_i \mathbf{P}_{1i} \mathbf{x} + \Gamma_1 = \sum_{i=1}^p \sum_{j=1}^p h_{ij} \mathbf{z}_1^T \Theta_{ij} \mathbf{z}_1 \quad (15)$$

where

$$\Theta_{ij} = \begin{bmatrix} \Theta_{ij}^{(11)} & * \\ \Theta_{ij}^{(21)} & \Theta_{ij}^{(22)} \end{bmatrix}$$

$$\Theta_{ij}^{(11)} = \mathbf{P}_{1i} + \mu_1 \mathbf{M}(\mathbf{A}_i + \mathbf{B}_i \mathbf{G}_j) + \mu_1(\mathbf{A}_i + \mathbf{B}_i \mathbf{G}_j)^T \mathbf{M}^T$$

$$\Theta_{ij}^{(21)} = \mathbf{P}_2 - \mu_1 \mathbf{M}^T + \mu_2 \mathbf{M}(\mathbf{A}_i + \mathbf{B}_i \mathbf{G}_j)$$

$$\Theta_{ij}^{(22)} = -\mu_2(\mathbf{M} + \mathbf{M}^T)$$

and  $\mu_1$  and  $\mu_2$  are arbitrary scalars.

Therefore, condition (4) holds if  $\sum_{i=1}^p \sum_{j=1}^p h_{ij} \Theta_{ij} > 0$ . By congruence transform with premultiplying  $\text{diag}\{\mathbf{X}, \mathbf{X}\}$  and post-multiplying  $\text{diag}\{\mathbf{X}^T, \mathbf{X}^T\}$ , where  $\mathbf{X} = \mathbf{M}^{-1}$ , denoting  $\mathbf{N}_j = \mathbf{G}_j \mathbf{X}^T$ ,  $\tilde{\mathbf{P}}_{1i} = \mathbf{X} \mathbf{P}_{1i} \mathbf{X}^T$ ,  $\tilde{\mathbf{P}}_2 = \mathbf{X} \mathbf{P}_2 \mathbf{X}^T$ , and grouping the terms with the same membership functions, we obtain the stability condition (7).

To eliminate the term  $\dot{w}_i$  in the following analysis, using property (13) and assuming  $\underline{\phi}_i \leq \dot{w}_i \leq \bar{\phi}_i$ ,  $\mathbf{P}_{1i} - \mathbf{Y}_1 \leq \mathbf{S}_i \forall i$  where  $\mathbf{S}_i \geq 0$ , the time derivative of  $W(\mathbf{x})$  is:

$$\begin{aligned}
 \dot{W}(\mathbf{x}) &= \Lambda + \mathbf{x}^T \left( \sum_{r=1}^p \dot{w}_r \mathbf{P}_{1r} - \Gamma_3 \right) \mathbf{x} \\
 &= \Lambda + \mathbf{x}^T \left( \sum_{r=1}^p (\dot{w}_r - \underline{\phi}_r) (\mathbf{P}_{1r} - \mathbf{Y}_1) \right. \\
 &\quad \left. + \sum_{i=1}^p \underline{\phi}_i (\mathbf{P}_{1i} - \mathbf{Y}_1) \right) \mathbf{x} \\
 &\leq \Lambda + \mathbf{x}^T \left( \sum_{r=1}^p (\bar{\phi}_r - \underline{\phi}_r) \mathbf{S}_r \right. \\
 &\quad \left. + \sum_{i=1}^p \underline{\phi}_i (\mathbf{P}_{1i} - \mathbf{Y}_1) \right) \mathbf{x} \quad (16)
 \end{aligned}$$

where  $\Lambda = 2\dot{\mathbf{x}}^T \mathbf{P}_2 \mathbf{x} + 2\dot{\mathbf{x}}^T \mathbf{P}_2 \dot{\mathbf{x}} + 2\dot{\mathbf{x}}^T \sum_{i=1}^p w_i \mathbf{P}_{1i} \mathbf{x}$ .

*Remark 3:* In [15], it is required that  $-\bar{\phi}_i \leq \dot{w}_i \leq \bar{\phi}_i \forall i$ . However, it is not necessary to require the lower bound of  $\dot{w}_i$  to be  $\underline{\phi}_i = -\bar{\phi}_i$ . Therefore, in this paper, we consider a more general case that  $\underline{\phi}_i \leq \dot{w}_i \leq \bar{\phi}_i$ . By introducing the information of the lower bound  $\underline{\phi}_i$  and corresponding slack matrix  $\mathbf{S}_i$  in (16), more relaxed stability conditions can be obtained.

Defining the augmented vector  $\mathbf{z}_2 = [\mathbf{x}^T \quad \dot{\mathbf{x}}^T \quad \ddot{\mathbf{x}}^T]^T$  and using properties (11), (12), and (14) on (16) with  $k_2 = 3$ ,  $k_3 = 4$ ,  $k_4 = 5$ ,  $k_5 = 6$ , and  $\mu_{k_1} = 1$  (same as [15], it is redundant to keep all  $\mu_{k_l}$  as variables due to the existence of matrix variable  $\mathbf{M}$ ), we have

$$\dot{W}(\mathbf{x}) \leq \sum_{i=1}^p \sum_{j=1}^p h_{ij} \mathbf{z}_2^T \left( \Xi_{ij} + \sum_{r=1}^p \sum_{s=1}^p (\Upsilon^T \Omega_{rs} + \Omega_{rs}^T \Upsilon) \right) \mathbf{z}_2 \quad (17)$$

where

$$\mathbf{E}_{ij} = \begin{bmatrix} \mathbf{E}_{ij}^{(11)} & * & * \\ \mathbf{E}_{ij}^{(21)} & \mathbf{E}_{ij}^{(22)} & * \\ \mathbf{E}_{ij}^{(31)} & \mathbf{E}_{ij}^{(32)} & \mathbf{E}_{ij}^{(33)} \end{bmatrix}$$

$$\mathbf{E}_{ij}^{(11)} = \sum_{r=1}^p (\bar{\phi}_r - \underline{\phi}_r) \mathbf{S}_r + \sum_{r=1}^p \underline{\phi}_r (\mathbf{P}_{1r} - \mathbf{Y}_1) + \mathbf{M}(\mathbf{A}_i + \mathbf{B}_i \mathbf{G}_j) + (\mathbf{A}_i + \mathbf{B}_i \mathbf{G}_j)^T \mathbf{M}^T$$

$$\mathbf{E}_{ij}^{(21)} = \mathbf{P}_{1i} - \mathbf{M}^T + \mu_3 \mathbf{M}(\mathbf{A}_i + \mathbf{B}_i \mathbf{G}_j) + \mu_4 (\mathbf{A}_i + \mathbf{B}_i \mathbf{G}_j)^T \mathbf{M}^T$$

$$\mathbf{E}_{ij}^{(22)} = 2\mathbf{P}_2 - \mu_3 (\mathbf{M} + \mathbf{M}^T) + \mu_5 \mathbf{M}(\mathbf{A}_i + \mathbf{B}_i \mathbf{G}_j) + \mu_5 (\mathbf{A}_i + \mathbf{B}_i \mathbf{G}_j)^T \mathbf{M}^T$$

$$\mathbf{E}_{ij}^{(31)} = \mathbf{P}_2 - \mu_4 \mathbf{M}^T$$

$$\mathbf{E}_{ij}^{(32)} = \mu_6 \mathbf{M}(\mathbf{A}_i + \mathbf{B}_i \mathbf{G}_j) - \mu_5 \mathbf{M}^T$$

$$\mathbf{E}_{ij}^{(33)} = -\mu_6 (\mathbf{M} + \mathbf{M}^T)$$

$$\mathbf{\Upsilon} = [\mu_4 \mathbf{M}^T \quad \mu_5 \mathbf{M}^T \quad \mu_6 \mathbf{M}^T]$$

$$\mathbf{\Omega}_{ij} = [\dot{h}_{ij}(\mathbf{A}_i + \mathbf{B}_i \mathbf{G}_j - \mathbf{Y}_2) \quad \mathbf{0} \quad \mathbf{0}]$$

and  $\mu_3 - \mu_6$  are arbitrary scalars.

To eliminate the term  $\dot{h}_{ij}$  in  $\mathbf{\Omega}_{ij}$ , assuming  $|\dot{h}_{ij}| \leq \rho_{ij}$  and using Lemma 2 and the property that  $(\mathbf{A}_i + \mathbf{B}_i \mathbf{G}_j - \mathbf{Y}_2)^T (\mathbf{A}_i + \mathbf{B}_i \mathbf{G}_j - \mathbf{Y}_2) \geq \mathbf{0} \forall i, j$ , condition (5) holds if

$$\begin{aligned} & \sum_{i=1}^p \sum_{j=1}^p h_{ij} \left( \mathbf{E}_{ij} + \sum_{r=1}^p \sum_{s=1}^p (\mathbf{\Upsilon}^T \mathbf{\Omega}_{rs} + \mathbf{\Omega}_{rs}^T \mathbf{\Upsilon}) \right) \\ & \leq \sum_{i=1}^p \sum_{j=1}^p h_{ij} \left( \mathbf{E}_{ij} + \sum_{r=1}^p \sum_{s=1}^p \left( \beta_{rs} \mathbf{\Upsilon}^T \mathbf{\Upsilon} + \frac{1}{\beta_{rs}} \mathbf{\Omega}_{rs}^T \mathbf{\Omega}_{rs} \right) \right) \\ & \leq \sum_{i=1}^p \sum_{j=1}^p h_{ij} \left( \mathbf{E}_{ij} + \sum_{r=1}^p \sum_{s=1}^p \left( \beta_{rs} \mathbf{\Upsilon}^T \mathbf{\Upsilon} + \frac{1}{\beta_{rs}} \hat{\mathbf{\Omega}}_{rs}^T \hat{\mathbf{\Omega}}_{rs} \right) \right) < \mathbf{0} \end{aligned} \quad (18)$$

where

$$\hat{\mathbf{\Omega}}_{ij} = [\rho_{ij}(\mathbf{A}_i + \mathbf{B}_i \mathbf{G}_j - \mathbf{Y}_2) \quad \mathbf{0} \quad \mathbf{0}]$$

and  $\beta_{ij} > 0 \forall i, j$ .

*Remark 4:* We have the relation that  $|\dot{h}_{ij}| = |\dot{w}_i w_j + w_i \dot{w}_j| \leq |\dot{w}_i w_j| + |w_i \dot{w}_j| \leq |\dot{w}_i| + |\dot{w}_j|$ . The upper bound of  $|\dot{h}_{ij}|$  can be approximated by the bounds of  $\dot{w}_i$ . However, it is very conservative to apply this relation to choose  $\rho_{ij}$ . More relaxed stability conditions can be obtained by choosing smaller  $\rho_{ij}$ . The assumption  $|\dot{h}_{ij}| \leq \rho_{ij}$  as well as  $\underline{\phi}_i \leq \dot{w}_i \leq \bar{\phi}_i$  can be verified after the stability analysis.

By congruence transform with premultiplying  $\text{diag}\{\mathbf{X}, \mathbf{X}, \mathbf{X}\}$  and post-multiplying  $\text{diag}\{\mathbf{X}^T, \mathbf{X}^T, \mathbf{X}^T\}$  to (18), denoting  $\tilde{\mathbf{Y}}_1 = \mathbf{X} \mathbf{Y}_1 \mathbf{X}^T$ ,  $\tilde{\mathbf{Y}}_2 = \mathbf{Y}_2 \mathbf{X}^T$ ,  $\tilde{\mathbf{S}}_i = \mathbf{X} \mathbf{S}_i \mathbf{X}^T$ , using Schur complement and grouping the terms with the same membership functions, we obtain stability condition (10).

This completes the proof.  $\blacksquare$

## B. Design of Fuzzy Functional Observer

In this section, the fuzzy functional observer is proposed to estimate the control input when only system output  $\mathbf{y}$  is measurable instead of system state  $\mathbf{x}$ . The T-S fuzzy model (1) is assumed to be in the following form:

$$\begin{aligned} \dot{\mathbf{x}} &= \sum_{i=1}^p w_i(\mathbf{y}) (\mathbf{A}_i \mathbf{x} + \mathbf{B}_i \mathbf{u}) \\ \mathbf{y} &= \mathbf{C} \mathbf{x} \end{aligned} \quad (19)$$

where  $\mathbf{y} \in \mathfrak{R}^l$  is the system output and  $\mathbf{C} \in \mathfrak{R}^{l \times n}$  is the output matrix. Moreover, the fuzzy controller (2) is considered to be

$$\mathbf{u} = \sum_{j=1}^p w_j(\mathbf{y}) \mathbf{u}_j = \sum_{j=1}^p w_j(\mathbf{y}) \mathbf{G}_j \mathbf{x} \quad (20)$$

where  $\mathbf{u}_j = \mathbf{G}_j \mathbf{x} \in \mathfrak{R}^m$  is the control input in the  $j$ th rule. Without loss of generality, we assume  $\text{rank}(\mathbf{C}) = l$  and  $\text{rank}(\mathbf{G}_j) = m$  [41], which means  $\mathbf{C}$  and  $\mathbf{G}_j$  are of full row rank.

The following fuzzy functional observer is proposed to estimate the control input  $\mathbf{u}$  in (20):

$$\begin{aligned} \dot{\mathbf{z}}_j &= \sum_{i=1}^p w_i(\mathbf{y}) (\mathbf{N}_{ij} \mathbf{z}_j + \mathbf{J}_{ij} \mathbf{y} + \mathbf{H}_{ij} \check{\mathbf{u}}) \quad \forall j \\ \check{\mathbf{u}}_j &= \mathbf{z}_j + \mathbf{E}_j \mathbf{y} \quad \forall j \\ \check{\mathbf{u}} &= \sum_{j=1}^p w_j(\mathbf{y}) \check{\mathbf{u}}_j \end{aligned} \quad (21)$$

where  $\mathbf{z}_j \in \mathfrak{R}^m$  is the observer state;  $\check{\mathbf{u}}_j \in \mathfrak{R}^m$  is the estimated control input in the  $j$ th rule;  $\check{\mathbf{u}} \in \mathfrak{R}^m$  is the estimated control input;  $\mathbf{N}_{ij} \in \mathfrak{R}^{m \times m}$ ,  $\mathbf{J}_{ij} \in \mathfrak{R}^{m \times l}$ ,  $\mathbf{H}_{ij} \in \mathfrak{R}^{m \times m}$ , and  $\mathbf{E}_j \in \mathfrak{R}^{m \times l}$  are observer gains to be designed.

*Remark 5:* The proposed form of fuzzy functional observer is different from those in [40] and [42]. In what follows, the separation principle [39] will be applied to separately design the fuzzy controller and fuzzy functional observer. Furthermore, the technique in [41] and [45] for linear functional observer will be extended to design the fuzzy functional observer. To achieve these two tasks, we choose such form of fuzzy functional observer.

For brevity, the membership function  $w_i(\mathbf{y})$  is denoted as  $w_i$ . The estimation error is defined as  $\mathbf{e}_j = \mathbf{u}_j - \check{\mathbf{u}}_j = \mathbf{G}_j \mathbf{x} - (\mathbf{z}_j + \mathbf{E}_j \mathbf{y}) = \mathbf{Q}_j \mathbf{x} - \mathbf{z}_j$ , where  $\mathbf{Q}_j = \mathbf{G}_j - \mathbf{E}_j \mathbf{C}$ , and then we have the closed-loop system consisting of the T-S fuzzy model (19), the fuzzy controller (20), and the fuzzy functional observer (21) as follows:

$$\begin{aligned} \dot{\mathbf{x}} &= \sum_{i=1}^p w_i \left( \mathbf{A}_i \mathbf{x} + \mathbf{B}_i \sum_{k=1}^p w_k \check{\mathbf{u}}_k \right) \\ &= \sum_{i=1}^p w_i \left( \mathbf{A}_i \mathbf{x} + \mathbf{B}_i \sum_{k=1}^p w_k (\mathbf{u}_k - \mathbf{e}_k) \right) \\ &= \sum_{i=1}^p \sum_{l=1}^p h_{il} \left( \mathbf{A}_i \mathbf{x} + \mathbf{B}_i \mathbf{G}_l \mathbf{x} - \mathbf{B}_i \sum_{k=1}^p w_k \mathbf{e}_k \right) \end{aligned} \quad (22)$$

$$\begin{aligned}
\dot{\mathbf{e}}_j &= \mathbf{Q}_j \dot{\mathbf{x}} - \dot{\mathbf{z}}_j \\
&= \sum_{i=1}^p \sum_{l=1}^p h_{il} \left( \mathbf{Q}_j \left( \mathbf{A}_i \mathbf{x} + \mathbf{B}_i \mathbf{G}_l \mathbf{x} - \mathbf{B}_i \sum_{k=1}^p w_k \mathbf{e}_k \right) \right. \\
&\quad \left. - \left( \mathbf{N}_{ij} (\mathbf{Q}_j \mathbf{x} - \mathbf{e}_j) + \mathbf{J}_{ij} \mathbf{C} \mathbf{x} \right. \right. \\
&\quad \left. \left. + \mathbf{H}_{ij} \sum_{k=1}^p w_k (\mathbf{G}_k \mathbf{x} - \mathbf{e}_k) \right) \right) \\
&= \sum_{i=1}^p \sum_{l=1}^p h_{il} \left( (\Phi_{ij} + \Lambda_{ij} \mathbf{G}_l) \mathbf{x} + \mathbf{N}_{ij} \mathbf{e}_j - \Lambda_{ij} \sum_{k=1}^p w_k \mathbf{e}_k \right) \forall j
\end{aligned} \tag{23}$$

where  $h_{il} \equiv w_i w_l$ ,  $\Phi_{ij} = \mathbf{Q}_j \mathbf{A}_i - \mathbf{N}_{ij} \mathbf{Q}_j - \mathbf{J}_{ij} \mathbf{C}$ ,  $\Lambda_{ij} = \mathbf{Q}_j \mathbf{B}_i - \mathbf{H}_{ij}$ .

The control objective is to make the augmented FMB functional observer-control system [formed by (22) and (23)] asymptotically stable, i.e.,  $\mathbf{x} \rightarrow 0$  and  $\mathbf{e}_j \rightarrow 0 \forall j$  as time  $t \rightarrow \infty$ , by determining the controller gain  $\mathbf{G}_j$  and observer gains  $\mathbf{N}_{ij}$ ,  $\mathbf{J}_{ij}$ ,  $\mathbf{H}_{ij}$ , and  $\mathbf{E}_j$ .

In order to apply the separation principle [39] to design the controller and observer separately, the following constraints can be imposed:

$$\Phi_{ij} = \mathbf{0} \quad \forall i, j \tag{24}$$

$$\Lambda_{ij} = \mathbf{0} \quad \forall i, j. \tag{25}$$

Defining the augmented vector  $\mathbf{x}_a = [\mathbf{x}^T \ \mathbf{e}_1^T \ \mathbf{e}_2^T \ \cdots \ \mathbf{e}_p^T]^T$ , the augmented FMB functional observer-control system is written as

$$\dot{\mathbf{x}}_a = \sum_{i=1}^p \sum_{l=1}^p h_{il} \Gamma_{il} \mathbf{x}_a \tag{26}$$

where

$$\Gamma_{il} = \begin{bmatrix} \mathbf{A}_i + \mathbf{B}_i \mathbf{G}_l & -\mathbf{B}_i w_1 & -\mathbf{B}_i w_2 & \cdots & -\mathbf{B}_i w_p \\ \mathbf{0} & \mathbf{N}_{i1} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{N}_{i2} & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{N}_{ip} \end{bmatrix}.$$

*Remark 6:* It has been justified in [39] that the separation principle can be applied to the system in triangular form (26). In other words, the fuzzy controller and the fuzzy functional observer can be designed separately. Consequently, we begin by designing the fuzzy controller using Theorem 1. Then the obtained controller gain  $\mathbf{G}_j$  is employed to design the fuzzy functional observer.

To design the fuzzy functional observer, the objective is to find observer gains  $\mathbf{N}_{ij}$ ,  $\mathbf{J}_{ij}$ ,  $\mathbf{H}_{ij}$ , and  $\mathbf{E}_j$  such that the error systems

$$\dot{\mathbf{e}}_j = \sum_{i=1}^p w_i \mathbf{N}_{ij} \mathbf{e}_j \quad \forall j \tag{27}$$

are asymptotically stable and the constraints (24) and (25) are satisfied.

In what follows, we first propose the stability conditions ensuring the stability of the error systems (27). Then a design

procedure is presented to obtain all observer gains while satisfying constraints (24) and (25).

*Theorem 2:* The error systems (27) are guaranteed to be asymptotically stable if there exist matrices  $\mathbf{X} = \mathbf{X}^T \in \Re^{m \times m}$ ,  $\mathbf{Y}_{ij} \in \Re^{m \times 2l}$ ,  $i, j = 1, 2, \dots, p$  such that the following LMI-based conditions are satisfied:

$$\mathbf{X} > \mathbf{0} \tag{28}$$

$$\mathbf{X} \mathbf{F}_{ij} - \mathbf{Y}_{ij} \mathbf{M}_{ij} + \mathbf{F}_{ij}^T \mathbf{X} - \mathbf{M}_{ij}^T \mathbf{Y}_{ij}^T < \mathbf{0} \quad \forall i, j \tag{29}$$

$$\tilde{\mathbf{E}}_{i_1 j} = \tilde{\mathbf{E}}_{i_2 j} \quad \forall i_1 < i_2, j \tag{30}$$

where

$$\mathbf{F}_{ij} = \mathbf{G}_j \mathbf{A}_i \mathbf{G}_j^+ - \mathbf{G}_j \bar{\mathbf{A}}_{ij} \Sigma_{ij}^+ \begin{bmatrix} \mathbf{C} \mathbf{A}_i \mathbf{G}_j^+ \\ \mathbf{C} \mathbf{G}_j^+ \end{bmatrix} \tag{31}$$

$$\mathbf{M}_{ij} = (\mathbf{I} - \Sigma_{ij} \Sigma_{ij}^+) \begin{bmatrix} \mathbf{C} \mathbf{A}_i \mathbf{G}_j^+ \\ \mathbf{C} \mathbf{G}_j^+ \end{bmatrix} \tag{32}$$

$$\bar{\mathbf{A}}_{ij} = \mathbf{A}_i (\mathbf{I} - \mathbf{G}_j^+ \mathbf{G}_j) \tag{33}$$

$$\Sigma_{ij} = \begin{bmatrix} \mathbf{C} \bar{\mathbf{A}}_{ij} \\ \bar{\mathbf{C}}_j \end{bmatrix} \tag{34}$$

$$\bar{\mathbf{C}}_j = \mathbf{C} (\mathbf{I} - \mathbf{G}_j^+ \mathbf{G}_j) \tag{35}$$

the controller gain  $\mathbf{G}_j$  is determined by Theorem 1;  $\tilde{\mathbf{E}}_{ij}$  in (30) is obtained by  $[\tilde{\mathbf{E}}_{ij} \ \tilde{\mathbf{K}}_{ij}] = \mathbf{X} \mathbf{G}_j \bar{\mathbf{A}}_{ij} \Sigma_{ij}^+ + \mathbf{Y}_{ij} (\mathbf{I} - \Sigma_{ij} \Sigma_{ij}^+)$ ; and  $\mathbf{Z}_{ij} = \mathbf{X}^{-1} \mathbf{Y}_{ij}$ .

*Proof:* The constraint (24) is equivalent to

$$\Phi_{ij} [\mathbf{G}_j^+ \ \mathbf{I} - \mathbf{G}_j^+ \mathbf{G}_j] = \mathbf{0} \quad \forall i, j \tag{36}$$

where  $\mathbf{G}_j^+$  is the Moore–Penrose generalized inverse of  $\mathbf{G}_j$ . The proof of (36) is shown in the Appendix. From (36), we get  $\Phi_{ij} \mathbf{G}_j^+ = \mathbf{0}$  and  $\Phi_{ij} (\mathbf{I} - \mathbf{G}_j^+ \mathbf{G}_j) = \mathbf{0}$ . Substituting  $\mathbf{Q}_j = \mathbf{G}_j - \mathbf{E}_j \mathbf{C}$  into  $\Phi_{ij}$ , we have

$$((\mathbf{G}_j - \mathbf{E}_j \mathbf{C}) \mathbf{A}_i - \mathbf{N}_{ij} (\mathbf{G}_j - \mathbf{E}_j \mathbf{C}) - \mathbf{J}_{ij} \mathbf{C}) \mathbf{G}_j^+ = \mathbf{0} \quad \forall i, j \tag{37}$$

$$((\mathbf{G}_j - \mathbf{E}_j \mathbf{C}) \mathbf{A}_i - \mathbf{N}_{ij} (\mathbf{G}_j - \mathbf{E}_j \mathbf{C}) - \mathbf{J}_{ij} \mathbf{C}) \times (\mathbf{I} - \mathbf{G}_j^+ \mathbf{G}_j) = \mathbf{0} \quad \forall i, j. \tag{38}$$

Since  $\mathbf{G}_j$  is of full row rank, we have  $\mathbf{G}_j \mathbf{G}_j^+ = \mathbf{I}$ . Using this property, we simplify (37) and (38) to

$$\mathbf{N}_{ij} = \mathbf{G}_j \mathbf{A}_i \mathbf{G}_j^+ - \mathbf{E}_j \mathbf{C} \mathbf{A}_i \mathbf{G}_j^+ - (\mathbf{J}_{ij} - \mathbf{N}_{ij} \mathbf{E}_j) \mathbf{C} \mathbf{G}_j^+ \quad \forall i, j \tag{39}$$

$$\begin{aligned}
&\mathbf{E}_j \mathbf{C} \mathbf{A}_i (\mathbf{I} - \mathbf{G}_j^+ \mathbf{G}_j) + (\mathbf{J}_{ij} - \mathbf{N}_{ij} \mathbf{E}_j) \mathbf{C} (\mathbf{I} - \mathbf{G}_j^+ \mathbf{G}_j) \\
&= \mathbf{G}_j \mathbf{A}_i (\mathbf{I} - \mathbf{G}_j^+ \mathbf{G}_j) \quad \forall i, j. \tag{40}
\end{aligned}$$

Writing (39) and (40) into compact forms, we obtain

$$\mathbf{N}_{ij} = \mathbf{G}_j \mathbf{A}_i \mathbf{G}_j^+ - [\mathbf{E}_j \ \mathbf{K}_{ij}] \begin{bmatrix} \mathbf{C} \mathbf{A}_i \mathbf{G}_j^+ \\ \mathbf{C} \mathbf{G}_j^+ \end{bmatrix} \quad \forall i, j \tag{41}$$

$$[\mathbf{E}_j \ \mathbf{K}_{ij}] \Sigma_{ij} = \mathbf{G}_j \bar{\mathbf{A}}_{ij} \quad \forall i, j \tag{42}$$

where

$$\mathbf{K}_{ij} = \mathbf{J}_{ij} - \mathbf{N}_{ij} \mathbf{E}_j \tag{43}$$

and  $\bar{\mathbf{A}}_{ij}$  and  $\Sigma_{ij}$  are defined in (33) and (34), respectively.

According to [46], the general solution of linear matrix equation (42) is

$$[\mathbf{E}_j \quad \mathbf{K}_{ij}] = \mathbf{G}_j \bar{\mathbf{A}}_{ij} \Sigma_{ij}^+ + \mathbf{Z}_{ij} (\mathbf{I} - \Sigma_{ij} \Sigma_{ij}^+) \quad \forall i, j \quad (44)$$

where  $\mathbf{Z}_{ij} \in \mathfrak{R}^{m \times 2l}$   $\forall i, j$  are arbitrary matrices.

*Remark 7:* In [41],  $\mathbf{E}_j$  and  $\mathbf{K}_{ij}$  can be obtained in (44) once  $\mathbf{Z}_{ij}$  is determined for linear functional observer. However, in fuzzy functional observer case, since  $\mathbf{Z}_{ij}$  varies with rule  $i$  and  $\mathbf{E}_j$  is obtained from  $\mathbf{Z}_{ij}$ ,  $\mathbf{E}_j$  will also vary with rule  $i$ . That is to say, we will get  $\mathbf{E}_{ij}$  rather than  $\mathbf{E}_j$  as follows:

$$[\mathbf{E}_{ij} \quad \mathbf{K}_{ij}] = \mathbf{G}_j \bar{\mathbf{A}}_{ij} \Sigma_{ij}^+ + \mathbf{Z}_{ij} (\mathbf{I} - \Sigma_{ij} \Sigma_{ij}^+) \quad \forall i, j \quad (45)$$

where  $\mathbf{E}_{ij}$  is obtained by giving  $\mathbf{Z}_{ij}$ . In order to make  $\mathbf{E}_j$  not vary with rule  $i$ , the following constraints need to be imposed:  $\mathbf{E}_{i_1 j} = \mathbf{E}_{i_2 j}$ ,  $\forall i_1, i_2$ . Defining  $0 < \mathbf{X} = \mathbf{X}^T \in \mathfrak{R}^{m \times m}$ , then  $\mathbf{E}_{i_1 j} = \mathbf{E}_{i_2 j}$  is equivalent to stability condition (30), where  $\tilde{\mathbf{E}}_{ij} = \mathbf{X} \mathbf{E}_{ij}$ .  $\tilde{\mathbf{E}}_{ij}$  in (30) is obtained by  $[\tilde{\mathbf{E}}_{ij} \quad \tilde{\mathbf{K}}_{ij}] = \mathbf{X} \mathbf{G}_j \bar{\mathbf{A}}_{ij} \Sigma_{ij}^+ + \mathbf{Y}_{ij} (\mathbf{I} - \Sigma_{ij} \Sigma_{ij}^+)$ , where  $\mathbf{Y}_{ij} = \mathbf{X} \mathbf{Z}_{ij}$ .

Substituting (44) to (41), we have

$$\begin{aligned} \mathbf{N}_{ij} &= \mathbf{G}_j \mathbf{A}_i \mathbf{G}_j^+ \\ &- \left( \mathbf{G}_j \bar{\mathbf{A}}_{ij} \Sigma_{ij}^+ + \mathbf{Z}_{ij} (\mathbf{I} - \Sigma_{ij} \Sigma_{ij}^+) \right) \begin{bmatrix} \mathbf{C} \mathbf{A}_i \mathbf{G}_j^+ \\ \mathbf{C} \mathbf{G}_j^+ \end{bmatrix} \quad \forall i, j. \end{aligned} \quad (46)$$

Writing (46) into a compact form, we obtain

$$\mathbf{N}_{ij} = \mathbf{F}_{ij} - \mathbf{Z}_{ij} \mathbf{M}_{ij} \quad \forall i, j \quad (47)$$

where  $\mathbf{F}_{ij}$  and  $\mathbf{M}_{ij}$  are defined in (31) and (32), respectively. Therefore, the error system (27) becomes

$$\dot{\mathbf{e}}_j = \sum_{i=1}^p w_i (\mathbf{F}_{ij} - \mathbf{Z}_{ij} \mathbf{M}_{ij}) \mathbf{e}_j \quad \forall j. \quad (48)$$

Applying the Lyapunov function  $V(\mathbf{e}_j) = \mathbf{e}_j^T \mathbf{X} \mathbf{e}_j$  to investigate the stability of (48) where  $\mathbf{X} \in \mathfrak{R}^{m \times m}$  and  $\mathbf{X} > 0$ , we have the time derivative of  $V(\mathbf{e}_j)$  as follows:

$$\dot{V}(\mathbf{e}_j) = \sum_{i=1}^p w_i \mathbf{e}_j^T (\mathbf{X} \mathbf{F}_{ij} - \mathbf{Y}_{ij} \mathbf{M}_{ij} + \mathbf{F}_{ij}^T \mathbf{X} - \mathbf{M}_{ij}^T \mathbf{Y}_{ij}^T) \mathbf{e}_j$$

where  $\mathbf{Y}_{ij} = \mathbf{X} \mathbf{Z}_{ij}$ .  $\dot{V}(\mathbf{e}_j) < 0$  holds if the stability condition (29) is satisfied.

This completes the proof.  $\blacksquare$

With  $\mathbf{Z}_{ij}$  obtained from Theorem 2, the following procedure [41] is employed to determine the observer gains such that the constraints (24) and (25) are satisfied.

- 1)  $\mathbf{N}_{ij}$  can be obtained from (47).
- 2)  $\mathbf{E}_j$  and the intermediate variable  $\mathbf{K}_{ij}$  are given by (44).
- 3)  $\mathbf{J}_{ij}$  can be obtained from (43).
- 4)  $\mathbf{H}_{ij}$  is given by (25).

*Remark 8:* The stability conditions in Theorem 2 are convex, which can be numerically solved by convex programming techniques. Once  $\mathbf{Z}_{ij}$  is obtained from Theorem 2, all observer gains are determined and the stability of the error system is guaranteed. Compared with [40] and [42], in this paper, there is no need to manually design any observer gains or check the stability after designing the gains.

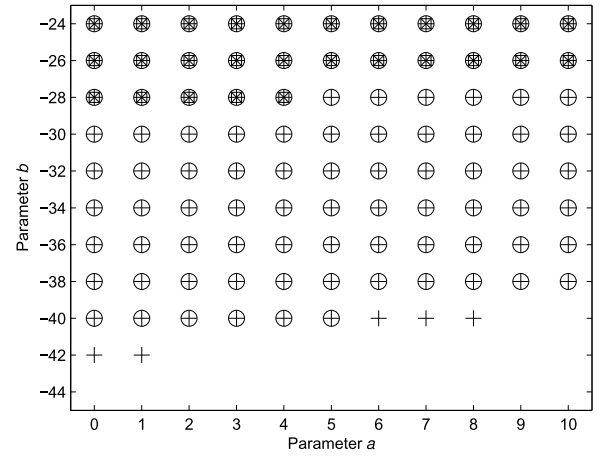


Fig. 1. Stabilization regions obtained from Theorem 1 (○ and +), [15, Th. 6] (×), and [15, Th. 7] (□).

#### IV. SIMULATION EXAMPLES

In this section, two examples are provided to demonstrate the feasibility of the proposed design procedure. A numerical model is presented first for comparison of the conservativeness. Then an inverted pendulum is considered to test the proposed fuzzy functional observer-controller.

##### A. Numerical Example

Consider the following 2-rule T-S fuzzy model [15]:

$$\begin{aligned} \mathbf{A}_1 &= \begin{bmatrix} 3.6 & -1.6 \\ 6.2 & -4.3 \end{bmatrix}, \mathbf{A}_2 = \begin{bmatrix} -a & -1.6 \\ 6.2 & -4.3 \end{bmatrix} \\ \mathbf{B}_1 &= [-0.45 \quad -3]^T, \mathbf{B}_2 = [-b \quad -3]^T \end{aligned}$$

where  $a$  and  $b$  are constant parameters to be determined. The region of stabilization will be revealed with  $a$  and  $b$  being chosen in the range of  $0 \leq a \leq 10$  and  $-44 \leq b \leq -24$  at the interval of 1 and 2, respectively.

The region of interest is defined as  $x_1 \in [-0.2, 0.2]$  and  $x_2 \in [-0.5, 0.5]$  where  $\mathbf{x} = [x_1 \quad x_2]^T$  are the system states. The membership functions are chosen as  $w_1(x_1) = e^{-(x_1^2)/18}$  and  $w_2(x_1) = 1 - w_1(x_1)$ .

In this example, since we aim to show the relaxation of stability conditions in Theorem 1, only Theorem 1 is employed to design the fuzzy controller to stabilize the system. We choose  $\beta_{ij} = 1$ ,  $\mu_1 = -10^{-2}$ ,  $\mu_2 = -10^{-4}$ ,  $\mu_3 = 0.04$ ,  $\mu_4 = -10^{-2}$ ,  $\mu_5 = 10^{-4}$ ,  $\mu_6 = 10^{-6}$ ,  $\phi_i = -1$ ,  $\bar{\phi}_i = 1$ ,  $\rho_{11} = \rho_{22} = 1$ ,  $\rho_{12} = \rho_{21} = 0.1$ ,  $i$ , and  $j = 1, 2$ . Finding the solution using MATLAB LMI toolbox, the stabilization region is indicated by “○” in Fig. 1.

*Remark 9:* To our experience, the predefined parameters  $\beta_{ij} > 0$ ,  $\mu_1, \mu_2, \dots, \mu_6$ ,  $i, j = 1, 2, \dots, p$  in Theorem 1 can be determined in the following way in order to obtain more relaxed results. According the conditions in Theorem 1, the sign of  $\mu_2$  should be opposite to those of  $\mu_3$  and  $\mu_6$ . Users can start choosing the magnitudes of  $\mu_1, \mu_2, \dots, \mu_6$  very small such as  $10^{-6}$ , and then gradually increase the magnitudes. For  $\beta_{ij}$ , start from large values and gradually reduce them. The reason is that by starting with these settings, the



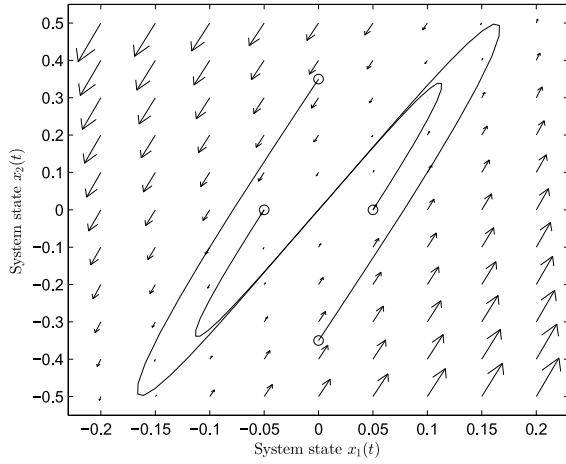


Fig. 2. Phase plot of  $x_1(t)$  and  $x_2(t)$  for  $a = 5$  and  $b = -40$  in  $\circ$  where the initial conditions are indicated by  $\circ$ .

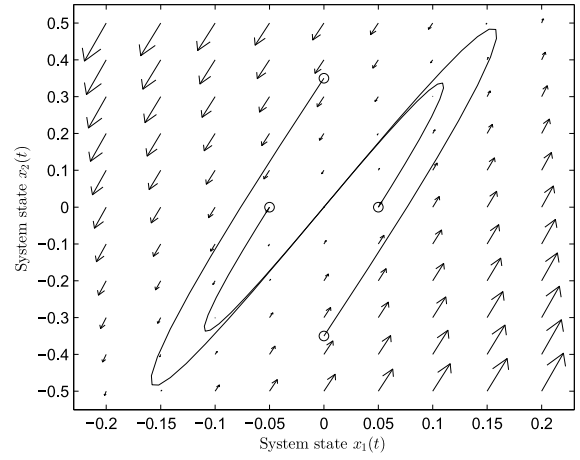


Fig. 3. Phase plot of  $x_1(t)$  and  $x_2(t)$  for  $a = 1$  and  $b = -42$  in  $+$  where the initial conditions are indicated by  $\circ$ .

conditions are similar to those in [15] using fuzzy Lyapunov function. Then the adjustment enlarges the effect of HODLF.

To show the influence of adding the information of the lower bound  $\underline{\phi}_i$  and corresponding slack matrix  $\mathbf{S}_i$  in the proposed analysis, we consider another case where  $\underline{\phi}_i \neq -\bar{\phi}_i$ . We choose  $\underline{\phi}_1 = -0.4$ ,  $\bar{\phi}_2 = 0.4$  and keep other parameters the same, the corresponding stabilization region is obtained as  $+$  in Fig. 1. It can be seen that the slack matrix leads to more relaxed results.

*Remark 10:* To compare the proposed stability conditions with those derived from the fuzzy Lyapunov function [15], we consider two set of conditions: 1) time-derivative dependent conditions [15, Th. 6] and 2) time-derivative independent conditions [15, Th. 7]. We apply [15, Th. 6] by choosing  $\mu = 0.04$  and  $\phi_{1,2} = 1$ . Also, [15, Th. 7] is employed by choosing  $\mu = 0.04$  and all possible substructures of decision matrices. Finding the solution using MATLAB LMI toolbox, the stabilization region is obtained and indicated by “ $\times$ ” and “ $\square$ ” in Fig. 1. It is shown that the HODLF in this paper provides more relaxed stability conditions than fuzzy Lyapunov function [15].

To verify the stabilization results, we consider two cases by choosing  $a = 5$  and  $b = -40$  in  $\circ$  and  $a = 1$  and  $b = -42$  in  $+$ . The controller feedback gains are obtained as  $\mathbf{G}_1 = [-1.0335 \times 10^{-2} \quad 2.6297]$ ,  $\mathbf{G}_2 = [8.0449 \times 10^{-2} \quad 3.2949 \times 10^{-2}]$  and  $\mathbf{G}_1 = [-1.1608 \times 10^{-1} \quad 2.9394]$ ,  $\mathbf{G}_2 = [-1.9578 \times 10^{-1} \quad 7.6306 \times 10^{-2}]$ , respectively, for both cases. With the initial conditions indicated by “ $\circ$ ,” the phase plots of  $x_1(t)$  and  $x_2(t)$  are shown in Figs. 2 and 3. It can be seen that all the trajectories asymptotically reach the equilibrium point  $\mathbf{x} = \mathbf{0}$ . Furthermore, we check that the constraints  $\underline{\phi}_i \leq \dot{w}_i \leq \bar{\phi}_i$  and  $|\dot{h}_{ij}| \leq \rho_{ij}$  are satisfied for these two cases. For  $a = 5$  and  $b = -40$  in  $\circ$ , we have  $-6.0569 \times 10^{-2} \leq \dot{w}_1 \leq 6.4402 \times 10^{-3}$ ,  $-6.4402 \times 10^{-3} \leq \dot{w}_2 \leq 6.0569 \times 10^{-2}$ ,  $|\dot{h}_{11}| \leq 1.2087 \times 10^{-1}$ ,  $|\dot{h}_{12}|, |\dot{h}_{21}| \leq 6.0300 \times 10^{-2}$ , and  $|\dot{h}_{22}| \leq 2.6890 \times 10^{-4}$ . Similarly for  $a = 1$  and  $b = -42$  in  $+$ , we have  $-6.3499 \times 10^{-2} \leq \dot{w}_1 \leq 6.6410 \times 10^{-3}$ ,  $-6.6410 \times 10^{-3} \leq \dot{w}_2 \leq 6.3499 \times 10^{-2}$ ,  $|\dot{h}_{11}| \leq 1.2672 \times 10^{-1}$ ,  $|\dot{h}_{12}|, |\dot{h}_{21}| \leq 6.3217 \times 10^{-2}$ , and  $|\dot{h}_{22}| \leq 2.8191 \times 10^{-4}$ . Therefore, the constraints are all satisfied for these two cases.

### B. Inverted Pendulum

In this example, we consider an inverted pendulum on a cart in the following state-space form [4]:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{g \sin(x_1) - am_p L x_2^2 \sin(x_1) \cos(x_1) - a \cos(x_1) u}{4L/3 - am_p L \cos^2(x_1)} \end{aligned} \quad (49)$$

where  $\mathbf{x} = [x_1 \quad x_2]^T$  are the system states;  $g = 9.8 \text{ m/s}^2$  is the acceleration of gravity;  $m_p = 2 \text{ kg}$  and  $M_c = 8 \text{ kg}$  are the mass of the pendulum and the cart, respectively;  $a = 1/(m_p + M_c)$ ;  $2L = 1 \text{ m}$  is the length of the pendulum;  $u$  is the control input force imposed on the cart.

The region of interest is defined as  $x_1 \in [-(80\pi/180), (80\pi/180)]$ . The dynamics of the inverted pendulum (49) is represented by a 2-rule fuzzy model [4] with the following parameters:

$$\begin{aligned} \mathbf{A}_1 &= \begin{bmatrix} 0 & 1 \\ \frac{g}{\frac{4L}{3} - am_p L} & 0 \end{bmatrix}, \mathbf{A}_2 = \begin{bmatrix} 0 & 1 \\ \pi \left( \frac{4L}{3} - am_p L \beta^2 \right) & 0 \end{bmatrix} \\ \mathbf{B}_1 &= \begin{bmatrix} 0 & -\frac{a}{\frac{4L}{3} - am_p L} \end{bmatrix}^T, \mathbf{B}_2 = \begin{bmatrix} 0 & -\frac{a\beta}{\frac{4L}{3} - am_p L \beta^2} \end{bmatrix}^T \\ \mathbf{C} &= [1 \quad 0] \end{aligned}$$

where  $\beta = \cos(80\pi/180)$ . The membership functions are chosen as  $w_1(x_1) = e^{-(x_1^2/0.32)}$  and  $w_2(x_1) = 1 - w_1(x_1)$ .

To design the proposed fuzzy functional observer-controller, Theorem 1 is employed to design the fuzzy controller first. We choose  $\beta_{ij} = 1$ ,  $i, j = 1, 2$ ,  $\mu_1 = -10^{-2}$ ,  $\mu_2 = -10^{-4}$ ,  $\mu_3 = 0.04$ ,  $\mu_4 = -10^{-3}$ ,  $\mu_5 = 10^{-4}$ ,  $\mu_6 = 10^{-6}$ ,  $\underline{\phi}_1 = -5$ ,  $\bar{\phi}_1 = 10$ ,  $\underline{\phi}_2 = -10$ ,  $\bar{\phi}_2 = 5$ ,  $\rho_{11} = \rho_{22} = 15$ , and  $\rho_{12} = \rho_{21} = 6$ . The controller feedback gains are obtained as  $\mathbf{G}_1 = [3.8334 \times 10^2 \quad 1.1677 \times 10^2]$  and  $\mathbf{G}_2 = [1.2092 \times 10^3 \quad 4.0828 \times 10^2]$  using MATLAB LMI toolbox.

After obtaining the controller feedback gains, Theorem 2 and the procedure at the end of Section III-B are employed to design the fuzzy functional observer using MATLAB

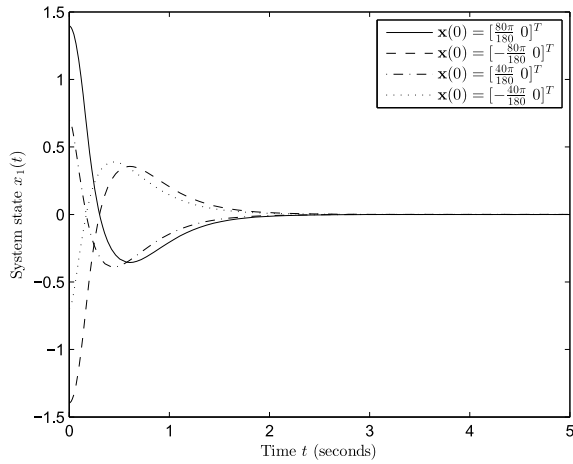


Fig. 4. Time response of system state  $x_1(t)$  with  $\mathbf{z}_1(0) = \mathbf{z}_2(0) = 0$ .

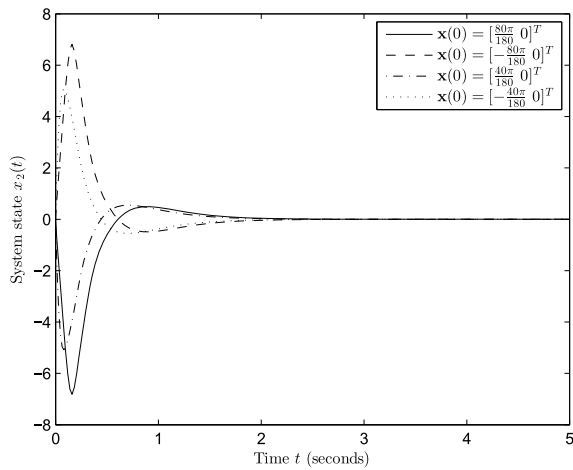


Fig. 5. Time response of system state  $x_2(t)$  with  $\mathbf{z}_1(0) = \mathbf{z}_2(0) = 0$ .

toolbox SOSTOOLS [6]. The observer gains are obtained as  $\mathbf{N}_{11} = -5.4762$ ,  $\mathbf{N}_{12} = -2.3765$ ,  $\mathbf{N}_{21} = -5.4762$ ,  $\mathbf{N}_{22} = -2.3765$ ,  $\mathbf{H}_{11} = -2.0606 \times 10$ ,  $\mathbf{H}_{12} = -7.2049 \times 10$ ,  $\mathbf{H}_{21} = -3.0554$ ,  $\mathbf{H}_{22} = -1.0683 \times 10$ ,  $\mathbf{J}_{11} = -1.4823 \times 10^3$ ,  $\mathbf{J}_{12} = 4.7550 \times 10^3$ ,  $\mathbf{J}_{21} = -2.4040 \times 10^3$ ,  $\mathbf{J}_{22} = 1.5323 \times 10^3$ ,  $\mathbf{E}_1 = 1.0228 \times 10^3$ , and  $\mathbf{E}_2 = 2.1795 \times 10^3$ . In this example, we verify the satisfaction of constraints (24) and (25). By substituting these gains into constraints (24) and (25), we have  $\Phi_{ij} \approx \mathbf{0}$  (the magnitude of all values is less than  $10^{-6}$ ) and  $\Lambda_{ij} = 0 \forall i, j$ . Accordingly, these constraints are satisfied as proved in the theory.

The designed controller gains and observer gains are applied to the original dynamic system of the inverted pendulum (49). Considering four different initial conditions, the time response of system states are shown in Figs. 4 and 5. The initial conditions for the observer states are chosen as  $\mathbf{z}_1(0) = \mathbf{z}_2(0) = 0$ . It is demonstrated that the inverted pendulum can be successfully stabilized by the proposed fuzzy functional observer-controller.

Choosing the initiation conditions  $\mathbf{x}(0) = [(80\pi/180) \ 0]^T$  for further demonstration, the objective control input  $\mathbf{u}(t)$  and estimated control input  $\hat{\mathbf{u}}(t)$  are shown in Fig. 6. Under this case, we also check that the constraints  $\underline{\phi}_i \leq \dot{w}_i \leq \bar{\phi}_i$  and

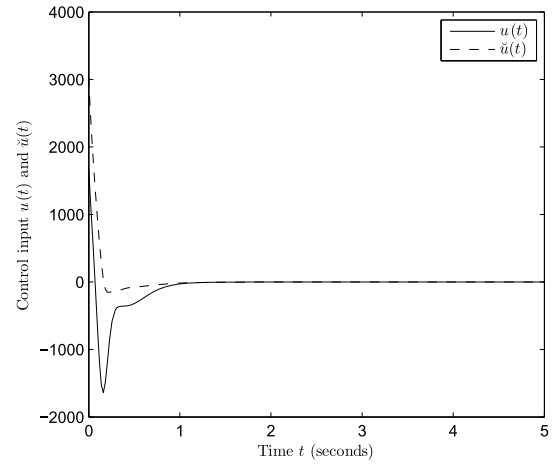


Fig. 6. Time response of objective control input  $\mathbf{u}(t)$  and estimated control input  $\hat{\mathbf{u}}(t)$  with  $\mathbf{x}(0) = [(80\pi/180) \ 0]^T$  and  $\mathbf{z}_1(0) = \mathbf{z}_2(0) = 0$ .

$|\dot{h}_{ij}| \leq \rho_{ij}$  are satisfied. It can be numerically calculated that  $-1.9179 \leq \dot{w}_1 \leq 8.9285$ ,  $-8.9285 \leq \dot{w}_2 \leq 1.9179$ ,  $|\dot{h}_{11}| \leq 1.1197 \times 10$ ,  $|\dot{h}_{12}|, |\dot{h}_{21}| \leq 3.8055$ , and  $|\dot{h}_{22}| \leq 1.0785 \times 10$ . Therefore, the constraints are satisfied according to the previous settings.

*Remark 11:* Instead of estimating the system states, the fuzzy functional observer can estimate the control input directly, which reduces the order of fuzzy observer [5], [31]–[34] from 2 to 1. Additionally, we compare the proposed fuzzy functional observer with the existing one in [42]. The closed-loop poles are chosen as  $-2$  and  $-5$  for controller design and  $-3$  for observer design in all rules. The controller gains are obtained as  $\mathbf{K}_1 = [1.5467 \times 10^2 \ 3.9667 \times 10]$  and  $\mathbf{K}_2 = [7.4146 \times 10^2 \ 2.6753 \times 10^2]$ . The observer gains are  $\mathbf{F}_1 = \mathbf{F}_2 = -3$ . Applying [42, Th. 2], however, no feasible common matrix  $\mathbf{P}$  is found. Consequently, the stability cannot be guaranteed. This comparison demonstrates the superiority of the proposed method that the stability is guaranteed while the feedback gains are obtained.

## V. CONCLUSION

In this paper, the applicability of FMB control scheme has been improved by relaxing stability conditions and considering unmeasurable system states. First, the fuzzy controller has been designed via HODLF to obtain relaxed stability conditions. To derive convex conditions, the properties of membership functions and the dynamics of the FMB control system have been exploited. More information of the derivative of membership functions has been utilized to relax the stability conditions. Next, the fuzzy functional observer has been designed to estimate the control input rather than the system states, which can reduce the order of the observer. A new form of fuzzy functional observer has been proposed which is in favor of applying the separation principle and deriving convex stability conditions. Based on the proposed fuzzy functional observer, users can easily obtain the observer gains while ensuring the stability. Simulation examples have been presented to verify the relaxation and the validity of

designed fuzzy functional observer-controller. In the future, more advanced techniques may be applied to meet the boundary requirement of the derivative of membership functions or to provide more relaxed conditions. The discrete-time fuzzy functional observer can also be investigated by extending the technique in discrete-time linear functional observer.

#### APPENDIX PROOF OF (36)

Consider the following two matrices:

$$\mathbf{P} = \begin{bmatrix} \mathbf{G}_j^+ & \mathbf{I}_n \end{bmatrix}$$

$$\mathbf{Q} = \begin{bmatrix} \mathbf{I}_m & -\mathbf{G}_j \\ \mathbf{0} & \mathbf{I}_n \end{bmatrix}$$

where  $\mathbf{I}_n$  is  $n \times n$  identity matrix. Due to  $\mathbf{G}_j \in \mathfrak{R}^{m \times n}$  and  $\mathbf{G}_j^+ \in \mathfrak{R}^{n \times m}$ , we have  $\text{rank}(\mathbf{P}) = n$  and  $\text{rank}(\mathbf{Q}) = m + n$  where  $\mathbf{Q}$  is of full rank.

Therefore, we have

$$\text{rank}(\mathbf{PQ}) = \text{rank}(\mathbf{P}) = n$$

where  $\mathbf{PQ} = [\mathbf{G}_j^+ \quad \mathbf{I}_n - \mathbf{G}_j^+ \mathbf{G}_j]$ . Due to  $[\mathbf{G}_j^+ \quad \mathbf{I}_n - \mathbf{G}_j^+ \mathbf{G}_j] \in \mathfrak{R}^{n \times (m+n)}$ ,  $[\mathbf{G}_j^+ \quad \mathbf{I}_n - \mathbf{G}_j^+ \mathbf{G}_j]$  is of full row rank.

According to the rank-nullity property [47], the rank of the left nullspace of  $[\mathbf{G}_j^+ \quad \mathbf{I}_n - \mathbf{G}_j^+ \mathbf{G}_j]$  is 0. Then, we can get the equivalent relation

$$\Phi_{ij} \begin{bmatrix} \mathbf{G}_j^+ & \mathbf{I}_n - \mathbf{G}_j^+ \mathbf{G}_j \end{bmatrix} = \mathbf{0} \iff \Phi_{ij} = \mathbf{0}.$$

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