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Static Output Feedback Stabilization of Positive Polynomial Fuzzy Systems

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***Abstract*—This paper deals with the static output feedback stabilization of positive polynomial fuzzy-model-based (PPFMB) control systems. The positive polynomial fuzzy model does not need to share the same premise membership functions with the static output feedback polynomial fuzzy controller. Unlike the state feedback control case, the static output feedback control usually leads to non-convex stability conditions. To circumvent the problem, an approach is employed to transform the non-convex stability conditions into convex ones by introducing a nonsingular transformation matrix. Initially, the conditions guaranteeing the resultant closed-loop systems to be positive and asymptotically stable are obtained. Moreover, the divisional approximated membership functions which carry the local information of the membership functions are employed to facilitate the stability analysis and controller synthesis. The relaxed stability conditions in terms of sum of squares (SOS) are obtained based on Lyapunov stability theory. Finally, a simulation example is given to testify the validity of the analysis result.**

***Index Terms*—Positive Polynomial Fuzzy-Model-Based (PPFM-B) Control Systems, Static Output Feedback Control, Stability Analysis, Sum of Squares (SOS).**

I. INTRODUCTION

IN practice, positive systems, which have the inherent constraint that the state is non-negative whenever the initial conditions are non-negative, are connected to a wide real world applications. For example, heat exchangers, absolute temperatures and level of liquids in tanks, etc. [1]–[4]. In recent years, positive systems have been enormously attractive to many researchers and many results have been obtained for positive linear systems such as state feedback controller design [5], positive observer and dynamic output feedback controller design [6], filter design [7] and positive nonlinear systems such as observer design [8] and filter design [9].

When designing controller for positive systems, the positivity constraint that the states of positive system are located in the positive orthant rather than the whole space must be taken into account [10]. However, the positivity constraint will lead to some challenges in stability analysis and control synthesis [5]. At present, most of the results are mainly for positive linear systems. For example, the work in [11] considered the stabilization problem and control design based on state feedback compensation technique for positive linear systems. While the state variables of positive linear systems usually cannot be measured directly, an output feedback controller was considered in [6], [12] to control positive linear systems. In addition, by using the input and output measurements to estimate a linear combination of system states, a positive linear filter was proposed for positive linear systems in [7]. However, many real-world positive systems are nonlinear in nature that the above mentioned results are no longer suitable. Thereby, it is essential to investigate the positive nonlinear systems.

It is thankful to the Takage-Sugeno (T-S) fuzzy model, a complex nonlinear system can be represented by a set of local linear models that are smoothly connected by the membership functions [13]–[17]. In the past few decades, many researchers paid close attention to the fuzzy systems and a great deal of results have been obtained. In [18], the fuzzy-model-based predictive control was investigated and the stability analysis results were obtained. For discrete-time single-input single-output (SISO) systems, the work in [19] investigated the stable and convergent iterative feedback tuning of fuzzy controllers. Based on the fuzzy Lyapunov function approach, the paper [20] discussed the local stability analysis of continuous-time T-S fuzzy systems. Some work in [21]–[24] investigated adaptive output feedback fuzzy control methods. However, the above results are mainly for general T-S fuzzy systems instead of positive T-S fuzzy systems. Positive T-S fuzzy systems are a kind of special T-S fuzzy systems which can be ensured if all subsystem matrices are Metzler matrices [25]–[27]. In [28], the stability analysis of positive interval type-2 TSK systems with application to energy markets was studied. Considering some system state variables are often unmeasurable, the work in [8] designed an observer for positive T-S fuzzy model. However, for static output feedback control, when considering the stability analysis of the closed-loop positive system, the stability conditions are usually non-convex. Due to the difficulty of transforming the non-convex stability conditions into the convex stability conditions, until now, the research outputs of the static output feedback control for positive T-S fuzzy systems are relatively less.

Although the T-S fuzzy model can systematically represent a nonlinear plant, it is found that the stability conditions of the positive T-S fuzzy model are very conservative since the lack of the information carried by the membership functions. Moreover, in the literature, the design of T-S fuzzy controller relies on parallel distributed compensation (PDC) technique [29]. Because the PDC technique requires that the T-S fuzzy controller shares the same premise membership functions as those of the T-S fuzzy model, the structural complexity of the

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T-S fuzzy controller may be increased under such a design criterion especially when the membership functions of the T-S fuzzy model are complex and/or the number of rules is large.

Relatively speaking, polynomial fuzzy model is more effective to represent actual systems than T-S fuzzy model [30]–[34]. However, due to the existence of polynomials, it becomes more complicated to analyse positivity and stability of positive nonlinear systems, resulting in less attention paid to stability analysis of positive polynomial fuzzy-model-based (PPFMB) control systems. From another perspective, it motivates us to make an effort to carry out research on this topic.

In this paper, the positive nonlinear systems are represented by the positive polynomial fuzzy models which allow polynomials to present in the consequent part of the fuzzy rules and be handled in stability conditions. On one hand, the membership functions of polynomial fuzzy controller are allowed to be different from the membership functions of polynomial fuzzy model. It will make the controller design more flexible and lower the implementation cost when simple membership functions and/or smaller number of rules are considered. On the other hand, the information of the shape of membership functions will be brought into stability analysis that will contribute to the relaxation of stability conditions [32], [35], [36]. To obtain useful information of membership functions, we divide the operating domain of membership functions into some operating subdomains and the original membership functions are represented by approximated membership functions in each operating subdomain where the approximation error will be considered in the stability analysis.

As far as the stability of positive T-S fuzzy model is concerned, there are two main methods to investigate the stability of positive systems: first and foremost, the linear-matrix-inequality (LMI) approach is widely used to derive stability conditions which are used to guarantee the positivity and stability of positive T-S fuzzy systems based on the Lyapunov stability theory [26], [37]. Second, some works [5], [27] employed the linear programming (LP) approach to investigate the stability of positive linear or nonlinear systems based on the Lyapunov stability theory. However, when the positive polynomial fuzzy model is considered, due to the existence of polynomials in stability conditions, the LMI approach and LP approach become powerless, but the sum of squares (SOS) based approach plays an effective role to guarantee the system stability and facilitate the control synthesis.

Under the SOS-based approach, the work [38] has made the first attempt on the stability analysis of an open-loop PPFMB system. The stability conditions are obtained based on the polynomial Lyapunov function and piecewise linear membership functions (PLMFs) are taken into consideration for relaxing the stability conditions. The follow-up work considering full state feedback control was studied based on SOS-based approach in [39]. However, in practical applications, it is often impossible to obtain the full information of state variables [40]–[42]. Thus, it is particularly significant to design an output feedback controller for the PPFMB systems where control can be realized using only the system output. To the best knowledge of the authors, there is no work focusing on designing a static output feedback controller for the PPFMB systems, which motivates us to work on this problem.

In this paper, we investigate the static output feedback polynomial fuzzy controller design and stabilization analysis of PPFMB control systems, aiming to relax the stability conditions. To realize the objective, some efforts need to be made to crack the hard nut that the stability conditions are of non-convex due to the involvement of output matrix. We first employ a positive polynomial fuzzy model to represent a positive nonlinear plant in support of stability analysis and control synthesis. A static output feedback polynomial fuzzy controller is then introduced for the control of the positive nonlinear plant. To obtain stability conditions in convex form, a nonsingular transformation matrix is considered to play a mathematical trick. To achieve relaxed stability analysis results, membership-function-dependent (MFD) analysis techniques [43] are employed that approximated membership functions are used to represent the original membership functions such that some useful information of membership functions can be taken into the stability conditions with the consideration of approximation error. Based on Lyapunov stability theory, MFD SOS-based conditions are developed to guarantee the system stability and facilitate the control synthesis. A feasible solution to the SOS-based stability conditions (if any) for the static output feedback PPFMB control system can be found numerically by using software package such as the third-party Matlab toolbox SOSTOOLS [44].

II. Preliminaries

Standard notations and fundamental technical concepts of positive polynomial fuzzy model and static output feedback polynomial fuzzy controller are introduced in this section.

A. Notation

Throughout this paper, the following notations are employed [45]. The monomial in $\mathbf{x}(t) = [x_1(t), \dots, x_n(t)]^T$ is defined as $x_1^{d_1}(t), \dots, x_n^{d_n}(t)$, where d_k, $k \in \{1, \dots, n\}$, is a non-negative integer. The degree of a monomial is defined as $d = \sum_{k=1}^{n} d_k$. A polynomial $\mathbf{p}(\mathbf{x}(t))$ is shown as finite linear combination of monomials with real coefficients. If a polynomial $\mathbf{p}(\mathbf{x}(t))$ can be expressed as $\mathbf{p}(\mathbf{x}(t)) = \sum_{j=1}^{m} \mathbf{q}_j(\mathbf{x}(t))^2$, where m is a non-zero positive integer and $\mathbf{q}_j(\mathbf{x}(t))$ is a polynomial for all j, we may safely draw the conclusion that $\mathbf{p}(\mathbf{x}(t))$ is a SOS and $\mathbf{p}(\mathbf{x}(t)) \geq 0$. For a matrix $\mathbf{N} \in \Re^{m \times n}$ where n_{rs} denotes the element located at the r-th row and s-th column, the expressions $\mathbf{N} \succeq 0$, $\mathbf{N} \succ 0$, $\mathbf{N} \preceq 0$ and $\mathbf{N} \prec 0$ mean that each element n_{rs} is non-negative, positive, non-positive and negative, respectively. The expression $\mathbf{Q}(\mathbf{x}) = \text{diag}(x_1, \dots, x_n)$ means that the matrix $\mathbf{Q}(\mathbf{x})$ is a diagonal matrix whose diagonal elements are $x_1, \dots, x_n$. $\mathbf{A}^T$ is the transpose of matrix $\mathbf{A}$. Some other notations used in this paper are listed in Table I.

B. Positive Polynomial Fuzzy Model

A positive polynomial fuzzy model is described by p rules. The i-th rule is of the following format:

$$\begin{aligned}\text{Rule } i: &\text{IF } f_1(\mathbf{x}(t)) \text{ is } M_1^i \text{ AND} \cdots \text{AND } f_\Psi(\mathbf{x}(t)) \text{ is } M_\Psi^i \\ &\text{THEN } \dot{\mathbf{x}}(t) = \mathbf{A}_i(\mathbf{x}(t))\mathbf{x}(t) + \mathbf{B}_i(\mathbf{x}(t))\mathbf{u}(t), \qquad (1)\end{aligned}$$

TABLE I
DESCRIPTION OF NOTATIONS.

Notation	Description	Notation	Description
$\mathbf{A}_i(\mathbf{x}(t))$	polynomial system matrix	$\mathbf{B}_i(\mathbf{x}(t))$	polynomial input matrix
$\mathbf{x}(t)$	system state vector	$\mathbf{u}(t)$	input vector
$\mathbf{y}(t)$	output vector	$\mathbf{C}$	constant output matrix
$\mathbf{K}_j(\mathbf{y}(t))$	static output feedback gain	$\mathbf{v}$	constant vector
$\mathbf{\Gamma}$	nonsingular transformation matrix	$\mathbf{P}$	inverse matrix of $\mathbf{\Gamma}$
$\eta_{ij,s_\tau}(\mathbf{x})$	approximated polynomial	$\Delta\eta_{ij,s_\tau}(\mathbf{x})$	error term
γ_{ij,s_τ}	lower bound of error term	β_{ij,s_τ}	upper bound of error term
$\mathbf{G}_{ij,s_\tau}(\mathbf{x})$	slack polynomial matrices	$\mathbf{R}(\mathbf{x})$	polynomial vector

where $f_l(\mathbf{x}(t))$, $l \in \{1,\dots,\Psi\}$, is the premise variable, Ψ is a positive integer; M_l^i, $i \in \{1,\dots,p\}$, is the fuzzy set of the i-th rule corresponding to the function $f_l(\mathbf{x}(t))$; $\mathbf{x}(t) \in \Re^n$ is the system state vector; $\mathbf{u}(t) \in \Re^m$ is the input vector; $\mathbf{A}_i(\mathbf{x}(t)) \in \Re^{n\times n}$ is the polynomial system matrix; $\mathbf{B}_i(\mathbf{x}(t)) = [\mathbf{b}_{i1}^T(\mathbf{x}(t)),\dots,\mathbf{b}_{in}^T(\mathbf{x}(t))]^T \in \Re^{n\times m}$ is the polynomial input matrix where $\mathbf{b}_{ir}(\mathbf{x}(t)) \in \Re^{1\times m}$, $i \in \{1,\dots,p\}$, $r \in \{1,\dots,n\}$, is the r-th row vector of $\mathbf{B}_i(\mathbf{x}(t))$ for all i.

The overall system dynamics is shown as follows:

$$\dot{\mathbf{x}}(t) = \sum_{i=1}^{p} w_i(\mathbf{x}(t))\Big(\mathbf{A}_i(\mathbf{x}(t))\mathbf{x}(t) + \mathbf{B}_i(\mathbf{x}(t))\mathbf{u}(t)\Big),$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t), \tag{2}$$

where

$$w_i(\mathbf{x}(t)) = \frac{\prod_{l=1}^{\Psi}\mu_{M_l^i}(f_l(\mathbf{x}(t)))}{\sum_{k=1}^{p}\prod_{l=1}^{\Psi}\mu_{M_l^k}(f_l(\mathbf{x}(t)))} \in [0,1], \forall i,$$
$$\sum_{i=1}^{p} w_i(\mathbf{x}(t)) = 1, w_i(\mathbf{x}(t)) \geq 0, \forall i, \tag{3}$$

$w_i(\mathbf{x}(t))$ is the normalized grade of membership; $\mu_{M_l^i}(f_l(\mathbf{x}(t)))$ is the grade of membership corresponding to the fuzzy term M_l^i; $\mathbf{y}(t) \in \Re^l$ is the output vector; $\mathbf{C} = [\mathbf{c}_1,\dots,\mathbf{c}_n] \in \Re^{l\times n}$ is an output matrix where $\mathbf{c}_s \in \Re^l$, $s \in \{1,\dots,n\}$, is the s-th column vector of matrix $\mathbf{C}$.

Definition 1: [5] A system is deemed to be positive if the initial condition $\mathbf{x}(0) = \mathbf{x}_0 \succeq 0$ holds and the corresponding trajectory $\mathbf{x}(t) \succeq 0$ for all $t \geq 0$ is satisfied.

Definition 2: [5] A matrix $\mathbf{M}$ is called a Metzler matrix if its off-diagonal elements are non-negative: $m_{rs} \succeq 0$, $r \neq s$.

Lemma 1: [46], [47] System (2) is a positive system if $\mathbf{A}_i(\mathbf{x}(t))$ is a Metzler matrix, $\mathbf{B}_i(\mathbf{x}(t)) \succeq 0\ \forall i$ and $\mathbf{C} \succeq 0$.

Assumption 1: It is assumed that the positive nonlinear system can be represented by a positive polynomial fuzzy model (2) satisfying the conditions in Lemma 1.

C. Static Output Feedback Polynomial Fuzzy Controller

In order to close the feedback loop of the positive polynomial fuzzy model (2), a static output feedback polynomial fuzzy controller described by c rules is employed. The j-th rule is presented as the following format:

$$\text{Rule } j: \text{IF } g_1(\mathbf{y}(t)) \text{ is } N_1^j \text{ AND} \dots \text{AND } g_\Omega(\mathbf{y}(t)) \text{ is } N_\Omega^j$$
$$\text{THEN } \mathbf{u}(t) = \mathbf{K}_j(\mathbf{y}(t))\mathbf{v}\mathbf{y}(t), \tag{4}$$

where $g_\beta(\mathbf{y}(t))$, $\beta \in \{1,\dots,\Omega\}$, is the premise variable, Ω is a positive integer; N_β^j, $j \in \{1,\dots,c\}$, is the fuzzy set of j-th rule corresponding to the function $g_\beta(\mathbf{y}(t))$; $\mathbf{K}_j(\mathbf{y}(t)) \in \Re^{m\times 1}$ is the static output polynomial feedback gain to be determined; $\mathbf{v} \in \Re^{1\times l}$ is a user-chosen non-zero constant vector which is satisfied with $\mathbf{v}\mathbf{C} = \bar{\mathbf{C}} \succeq 0$ and $\bar{\mathbf{C}} \in \Re^{1\times n}$. The static output feedback polynomial fuzzy controller is defined as:

$$\mathbf{u}(t) = \sum_{j=1}^{c} m_j(\mathbf{y}(t))\mathbf{K}_j(\mathbf{y}(t))\mathbf{v}\mathbf{y}(t), \tag{5}$$

where

$$m_j(\mathbf{y}(t)) = \frac{\prod_{\beta=1}^{\Omega}\mu_{N_\beta^j}(g_\beta(\mathbf{y}(t)))}{\sum_{k=1}^{c}\prod_{\beta=1}^{\Omega}\mu_{N_\beta^k}(g_\beta(\mathbf{y}(t)))} \in [0,1], \forall j,$$
$$\sum_{j=1}^{c} m_j(\mathbf{y}(t)) = 1, m_j(\mathbf{y}(t)) \geq 0, \forall j, \tag{6}$$

$m_j(\mathbf{y}(t))$ is the normalized grade of membership; $\mu_{N_\beta^j}(g_\beta(\mathbf{y}(t)))$ is the grade of membership corresponding to the fuzzy term of N_β^j.

The static output feedback polynomial fuzzy controller (5) can be rewritten as:

$$\mathbf{u}(t) = \sum_{j=1}^{c} m_j(\mathbf{y}(t))\mathbf{K}_j(\mathbf{y}(t))\bar{\mathbf{C}}\mathbf{x}(t). \tag{7}$$

Remark 1: It is worth noting that, in some papers [39], [48], the open-loop systems are not necessarily of positive but the closed-loop systems need to be positive and asymptotically stable. While, in other papers [6], [8], [47], not only the open-loop systems is positive but also the closed-loop systems is positive. In this paper, we consider the latter case that it requires the polynomial fuzzy model (the original nonlinear plant) to be positive and the static output feedback PPFMB control system to maintain positivity and stability.

III. STABILITY ANALYSIS

From the positive polynomial fuzzy model (2) and the static output feedback polynomial fuzzy controller (7), using the property of the membership functions given by (3) and (6), i.e., $\sum_{i=1}^{p}\sum_{j=1}^{c} w_i(\mathbf{x}(t))m_j(\mathbf{y}(t)) = 1$, we have the following static output feedback PPFMB control system:

$$\dot{\mathbf{x}}(t) = \sum_{i=1}^{p}\sum_{j=1}^{c} w_i(\mathbf{x}(t))m_j(\mathbf{y}(t)) \times \big(\mathbf{A}_i(\mathbf{x}(t)) + \mathbf{B}_i(\mathbf{x}(t))\mathbf{K}_j(\mathbf{y}(t))\bar{\mathbf{C}}\big)\mathbf{x}(t). \tag{8}$$

A system whose matrices are obtained by transposition of original system is called dual system of the original system. It is advantageous [10], [38] to obtain the stability results using the dual system and the stability analysis results between the two systems under duality will not change. So we also study the property of the static output feedback PPFMB control system (8) by its dual system which is shown as follows:

$$\dot{\mathbf{x}}(t) = \sum_{i=1}^{p}\sum_{j=1}^{c} w_i(\mathbf{x}(t))m_j(\mathbf{y}(t)) \times \big(\mathbf{A}_i(\mathbf{x}(t)) + \mathbf{B}_i(\mathbf{x}(t))\mathbf{K}_j(\mathbf{y}(t))\bar{\mathbf{C}}\big)^T\mathbf{x}(t). \quad (9)$$

Remark 2: According to Lemma 1, the dual PPFMB control system (9) is positive if $\mathbf{A}_i(\mathbf{x}(t)) + \mathbf{B}_i(\mathbf{x}(t))\mathbf{K}_j(\mathbf{y}(t))\bar{\mathbf{C}}$ is a Metzler matrix for all i and j and $\mathbf{C} \succeq 0$.

Hereinafter, we shall investigate the stability of the dual PPFMB control system (9). Basic stability conditions are first derived without the consideration of the information of membership functions. For sake of relaxation of stability analysis, MFD stability analysis will then be conducted through the use of polynomial functions approximating the membership functions with the consideration of approximation error.

For brevity, in the following, $w_i(\mathbf{x}(t))$, $m_j(\mathbf{y}(t))$, $\mathbf{x}(t)$ and $\mathbf{y}(t)$ are denoted as $w_i(\mathbf{x})$, $m_j(\mathbf{y})$, $\mathbf{x}$ and $\mathbf{y}$, respectively.

A. Basic Stability Analysis of Static Output Feedback PPFMB Control Systems

To study the stability of the dual PPFMB control system (9), the Lyapunov function candidate [47] is employed:

$$V(t) = \mathbf{x}^T\lambda, \quad (10)$$

where $\lambda = [\lambda_1, \dots, \lambda_n]^T \succ 0$ is a vector to be determined.

Based on the Lyapunov stability theory, if $V(t) > 0$ and $\dot{V}(t) < 0$, then the dual PPFMB control system (9) is asymptotically stable. From (9) and (10), the time derivative of the Lyapunov function candidate is obtained as follows:

$$\dot{V}(t) = \dot{\mathbf{x}}^T\lambda = \sum_{i=1}^{p}\sum_{j=1}^{c} w_i(\mathbf{x})m_j(\mathbf{y}) \times \mathbf{x}^T\big(\mathbf{A}_i(\mathbf{x}) + \mathbf{B}_i(\mathbf{x})\mathbf{K}_j(\mathbf{y})\bar{\mathbf{C}}\big)\lambda. \quad (11)$$

The inequality $\dot{V}(t) < 0$ holds if $(\mathbf{A}_i(\mathbf{x}) + \mathbf{B}_i(\mathbf{x})\mathbf{K}_j(\mathbf{y})\bar{\mathbf{C}})\lambda \prec 0$ for all i and j is satisfied. But the matrix $\bar{\mathbf{C}}$ will lead to non-convex stability conditions. To transform the non-convex stability conditions into convex stability conditions, we concentrate on the handling of matrix $\bar{\mathbf{C}}$. Inspired by the literature [49], the nonsingular transformation matrix $\mathbf{\Gamma} \in \Re^{n\times n}$ is considered as follows:

$$\mathbf{\Gamma} = [\bar{\mathbf{C}}^T(\bar{\mathbf{C}}\bar{\mathbf{C}}^T)^{-1} \quad \mathrm{ortc}(\bar{\mathbf{C}}^T)], \quad (12)$$

where $\mathrm{ortc}(\bar{\mathbf{C}}^T) \in \Re^{n\times(n-1)}$ denotes the orthogonal complement of $\bar{\mathbf{C}}^T$. From (12), it is obtained that

$$\bar{\mathbf{C}}\mathbf{\Gamma} = [1 \quad \mathbf{0}_{n-1}] \quad (13)$$

where $\mathbf{0}_{n-1}$ is a all-zero row vector of $n-1$ elements.

Defining $\mathbf{\Gamma}^{-1} = \mathbf{P} = [\mathbf{p}_1^T, \dots, \mathbf{p}_n^T]^T$, in which $\mathbf{p}_r \in \Re^{1\times n}$ is the r-th row vector of $\mathbf{P}$, then we have $\mathbf{\Gamma}\mathbf{P} = \mathbf{P}\mathbf{\Gamma} = \mathbf{I}$. It follows from (11) that

$$\dot{V}(t) = \sum_{i=1}^{p}\sum_{j=1}^{c} w_i(\mathbf{x})m_j(\mathbf{y}) \times \mathbf{x}^T\big(\mathbf{A}_i(\mathbf{x})\lambda + \mathbf{B}_i(\mathbf{x})\mathbf{K}_j(\mathbf{y})\bar{\mathbf{C}}\mathbf{\Gamma}\mathbf{P}\lambda\big). \quad (14)$$

Denoting $\mathbf{P}\lambda = \mathbf{h}$, we have

$$\lambda = \mathbf{P}^{-1}\mathbf{h} = \mathbf{\Gamma}\mathbf{h} \succ 0, \quad (15)$$

in which $\mathbf{h} = [h_1, \dots, h_n]^T \in \Re^n$. From (13), (14) and (15), we obtain $\mathbf{K}_j(\mathbf{y})\bar{\mathbf{C}}\mathbf{\Gamma}\mathbf{P}\lambda = \mathbf{K}_j(\mathbf{y})h_1$ and denote

$$\mathbf{Z}_j(\mathbf{y}) = \mathbf{K}_j(\mathbf{y})h_1, \quad (16)$$

where $\mathbf{Z}_j(\mathbf{y}) \in \Re^m$. Regarding (15) and (16), (14) can be written as follows:

$$\dot{V}(t) = \sum_{i=1}^{p}\sum_{j=1}^{c} w_i(\mathbf{x})m_j(\mathbf{y})\mathbf{x}^T\big(\mathbf{A}_i(\mathbf{x})\mathbf{\Gamma}\mathbf{h} + \mathbf{B}_i(\mathbf{x})\mathbf{Z}_j(\mathbf{y})\big). \quad (17)$$

From (17), it can be seen that $\dot{V}(t) < 0$ can be achieved by the following inequality:

$$\mathbf{A}_i(\mathbf{x})\mathbf{\Gamma}\mathbf{h} + \mathbf{B}_i(\mathbf{x})\mathbf{Z}_j(\mathbf{y}) \prec 0 \quad \forall i, j. \quad (18)$$

To ensure the positivity of the dual PPFMB control system (9), by Remark 2, the matrix $\mathbf{A}_i(\mathbf{x}) + \mathbf{B}_i(\mathbf{x})\mathbf{K}_j(\mathbf{y})\bar{\mathbf{C}}$ is required to be a Metzler matrix, that is

$$a_{irs}(\mathbf{x}) + \mathbf{b}_{ir}(\mathbf{x})\mathbf{K}_j(\mathbf{y})\mathbf{v}\mathbf{c}_s \succeq 0 \quad \forall\, r \neq s, i, j \quad (19)$$

where $a_{irs}(\mathbf{x})$ is the element of r-th row and s-th column of $\mathbf{A}_i(\mathbf{x})$, $r, s \in \{1, \dots, n\}$.

Combining (13) and (15), we have $h_1 = [1 \quad \mathbf{0}_{n-1}]\mathbf{P}\lambda = \mathbf{p}_1\lambda$. With regard to (12), we can obtain $\mathbf{p}_1 = \bar{\mathbf{C}}$ where the proof is given in the Appendix. As $\bar{\mathbf{C}} \succeq 0$, it leads to $\mathbf{p}_1 \succeq 0$. Considering $\mathbf{p}_1 \succeq 0$ and $\lambda \succ 0$, it can conclude that $h_1 \succ 0$. Then multiplying h_1 to the left side of (19), it follows that

$$a_{irs}(\mathbf{x})h_1 + \mathbf{b}_{ir}(\mathbf{x})\mathbf{Z}_j(\mathbf{y})\mathbf{v}\mathbf{c}_s \succeq 0 \quad \forall\, r \neq s, i, j. \quad (20)$$

Remark 3: If there exists a solution to (15), (18) and (20), we can conclude that the dual PPFMB control system (9) (as the original PPFMB control system (8)) is positive and asymptotically stable.

Remark 4: Based on the nonsingular transformation matrix $\mathbf{\Gamma}$, we can obtain convex stability conditions (18) and positivity conditions (20) which are linear in the decision variables $\mathbf{h}$ and $\mathbf{Z}_j(\mathbf{y})$. Once there exist feasible $\mathbf{h}$ and $\mathbf{Z}_j(\mathbf{y})$, λ can be obtained by (15) and the polynomial feedback gains $\mathbf{K}_j(\mathbf{y})$ can be obtained by (16).

The above positivity and stability analysis results are summarized in the following theorem.

Theorem 1: Consider that the nonlinear plant represented by the polynomial fuzzy model (2) is a positive system satisfying the conditions in Lemma 1 and the vector $\mathbf{v} \in \Re^{1\times l}$ is a user-chosen non-zero constant vector such that $\mathbf{v}\mathbf{C} = \bar{\mathbf{C}} \succeq 0$. The static output feedback PPFMB control system (8) or its dual system (9) with the initial condition $\mathbf{x}_0 \succeq 0$ is positive and asymptotically stable by the static output feedback polynomial fuzzy controller (7) if there exist vectors $\mathbf{Z}_j(\mathbf{y}) \in \Re^m$ and $\mathbf{h} = [h_1, \dots, h_n]^T \in \Re^n$, $i \in \{1, \dots, p\}$ and $j \in \{1, \dots, c\}$, such that the following SOS-based positivity and stability conditions are satisfied:

$$a_{irs}(\mathbf{x})h_1 + \mathbf{b}_{ir}(\mathbf{x})\mathbf{Z}_j(\mathbf{y})\mathbf{v}\mathbf{c}_s \text{ is SOS } \forall\, r \neq s, i, j; \tag{21}$$

$$\upsilon^T\big(\text{diag}(\mathbf{\Gamma h}) - \epsilon_1 \mathbf{I}\big)\upsilon \text{ is SOS}; \tag{22}$$

$$-\,\upsilon^T\big(\text{diag}(\mathbf{A}_i(\mathbf{x})\mathbf{\Gamma h} + \mathbf{B}_i(\mathbf{x})\mathbf{Z}_j(\mathbf{y})) + \epsilon_2(\mathbf{x})\mathbf{I}\big)\upsilon \text{ is SOS } \forall\, i, j, \tag{23}$$

where $\upsilon \in \Re^n$ is an arbitrary vector independent of $\mathbf{x}$ and $\mathbf{y}$; $\epsilon_1 > 0$ is a predefined scalar and $\epsilon_2(\mathbf{x}) > 0$ for $\mathbf{x} \neq 0$ is a predefined scalar polynomial; $\mathbf{\Gamma}$ is the nonsingular transformation matrix which is defined in (12). The static output polynomial feedback gain can be obtained as $\mathbf{K}_j(\mathbf{y}) = \mathbf{Z}_j(\mathbf{y})/h_1$ according to (16).

Remark 5: The positivity condition (21) is used to ensure the matrix $\mathbf{A}_i(\mathbf{x}) + \mathbf{B}_i(\mathbf{x})\mathbf{K}_j(\mathbf{y})\bar{\mathbf{C}}$ to be a Metzler matrix for all i and j according to (20). The stability condition (22) is used to ensure $V(t) > 0$ (excluding $\mathbf{x} = \mathbf{0}$) by meeting $\lambda \succ 0$. The stability condition (23) is used to ensure $\dot{V}(t) < 0$ (excluding $\mathbf{x} = \mathbf{0}$) by meeting the condition (18).

B. Stability Analysis of Static Output Feedback PPFMB Control System via Approximate Membership Functions

It is worth mentioning that the stability conditions in Theorem 1 are membership-function-independent (MFI) and thus valid for any shape of membership functions, which will usually lead to conservative results. In this section, to obtain relaxed stability conditions, we will take the information of membership functions into the stability analysis. Relaxed SOS-based stability analysis for static output feedback control is investigated and MFD stability conditions with embedded information of membership functions represented by approximated polynomial functions are developed.

As the membership functions $w_i(\mathbf{x})$ and $m_j(\mathbf{y})$ depend on different variables, i.e., $\mathbf{x}$ and $\mathbf{y}$, respectively, it is not advantageous to analyze and relax the stability conditions when applying MFD stability analysis technique [43]. However, due to $\mathbf{y} = \mathbf{C}\mathbf{x}$, $m_j(\mathbf{y})$ is actually related to $\mathbf{x}$. Similarly, $\mathbf{K}_j(\mathbf{y})$ can also be seen as a function of $\mathbf{x}$.

Considering the product term $w_i(\mathbf{x})m_j(\mathbf{y})$ and the operating domain, the information of membership functions and states boundary will be utilized in the stability conditions. To facilitate the stability analysis, polynomial functions approximating $w_i(\mathbf{x})m_j(\mathbf{y})$ [33], [50] are first constructed. However, it may require a high degree of polynomial for good approximation, which will demand high computational power for solving a feasible solution. Hence, we divide the global operating domain into D connected subdomains, where D denotes the number of the subdomains. In each subdomain s_τ, the membership function $w_i(\mathbf{x})m_j(\mathbf{y})$ can be approximated by a relatively simpler polynomial of lower degrees.

To realize the stability analysis, we introduce a scalar function $\zeta_\tau(\mathbf{x})$ indicating active/inactive subdomains, which has the following properties for $\tau \in \{1, \ldots, D\}$:

$$\begin{cases} \zeta_\tau(\mathbf{x}) = 1 & \forall \mathbf{x} \in s_\tau, \\ \zeta_\tau(\mathbf{x}) = 0 & \forall \mathbf{x} \notin s_\tau. \end{cases} \tag{24}$$

When the system state $\mathbf{x}$ is working at the operating subdomain s_τ, the scalar function will satisfy $\zeta_\tau(\mathbf{x}) = 1$, otherwise the scalar function will satisfy $\zeta_\tau(\mathbf{x}) = 0$.

The product term $w_i(\mathbf{x})m_j(\mathbf{y})$ in each subdomain s_τ is expressed as an approximated polynomial function with an error term shown below [50]:

$$w_i(\mathbf{x})m_j(\mathbf{y}) = \eta_{ij,s_\tau}(\mathbf{x}) + \Delta\eta_{ij,s_\tau}(\mathbf{x}), \forall i, j, \tau, \tag{25}$$

where $\eta_{ij,s_\tau}(\mathbf{x})$ is the approximated polynomial; $\Delta\eta_{ij,s_\tau}(\mathbf{x})$ is the error term for all i, j and τ.

Therefore, based on (17), (24) and (25), the derivative of Lyapunov function can be written as:

$$\dot{V}(t) = \sum_{\tau=1}^{D}\sum_{i=1}^{p}\sum_{j=1}^{c}\zeta_\tau(\mathbf{x})\mathbf{x}^T \times \Big(\big(\eta_{ij,s_\tau}(\mathbf{x}) + \Delta\eta_{ij,s_\tau}(\mathbf{x})\big)\mathbf{Q}_{ij}(\mathbf{x})\Big), \tag{26}$$

where $\mathbf{Q}_{ij}(\mathbf{x})$ denotes $\mathbf{A}_i(\mathbf{x})\mathbf{\Gamma h} + \mathbf{B}_i(\mathbf{x})\mathbf{Z}_j(\mathbf{y})$.

Next, the information of error term $\Delta\eta_{ij,s_\tau}(\mathbf{x})$ will be introduced into the stability analysis by considering its lower and upper bounds given as follows:

$$\gamma_{ij,s_\tau} \leq \Delta\eta_{ij,s_\tau}(\mathbf{x}) \leq \beta_{ij,s_\tau}, \tag{27}$$

where γ_{ij,s_τ} is the constant lower bound of the error term $\Delta\eta_{ij,s_\tau}(\mathbf{x})$ and β_{ij,s_τ} is the constant upper bound of the error term $\Delta\eta_{ij,s_\tau}(\mathbf{x})$. Both of them are to be determined.

To facilitate the stability analysis, based on the property of the error term $\Delta\eta_{ij,s_\tau}(\mathbf{x})$ in (27), we introduce the slack polynomial matrices $0 \preceq \mathbf{G}_{ij,s_\tau}(\mathbf{x}) = [g_1^{ij,s_\tau}(\mathbf{x}), \ldots, g_n^{ij,s_\tau}(\mathbf{x})]^T \in \Re^n$, which satisfies

$$\mathbf{G}_{ij,s_\tau}(\mathbf{x}) \succeq \mathbf{Q}_{ij}(\mathbf{x}) \;\forall\, i, j, \tau. \tag{28}$$

Combining (26) and (27), we have

$$\dot{V}(t) = \sum_{\tau=1}^{D}\sum_{i=1}^{p}\sum_{j=1}^{c}\zeta_\tau(\mathbf{x})\mathbf{x}^T\Big(\big(\eta_{ij,s_\tau}(\mathbf{x}) + \gamma_{ij,s_\tau}\big)\mathbf{Q}_{ij}(\mathbf{x}) + \big(\Delta\eta_{ij,s_\tau}(\mathbf{x}) - \gamma_{ij,s_\tau}\big)\mathbf{Q}_{ij}\Big). \tag{29}$$

Recalling the property of error term $\Delta\eta_{ij,s_\tau}(\mathbf{x})$ in (27), we have $0 \leq \Delta\eta_{ij,s_\tau}(\mathbf{x}) - \gamma_{ij,s_\tau} \leq \beta_{ij,s_\tau} - \gamma_{ij,s_\tau}$. Furthermore, the slack matrix $\mathbf{G}_{ij,s_\tau}(\mathbf{x})$ satisfies with $\mathbf{G}_{ij,s_\tau}(\mathbf{x}) \succeq 0$ and (28). Thereby, the inequality $\big(\Delta\eta_{ij,s_\tau}(\mathbf{x}) - \gamma_{ij,s_\tau}\big)\mathbf{Q}_{ij}(\mathbf{x}) \preceq \big(\beta_{ij,s_\tau} - \gamma_{ij,s_\tau}\big)\mathbf{G}_{ij,s_\tau}(\mathbf{x})$ is obtained, then we have

$$\dot{V}(t) \leq \sum_{\tau=1}^{D}\sum_{i=1}^{p}\sum_{j=1}^{c}\zeta_\tau(\mathbf{x})\mathbf{x}^T\Big(\big(\eta_{ij,s_\tau}(\mathbf{x}) + \gamma_{ij,s_\tau}\big)\mathbf{Q}_{ij}(\mathbf{x}) + \big(\beta_{ij,s_\tau} - \gamma_{ij,s_\tau}\big)\mathbf{G}_{ij,s_\tau}(\mathbf{x})\Big). \tag{30}$$

In the following, the information of states boundary in each subdomain s_τ is taken into consideration, which provides additional information for the relaxation of stability analysis. Inspired by [30], we introduce a user-chosen scalar function $L_{s_\tau}(\mathbf{x})$ which exhibits the following properties:

$$\begin{cases} L_{s_\tau}(\mathbf{x}) \leq 0 & \forall \mathbf{x} \in s_\tau, \\ L_{s_\tau}(\mathbf{x}) > 0 & \forall \mathbf{x} \notin s_\tau. \end{cases} \tag{31}$$

From (31), it can be seen that when the system state $\mathbf{x}$ is working at the operating subdomain s_τ, the scalar function satisfies $L_{s_\tau}(\mathbf{x}) \leq 0$, otherwise it will satisfy $L_{s_\tau}(\mathbf{x}) > 0$.

In order to bring the information $L_{s_\tau}(\mathbf{x})$ in (31) to relax the stability condition (30), we first consider a polynomial vector $\mathbf{R}(\mathbf{x}) = [R_1(\mathbf{x}), \ldots, R_n(\mathbf{x})] \in \Re^n$ satisfying $\mathbf{R}(\mathbf{x}) \succeq 0$ and recall the property of $L_{s_\tau}(\mathbf{x})$ in (31). Consequently, we have $L_{s_\tau}(\mathbf{x})\mathbf{R}(\mathbf{x}) \preceq 0$ for $\mathbf{x} \in s_\tau$ and $L_{s_\tau}(\mathbf{x})\mathbf{R}(\mathbf{x}) \succeq 0$ for $\mathbf{x} \notin s_\tau$, which is considered in (30) with (24), and obtain

$$\dot{V}(t) \leq \sum_{\tau=1}^{D} \zeta_\tau(\mathbf{x})\mathbf{x}^T \Bigg(\sum_{i=1}^{p}\sum_{j=1}^{c} \big((\eta_{ij,s_\tau}(\mathbf{x}) + \gamma_{ij,s_\tau})\mathbf{Q}_{ij}(\mathbf{x}) + (\beta_{ij,s_\tau} - \gamma_{ij,s_\tau})\mathbf{G}_{ij,s_\tau}(\mathbf{x})\big) - L_{s_\tau}(\mathbf{x})\mathbf{R}(\mathbf{x}) \Bigg). \quad (32)$$

According to Lyapunov stability theory, $\dot{V}(t) < 0$ (excluding $\mathbf{x} = \mathbf{0}$) implies the static output feedback PPFMB control system (8) or its dual system (9) to be asymptotically stable, which can be achieved by satisfying the following condition:

$$\sum_{i=1}^{p}\sum_{j=1}^{c} \big((\eta_{ij,s_\tau}(\mathbf{x}) + \gamma_{ij,s_\tau})\mathbf{Q}_{ij}(\mathbf{x}) + (\beta_{ij,s_\tau} - \gamma_{ij,s_\tau}) \times \mathbf{G}_{ij,s_\tau}(\mathbf{x})\big) - L_{s_\tau}(\mathbf{x})\mathbf{R}(\mathbf{x}) \prec 0, \quad \forall\, i, j, \tau. \quad (33)$$

Since the information of membership functions and system states is taken into stability conditions, the conservativeness can be relaxed. The positivity and stability analysis results are summarized in the following theorem.

Theorem 2: Consider that the nonlinear plant represented by the polynomial fuzzy model (2) is a positive system satisfying the conditions in Lemma 1 and the vector $\mathbf{v} \in \Re^{1\times l}$ is a user-chosen non-zero constant vector such that $\mathbf{vC} = \bar{\mathbf{C}} \succeq 0$. Let the global operating domain be divided into D connected subdomains. In each subdomain s_τ, the product term $w_i(\mathbf{x})m_j(\mathbf{y})$ is replaced by the polynomial $\eta_{ij,s_\tau}(\mathbf{x})$ and the error term $\Delta\eta_{ij,s_\tau}(\mathbf{x})$, $i \in \{1,\ldots,p\}$, $j \in \{1,\ldots,c\}$ and $\tau \in \{1,\ldots,D\}$, satisfying (25). The static output feedback PPFMB control system (8) or its dual system (9) with the initial condition $\mathbf{x}_0 \succeq 0$ is positive and asymptotically stable by the static output feedback polynomial fuzzy controller (7) if there exist vectors $\mathbf{h} = [h_1,\ldots,h_n]^T \in \Re^n$, $\mathbf{Z}_j(\mathbf{y}) \in \Re^m$, $\mathbf{G}_{ij,s_\tau}(\mathbf{x}) \in \Re^n$ and the scalar polynomial $\mathbf{R}(\mathbf{x}) \in \Re^n$ such that the following SOS-based positivity and stability conditions are satisfied:

$$a_{irs}(\mathbf{x})h_1 + \mathbf{b}_{ir}(\mathbf{x})\mathbf{Z}_j(\mathbf{y})\mathbf{vc}_s \text{ is SOS } \forall\, r \neq s, i, j; \quad (34)$$

$$\upsilon^T\big(\text{diag}(\mathbf{\Gamma h}) - \epsilon_1\mathbf{I}\big)\upsilon \text{ is SOS}; \quad (35)$$

$$\upsilon^T\text{diag}(\mathbf{G}_{ij,s_\tau}(\mathbf{x}))\upsilon \text{ is SOS } \forall\, i, j, \tau; \quad (36)$$

$$\upsilon^T\big(\text{diag}(\mathbf{G}_{ij,s_\tau}(\mathbf{x}) - \mathbf{Q}_{ij}(\mathbf{x}))\big)\upsilon \text{ is SOS } \forall\, i, j, \tau; \quad (37)$$

$$\upsilon^T\text{diag}(\mathbf{R}(\mathbf{x}))\upsilon \text{ is SOS}; \quad (38)$$

$$-\upsilon^T\Bigg(\text{diag}\Big(\sum_{i=1}^{p}\sum_{j=1}^{c}\big((\eta_{ij,s_\tau}(\mathbf{x}) + \gamma_{ij,s_\tau})\mathbf{Q}_{ij}(\mathbf{x}) + (\beta_{ij,s_\tau} - \gamma_{ij,s_\tau})\mathbf{G}_{ij,s_\tau}(\mathbf{x})\big) - L_{s_\tau}(\mathbf{x})\mathbf{R}(\mathbf{x})\Big) + \epsilon_2(\mathbf{x})\mathbf{I}\Bigg)\upsilon \text{ is SOS } \forall\, i, j, \tau; \quad (39)$$

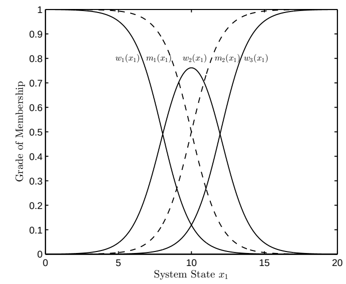

Fig. 1. Membership functions of positive polynomial fuzzy model and static output feedback polynomial fuzzy controller.

where $\upsilon \in \Re^n$ is an arbitrary vector independent of $\mathbf{x}$ and $\mathbf{y}$; $\epsilon_1 > 0$ is a predefined scalar and $\epsilon_2(\mathbf{x}) > 0$ for $\mathbf{x} \neq 0$ is a predefined scalar polynomial; $\mathbf{\Gamma}$ is the nonsingular transformation matrix which is defined in (12); $\mathbf{Q}_{ij}(\mathbf{x})$ is defined in (26); γ_{ij,s_τ} and β_{ij,s_τ} are the pre-determined lower and upper bounds of error term satisfying (27); $L_{s_\tau}(\mathbf{x})$ is the user-chosen scalar function which satisfies the property in (31); the static output polynomial feedback gain is obtained as $\mathbf{K}_j(\mathbf{y}) = \mathbf{Z}_j(\mathbf{y})/h_1$ according to (16).

Remark 6: To improve the computational efficiency, $\mathbf{G}_{ij,s_\tau}(\mathbf{x})$ can be chosen to be the same matrix for all τ, in other words, $\mathbf{G}_{ij}(\mathbf{x}) = \mathbf{G}_{ij,s_\tau}(\mathbf{x})$, for all τ.

Remark 7: By removing the positivity conditions in Lemma 1 in Theorem 1 or 2, the rest conditions of the corresponding theorem can be used to make sure that the static output feedback PPFMB control system (8) is controlled to be positive and asymptotically stable but without requiring that the nonlinear plant represented by the polynomial fuzzy model (2) to be open-loop positive.

The control design procedures and the guideline of the parameter selections mainly consist of the following steps: 1) Construct a polynomial fuzzy system for the nonlinear system, which are characterized by the number of rules ("p"), membership functions $w_i(\mathbf{x})$ and matrices $\mathbf{A}_i(\mathbf{x})$, $\mathbf{B}_i(\mathbf{x})$ and $\mathbf{C}$. 2) Design a static output feedback polynomial fuzzy controller in the form of (5) by choosing the number of rules ("c"), membership functions $m_j(\mathbf{x})$ and the vector $\mathbf{v}$ satisfying $\mathbf{vC} = \bar{\mathbf{C}} \succeq 0$. 3) Obtain the approximated membership function $\eta_{ij,s_\tau}(\mathbf{x})$ for $w_i(\mathbf{x})m_j(\mathbf{x})$, say, using the nonlinear least squares data fitting method. According to equation (25) in the paper, obtain the error term $\Delta\eta_{ij,s_\tau}(\mathbf{x})$ and its maximums β_{ij,s_τ} and minimums γ_{ij,s_τ}. 4) Choose the parameters, using convex programming software to solve numerically the positivity and stability conditions according to the theorem.

IV. Simulation Example

An example is provided to demonstrate and verify the analysis results. In this example, three main areas are discussed: 1) whether the number of subdomains, D, will influence the stability region, 2) whether the degree of slack polynomial matrix, $\mathbf{G}_{ij,s_\tau}(\mathbf{x})$, will influence the stability region, 3) whether the existence of scalar function, $L_{s_\tau}(\mathbf{x})$, in the relaxed stability conditions of Theorem 2 will impact on the stability region.

A. Scenario

Let us consider a three-rule positive polynomial fuzzy model with the following system and input matrices:

$$\mathbf{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T, \mathbf{A}_1(x_1) = \begin{bmatrix} 1 & 1.8 + x_1^2 \\ 0.1b & -1.9 \end{bmatrix},$$

$$\mathbf{A}_2(x_1) = \begin{bmatrix} 0.8 + x_1^2 & 1.5 \\ 0.9 & -0.1a \end{bmatrix},$$

$$\mathbf{A}_3(x_1) = \begin{bmatrix} 0.5 + x_1^2 + x_1 & 1 \\ 0.5 & -0.2a \end{bmatrix},$$

$$\mathbf{B}_1(x_1) = \begin{bmatrix} 1 + 0.09x_1^2 \\ 0.15 \end{bmatrix},$$

$$\mathbf{B}_2(x_1) = \begin{bmatrix} 0.1b + 3 + 1.5x_1^2 + x_1 \\ 0 \end{bmatrix},$$

$$\mathbf{B}_3(x_1) = \begin{bmatrix} 0.1b + 1.75x_1^2 \\ 0 \end{bmatrix}, \mathbf{C} = \begin{bmatrix} 2 & 0 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 1 \end{bmatrix}$$

where a and b are constant scalars.

It can be seen that the system matrices $\mathbf{A}_i(x_1), i \in \{1,2,3\}$ are Metzler matrices; the elements of the input matrices $\mathbf{B}_i(x_1), i \in \{1,2,3\}$ are non-negative; the elements of output matrix $\mathbf{C}$ are non-negative. Hence, in the light of Lemma 1, the polynomial fuzzy system is positive. Meanwhile, $\mathbf{v}$ is a user-chosen non-zero constant vector and chosen arbitrarily in the simulation satisfying $\mathbf{v}\mathbf{C} = \bar{\mathbf{C}} = [2 \quad 0] \succeq 0$. When we derive the positivity conditions, e.g., (20), we need to ensure that $\mathbf{p}_1\lambda = h_1 \succ 0$. Due to $\lambda \succeq 0$ and $\mathbf{p}_1 = \bar{\mathbf{C}}$, hence, $\bar{\mathbf{C}} \succeq 0$ can be achieved, which lead to $\mathbf{v}\mathbf{C} = \bar{\mathbf{C}} \succeq 0$ by picking $\mathbf{v} \succeq 0$. Therefore, in this example, we choose the vector $\mathbf{v} = 1$ arbitrarily to satisfy $\mathbf{v}\mathbf{C} = \bar{\mathbf{C}} = [2 \quad 0] \succeq 0$.

A two-rule static output feedback polynomial fuzzy controller in the form (7) is employed to stabilize the positive polynomial fuzzy model. Considering $x_1 \in [0 \quad 20]$, the membership functions of the positive polynomial fuzzy model and static output feedback polynomial fuzzy controller are chosen as follows: $w_1(x_1) = 1 - \frac{1}{1+e^{-(x_1-8)}}$, $w_2(x_1) = 1 - w_1(x_1) - w_3(x_1)$, $w_3(x_1) = \frac{1}{1+e^{-(x_1-12)}}$, $m_1(x_1) = 1 - \frac{1}{1+e^{-(x_1-10)}}$, $m_2(x_1) = 1 - m_1(x_1)$, which are shown in Fig. 1. In this example, each Theorem is investigated with $10 \leq a \leq 40$ at the interval of 5 and $3 \leq b \leq 16$ at the interval of 0.5.

B. Settings when Applying Theorems

To apply the stability conditions in Theorem 1 for comparison purposes, which are without considering the information of membership functions and system states, we choose $\epsilon_1 = \epsilon_2(\mathbf{x}) = 0.001$ and the highest degree of $\mathbf{Z}_j(x_1)$ to be 2.

In Theorem 2, relaxed stability conditions are obtained with the consideration of the membership-function information and system states in each subdomain. We choose $\epsilon_1 = \epsilon_2(\mathbf{x}) = 0.001$, the highest degree of $\mathbf{Z}_j(x_1)$ to be 2 which is the same as in applying Theorem 1 and the degree of $\mathbf{R}(x_1)$ to be 2.

Theorem 2 will be tested with different number of subdomains, D, such that the influence to stability region can be revealed with the constant parameters a and b for the polynomial fuzzy model. In addition, different highest degrees of $\mathbf{G}_{ij,s_\tau}(x_1)$ will be discussed to study whether the degree of slack polynomial matrix, $\mathbf{G}_{ij,s_\tau}(x_1)$, will influence the stability region. To reduce computational demand on searching for a feasible solution, $\mathbf{G}_{ij,s_\tau}(x_1)$ is chosen to be the same for all subdomains, i.e., $\mathbf{G}_{ij}(x_1) = \mathbf{G}_{ij,s_\tau}(x_1)$ for all τ.

C. Effect of Number of Subdomains D and Degree of $\mathbf{G}_{ij}(x_1)$

The operating domain of membership functions is divided uniformly into D subdomains. For investigation purposes, the number of subdomains is chosen, in turn, as $D = 5$, $D = 7$ and $D = 9$ when applying Theorem 2. For the three cases, the information of the operating domain division is given in Table II. According to (25), by using the nonlinear least squares data fitting method, each product term $w_i(x_1)m_j(x_1)$ is approximated by a polynomial function $\eta_{ij,s_\tau}(x_1)$ in each subdomain and these polynomial functions are shown in Tables III-V in the Appendix. According to the original membership functions and the approximated membership functions, the error term $\Delta\eta_{ij,s_\tau}(x_1)$ can be obtained, meanwhile, the lower bound γ_{ij,s_τ} and upper bound β_{ij,s_τ} of the error terms $\Delta\eta_{ij,s_\tau}(x_1)$ satisfying (27) are found numerically which can be seen in Tables I to IV in the supplemental file. $L_{s_\tau}(x_1)$ are presented in the form of $L_{s_\tau}(x_1) = (x_1 - x_{min,s_\tau})(x_1 - x_{max,s_\tau})$ where x_{min,s_τ} and x_{max,s_τ} take the left and right boundary values of x_1 in the operating subdomain s_τ. For example, referring to Region 1 ($0 \leq x_1 \leq 3$) of Case 1 in Table II, the left and right boundary values of x_1 are 0 and 3, respectively, namely $x_{min,s_1} = 0$ and $x_{max,s_1} = 3$ which gives $L_{s_1}(x_1) = (x_1 - 0)(x_1 - 3)$.

When the highest degree of $\mathbf{G}_{ij}(x_1)$ is chosen as 0 or 2 for the cases of $D = 5$, $D = 7$ or $D = 9$, the stability regions obtained using Theorem 2 with $L_{s_\tau}(x_1)$ are shown in Figs. 2, 3 and 4, respectively. From the point of view of the number of subdomains D, by keeping the highest degree of $\mathbf{G}_{ij}(x_1)$ to be the same, it can be seen from Figs. 2, 3 and 4 that larger stability region can be obtained with more number of subdomains D is considered which is due to smaller approximated error $\Delta\eta_{ij,s_\tau}(x_1)$ of the membership functions making the stability conditions to be satisfied easier.

From the point of view of the degree of $\mathbf{G}_{ij}(x_1)$, by keeping the same number of subdomains D, it can be seen from Figs. 2, 3 and 4 that higher degree of $\mathbf{G}_{ij}(x_1)$ can offer larger stability region because higher degree of $\mathbf{G}_{ij}(x_1)$ demonstrates better feedback compensation capability, which offers lager degree of freedom when searching for a feasible solution.

D. Effect of $L_{s_\tau}(x_1)$

For demonstrating the effectiveness of the scalar function $L_{s_\tau}(x_1)$ in the stability conditions, relaxed stability conditions in Theorem 2 without $L_{s_\tau}(x_1)$ for the case of $D = 5$ and $\mathbf{G}_{ij}(x_1)$ of degree 0 are employed to find the stability region which is shown in Fig. 5 indicated by "+" while the stability region with $L_{s_\tau}(x_1)$ is indicated by "×". Compared with the two stability regions, it can be seen that the relaxed stability conditions in Theorem 2 with $L_{s_\tau}(x_1)$ are capable of obtaining a larger stability region, which demonstrate the significance of the information embedded in $L_{s_\tau}(x_1)$. In order to investigate the advantages of the relaxed stability conditions in Theorem 2, the stability region obtained according to the basic stability

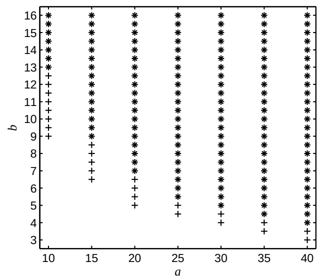

Fig. 2. Stability region given by Theorem 2 with $L_{s_\tau}(x_1)$ for $D = 5$, $\mathbf{G}_{ij}(x_1)$ of degrees 0 ("×") and 2 ("+").

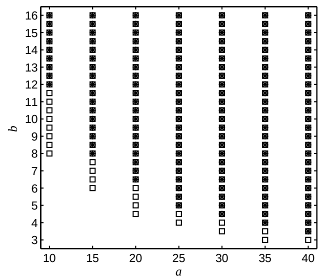

Fig. 3. Stability region given by Theorem 2 with $L_{s_\tau}(x_1)$ for $D = 7$, $\mathbf{G}_{ij}(x_1)$ of degrees 0 ("∗") and 2 ("□").

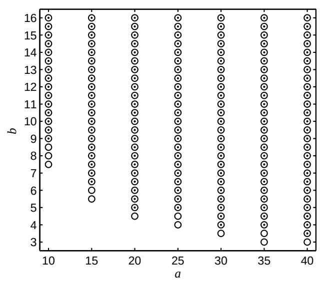

Fig. 4. Stability region given by Theorem 2 with $L_{s_\tau}(x_1)$ for $D = 9$, $\mathbf{G}_{ij}(x_1)$ of degrees 0 ("•") and 2 ("○").

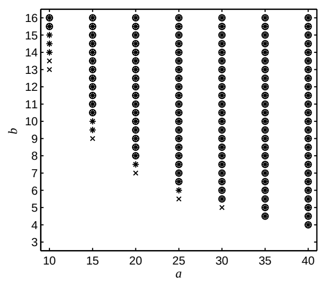

Fig. 5. Stability region given by Theorem 1 ("○"). Stability region given by Theorem 2 without $L_{s_\tau}(x_1)$ for $D = 5$, $\mathbf{G}_{ij}(x_1)$ of degrees 0 ("+"). Stability region given by Theorem 2 with $L_{s_\tau}(x_1)$ for $D = 5$, $\mathbf{G}_{ij}(x_1)$ of degrees 0 ("×").

conditions in Theorem 1 also is shown as "○" in Fig. 5. It is obviously that the relaxed stability conditions in Theorem 2 are able to provide a larger stability region compared with that given by the stability conditions in Theorem 1.

E. Phase Plot of System States

In order to verify the stability analysis results, the phase plots of x_1 and x_2 subject to various initial conditions are obtained. In Fig. 2, we choose $a = 40$, $b = 5.5$ and $a = 40$, $b = 3$ indicated by "×" and "+", respectively. In Fig. 3, we

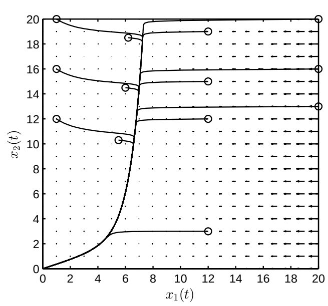

Fig. 6. Phase plot of the states x_1 and x_2 for $D = 5$, $a = 40, b = 5.5$ and $\mathbf{G}_{ij}(x_1)$ of degree 0 given by Theorem 2 with $L_{s_\tau}(x_1)$.

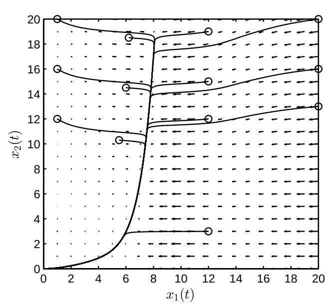

Fig. 7. Phase plot of the states x_1 and x_2 for $D = 5$, $a = 40, b = 3$ and $\mathbf{G}_{ij}(x_1)$ of degree 2 given by Theorem 2 with $L_{s_\tau}(x_1)$.

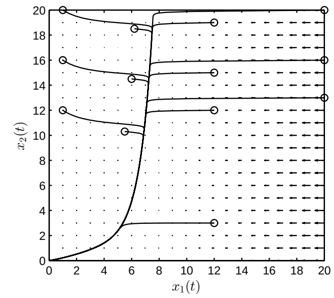

Fig. 8. Phase plot of the states x_1 and x_2 for $D = 7$, $a = 40, b = 4.5$ and $\mathbf{G}_{ij}(x_1)$ of degree 0 given by Theorem 2 with $L_{s_\tau}(x_1)$.

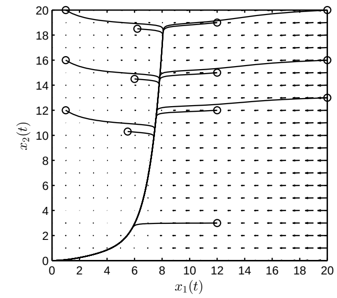

Fig. 9. Phase plot of the states x_1 and x_2 for $D = 7$, $a = 40, b = 3$ and $\mathbf{G}_{ij}(x_1)$ of degree 2 given by Theorem 2 with $L_{s_\tau}(x_1)$.

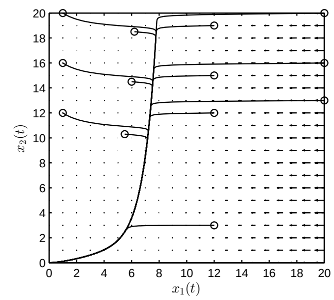

Fig. 10. Phase plot of the states x_1 and x_2 for $D = 9$, $a = 40, b = 3.5$ and $\mathbf{G}_{ij}(x_1)$ of degree 0 given by Theorem 2 with $L_{s_\tau}(x_1)$.

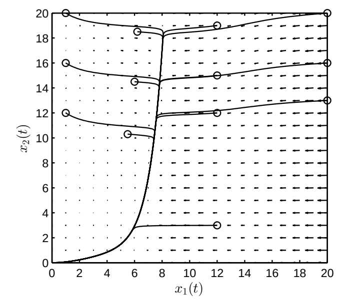

Fig. 11. Phase plot of the states x_1 and x_2 for $D = 9$, $a = 40, b = 3$ and $\mathbf{G}_{ij}(x_1)$ of degree 2 given by Theorem 2 with $L_{s_\tau}(x_1)$.

choose $a = 40$, $b = 4.5$ and $a = 40$, $b = 3$ indicated by "∗" and "□", respectively. In Fig. 4, we choose $a = 40$, $b = 3.5$ and $a = 40$, $b = 3$ indicated by "•" and "○", respectively. Corresponding to different values of a and b, λ are obtained and shown in Table V in the supplemental file, static output polynomial feedback gain matrices $\mathbf{K}_j(x_1)$ are listed in Table VI in the supplemental file and slack polynomial matrices $\mathbf{G}_{ij}(x_1)$ are presented in Table VII in the supplemental file. The corresponding phase plots are shown in Figs. 6 to 11.

As can be seen from Figs. 6 to 11, the static output feedback PPFMB control system is positive and asymptotically stable, and the static output feedback polynomial fuzzy controller can drive the positive polynomial fuzzy system to the origin.

V. Conclusion

The static output feedback control problem for PPFMB control systems has been investigated. A static output feedback polynomial fuzzy controller guaranteeing the static output feedback PPFMB control system to be positive and asymptotically stable has been designed. Basic SOS-based stability conditions have been obtained based on the Lyapunov stability theory. To facilitate the stability analysis, relaxed SOS-based stability conditions have been obtained by dividing the global operating domain of membership functions into subdomains and introducing the information of approximate membership functions and states boundary. A simulation example has been presented to verify the analysis results, and illustrated the effectiveness of the proposed control design and strategy.

TABLE II
OPERATING SUBDOMAINS OF x_1 FOR THE THREE CASES.

Region τ	Case 1:$D=5$	Case 2:$D=7$	Case 3:$D=9$
Region 1	$0 \leq x_1 \leq 3$	$0 \leq x_1 \leq 3$	$0 \leq x_1 \leq 3$
Region 2	$3 \leq x_1 \leq 8.9$	$3 \leq x_1 \leq 5.7$	$3 \leq x_1 \leq 5.7$
Region 3	$8.9 \leq x_1 \leq 11.1$	$5.7 \leq x_1 \leq 8.9$	$5.7 \leq x_1 \leq 7$
Region 4	$11.1 \leq x_1 \leq 17$	$8.9 \leq x_1 \leq 11.1$	$7 \leq x_1 \leq 8.9$
Region 5	$17 \leq x_1 \leq 20$	$11.1 \leq x_1 \leq 14.3$	$8.9 \leq x_1 \leq 11.1$
Region 6	–	$14.3 \leq x_1 \leq 17$	$11.1 \leq x_1 \leq 13$
Region 7	–	$17 \leq x_1 \leq 20$	$13 \leq x_1 \leq 14.3$
Region 8	–	–	$14.3 \leq x_1 \leq 17$
Region 9	–	–	$17 \leq x_1 \leq 20$

VI. APPENDIX

A. Proof

According to the nonsingular transformation matrix $\mathbf{\Gamma} = [\bar{\mathbf{C}}^T(\bar{\mathbf{C}}\bar{\mathbf{C}}^T)^{-1} \quad \mathrm{ortc}(\bar{\mathbf{C}}^T)]$ in (12), where $\mathrm{ortc}(\bar{\mathbf{C}}^T)$ denotes the orthogonal complement of $\bar{\mathbf{C}}^T$, we can obtain that $\mathrm{ortc}(\bar{\mathbf{C}}^T) \in \Re^{n\times(n-1)}$. Now, we might as well decompose $\mathrm{ortc}(\bar{\mathbf{C}}^T)$ into $\mathrm{ortc}(\bar{\mathbf{C}}^T) = \mathbf{F} = [\mathbf{f}_1 \quad \dots \quad \mathbf{f}_{n-1}]$, where $\mathbf{f}_k \in \Re^n$ is a column vector, for $k \in \{1, \dots, n-1\}$. Meanwhile they meet the properties of orthogonal complement [51] such that $\mathbf{f}_k^T\bar{\mathbf{C}}^T = 0$, $\mathbf{f}_k^T\mathbf{f}_k = 1$ and $\mathbf{f}_k^T\mathbf{f}_l = 0$, for k, $l \in \{1, \dots, n-1\}$, $k \neq l$. By using the properties of orthogonal complement we can obtain that

$$\begin{aligned}\mathbf{f}_k^T\mathbf{\Gamma} =&[\mathbf{f}_k^T\bar{\mathbf{C}}^T(\bar{\mathbf{C}}\bar{\mathbf{C}}^T)^{-1} \quad \mathbf{f}_k^T\mathbf{f}_1 \quad \dots \quad \mathbf{f}_k^T\mathbf{f}_k \quad \dots \quad \mathbf{f}_k^T\mathbf{f}_{n-1}]\\ =&[0 \quad 0 \quad \dots \quad 1_k \quad \dots \quad 0].\end{aligned}$$

Because of $\bar{\mathbf{C}}\mathbf{\Gamma} = [1 \quad \mathbf{0}_{n-1}]$ and $\mathbf{f}_k^T\mathbf{\Gamma} = [0 \quad 0 \quad \dots \quad 1_k \quad \dots \quad 0]$, therefore, we can derive that $[\bar{\mathbf{C}}^T \quad \mathbf{F}]^T\mathbf{\Gamma} = \mathbf{I}$, where $\mathbf{I}$ is an identity matrix. In other words, $[\bar{\mathbf{C}}^T \quad \mathbf{F}]^T = \mathbf{P}$ and $\mathbf{p}_1 = \bar{\mathbf{C}}$. The proof is completed.

B. Tables

REFERENCES

[1] B. D. O. Anderson, M. Deistler, L. Farina, and L. Benvenuti, "Nonnegative realization of a linear system with nonnegative impulse response," *IEEE Trans. Circuits Syst. I: Fundam. Theory Appl.*, vol. 43, no. 2, pp. 134–142, Feb. 1996.

[2] R. Rouhani and E. Tse, "Structural design for classes of positive linear systems," *IEEE Trans. Syst., Man Cybern.*, vol. 11, no. 2, pp. 126–134, Feb. 1981.

[3] C. H. Huang, P. A. Ioannou, J. Maroulas, and M. G. Safonov, "Design of strictly positive real systems using constant output feedback," *IEEE Trans. Autom. Control*, vol. 44, no. 3, pp. 569–573, Mar. 1999.

[4] E. Fornasini and M. E. Valcher, "Linear copositive Lyapunov functions for continuous-time positive switched systems," *IEEE Trans. Autom. Control*, vol. 55, no. 8, pp. 1933–1937, Aug. 2010.

[5] M. A. Rami and F. Tadeo, "Controller synthesis for positive linear systems with bounded controls," *IEEE Trans. Circuits Syst.*, vol. 54, no. 2, pp. 151–155, Feb. 2007.

[6] Z. Shu, J. Lam, H. Gao, and B. Du, "Positive observers and dynamic output-feedback controllers for interval positive linear systems," *IEEE Trans. Circuits Syst.*, vol. 55, no. 10, pp. 3209–3222, Apr. 2008.

[7] M. Ezzine, M. Darouach, H. S. Ali, and H. Messaoud, "A new positive linear functional filters design for positive linear systems," in *Proc. of 22nd Mediter. Autom. Control Conf.*, Jun. 2014, pp. 407–411.

[8] X. Zhao, T. Wu, X. Zheng, and R. Li, "Discussions on observer design of nonlinear positive systems via T-S fuzzy modeling," *Neurocomputing*, vol. 157, pp. 70–75, Jun. 2015.

[9] D. Wang, Z. Wang, G. Li, and W. Wang, "Distributed filtering for switched nonlinear positive systems with missing measurements over sensor networks," *IEEE Sensors J.*, vol. 16, no. 12, pp. 4940–4948, Jun. 2016.

[10] Y. Mao, H. Zhang, and C. Dang, "Stability analysis and constrained control of a class of fuzzy positive systems with delays using linear copositive lyapunov functional," *Circuits Syst. Signal Process.*, vol. 31, no. 5, pp. 1863–1875, Oct. 2012.

[11] H. Gao, J. Lam, C. Wang, and S. Xu, "Control for stability and positivity: Equivalent conditions and computation," *IEEE Trans. Circuits Syst.*, vol. 52, no. 9, pp. 540–544, Sep. 2005.

[12] J. Shen and J. Lam, "Static output-feedback stabilization with optimal L_1-gain for positive linear systems," *Automatica*, vol. 63, pp. 248–253, Jan. 2016.

[13] T. Zhao, J. Xiao, H. Sheng, and W. Tao, "H_∞ control of continuous-time interval type-2 T-S fuzzy systems via dynamic output feedback controllers," *Neurocomputing*, vol. 165, pp. 133–143, Oct. 2015.

[14] C. Lin, B. Chen, and Q. G. Wang, "Static output feedback stabilization for fractional-order systems in T-S fuzzy models," *Neurocomputing*, vol. 218, pp. 354–358, Dec. 2016.

[15] T. Wang and S. Tong, "Observer-based output-feedback asynchronous control for switched fuzzy systems," *IEEE Trans. Cybern.*, May. 2016. [Online]. Available: http://dx.doi.org/10.1109/TCYB.2016.2558821

[16] E. T. Jeung and K. R. Lee, "Static output feedback control for continuous-time T-S fuzzy systems: An LMI approach," *Int. J. Control Auto. Syst.*, vol. 12, no. 3, pp. 703–708, Jun. 2014.

[17] H. K. Lam and L. D. Seneviratne, "LMI-based stability design of fuzzy controller for nonlinear systems," *IET Control Theory Appl.*, vol. 1, no. 1, pp. 393–401, Jan. 2007.

[18] S. Blai and I. krjanc, "Design and stability analysis of fuzzy model-based predictive control a case study," *Int.J. Robotic Syst.*, vol. 49, no. 3, pp. 279–292, Jul. 2007.

[19] R. E. Precup, M. B. Rdac, M. L. Tomescu, E. M. Petriu, and S. Preitl, "Stable and convergent iterative feedback tuning of fuzzy controllers for discrete-time siso systems," *Expert Syst. Appl.*, vol. 40, no. 1, pp. 188–199, Jan. 2013.

[20] D. H. Lee, Y. H. Joo, and M. H. Tak, *Local stability analysis of continuous-time Takagi-Sugeno fuzzy systems: A fuzzy Lyapunov function approach*. Elsevier Science Inc., 2014.

[21] S. Tong, T. Wang, Y. Li, and B. Chen, "A combined backstepping and stochastic small-gain approach to robust adaptive fuzzy output feedback control," *IEEE Trans. Fuzzy Syst.*, vol. 21, no. 2, pp. 314–327, Apr. 2013.

[22] S. C. Tong, Y. M. Li, G. Feng, and T. S. Li, "Observer-based adaptive fuzzy backstepping dynamic surface control for a class of MIMO nonlinear systems," *IEEE Trans. Syst. Man Cybern. Part B (Cybernetics)*, vol. 41, no. 4, pp. 1124–1135, Aug. 2011.

[23] Y. Li, S. Sui, and S. Tong, "Adaptive fuzzy control design for stochastic nonlinear switched systems with arbitrary switchings and unmodeled dynamics," *IEEE Trans. Cybern.*, vol. 47, no. 2, pp. 403–414, Feb. 2017.

[24] Y. Li, S. Tong, and T. Li, "Hybrid fuzzy adaptive output feedback control design for uncertain mimo nonlinear systems with time-varying delays and input saturation," *IEEE Trans. Fuzzy Syst.*, vol. 24, no. 4, pp. 841–853, Aug. 2016.

[25] A. Benzaouia, A. Hmamed, and A. E. Hajjaji, "Stabilization of controlled positive discrete-time T-S fuzzy systems by state feedback control," *Adap. Control and Signal Process*, vol. 24, no. 12, pp. 1091–1106, Dec. 2010.

[26] M. S. Fadali and S. Jafarzadeh, "Fuzzy TSK positive systems: Stability and control," in *Proc. of Amer. Control Conf.*, Aug. 2011, pp. 4964–4969.

[27] A. Benzaouia, F. Mesquine, M. Benhayoun, H. Schulte, and S. Georg, "Stabilization of positive constrained T-S fuzzy systems: Application to a buck converter ," *J. Franklin Institute.*, vol. 351, no. 8, pp. 4111–4123, Aug. 2014.

[28] M. S. Fadali and S. Jafarzadeh, "Stability analysis of positive interval type-2 tsk systems with application to energy markets," *IEEE Trans. Fuzzy Syst.*, vol. 22, no. 4, pp. 1031–1038, Aug. 2014.

[29] K. Tanaka and H. O. Wang, *Fuzzy Control Systems Design and Analysis: A Linear Matrix Inequality Approach*. New York: Wiley-Interscience, 2001.

[30] H. K. Lam, M. Narimani, H. Li, and H. Liu, "Stability analysis of polynomial-fuzzy-model-based control systems using switching polynomial Lyapunov function," *IEEE Trans. Fuzzy Syst.*, vol. 21, no. 5, pp. 800–813, Oct. 2013.

[31] H. K. Lam and H. Li, "Output-feedback tracking control for polynomial fuzzy-model-based control systems," *IEEE Trans. Ind. Electron.*, vol. 60, no. 12, pp. 5830–5840, Dec. 2013.

[32] C. Liu and H. K. Lam, "Design of a polynomial fuzzy observer controller with sampled-output measurements for nonlinear systems considering

TABLE III
APPROXIMATED POLYNOMIALS FOR PRODUCT TERMS OF MEMBERSHIP FUNCTION FOR $D = 5$.

$\eta_{ij,s_\tau}(x_1)$	$i=1,j=1$	$i=1,j=2$	$i=2,j=1$	$i=2,j=2$	$i=3,j=1$	$i=3,j=2$
s_1	0.99962	$\frac{1}{16800}(x_1+0.9)^2$	$\frac{1}{2315}(x_1+0.9)^2$	0	$\frac{1}{123084}(x_1+0.9)^2$	0
s_2	$\frac{1}{122}(x_1-14)^2$	$\frac{1}{404}(x_1-3.5)^2$	$\frac{1}{60.4}(x_1-3.4)^2$	$\frac{1}{238}(x_1-2.6)^2$	$\frac{1}{1226}(x_1-2.6)^2$	$\frac{1}{3682}(x_1-2.6)^2$
s_3	$\frac{1}{38.7}(x_1-11.8)^2$	$\frac{1}{604}(x_1-15.5)^2$	$\frac{1}{54}(x_1-14.1)^2$	$\frac{1}{54}(x_1-5.9)^2$	$\frac{1}{604}(x_1-4.5)^2$	$\frac{1}{38.7}(x_1-8.2)^2$
s_4	$\frac{1}{3682}(x_1-17.4)^2$	$\frac{1}{1226}(x_1-17.4)^2$	$\frac{1}{238}(x_1-17.4)^2$	$\frac{1}{60.4}(x_1-16.6)^2$	$\frac{1}{404}(x_1-16.5)^2$	$\frac{1}{122}(x_1-6)^2$
s_5	0	$\frac{1}{123084}(x_1-20.9)^2$	0	$\frac{1}{2315}(x_1-20.9)^2$	$\frac{1}{16800}(x_1-20.9)^2$	0.99962

TABLE IV
APPROXIMATED POLYNOMIALS FOR PRODUCT TERMS OF MEMBERSHIP FUNCTION FOR $D = 7$.

$\eta_{ij,s_\tau}(x_1)$	$i=1,j=1$	$i=1,j=2$	$i=2,j=1$	$i=2,j=2$	$i=3,j=1$	$i=3,j=2$
s_1	0.99962	$\frac{1}{16800}(x_1+0.9)^2$	$\frac{1}{2315}(x_1+0.9)^2$	0	$\frac{1}{123084}(x_1+0.9)^2$	0
s_2	$\frac{1}{3051}(x_1-58)^2$	$\frac{1}{1112}(x_1-2)^2$	$\frac{1}{154}(x_1-2)^2$	$\frac{1}{7000}(x_1-2.8)^2$	$\frac{1}{7568}(x_1-2)^2$	0
s_3	$\frac{1}{44.3}(x_1-12)^2$	$\frac{1}{404}(x_1-3.5)^2$	$\frac{1}{60.4}(x_1-3.4)^2$	$\frac{1}{73.2}(x_1-5.4)^2$	$\frac{1}{545}(x_1-4.7)^2$	$\frac{1}{1072}(x_1-5.5)^2$
s_4	$\frac{1}{38.7}(x_1-11.8)^2$	$\frac{1}{604}(x_1-15.5)^2$	$\frac{1}{54}(x_1-14.1)^2$	$\frac{1}{54}(x_1-5.9)^2$	$\frac{1}{604}(x_1-4.5)^2$	$\frac{1}{38.7}(x_1-8.2)^2$
s_5	$\frac{1}{1072}(x_1-14.5)^2$	$\frac{1}{545}(x_1-15.3)^2$	$\frac{1}{73.2}(x_1-14.6)^2$	$\frac{1}{60.4}(x_1-16.6)^2$	$\frac{1}{404}(x_1-16.5)^2$	$\frac{1}{44.3}(x_1-8)^2$
s_6	0	$\frac{1}{7568}(x_1-18)^2$	$\frac{1}{7000}(x_1-17.2)^2$	$\frac{1}{154}(x_1-18)^2$	$\frac{1}{1112}(x_1-18)^2$	$\frac{1}{3051}(x_1+38)^2$
s_7	0	$\frac{1}{123084}(x_1-20.9)^2$	0	$\frac{1}{2315}(x_1-20.9)^2$	$\frac{1}{16800}(x_1-20.9)^2$	0.99962

TABLE V
APPROXIMATED POLYNOMIALS FOR PRODUCT TERMS OF MEMBERSHIP FUNCTION FOR $D = 9$.

$\eta_{ij,s_\tau}(x_1)$	$i=1,j=1$	$i=1,j=2$	$i=2,j=1$	$i=2,j=2$	$i=3,j=1$	$i=3,j=2$
s_1	0.99962	$\frac{1}{16800}(x_1+0.9)^2$	$\frac{1}{2315}(x_1+0.9)^2$	0	$\frac{1}{123084}(x_1+0.9)^2$	0
s_2	$\frac{1}{3051}(x_1-58)^2$	$\frac{1}{1112}(x_1-2)^2$	$\frac{1}{154}(x_1-2)^2$	$\frac{1}{7000}(x_1-2.8)^2$	$\frac{1}{7568}(x_1-2)^2$	0
s_3	$\frac{1}{135.1}(x_1-16.7)^2$	$\frac{1}{295.2}(x_1-3.8)^2$	$\frac{1}{41}(x_1-3.8)^2$	$\frac{1}{290}(x_1-5.1)^2$	$\frac{1}{1228}(x_1-4.2)^2$	$\frac{1}{10192}(x_1-5.2)^2$
s_4	$\frac{1}{26.6}(x_1-11.3)^2$	$\frac{1}{534}(x_1-2.7)^2$	$\frac{1}{84.8}(x_1-2.4)^2$	$\frac{1}{43.5}(x_1-6.2)^2$	$\frac{1}{354}(x_1-5.5)^2$	$\frac{1}{490}(x_1-6.6)^2$
s_5	$\frac{1}{38.7}(x_1-11.8)^2$	$\frac{1}{604}(x_1-15.5)^2$	$\frac{1}{54}(x_1-14.1)^2$	$\frac{1}{54}(x_1-5.9)^2$	$\frac{1}{604}(x_1-4.5)^2$	$\frac{1}{38.7}(x_1-8.2)^2$
s_6	$\frac{1}{490}(x_1-13.4)^2$	$\frac{1}{354}(x_1-14.5)^2$	$\frac{1}{43.5}(x_1-13.8)^2$	$\frac{1}{84.8}(x_1-17.6)^2$	$\frac{1}{534}(x_1-17.3)^2$	$\frac{1}{26.6}(x_1-8.7)^2$
s_7	$\frac{1}{10192}(x_1-14.8)^2$	$\frac{1}{1228}(x_1-15.8)^2$	$\frac{1}{290}(x_1-14.9)^2$	$\frac{1}{41}(x_1-16.2)^2$	$\frac{1}{295.2}(x_1-16.2)^2$	$\frac{1}{135.1}(x_1-3.3)^2$
s_8	0	$\frac{1}{7568}(x_1-18)^2$	$\frac{1}{7000}(x_1-17.2)^2$	$\frac{1}{154}(x_1-18)^2$	$\frac{1}{1112}(x_1-18)^2$	$\frac{1}{3051}(x_1+38)^2$
s_9	0	$\frac{1}{123084}(x_1-20.9)^2$	0	$\frac{1}{2315}(x_1-20.9)^2$	$\frac{1}{16800}(x_1-20.9)^2$	0.99962

unmeasurable premise variables," *IEEE Trans. Fuzzy Syst.*, vol. 23, no. 6, pp. 2067–2079, Dec. 2015.

[33] H. K. Lam and S. H. Tsai, "Stability analysis of polynomial-fuzzy-model-based control systems with mismatched premise membership functions," *IEEE Trans. Fuzzy Syst.*, vol. 22, no. 1, pp. 223–229, Feb. 2014.

[34] H. K. Lam, L. Wu, and J. Lam, "Two-step stability analysis for general polynomial-fuzzy-model-based control systems," *IEEE Trans. Fuzzy Syst.*, vol. 23, no. 3, pp. 511–524, Jun. 2015.

[35] H. K. Lam, "Polynomial fuzzy-model-based control systems: Stability analysis via piecewise-linear membership functions," *IEEE Trans. Fuzzy Syst.*, vol. 19, no. 3, pp. 588–593, Jun. 2011.

[36] H. K. Lam, C. Liu, L. Wu, and X. Zhao, "Polynomial fuzzy-model-based control systems: Stability analysis via approximated membership functions considering sector nonlinearity of control input," *IEEE Trans. Fuzzy Syst.*, vol. 23, no. 6, pp. 2202–2214, Dec. 2015.

[37] Y. Yin, X. Zheng, H. Yang, and X. Zhao, "Stabilization for delayed positive nonlinear systems via T-S fuzzy modeling," in *Proc. of IEEE Int. Conf. Autom. Control Technology and Intelligent Syst.*, Jun. 2015, pp. 1687–1692.

[38] X. Li, C. Liu, H. K. Lam, F. Liu, and X. Zhao, "Stability analysis of fuzzy polynomial positive systems with time delay," in *Proc. of Int. Conf. Fuzzy Theory and Its Appl.*, Nov. 2014, pp. 24–28.

[39] X. Li, H. K. Lam, F. Liu, and X. Zhao, "Stability and stabilization analysis of positive polynomial fuzzy systems with time delay considering piecewise membership functions," *IEEE Trans. Fuzzy Syst.*, 2016. [Online]. Available: http://dx.doi.org/10.1109/TFUZZ.2016.2593494

[40] J. Chen, C. Lin, B. Chen, and Q. G. Wang, "Mixed H_∞ and passive control for singular systems with time delay via static output feedback," *Appl. Math. Comput.*, vol. 293, no. C, pp. 244–253, Jan. 2017.

[41] M. S. Sadabadi and D. Peaucelle, "From static output feedback to structured robust static output feedback: A survey," *Annual Reviews in Control*, vol. 42, pp. 11–26, 2016. [Online]. Available: http://www.sciencedirect.com/science/article/pii/S1367578816300852

[42] M. Abbaszadeh and H. J. Marquez, "Static output feedback control for nonlinear systems subject to parametric and nonlinear uncertainties," *CoRR*, 2016.

[43] H. K. Lam, *Polynomial Fuzzy Model-Based Control Systems: Stability Analysis and Control Synthesis Using Membership Function Dependent Techniques*. Switzerland: Springer International Publishing, 2016.

[44] S. Prajna, A. Papachristodoulou, and P. A. Parrilo, "Introducing SOSTOOLS: A general purpose sum of squares programming solver," in *Proc. of 41st IEEE Conf. Decision Control.*, vol. 1, Dec. 2002, pp. 741–746.
[45] S. Prajna, A. Papachristodoulou, and F. Wu, "Nonlinear control synthesis by sum of squares optimization: A Lyapunov-based approach," in *Proc. of 5th Asian Control Conf.*, vol. 1, Jul. 2004, pp. 157–165.
[46] L. Farina and S. Rinaldi, *Positive Linear Systems: Theory and Applications*. New York: John Wiley and Sons Inc., 2000.
[47] J. Zhang, Z. Han, F. Zhu, and J. Huang, "Brief paper: Feedback control for switched positive linear systems," *IET Control Theory Appl.*, vol. 7, no. 3, pp. 464–469, Feb. 2013.
[48] M. Min, Z. Shuqian, and Z. Chenghui, "Static output feedback control for positive systems via LP approach," in *Prof. of 31st Chin. Control Conf.*, Jul. 2012, pp. 1435–1440.
[49] H. K. Lam, "Stabilization of nonlinear systems using sampled-data output-feedback fuzzy controller based on polynomial-fuzzy-model-based control approach," *IEEE Trans. Syst. Man Cybern. B. Cybern.*, vol. 42, no. 1, pp. 258–267, Feb. 2012.
[50] M. Narimani and H. K. Lam, "SOS-based stability analysis of polynomial fuzzy-model-based control systems via polynomial membership functions," *IEEE Trans. Fuzzy Syst.*, vol. 18, no. 5, pp. 862–871, Oct. 2010.
[51] X. Liang, "Discussion on the properties of hilbert space adjoint operator on the orthogonal complement," in *Prof. of 2011 Int. Conf. Elec. and Opto.*, vol. 1, Jul. 2011, pp. V1–502–V1–504.

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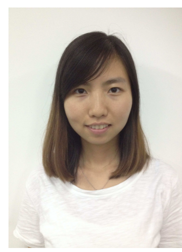

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