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Membership-Function-Dependent Stability Analysis of Interval Type-2 Polynomial Fuzzy-Model-Based Control Systems

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Abstract: In this paper, the stability analysis for interval type-2 (IT2) polynomial fuzzy-model-based (PFMB) control system using the information of membership functions is investigated. In order to tackle uncertainties, IT2 membership functions are used in the IT2 polynomial fuzzy model and IT2 polynomial fuzzy controller. To improve the design flexibility and reduce the implementation costs, the IT2 polynomial fuzzy controller does not need to share the same premise membership functions nor the same number of rules with the IT2 polynomial fuzzy model. The stability of IT2 PFMB control system is investigated based on the Lyapunov stability theory and both sets of membership function independent (MFI) and membership function dependent (MFD) stability conditions are derived on the basis of the sum-of-squares (SOS) approach. To make the stability conditions membership function dependent, the boundary information of IT2 membership functions is used in the stability analysis. To extract richer information of IT2 membership functions, the operating domain is partitioned into sub-domains. In each sub-domain, the boundary information of IT2 membership functions and those of the upper and lower membership function are obtained. Furthermore, to further relax the conservativeness, a switching polynomial fuzzy controller, together with the informations obtained in each sub-domain, is employed in investigating the stability analysis. Numerical Examples and simulation results are given to demonstrate the validity of MFD and MFD switching methods.

1 Introduction

The Takagi-Sugeno (T-S) fuzzy model, which was firstly proposed in [1, 2], is an effective method to represent the nonlinear systems. It represents the nonlinear plant by an average weighted summation of local linear systems. The weights of local linear systems, defined by membership functions [3], embed the system's nonlinearity. In order to control the nonlinear system representing by T-S fuzzy model, a fuzzy controller, which is the average weighted summation of local linear controllers, is employed. In general, there are three types of fuzzy controllers [4–7], e.g. parallel distributed compensation (PDC) fuzzy controller, in which the fuzzy controller shares the same premise membership functions as the fuzzy model; partially matched fuzzy controller, in which it has the same number of rules as but different premise membership functions from the fuzzy model; and imperfectly matched fuzzy controller, in which neither the number of rules nor the premise membership functions are the same as the fuzzy model. In terms of imperfectly matched fuzzy controller, since it does not need to share the same premise membership functions with the fuzzy model and the number of rules can be freely chosen, design flexibility and robustness property can be enhanced [5]. Connecting a fuzzy controller with the nonlinear plant (represented by the T-S fuzzy model) in a closed loop forms the T-S fuzzy-model-based (FMB) control system. The stability of T-S FMB control system can be analyzed and guaranteed by stability conditions in terms of linear-matrix inequalities (LMIs) obtained on the basis of quadratic Lyapunov function. The T-S FMB control system faces great success and has a wide range of applications on, to name a few, tracking control systems, time-delay control systems, chaotic control systems etc. [8–14]. However, issues related to stability analysis and control design still exist on using the T-S FMB control systems [15], such as stability conditions are non-convex, or the analysis results are too conservative. In the above discussion, since the membership functions contain no uncertainty information, the control system cannot deal with the system uncertainties directly. Once uncertainties appear, the grades of membership will become

uncertain in value leading to conservative stability conditions when uncertainty is not considered in the stability analysis.

Recently, the T-S fuzzy model is extended to a more general polynomial fuzzy model [16–19]. The nonlinear plant can be represented more effectively because polynomials are allowed in the consequent part. Due to the existence of polynomials in the consequent part, a better expressing capability compared with the T-S fuzzy model is demonstrated, and thus the number of rules in polynomial fuzzy model is generally fewer than that in the T-S fuzzy model. The polynomial fuzzy model together with the polynomial fuzzy controller connected in a closed loop forms the polynomial fuzzy-model-based (PFMB) control system. To facilitate the stability analysis of PFMB control system, a more general Lyapunov function, namely polynomial Lyapunov function, is used and it can provide more relaxed analysis and stability conditions than the results obtained in quadratic Lyapunov function. However, because of the existence of monomials, the LMI-based approach cannot be used to get the solution for the stability conditions. Then, a SOS-based approach [20] is employed [16, 21], where a feasible solution (if any) can be found by, for example, SOSTOOLS [22]. In the literature such as the work mentioned above, most of the stability analysis are MFI that the information of membership functions is not taken into account and thus it potentially leads to conservative analysis results. To relax the conservativeness of stability analysis results through the use of membership function information, different types of membership functions such as piecewise-linear membership functions [18], polynomial membership functions [17] and mismatched premise membership functions [23] have been proposed for the MFD stability analysis.

In terms of uncertainties which can be a form of parameter uncertainty, measurement uncertainty, saturation etc, a type-2 fuzzy set was proposed to represent uncertain grade of membership functions [24–26]. Type-2 fuzzy sets can be regarded as a collection of type-1 fuzzy sets that there are two membership functions, namely primary and secondary membership functions, in it and the additional information including system uncertainties can be captured by the

secondary membership functions. Hence, the type-2 fuzzy logic systems (FLSs) is useful when the nonlinear system faces uncertainties. For type-2 FLSs [27], it involves the operations of fuzzification, inference, and output processing. The crisp input is mapped into a fuzzy set, which is in general a type-2 fuzzy set in type-2 FLSs. Then, the input type-2 fuzzy sets are mapped to the output fuzzy sets by the inference engine combined with rules. The structure of “If-Then” rules in type-2 FLSs are similar to type-1 FLSs that the only difference is that some or all of the fuzzy sets involved are type-2 fuzzy sets. In the output process, the first step is to obtain a type-reduced set from the output type-2 fuzzy set by the type reduction process and then a crisp output from the type-2 FLS can be obtained by defuzzifying the type-reduced set. However, because of the computational complexity of using a general type-2 fuzzy set, an interval type-2 (IT2) fuzzy set whose secondary membership function is constant, which is expressed as a particular case of the general type-2 fuzzy set, has been used. In the context of FMB control system, the work in [28] considered an IT2 fuzzy model to represent a nonlinear plant subject to uncertainties, and the uncertainties can be effectively captured by the region between the lower and upper membership functions, which is defined as the footprint of uncertainties (FOU). However, because the information contained in the FOU eliminates the favorable property of type-1 FMB analysis approach to facilitate the stability analysis, conservative stability analysis results will usually be obtained. Thus to relax the stability analysis results, the information of lower and upper membership functions are utilized. So far, the concept of IT2 FMB control has been extended to various control strategy and systems [29–36] such as state and output feedback control [31, 34], control of nonlinear networked systems [30], filter design [32, 33, 35] of the IT2 fuzzy system, control of time-varying delay system [34, 36] etc. Yet, most of them focus on the control methodology and the stability analysis is approached by existing techniques. Furthermore, the T-S fuzzy model is the main stream on these work but the polynomial fuzzy model [19, 37] is rarely considered in the literature. Although [19, 37] investigated the stability analysis and tracking control of IT2 PFMB control systems, the method they used is on the basis of type-1 PFMB control system, i.e. approximating the IT2 membership functions by an embedded type-1 membership functions and considering the influences of the modeling errors between the embedded Type-1 membership functions and IT2 membership functions.

In this paper, the stability analysis of IT2 PFMB control system is conducted and the stability conditions are obtained in terms of SOS. By applying IT2 fuzzy sets into the polynomial fuzzy model, system uncertainties can be captured by the lower and upper membership functions. In addition, the information of IT2 membership functions such as the polynomial function or constant approximated upper and lower membership functions and the boundary informations of the approximation are considered in the analysis. With such information, slack matrices are introduced to relax the conservativeness of the stability analysis results. Based on the Lyapunov stability theory, stability conditions in terms of SOS are derived to achieve a stable IT2 PFMB control system. Moreover, to further relax the stability conditions, an IT2 switching polynomial fuzzy controller is employed in conducting the stability analysis. The whole operating domain are divided into sub-domains. In each sub-domain, a local IT2 polynomial fuzzy controller is designed with the local information of membership functions introduced by some slacks matrices to facilitate the stability analysis. Since the feedback gains can be designed separately in each sub-domains, the feedback compensation capability can be enhanced and the stability conditions are more relaxed.

The rest of the paper is organized as follows. In Section 2, an IT2 polynomial fuzzy model, an IT2 polynomial fuzzy controller, an IT2 switching polynomial fuzzy controller, an IT2 PFMB control system and an IT2 switching PFMB control system are presented. In Section 3, the stability issue is discussed based on the Lyapunov stability theory and the SOS-based stability conditions are derived. In Section 4, numerical examples and simulation results are given to illustrate the merits of proposed approach. Conclusion is given in Section 5.

2 Notations and Preliminaries

In this section, notations and preliminaries of the IT2 polynomial fuzzy model, the polynomial fuzzy controller including the IT2 polynomial fuzzy controller and the IT2 switching polynomial fuzzy controller, the IT2 PFMB control system and the IT2 switching PFMB control system are presented.

2.1 Notations

Throughout the paper, the following notations are applied [38]. The monomial in $\mathbf{x}(t) = [x_1(t), x_2(t), \dots, x_n(t)]^T$ is defined as $x_1^{\lambda_1}(t), x_2^{\lambda_2}(t), \dots, x_n^{\lambda_n}(t)$, where $\lambda_l, l=1, 2, \dots, n$, are non-negative integers. The degree of a monomial is defined as $\lambda = \sum_{l=1}^n \lambda_l$. A polynomial $\mathbf{p}(\mathbf{x}(t))$ is defined as a finite linear combination of monomials with real coefficients. A polynomial $\mathbf{p}(\mathbf{x}(t))$ is an SOS if it can be rewritten as $\mathbf{p}(\mathbf{x}(t)) = \sum_{j=1}^n \mathbf{q}_j(\mathbf{x}(t))^2$, where $\mathbf{q}_j(\mathbf{x}(t))$ is a polynomial. Hence, it can be seen that $\mathbf{p}(\mathbf{x}(t)) \geq 0$ if it is an SOS. If polynomial $\mathbf{p}(\mathbf{x}(t))$ is an SOS, then it can be represented in the form of $\hat{\mathbf{x}}(t)^T \mathbf{Q} \hat{\mathbf{x}}(t)$, where \mathbf{Q} is a positive semi-definite matrix. The problem of finding a \mathbf{Q} can be formulated as a semi-definite programme. The notations of $\mathbf{M} > 0, \mathbf{M} \geq 0, \mathbf{M} < 0$ and $\mathbf{M} \leq 0$ denote a positive, semi-positive, negative, semi-negative definite matrix \mathbf{M} , respectively.

2.2 IT2 Polynomial Fuzzy Model

The dynamics of the nonlinear plant with uncertainties can be represented by an IT2 polynomial fuzzy model with p rules of which the antecedents are of type-2 fuzzy sets, and the consequent part is a local polynomial dynamic system. The i -th rule is of the following format [39]:

$$\begin{aligned} \text{Rule } i : & \text{ if } f_1(\mathbf{x}(t)) \text{ is } \tilde{M}_1^i \text{ and } \dots, \text{ and } f_\Psi(\mathbf{x}(t)) \text{ is } \tilde{M}_\Psi^i \\ & \text{ then } \dot{\mathbf{x}}(t) = \mathbf{A}_i(\mathbf{x}(t)) \hat{\mathbf{x}}(\mathbf{x}(t)) + \mathbf{B}_i(\mathbf{x}(t)) \mathbf{u}(t), \end{aligned} \quad (1)$$

where \tilde{M}_α^i is an IT2 fuzzy set of rule i corresponding to the function $f_\alpha(\mathbf{x}(t)), \alpha=1, 2, \dots, \Psi; i=1, 2, \dots, p; \Psi$ is a positive integer; $\mathbf{A}_i(\mathbf{x}(t)) \in \mathbb{R}^{n \times N}$ and $\mathbf{B}_i(\mathbf{x}(t)) \in \mathbb{R}^{n \times m}$, which are polynomial matrices in $\mathbf{x}(t)$, are the known polynomial system and input matrices, respectively; $\mathbf{x}(t) \in \mathbb{R}^n$ is the system state vector and $\hat{\mathbf{x}}(\mathbf{x}(t)) \in \mathbb{R}^N$ is a vector of monomials in $\mathbf{x}(t)$; $\mathbf{u}(t) \in \mathbb{R}^m$ is the input vector. It is assumed that $\hat{\mathbf{x}}(\mathbf{x}(t)) = \mathbf{0}$ iff $\mathbf{x}(t) = \mathbf{0}$.

The firing strength of the i -th rule is in the following interval sets,

$$\tilde{w}_i(\mathbf{x}(t)) \in [\underline{w}_i(\mathbf{x}(t)), \bar{w}_i(\mathbf{x}(t))], \quad i = 1, 2, \dots, p, \quad (2)$$

where

$$\underline{w}_i(\mathbf{x}(t)) = \prod_{\alpha=1}^{\Psi} \underline{\mu}_{\tilde{M}_\alpha^i}(f_\alpha(\mathbf{x}(t))), \quad (3)$$

$$\bar{w}_i(\mathbf{x}(t)) = \prod_{\alpha=1}^{\Psi} \bar{\mu}_{\tilde{M}_\alpha^i}(f_\alpha(\mathbf{x}(t))), \quad (4)$$

in which $\underline{\mu}_{\tilde{M}_\alpha^i}(f_\alpha(\mathbf{x}(t))) \in [0, 1]$ and $\bar{\mu}_{\tilde{M}_\alpha^i}(f_\alpha(\mathbf{x}(t))) \in [0, 1]$ denote the lower and upper grades of membership governed by their lower and upper membership functions, respectively. The definition of IT2 membership functions makes the property $\bar{\mu}_{\tilde{M}_\alpha^i}(f_\alpha(\mathbf{x}(t))) \geq \underline{\mu}_{\tilde{M}_\alpha^i}(f_\alpha(\mathbf{x}(t)))$ hold, which leads to $1 \geq \bar{w}_i(\mathbf{x}(t)) \geq \underline{w}_i(\mathbf{x}(t)) \geq 0$ for all i .

Then the IT2 polynomial fuzzy model is defined as,

$$\begin{aligned} \dot{\mathbf{x}}(t) = & \sum_{i=1}^p \underline{w}_i(\mathbf{x}(t)) \underline{v}_i(\mathbf{x}(t)) (\mathbf{A}_i(\mathbf{x}(t)) \hat{\mathbf{x}}(\mathbf{x}(t)) + \mathbf{B}_i(\mathbf{x}(t)) \mathbf{u}(t)) \\ & + \sum_{i=1}^p \bar{w}_i(\mathbf{x}(t)) \bar{v}_i(\mathbf{x}(t)) (\mathbf{A}_i(\mathbf{x}(t)) \hat{\mathbf{x}}(\mathbf{x}(t)) + \mathbf{B}_i(\mathbf{x}(t)) \mathbf{u}(t)) \end{aligned}$$

$$= \sum_{i=1}^p \tilde{w}_i(\mathbf{x}(t))(\mathbf{A}_i(\mathbf{x}(t))\hat{\mathbf{x}}(\mathbf{x}(t)) + \mathbf{B}_i(\mathbf{x}(t))\mathbf{u}(t)), \quad (5)$$

where

$$\tilde{w}_i(\mathbf{x}(t)) = \underline{w}_i(\mathbf{x}(t))\underline{v}_i(\mathbf{x}(t)) + \bar{w}_i(\mathbf{x}(t))\bar{v}_i(\mathbf{x}(t)), \quad (6)$$

$$\sum_{i=1}^p \tilde{w}_i(\mathbf{x}(t)) = 1, \quad (7)$$

in which $\underline{v}_i(\mathbf{x}(t)) \in [0, 1]$ and $\bar{v}_i(\mathbf{x}(t)) \in [0, 1]$ are nonlinear functions, which are not necessary to be known but exist to represent the uncertainty, that have the property $\underline{v}_i(\mathbf{x}(t)) + \bar{v}_i(\mathbf{x}(t)) = 1$ for all i .

2.3 IT2 Polynomial Fuzzy Controller

An IT2 polynomial fuzzy controller with c rules is employed to stabilize the nonlinear plant represented by an IT2 polynomial fuzzy model (5). The j -th rule is of the following format [29]:

$$\begin{aligned} \text{Rule } j : & \text{ if } g_1(\mathbf{x}(t)) \text{ is } \tilde{N}_1^j \text{ and } \dots, \text{ and } g_\Omega(\mathbf{x}(t)) \text{ is } \tilde{N}_\Omega^j \\ & \text{ then } \mathbf{u}(t) = \mathbf{G}_j(\mathbf{x}(t))\hat{\mathbf{x}}(\mathbf{x}(t)), \end{aligned} \quad (8)$$

where \tilde{N}_β^j is the IT2 fuzzy term of rule j corresponding to the function $g_\beta(\mathbf{x}(t))$, $\beta=1, 2, \dots, \Omega$; $j=1, 2, \dots, c$; Ω is a positive integer; and $\mathbf{G}_j(\mathbf{x}(t)) \in \mathfrak{R}^{m \times N}$, $j=1, 2, \dots, c$, are polynomial feedback gains need to be determined.

The firing strength of the j -th rule is in the following interval sets,

$$\tilde{m}_j(\mathbf{x}(t)) \in [\underline{m}_j(\mathbf{x}(t)), \bar{m}_j(\mathbf{x}(t))], \quad j = 1, 2, \dots, c, \quad (9)$$

where

$$\underline{m}_j(\mathbf{x}(t)) = \prod_{\beta=1}^{\Omega} \underline{\nu}_{\tilde{N}_\beta^j}(g_\beta(\mathbf{x}(t))), \quad (10)$$

$$\bar{m}_j(\mathbf{x}(t)) = \prod_{\beta=1}^{\Omega} \bar{\nu}_{\tilde{N}_\beta^j}(g_\beta(\mathbf{x}(t))), \quad (11)$$

in which $\underline{\nu}_{\tilde{N}_\beta^j}(g_\beta(\mathbf{x}(t))) \in [0, 1]$ and $\bar{\nu}_{\tilde{N}_\beta^j}(g_\beta(\mathbf{x}(t))) \in [0, 1]$ denote the lower and upper grades of membership governed by the their lower and upper membership functions, respectively. The definition of IT2 membership functions makes the property $\bar{\nu}_{\tilde{N}_\beta^j}(g_\beta(\mathbf{x}(t))) \geq \underline{\nu}_{\tilde{N}_\beta^j}(g_\beta(\mathbf{x}(t)))$ hold, which leads to $1 \geq \bar{m}_j(\mathbf{x}(t)) \geq \underline{m}_j(\mathbf{x}(t)) \geq 0$ for all j .

Then the IT2 polynomial fuzzy controller is defined as follows,

$$\mathbf{u}(t) = \sum_{j=1}^c \tilde{m}_j(\mathbf{x}(t))\mathbf{G}_j(\mathbf{x}(t))\hat{\mathbf{x}}(\mathbf{x}(t)), \quad (12)$$

where

$$\tilde{m}_j(\mathbf{x}(t)) = \frac{\underline{m}_j(\mathbf{x}(t))\underline{\lambda}_j(\mathbf{x}(t)) + \bar{m}_j(\mathbf{x}(t))\bar{\lambda}_j(\mathbf{x}(t))}{\sum_{k=1}^c (\underline{m}_k(\mathbf{x}(t))\underline{\lambda}_k(\mathbf{x}(t)) + \bar{m}_k(\mathbf{x}(t))\bar{\lambda}_k(\mathbf{x}(t)))}, \quad (13)$$

$$\sum_{j=1}^c \tilde{m}_j(\mathbf{x}(t)) = 1, \quad (14)$$

in which $\underline{\lambda}_j(\mathbf{x}(t)) \in [0, 1]$ and $\bar{\lambda}_j(\mathbf{x}(t)) \in [0, 1]$ are nonlinear functions that have the property $\underline{\lambda}_j(\mathbf{x}(t)) + \bar{\lambda}_j(\mathbf{x}(t)) = 1$ for all j , and (13) is the type reduction.

2.4 IT2 Switching Polynomial Fuzzy Controller

To enhance the IT2 polynomial fuzzy controller's capability of feedback compensation, the whole operation domain Φ is divided into D connected sub-domains Φ_d , i.e., $\Phi = \bigcup_{d=1}^D \Phi_d$, where $d=1, 2, \dots, D$. As an extension of the controller proposed in [40] which uses constant feedback gains, polynomial feedback gains are used. The j -th rule of IT2 switching polynomial fuzzy controller with c rules is of the following format:

$$\begin{aligned} \text{Rule } j : & \text{ if } g_1(\mathbf{x}(t)) \text{ is } \tilde{N}_1^j \text{ and } \dots, \text{ and } g_\Omega(\mathbf{x}(t)) \text{ is } \tilde{N}_\Omega^j \\ & \text{ then } \mathbf{u}(t) = \mathbf{G}_{jd}(\mathbf{x}(t))\hat{\mathbf{x}}(\mathbf{x}(t)), \end{aligned} \quad (15)$$

where \tilde{N}_β^j is the IT2 fuzzy term of rule j corresponding to the function $g_\beta(\mathbf{x}(t))$, $\beta=1, 2, \dots, \Omega$; $j=1, 2, \dots, c$; Ω is a positive integer and $\mathbf{G}_{jd}(\mathbf{x}(t)) \in \mathfrak{R}^{m \times N}$, $j=1, 2, \dots, c, d=1, 2, \dots, D$, are polynomial feedback gains need to be determined.

Then, the overall IT2 switching polynomial fuzzy controller is defined as follows:

$$\mathbf{u}(t) = \sum_{j=1}^c \tilde{m}_j(\mathbf{x}(t))\mathbf{G}_{jd}(\mathbf{x}(t))\hat{\mathbf{x}}(\mathbf{x}(t)), \quad \mathbf{x} \in \Phi_d, \forall d \quad (16)$$

where $\tilde{m}_j(\mathbf{x}(t))$ is defined by (13) and satisfies the property that $\sum_{j=1}^c \tilde{m}_j(\mathbf{x}(t)) = 1$.

2.5 IT2 PFMB Control System

The IT2 polynomial fuzzy controller (12) is considered to close the feedback loop of IT2 polynomial fuzzy model (5). Then the IT2 PFMB control system can be represented as follows:

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \sum_{i=1}^p \tilde{w}_i(\mathbf{x}(t))(\mathbf{A}_i(\mathbf{x}(t))\hat{\mathbf{x}}(\mathbf{x}(t)) + \mathbf{B}_i(\mathbf{x}(t))\sum_{j=1}^c \tilde{m}_j(\mathbf{x}(t)) \\ & \quad \mathbf{G}_j(\mathbf{x}(t))\hat{\mathbf{x}}(\mathbf{x}(t))) \\ &= \sum_{i=1}^p \sum_{j=1}^c \tilde{w}_i(\mathbf{x}(t))\tilde{m}_j(\mathbf{x}(t))(\mathbf{A}_i(\mathbf{x}(t)) + \mathbf{B}_i(\mathbf{x}(t))\mathbf{G}_j(\mathbf{x}(t))) \\ & \quad \hat{\mathbf{x}}(\mathbf{x}(t)). \end{aligned} \quad (17)$$

2.6 IT2 switching PFMB control system

When an IT2 switching polynomial fuzzy controller (16) is used to close the feedback loop of IT2 polynomial fuzzy model (5), an IT2 switching PFMB control system, which is of the following format, can be obtained.

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \sum_{d=1}^D \sum_{i=1}^p \sum_{j=1}^c \xi_d(\mathbf{x}(t))\tilde{w}_i(\mathbf{x}(t))\tilde{m}_j(\mathbf{x}(t))(\mathbf{A}_i(\mathbf{x}(t)) + \\ & \quad \mathbf{B}_i(\mathbf{x}(t))\mathbf{G}_{jd}(\mathbf{x}(t)))\hat{\mathbf{x}}(\mathbf{x}(t)), \end{aligned} \quad (18)$$

where $\xi_d(\mathbf{x}(t)) = 1$ for $\mathbf{x} \in \Phi_d$, otherwise $\xi_d(\mathbf{x}(t)) = 0$, $d=1, 2, \dots, D$.

3 Stability Analysis

The control objective is to determine the polynomial feedback gains $\mathbf{G}_j(\mathbf{x}(t))$ ($\mathbf{G}_{jd}(\mathbf{x}(t))$ when IT2 switching polynomial fuzzy controller (16) is considered) to make the IT2 PFMB control system (17) (IT2 switching PFMB control system (18)) asymptotically stable, i.e., $\mathbf{x}(t) \rightarrow 0$ as time $t \rightarrow \infty$. In this section, MFI and MFD stability conditions are developed.

In the following analysis, for brevity, $\tilde{w}_i(\mathbf{x}(t))$ is denoted as $\tilde{w}_i(\mathbf{x})$ and $\tilde{m}_j(\mathbf{x}(t))$ is denoted as $\tilde{m}_j(\mathbf{x})$. The time t associated

with the variables is dropped for the situation without ambiguity, e.g. $\mathbf{x}(t)$ and $\dot{\mathbf{x}}(\mathbf{x}(t))$ are denote as \mathbf{x} and $\dot{\mathbf{x}}(\mathbf{x})$, respectively.

From the IT2 PFMB control system (17), the relation between $\dot{\mathbf{x}}(\mathbf{x})$ and $\dot{\mathbf{x}}$ can be obtained as follows:

$$\begin{aligned}\dot{\mathbf{x}}(\mathbf{x}) &= \frac{\partial \dot{\mathbf{x}}(\mathbf{x})}{\partial \mathbf{x}} \frac{d\mathbf{x}}{dt} = \mathbf{T}(\mathbf{x})\dot{\mathbf{x}} \\ &= \sum_{i=1}^p \sum_{j=1}^c \tilde{w}_i(\mathbf{x})\tilde{m}_j(\mathbf{x})(\tilde{\mathbf{A}}_i(\mathbf{x}) + \tilde{\mathbf{B}}_i(\mathbf{x})\mathbf{G}_j(\mathbf{x}))\dot{\mathbf{x}}(\mathbf{x}),\end{aligned}\quad (19)$$

where $\tilde{\mathbf{A}}_i(\mathbf{x}) = \mathbf{T}(\mathbf{x})\mathbf{A}_i(\mathbf{x})$, $\tilde{\mathbf{B}}_i(\mathbf{x}) = \mathbf{T}(\mathbf{x})\mathbf{B}_i(\mathbf{x})$ and $\mathbf{T}(\mathbf{x}) \in \mathbb{R}^{N \times n}$ with its (i, j) th entry defined as follows,

$$T_{ij}(\mathbf{x}) = \frac{\partial \hat{x}_i(\mathbf{x})}{\partial x_j}, \quad i = 1, 2, \dots, N; \quad j = 1, 2, \dots, n. \quad (20)$$

To facilitate the stability analysis, let $\mathbf{A}_i^k(\mathbf{x})$ denote the k -th row of $\mathbf{A}_i(\mathbf{x})$, $\mathbf{K} = \{k_1, k_2, \dots, k_m\}$ denote the row indices of $\mathbf{B}_i(\mathbf{x})$ whose corresponding row is equal to zero, and define $\tilde{\mathbf{x}} = (x_{k_1}, x_{k_2}, \dots, x_{k_m})$ [15]. The following polynomial Lyapunov function is considered to investigate the system stability of (17).

$$V(t) = \hat{\mathbf{x}}(\mathbf{x})^T \mathbf{X}(\tilde{\mathbf{x}})^{-1} \hat{\mathbf{x}}(\mathbf{x}), \quad (21)$$

where $0 < \mathbf{X}(\tilde{\mathbf{x}}) = \mathbf{X}(\tilde{\mathbf{x}})^T \in \mathbb{R}^{N \times N}$ is a positive definite symmetric polynomial matrix.

According to the Lyapunov stability theory, if $\mathbf{X}(\tilde{\mathbf{x}})$ can be found that satisfies $V(t) > 0$ for $\mathbf{x} \neq \mathbf{0}$ and $\dot{V}(t) < 0$ for $\mathbf{x} \neq \mathbf{0}$, the IT2 PFMB control system (17) is asymptotically stable.

3.1 MFI Stability Analysis

The stability analysis of the IT2 PFMB control system (17) is conducted in this section. The MFI stability conditions in the form of SOS are derived using the Lyapunov-based approach.

Considering the polynomial Lyapunov function (21), the time derivative of $V(t)$ is obtained as follows:

$$\begin{aligned}\dot{V}(t) &= \dot{\mathbf{x}}(\mathbf{x})^T \mathbf{X}(\tilde{\mathbf{x}})^{-1} \hat{\mathbf{x}}(\mathbf{x}) + \hat{\mathbf{x}}(\mathbf{x})^T \frac{d\mathbf{X}(\tilde{\mathbf{x}})^{-1}}{dt} \hat{\mathbf{x}}(\mathbf{x}) \\ &\quad + \hat{\mathbf{x}}(\mathbf{x})^T \mathbf{X}(\tilde{\mathbf{x}})^{-1} \dot{\mathbf{x}}(\mathbf{x}) \\ &= \dot{\mathbf{x}}(\mathbf{x})^T \mathbf{X}(\tilde{\mathbf{x}})^{-1} \hat{\mathbf{x}}(\mathbf{x}) + \hat{\mathbf{x}}(\mathbf{x})^T \left(\sum_{k=1}^n \frac{\partial \mathbf{X}(\tilde{\mathbf{x}})^{-1}}{\partial x_k} \dot{x}_k \right) \hat{\mathbf{x}}(\mathbf{x}) \\ &\quad + \hat{\mathbf{x}}(\mathbf{x})^T \mathbf{X}(\tilde{\mathbf{x}})^{-1} \dot{\mathbf{x}}(\mathbf{x}).\end{aligned}\quad (22)$$

Since $\mathbf{B}_i^k(\mathbf{x}) = \mathbf{0}$ for $k \in \mathbf{K}$, we can obtain

$$\dot{x}_k = \sum_{i=1}^p \tilde{w}_i(\mathbf{x}) \mathbf{A}_i^k(\mathbf{x}) \hat{\mathbf{x}}(\mathbf{x}), \quad k \in \mathbf{K}. \quad (23)$$

Otherwise,

$$\frac{\partial \mathbf{X}(\tilde{\mathbf{x}})^{-1}}{\partial x_k} = 0, \quad k \notin \mathbf{K}. \quad (24)$$

By substituting (19), (23) and (24) into (22), it can be obtained that

$$\dot{V}(t) = \sum_{i=1}^p \sum_{j=1}^c \tilde{w}_i(\mathbf{x})\tilde{m}_j(\mathbf{x})[(\tilde{\mathbf{A}}_i(\mathbf{x}) + \tilde{\mathbf{B}}_i(\mathbf{x})\mathbf{N}_j(\mathbf{x})\mathbf{X}(\tilde{\mathbf{x}})^{-1})$$

$$\begin{aligned}& \hat{\mathbf{x}}(\mathbf{x})]^T \mathbf{X}(\tilde{\mathbf{x}})^{-1} \hat{\mathbf{x}}(\mathbf{x}) + \hat{\mathbf{x}}(\mathbf{x})^T \left(\sum_{k=1}^n \frac{\partial \mathbf{X}(\tilde{\mathbf{x}})^{-1}}{\partial x_k} \sum_{i=1}^p \sum_{j=1}^c \right. \\ & \left. \tilde{w}_i(\mathbf{x})\tilde{m}_j(\mathbf{x}) \mathbf{A}_i^k(\mathbf{x}) \hat{\mathbf{x}}(\mathbf{x}) \right) \hat{\mathbf{x}}(\mathbf{x}) + \hat{\mathbf{x}}(\mathbf{x})^T \mathbf{X}(\tilde{\mathbf{x}})^{-1} \\ & \sum_{i=1}^p \sum_{j=1}^c \tilde{w}_i(\mathbf{x})\tilde{m}_j(\mathbf{x}) (\tilde{\mathbf{A}}_i(\mathbf{x}) + \tilde{\mathbf{B}}_i(\mathbf{x})\mathbf{N}_j(\mathbf{x})\mathbf{X}(\tilde{\mathbf{x}})^{-1}) \hat{\mathbf{x}}(\mathbf{x}) \\ & = \sum_{i=1}^p \sum_{j=1}^c \tilde{h}_{ij}(\mathbf{x}) \hat{\mathbf{x}}(\mathbf{x})^T \mathbf{X}(\tilde{\mathbf{x}})^{-1} [\mathbf{X}(\tilde{\mathbf{x}}) \tilde{\mathbf{A}}_i(\mathbf{x})^T \\ & \quad + \mathbf{N}_j(\mathbf{x})^T \tilde{\mathbf{B}}_i(\mathbf{x})^T + \tilde{\mathbf{A}}_i(\mathbf{x})\mathbf{X}(\tilde{\mathbf{x}}) + \tilde{\mathbf{B}}_i(\mathbf{x})\mathbf{N}_j(\mathbf{x}) \\ & \quad + \mathbf{X}(\tilde{\mathbf{x}}) \sum_{k=1}^n \frac{\partial \mathbf{X}(\tilde{\mathbf{x}})^{-1}}{\partial x_k} \mathbf{A}_i^k(\mathbf{x}) \hat{\mathbf{x}}(\mathbf{x}) \mathbf{X}(\tilde{\mathbf{x}})] \mathbf{X}(\tilde{\mathbf{x}})^{-1} \hat{\mathbf{x}}(\mathbf{x}) \\ & = \sum_{i=1}^p \sum_{j=1}^c \tilde{h}_{ij}(\mathbf{x}) \mathbf{z}(\mathbf{x})^T [\mathbf{Q}_{ij}(\mathbf{x}) \\ & \quad + \mathbf{X}(\tilde{\mathbf{x}}) \sum_{k=1}^n \frac{\partial \mathbf{X}(\tilde{\mathbf{x}})^{-1}}{\partial x_k} \mathbf{A}_i^k(\mathbf{x}) \hat{\mathbf{x}}(\mathbf{x}) \mathbf{X}(\tilde{\mathbf{x}})] \mathbf{z}(\mathbf{x}),\end{aligned}\quad (25)$$

where $\tilde{h}_{ij}(\mathbf{x}) \equiv \tilde{w}_i(\mathbf{x})\tilde{m}_j(\mathbf{x})$, $\mathbf{z}(\mathbf{x}) = \mathbf{X}(\tilde{\mathbf{x}})^{-1} \hat{\mathbf{x}}(\mathbf{x})$, $\mathbf{Q}_{ij}(\mathbf{x}) = \mathbf{X}(\tilde{\mathbf{x}}) \tilde{\mathbf{A}}_i(\mathbf{x})^T + \mathbf{N}_j(\mathbf{x})^T \tilde{\mathbf{B}}_i(\mathbf{x})^T + \tilde{\mathbf{A}}_i(\mathbf{x})\mathbf{X}(\tilde{\mathbf{x}}) + \tilde{\mathbf{B}}_i(\mathbf{x})\mathbf{N}_j(\mathbf{x})$, $\mathbf{N}_j(\mathbf{x}) \in \mathbb{R}^{m \times N}$ is a polynomial matrix for all j , $\mathbf{G}_j(\mathbf{x}) = \mathbf{N}_j(\mathbf{x})\mathbf{X}(\tilde{\mathbf{x}})^{-1}$, and $\mathbf{A}_i^k(\mathbf{x})\hat{\mathbf{x}}(\mathbf{x})$ is a scalar.

Then, the last term in (25) can be rewritten by the following Lemma 1 [15].

Lemma 1. For any invertible polynomial matrix $\mathbf{X}(\tilde{\mathbf{x}})$, the following holds,

$$\frac{\partial \mathbf{X}(\tilde{\mathbf{x}})}{\partial x_k} = -\mathbf{X}(\tilde{\mathbf{x}}) \frac{\partial \mathbf{X}(\tilde{\mathbf{x}})^{-1}}{\partial x_k} \mathbf{X}(\tilde{\mathbf{x}}). \quad (26)$$

Proof: Since $\mathbf{X}(\tilde{\mathbf{x}})$ is invertible, we have $\mathbf{X}(\tilde{\mathbf{x}})\mathbf{X}(\tilde{\mathbf{x}})^{-1} = \mathbf{I}$. Differentiating both sides with respect to x_k , we have

$$\frac{\partial \mathbf{X}(\tilde{\mathbf{x}})}{\partial x_k} \mathbf{X}(\tilde{\mathbf{x}})^{-1} + \mathbf{X}(\tilde{\mathbf{x}}) \frac{\partial \mathbf{X}(\tilde{\mathbf{x}})^{-1}}{\partial x_k} = \mathbf{0}.$$

Hence, the following relation holds

$$\mathbf{X}(\tilde{\mathbf{x}}) \frac{\partial \mathbf{X}(\tilde{\mathbf{x}})^{-1}}{\partial x_k} \mathbf{X}(\tilde{\mathbf{x}}) = -\frac{\partial \mathbf{X}(\tilde{\mathbf{x}})}{\partial x_k}.$$

By substituting (26) into (25), it can be obtained that

$$\begin{aligned}\dot{V}(t) &= \sum_{i=1}^p \sum_{j=1}^c \tilde{h}_{ij}(\mathbf{x}) \mathbf{z}(\mathbf{x})^T [\mathbf{Q}_{ij}(\mathbf{x}) \\ & \quad - \sum_{k=1}^n \frac{\partial \mathbf{X}(\tilde{\mathbf{x}})}{\partial x_k} \mathbf{A}_i^k(\mathbf{x}) \hat{\mathbf{x}}(\mathbf{x})] \mathbf{z}(\mathbf{x}).\end{aligned}\quad (27)$$

□

In order to keep $\dot{V}(t) \leq 0$ (equality holds for $\mathbf{x} = \mathbf{0}$), $\mathbf{Q}_{ij}(\mathbf{x}) - \sum_{k=1}^n \frac{\partial \mathbf{X}(\tilde{\mathbf{x}})}{\partial x_k} \mathbf{A}_i^k(\mathbf{x}) \hat{\mathbf{x}}(\mathbf{x}) \leq 0$ is a sufficient condition that must be held. The MFI stability conditions can be summarized in the following theorem.

Theorem 1. The IT2 PFMB control system (17) consisting of IT2 polynomial fuzzy model (5) and IT2 polynomial fuzzy controller (12) connected in a closed loop, is asymptotically stable if there exist a symmetric polynomial matrix $\mathbf{X}(\tilde{\mathbf{x}}) = \mathbf{X}(\tilde{\mathbf{x}})^T \in \mathbb{R}^{N \times N}$

and a polynomial matrix $\mathbf{N}_j(\mathbf{x}) \in \mathbb{R}^{m \times N}$ such that the following SOS-based conditions are satisfied.

$$v^T (\mathbf{X}(\tilde{\mathbf{x}}) - \varepsilon_1(\tilde{\mathbf{x}})\mathbf{I})v \text{ is SOS}; \quad (28)$$

$$-v^T (\mathbf{Q}_{ij}(\mathbf{x}) - \sum_{k \in \mathbf{K}} \frac{\partial \mathbf{X}(\tilde{\mathbf{x}})}{\partial x_k} \mathbf{A}_i^k(\mathbf{x})\dot{\mathbf{x}}(\mathbf{x}) + \varepsilon_2(\mathbf{x})\mathbf{I})v \text{ is SOS};$$

$$\forall i, j. \quad (29)$$

where $\varepsilon_1(\tilde{\mathbf{x}}) > 0$ and $\varepsilon_2(\mathbf{x}) > 0$ are predefined polynomial scalars; $v \in \mathbb{R}^N$ is an arbitrary vector independent of \mathbf{x} ; $\mathbf{Q}_{ij}(\mathbf{x}) = \tilde{\mathbf{A}}_i(\mathbf{x})\mathbf{X}(\tilde{\mathbf{x}}) + \mathbf{X}(\tilde{\mathbf{x}})\tilde{\mathbf{A}}_i(\mathbf{x})^T + \tilde{\mathbf{B}}_i(\mathbf{x})\mathbf{N}_j(\mathbf{x}) + \mathbf{N}_j(\mathbf{x})^T\tilde{\mathbf{B}}_i(\mathbf{x})^T$ for $i=1, 2, \dots, p, j=1, 2, \dots, c$; $\mathbf{A}_i^k(\mathbf{x}) = \mathbf{T}(\mathbf{x})\mathbf{A}_i(\mathbf{x})$; $\tilde{\mathbf{B}}_i(\mathbf{x}) = \mathbf{T}(\mathbf{x})\mathbf{B}_i(\mathbf{x})$; $\mathbf{T}(\mathbf{x}) \in \mathbb{R}^{N \times n}$ is a polynomial matrix with its (i, j) th entry defined in (20); $\mathbf{A}_i^k(\mathbf{x}) \in \mathbb{R}^N$ and $\mathbf{B}_i^k(\mathbf{x}) \in \mathbb{R}^m, i=1, 2, \dots, p, k=1, 2, \dots, n$, denote the k -th row of $\mathbf{A}_i(\mathbf{x})$ and $\mathbf{B}_i(\mathbf{x})$, respectively; and the polynomial feedback gains are defined as $\mathbf{G}_j(\mathbf{x}) = \mathbf{N}_j(\mathbf{x})\mathbf{X}(\tilde{\mathbf{x}})^{-1}, j=1, 2, \dots, c$.

Remark 1. Since no information about membership functions is used in the above analysis, it is valid for all kinds of membership functions, therefore, this stability condition is conservative. In order to reduce conservativeness, the information of membership functions can be introduced into the stability analysis process to obtain more relaxed stability conditions, which is discussed in the following sections.

3.2 MFD Stability Analysis

Two types of stability analysis, namely MFD stability analysis and MFD switching stability analysis, are given in this section. In the MFD stability analysis, some slack matrices are used to introduce the information of IT2 membership functions and an IT2 polynomial fuzzy controller (12) can be obtained to control the nonlinear system represented by the IT2 polynomial fuzzy model (5). On the basis of the MFD stability analysis, the MFD switching stability analysis is developed to generate an IT2 switching polynomial fuzzy controller (16), which demonstrates stronger stabilizability due to the switching control strategy.

3.2.1 MFD Stability Analysis: In this section, some slack matrices are introduced on the basis of Theorem 1 to bring the information of IT2 membership functions into the stability conditions, which could effectively relaxed the conservativeness of the stability analysis results.

Recalling that, the whole operation domain Φ is divided into D sub-domains Φ_d , i.e., $\Phi = \bigcup_{d=1}^D \Phi_d$, where $d=1, 2, \dots, D$. In each sub-domain, the lower and upper bounds $\underline{h}_{ijd}(\mathbf{x})$ and $\bar{h}_{ijd}(\mathbf{x})$ of the IT2 membership functions $\tilde{h}_{ij}(\mathbf{x})$, the minimum and maximum value $\underline{\sigma}_{ijd}$ and $\bar{\sigma}_{ijd}$ of the upper bound $\bar{h}_{ijd}(\mathbf{x})$, and the minimum and maximum value $\underline{\gamma}_{ijd}$ and $\bar{\gamma}_{ijd}$ of the lower bound $\underline{h}_{ijd}(\mathbf{x})$, can be obtained. $\underline{h}_{ijd}(\mathbf{x})$ and $\bar{h}_{ijd}(\mathbf{x})$ are functions of \mathbf{x} , while $\underline{\sigma}_{ijd}, \bar{\sigma}_{ijd}, \underline{\gamma}_{ijd}$ and $\bar{\gamma}_{ijd}$ are constants.

According to the property of the IT2 membership functions, the following properties hold.

$$0 \leq \underline{h}_{ijd}(\mathbf{x}) \leq \tilde{h}_{ij}(\mathbf{x}) \leq \bar{h}_{ijd}(\mathbf{x}) \leq 1 \quad (30)$$

$$0 \leq \underline{\gamma}_{ijd} \leq \underline{h}_{ijd}(\mathbf{x}) \leq \bar{\gamma}_{ijd} \leq 1 \quad (31)$$

$$0 \leq \underline{\sigma}_{ijd} \leq \bar{h}_{ijd}(\mathbf{x}) \leq \bar{\sigma}_{ijd} \leq 1 \quad (32)$$

Then, we have $\tilde{h}_{ij}(\mathbf{x}) - \underline{h}_{ijd}(\mathbf{x}) \geq 0$; $\bar{h}_{ijd}(\mathbf{x}) - \tilde{h}_{ij}(\mathbf{x}) \geq 0$; $\underline{h}_{ijd}(\mathbf{x}) - \underline{\gamma}_{ijd} \geq 0$; $\bar{\gamma}_{ijd} - \bar{h}_{ijd}(\mathbf{x}) \geq 0$; $\bar{h}_{ijd}(\mathbf{x}) - \underline{\sigma}_{ijd} \geq 0$; $\bar{\sigma}_{ijd} - \bar{h}_{ijd}(\mathbf{x}) \geq 0$ for all d .

From (27), we have

$$\dot{V}(t) = \sum_{i=1}^p \sum_{j=1}^c \tilde{h}_{ij}(\mathbf{x})\mathbf{z}(\mathbf{x})^T [\mathbf{Q}_{ij}(\mathbf{x}) - \sum_{k=1}^n \frac{\partial \mathbf{X}(\tilde{\mathbf{x}})}{\partial x_k} \mathbf{A}_i^k(\mathbf{x})\dot{\mathbf{x}}(\mathbf{x})]$$

$$\mathbf{z}(\mathbf{x}) = \sum_{i=1}^p \sum_{j=1}^c \tilde{h}_{ij}(\mathbf{x})\mathbf{z}(\mathbf{x})^T \Theta_{ij}(\mathbf{x})\mathbf{z}(\mathbf{x}), \quad (33)$$

where $\Theta_{ij}(\mathbf{x}) = \mathbf{Q}_{ij}(\mathbf{x}) - \sum_{k \in \mathbf{K}} \frac{\partial \mathbf{X}(\tilde{\mathbf{x}})}{\partial x_k} \mathbf{A}_i^k(\mathbf{x})\dot{\mathbf{x}}(\mathbf{x})$ for all $i=1, 2, \dots, p, j=1, 2, \dots, c$.

Combining the slack matrices $0 \leq \underline{\mathbf{R}}_{ijd}(\mathbf{x}) = \underline{\mathbf{R}}_{ijd}(\mathbf{x})^T \in \mathbb{R}^{N \times N}, 0 \leq \bar{\mathbf{R}}_{ijd}(\mathbf{x}) = \bar{\mathbf{R}}_{ijd}(\mathbf{x})^T \in \mathbb{R}^{N \times N}, 0 \leq \underline{\mathbf{S}}_{ijd}(\mathbf{x}) = \underline{\mathbf{S}}_{ijd}(\mathbf{x})^T \in \mathbb{R}^{N \times N}, 0 \leq \bar{\mathbf{S}}_{ijd}(\mathbf{x}) = \bar{\mathbf{S}}_{ijd}(\mathbf{x})^T \in \mathbb{R}^{N \times N}, 0 \leq \underline{\mathbf{U}}_{ijd}(\mathbf{x}) = \underline{\mathbf{U}}_{ijd}(\mathbf{x})^T \in \mathbb{R}^{N \times N}, 0 \leq \bar{\mathbf{U}}_{ijd}(\mathbf{x}) = \bar{\mathbf{U}}_{ijd}(\mathbf{x})^T \in \mathbb{R}^{N \times N}$ with the inequalities obtained above, when $\mathbf{x} \in \Phi_d, d=1, 2, \dots, D$, the following inequality can be obtained by extending the terms in the right-hand side of (33),

$$\begin{aligned} \dot{V}(t) &\leq \sum_{i=1}^p \sum_{j=1}^c \tilde{h}_{ij}(\mathbf{x})\mathbf{z}(\mathbf{x})^T \Theta_{ij}(\mathbf{x})\mathbf{z}(\mathbf{x}) + \sum_{i=1}^p \sum_{j=1}^c \tilde{h}_{ij}(\mathbf{x}) \\ &\quad - \underline{h}_{ijd}(\mathbf{x})\mathbf{z}(\mathbf{x})^T \underline{\mathbf{R}}_{ijd}(\mathbf{x})\mathbf{z}(\mathbf{x}) + \sum_{i=1}^p \sum_{j=1}^c (\bar{h}_{ijd}(\mathbf{x}) \\ &\quad - \tilde{h}_{ij}(\mathbf{x}))\mathbf{z}(\mathbf{x})^T \bar{\mathbf{R}}_{ijd}(\mathbf{x})\mathbf{z}(\mathbf{x}) + \sum_{i=1}^p \sum_{j=1}^c (\underline{h}_{ijd}(\mathbf{x}) \\ &\quad - \underline{\gamma}_{ijd})\mathbf{z}(\mathbf{x})^T \underline{\mathbf{S}}_{ijd}(\mathbf{x})\mathbf{z}(\mathbf{x}) + \sum_{i=1}^p \sum_{j=1}^c (\bar{\gamma}_{ijd} \\ &\quad - \bar{h}_{ijd}(\mathbf{x}))\mathbf{z}(\mathbf{x})^T \bar{\mathbf{S}}_{ijd}(\mathbf{x})\mathbf{z}(\mathbf{x}) + \sum_{i=1}^p \sum_{j=1}^c (\bar{h}_{ijd}(\mathbf{x}) \\ &\quad - \underline{\sigma}_{ijd})\mathbf{z}(\mathbf{x})^T \underline{\mathbf{U}}_{ijd}(\mathbf{x})\mathbf{z}(\mathbf{x}) \\ &\quad + \sum_{i=1}^p \sum_{j=1}^c (\bar{\sigma}_{ijd} - \bar{h}_{ijd}(\mathbf{x}))\mathbf{z}(\mathbf{x})^T \bar{\mathbf{U}}_{ijd}(\mathbf{x})\mathbf{z}(\mathbf{x}) \\ &= \sum_{i=1}^p \sum_{j=1}^c \tilde{h}_{ij}(\mathbf{x})\mathbf{z}(\mathbf{x})^T \{ \Theta_{ij}(\mathbf{x}) + \underline{\mathbf{R}}_{ijd}(\mathbf{x}) - \bar{\mathbf{R}}_{ijd}(\mathbf{x}) \\ &\quad + \sum_{r=1}^p \sum_{s=1}^c [\underline{h}_{rsd}(\mathbf{x})(\underline{\mathbf{S}}_{rsd}(\mathbf{x}) - \underline{\mathbf{R}}_{rsd}(\mathbf{x}) - \bar{\mathbf{S}}_{rsd}(\mathbf{x})) \\ &\quad + \bar{h}_{rsd}(\mathbf{x})(\bar{\mathbf{R}}_{rsd}(\mathbf{x}) + \underline{\mathbf{U}}_{rsd}(\mathbf{x}) - \bar{\mathbf{U}}_{rsd}(\mathbf{x})) \\ &\quad + \bar{\gamma}_{rsd}\bar{\mathbf{S}}_{rsd}(\mathbf{x}) - \underline{\gamma}_{rsd}\underline{\mathbf{S}}_{rsd}(\mathbf{x}) + \bar{\sigma}_{rsd}\bar{\mathbf{U}}_{rsd}(\mathbf{x}) \\ &\quad - \underline{\sigma}_{rsd}\underline{\mathbf{U}}_{rsd}(\mathbf{x})] \} \mathbf{z}(\mathbf{x}), \forall \mathbf{x} \in \Phi_d; d=1, 2, \dots, D. \end{aligned} \quad (34)$$

According to Lyapunov stability theory, the nonlinear system is asymptotically stable if $V(t) > 0$ and $\dot{V}(t) < 0$ are achieved for all $\mathbf{x} \neq \mathbf{0}$. The stability analysis results are summarized in the following theorem.

Theorem 2. The IT2 PFMB control system (17) consisting of IT2 polynomial fuzzy model (5) and IT2 polynomial fuzzy controller (12) connected in a closed loop, is asymptotically stable if there exist symmetric polynomial matrices $\mathbf{X}(\tilde{\mathbf{x}}) \in \mathbb{R}^{N \times N}, \underline{\mathbf{R}}_{ijd}(\mathbf{x}) \in \mathbb{R}^{N \times N}, \bar{\mathbf{R}}_{ijd}(\mathbf{x}) \in \mathbb{R}^{N \times N}, \underline{\mathbf{S}}_{ijd}(\mathbf{x}) \in \mathbb{R}^{N \times N}, \bar{\mathbf{S}}_{ijd}(\mathbf{x}) \in \mathbb{R}^{N \times N}, \underline{\mathbf{U}}_{ijd}(\mathbf{x}) \in \mathbb{R}^{N \times N}, \bar{\mathbf{U}}_{ijd}(\mathbf{x}) \in \mathbb{R}^{N \times N}$ and a polynomial matrix $\mathbf{N}_j(\mathbf{x}) \in \mathbb{R}^{m \times N}$ such that the following SOS-based conditions are satisfied.

$$v^T (\mathbf{X}(\tilde{\mathbf{x}}) - \varepsilon_1(\tilde{\mathbf{x}})\mathbf{I})v \text{ is SOS}; \quad (35)$$

$$-v^T \{ \Theta_{ij}(\mathbf{x}) + \underline{\mathbf{R}}_{ijd}(\mathbf{x}) - \bar{\mathbf{R}}_{ijd}(\mathbf{x}) + \sum_{r=1}^p \sum_{s=1}^c [\underline{h}_{rsd}(\mathbf{x})$$

$$\begin{aligned}
& (\underline{\mathbf{S}}_{rsd}(\mathbf{x}) - \underline{\mathbf{R}}_{rsd}(\mathbf{x}) - \overline{\mathbf{S}}_{rsd}(\mathbf{x}) + \overline{h}_{rsd}(\mathbf{x})(\overline{\mathbf{R}}_{rsd}(\mathbf{x})) \\
& + \underline{\mathbf{U}}_{rsd}(\mathbf{x}) - \overline{\mathbf{U}}_{rsd}(\mathbf{x}) + \overline{\gamma}_{rsd}\overline{\mathbf{S}}_{rsd}(\mathbf{x}) - \underline{\gamma}_{rsd}\underline{\mathbf{S}}_{rsd}(\mathbf{x}) \\
& + \overline{\sigma}_{rsd}\overline{\mathbf{U}}_{rsd}(\mathbf{x}) - \underline{\sigma}_{rsd}\underline{\mathbf{U}}_{rsd}(\mathbf{x})] + \varepsilon_2(\mathbf{x})\mathbf{I}]v \text{ is SOS,} \\
& \quad \forall i, j, d; \quad (36)
\end{aligned}$$

$$\begin{aligned}
& v^T \underline{\mathbf{R}}_{ijd}(\mathbf{x})v \text{ is SOS,} \quad \forall i, j, d; \\
& v^T \overline{\mathbf{R}}_{ijd}(\mathbf{x})v \text{ is SOS,} \quad \forall i, j, d; \\
& v^T \underline{\mathbf{S}}_{ijd}(\mathbf{x})v \text{ is SOS,} \quad \forall i, j, d; \\
& v^T \overline{\mathbf{S}}_{ijd}(\mathbf{x})v \text{ is SOS,} \quad \forall i, j, d; \\
& v^T \underline{\mathbf{U}}_{ijd}(\mathbf{x})v \text{ is SOS,} \quad \forall i, j, d; \\
& v^T \overline{\mathbf{U}}_{ijd}(\mathbf{x})v \text{ is SOS,} \quad \forall i, j, d;
\end{aligned}$$

where $\varepsilon_1(\tilde{\mathbf{x}}) > 0$ and $\varepsilon_2(\mathbf{x}) > 0$ are predefined polynomial scalars; $v \in \mathbb{R}^N$ is an arbitrary vector independent of \mathbf{x} ; $\Theta_{ij}(\mathbf{x}) = \tilde{\mathbf{A}}_i(\mathbf{x})\mathbf{X}(\tilde{\mathbf{x}}) + \mathbf{X}(\tilde{\mathbf{x}})\tilde{\mathbf{A}}_i(\mathbf{x})^T + \tilde{\mathbf{B}}_i(\mathbf{x})\mathbf{N}_j(\mathbf{x}) + \mathbf{N}_j(\mathbf{x})\tilde{\mathbf{B}}_i(\mathbf{x})^T - \sum_{k \in \mathbf{K}} \frac{\partial \mathbf{X}(\tilde{\mathbf{x}})}{\partial x_k} \mathbf{A}_i^k(\mathbf{x})\hat{\mathbf{x}}(\mathbf{x})$ for $i=1, 2, \dots, p, j=1, 2, \dots, c$; $\tilde{\mathbf{A}}_i(\mathbf{x}) = \mathbf{T}(\mathbf{x})\mathbf{A}_i(\mathbf{x})$; $\tilde{\mathbf{B}}_i(\mathbf{x}) = \mathbf{T}(\mathbf{x})\mathbf{B}_i(\mathbf{x})$; $\mathbf{T}(\mathbf{x}) \in \mathbb{R}^{N \times n}$ is a polynomial matrix with its (i, j) th entry defined in (20); $\mathbf{A}_i^k(\mathbf{x}) \in \mathbb{R}^N$ and $\mathbf{B}_i^k(\mathbf{x}) \in \mathbb{R}^m$, $i=1, 2, \dots, p, k=1, 2, \dots, n$, denote the k -th row of $\mathbf{A}_i(\mathbf{x})$ and $\mathbf{B}_i(\mathbf{x})$, respectively; $\tilde{h}_{ijd}(\mathbf{x})$ and $\overline{h}_{ijd}(\mathbf{x})$ are the lower and upper bounds of the IT2 membership functions $\tilde{h}_{ij}(\mathbf{x})$, respectively, determined in prior and have property that $0 \leq \tilde{h}_{ijd}(\mathbf{x}) \leq \overline{h}_{ijd}(\mathbf{x}) \leq 1$, for all i, j, d ; similarly, $\underline{\gamma}_{ijd}$ and $\overline{\gamma}_{ijd}$ are the minimum and maximum value of the lower bound $\underline{h}_{ijd}(\mathbf{x})$ determined in prior and have the property that $0 \leq \underline{\gamma}_{ijd} \leq \overline{\gamma}_{ijd} \leq 1$ for all i, j, d ; $\underline{\sigma}_{ijd}$ and $\overline{\sigma}_{ijd}$ are the minimum and maximum values of the upper bound $\overline{h}_{ijd}(\mathbf{x})$ determined in prior and have the property that $0 \leq \underline{\sigma}_{ijd} \leq \overline{\sigma}_{ijd} \leq 1$ for all i, j, d ; and the polynomial feedback gains are defined as $\mathbf{G}_j(\mathbf{x}) = \mathbf{N}_j(\mathbf{x})\mathbf{X}(\tilde{\mathbf{x}})^{-1}$, $j=1, 2, \dots, c$.

Remark 2. Theorem 2 considers the information of membership functions in the stability conditions. The more information of membership functions are considered, i.e. the more sub-domains are used, the more relaxed stability conditions can be obtained. However, the computational complexity will be increased for solution solving when the number of sub-domains is increasing. Hence, it is suggested that a small number of sub-domains can be used in the beginning. If no feasible solution is found, try again by increasing the number of sub-domains gradually.

3.2.2 MFD Switching Stability Analysis: In this section, an IT2 switching polynomial fuzzy controller (16) is employed. Considering the polynomial Lyapunov function candidate (21), the time derivative of $V(t)$ is given by

$$\begin{aligned}
\dot{V}(t) &= \dot{\hat{\mathbf{x}}}(\mathbf{x})^T \mathbf{X}(\tilde{\mathbf{x}})^{-1} \hat{\mathbf{x}}(\mathbf{x}) + \hat{\mathbf{x}}(\mathbf{x})^T \left(\sum_{k=1}^n \frac{\partial \mathbf{X}(\tilde{\mathbf{x}})^{-1}}{\partial x_k} \dot{x}_k \right) \hat{\mathbf{x}}(\mathbf{x}) \\
&+ \hat{\mathbf{x}}(\mathbf{x})^T \mathbf{X}(\tilde{\mathbf{x}})^{-1} \dot{\hat{\mathbf{x}}}(\mathbf{x}) \\
&= \sum_{d=1}^D \sum_{i=1}^p \sum_{j=1}^c \xi_d(\mathbf{x}) \tilde{h}_{ij}(\mathbf{x}) \hat{\mathbf{x}}(\mathbf{x})^T \mathbf{X}(\tilde{\mathbf{x}})^{-1} [\mathbf{X}(\tilde{\mathbf{x}})\tilde{\mathbf{A}}_i(\mathbf{x})^T \\
&+ \mathbf{N}_{jd}(\mathbf{x})\tilde{\mathbf{B}}_i(\mathbf{x})^T + \tilde{\mathbf{A}}_i(\mathbf{x})\mathbf{X}(\tilde{\mathbf{x}}) + \tilde{\mathbf{B}}_i(\mathbf{x})\mathbf{N}_{jd}(\mathbf{x}) \\
&+ \mathbf{X}(\tilde{\mathbf{x}}) \sum_{k=1}^n \frac{\partial \mathbf{X}(\tilde{\mathbf{x}})^{-1}}{\partial x_k} \mathbf{A}_i^k(\mathbf{x})\hat{\mathbf{x}}(\mathbf{x})\mathbf{X}(\tilde{\mathbf{x}})^{-1} \hat{\mathbf{x}}(\mathbf{x}) \\
&= \sum_{d=1}^D \sum_{i=1}^p \sum_{j=1}^c \xi_d(\mathbf{x}) \tilde{h}_{ij}(\mathbf{x}) \mathbf{z}(\mathbf{x})^T [\mathbf{Q}_{ijd}(\mathbf{x})
\end{aligned}$$

$$- \sum_{k=1}^n \frac{\partial \mathbf{X}(\tilde{\mathbf{x}})}{\partial x_k} \mathbf{A}_i^k(\mathbf{x})\hat{\mathbf{x}}(\mathbf{x})] \mathbf{z}(\mathbf{x}), \quad (37)$$

where $\mathbf{Q}_{ijd}(\mathbf{x}) = \mathbf{X}(\tilde{\mathbf{x}})\tilde{\mathbf{A}}_i(\mathbf{x})^T + \mathbf{N}_{jd}(\mathbf{x})\tilde{\mathbf{B}}_i(\mathbf{x})^T + \tilde{\mathbf{A}}_i(\mathbf{x})\mathbf{X}(\tilde{\mathbf{x}}) + \tilde{\mathbf{B}}_i(\mathbf{x})\mathbf{N}_{jd}(\mathbf{x})$. $\mathbf{G}_{jd}(\mathbf{x}) = \mathbf{N}_{jd}(\mathbf{x})\mathbf{X}(\tilde{\mathbf{x}})$, where $\mathbf{N}_{jd}(\mathbf{x}) \in \mathbb{R}^{m \times N}$, $j=1, 2, \dots, c, d=1, 2, \dots, D$, are polynomial matrices to be determined.

From (37), it can be obtained that

$$\dot{V}(t) = \sum_{d=1}^D \sum_{i=1}^p \sum_{j=1}^c \xi_d(\mathbf{x}) \tilde{h}_{ij}(\mathbf{x}) \mathbf{z}(\mathbf{x})^T \Theta_{ijd}(\mathbf{x}) \mathbf{z}(\mathbf{x}), \quad (38)$$

where $\Theta_{ijd}(\mathbf{x}) = \mathbf{Q}_{ijd}(\mathbf{x}) - \sum_{k=1}^n \frac{\partial \mathbf{X}(\tilde{\mathbf{x}})}{\partial x_k} \mathbf{A}_i^k(\mathbf{x})\hat{\mathbf{x}}(\mathbf{x})$.

Then introducing the information of IT2 membership functions by some slack matrices to relax conservative. We use the same boundary information (30) (31) and (32) in Section 3.2.1. By combining these informations with the slack matrices $0 \leq \underline{\mathbf{R}}_{ijd}(\mathbf{x}) = \underline{\mathbf{R}}_{ijd}(\mathbf{x})^T \in \mathbb{R}^{N \times N}$, $0 \leq \overline{\mathbf{R}}_{ijd}(\mathbf{x}) = \overline{\mathbf{R}}_{ijd}(\mathbf{x})^T \in \mathbb{R}^{N \times N}$, $0 \leq \underline{\mathbf{S}}_{ijd}(\mathbf{x}) = \underline{\mathbf{S}}_{ijd}(\mathbf{x})^T \in \mathbb{R}^{N \times N}$, $0 \leq \overline{\mathbf{S}}_{ijd}(\mathbf{x}) = \overline{\mathbf{S}}_{ijd}(\mathbf{x})^T \in \mathbb{R}^{N \times N}$, $0 \leq \underline{\mathbf{U}}_{ijd}(\mathbf{x}) = \underline{\mathbf{U}}_{ijd}(\mathbf{x})^T \in \mathbb{R}^{N \times N}$, $0 \leq \overline{\mathbf{U}}_{ijd}(\mathbf{x}) = \overline{\mathbf{U}}_{ijd}(\mathbf{x})^T \in \mathbb{R}^{N \times N}$, the following inequality can be obtained.

$$\begin{aligned}
\dot{V}(t) &\leq \sum_{d=1}^D \sum_{i=1}^p \sum_{j=1}^c \xi_d(\mathbf{x}) \tilde{h}_{ij}(\mathbf{x}) \mathbf{z}(\mathbf{x})^T \{ \Theta_{ijd}(\mathbf{x}) + \underline{\mathbf{R}}_{ijd}(\mathbf{x}) \\
&- \overline{\mathbf{R}}_{ijd}(\mathbf{x}) + \sum_{r=1}^p \sum_{s=1}^c [\underline{h}_{rsd}(\mathbf{x})(\underline{\mathbf{S}}_{rsd}(\mathbf{x}) - \underline{\mathbf{R}}_{rsd}(\mathbf{x}) \\
&- \overline{\mathbf{S}}_{rsd}(\mathbf{x})) + \overline{h}_{rsd}(\mathbf{x})(\overline{\mathbf{R}}_{rsd}(\mathbf{x}) + \underline{\mathbf{U}}_{rsd}(\mathbf{x}) \\
&- \overline{\mathbf{U}}_{rsd}(\mathbf{x})) + \overline{\gamma}_{rsd}\overline{\mathbf{S}}_{rsd}(\mathbf{x}) - \underline{\gamma}_{rsd}\underline{\mathbf{S}}_{rsd}(\mathbf{x}) \\
&+ \overline{\sigma}_{rsd}\overline{\mathbf{U}}_{rsd}(\mathbf{x}) - \underline{\sigma}_{rsd}\underline{\mathbf{U}}_{rsd}(\mathbf{x})] \} \mathbf{z}(\mathbf{x}) \quad \forall i, j, d. \quad (39)
\end{aligned}$$

According to Lyapunov stability theory, the nonlinear system is asymptotically stable if $V(t) > 0$ and $\dot{V}(t) < 0$ for all $\mathbf{x} \neq \mathbf{0}$ are achieved. The stability analysis results are summarized in the following theorem.

Theorem 3. The IT2 switching PFMB control system (17) consisting of IT2 polynomial fuzzy model(5) and IT2 switching polynomial controller (16) connected in a closed loop, is asymptotically stable if there exist symmetric polynomial matrices $\mathbf{X}(\tilde{\mathbf{x}}) \in \mathbb{R}^{N \times N}$, $\underline{\mathbf{R}}_{ijd}(\mathbf{x}) \in \mathbb{R}^{N \times N}$, $\overline{\mathbf{R}}_{ijd}(\mathbf{x}) \in \mathbb{R}^{N \times N}$, $\underline{\mathbf{S}}_{ijd}(\mathbf{x}) \in \mathbb{R}^{N \times N}$, $\overline{\mathbf{S}}_{ijd}(\mathbf{x}) \in \mathbb{R}^{N \times N}$, $\underline{\mathbf{U}}_{ijd}(\mathbf{x}) \in \mathbb{R}^{N \times N}$, $\overline{\mathbf{U}}_{ijd}(\mathbf{x}) \in \mathbb{R}^{N \times N}$ and a polynomial matrix $\mathbf{N}_{jd}(\mathbf{x}) \in \mathbb{R}^{m \times N}$ such that the following SOS-based conditions are satisfied.

$$v^T (\mathbf{X}(\tilde{\mathbf{x}}) - \varepsilon_1(\tilde{\mathbf{x}})\mathbf{I})v \text{ is SOS;} \quad (40)$$

$$\begin{aligned}
& - v^T \{ \Theta_{ijd}(\mathbf{x}) + \underline{\mathbf{R}}_{ijd}(\mathbf{x}) - \overline{\mathbf{R}}_{ijd}(\mathbf{x}) + \sum_{r=1}^p \sum_{s=1}^c [\underline{h}_{rsd}(\mathbf{x}) \\
& (\underline{\mathbf{S}}_{rsd}(\mathbf{x}) - \underline{\mathbf{R}}_{rsd}(\mathbf{x}) - \overline{\mathbf{S}}_{rsd}(\mathbf{x})) + \overline{h}_{rsd}(\mathbf{x})(\overline{\mathbf{R}}_{rsd}(\mathbf{x}) \\
& + \underline{\mathbf{U}}_{rsd}(\mathbf{x}) - \overline{\mathbf{U}}_{rsd}(\mathbf{x})) + \overline{\gamma}_{rsd}\overline{\mathbf{S}}_{rsd}(\mathbf{x}) - \underline{\gamma}_{rsd}\underline{\mathbf{S}}_{rsd}(\mathbf{x}) \\
& + \overline{\sigma}_{rsd}\overline{\mathbf{U}}_{rsd}(\mathbf{x}) - \underline{\sigma}_{rsd}\underline{\mathbf{U}}_{rsd}(\mathbf{x})] + \varepsilon_2(\mathbf{x})\mathbf{I} \} v \text{ is SOS,} \\
& \quad \forall i, j, d; \quad (41)
\end{aligned}$$

$$v^T \underline{\mathbf{R}}_{ijd}(\mathbf{x})v \text{ is SOS,} \quad \forall i, j, d;$$

$$v^T \overline{\mathbf{R}}_{ijd}(\mathbf{x})v \text{ is SOS,} \quad \forall i, j, d;$$

$$v^T \underline{\mathbf{S}}_{ijd}(\mathbf{x})v \text{ is SOS,} \quad \forall i, j, d;$$

$$v^T \overline{\mathbf{S}}_{ijd}(\mathbf{x})v \text{ is SOS,} \quad \forall i, j, d;$$

$$v^T \underline{U}_{ijd}(\mathbf{x})v \text{ is SOS, } \forall i, j, d;$$

$$v^T \overline{U}_{ijd}(\mathbf{x})v \text{ is SOS, } \forall i, j, d;$$

where $\varepsilon_1(\tilde{\mathbf{x}}) > 0$ and $\varepsilon_2(\mathbf{x}) > 0$ are predefined polynomial scalars; $v \in \mathbb{R}^N$ is an arbitrary vector independent of \mathbf{x} ; $\Theta_{ijd}(\mathbf{x}) = \tilde{\mathbf{A}}_i(\mathbf{x})\mathbf{X}(\tilde{\mathbf{x}}) + \mathbf{X}(\tilde{\mathbf{x}})\tilde{\mathbf{A}}_i(\mathbf{x})^T + \tilde{\mathbf{B}}_i(\mathbf{x})\mathbf{N}_{jd}(\mathbf{x}) + \mathbf{N}_{jd}(\mathbf{x})^T\tilde{\mathbf{B}}_i(\mathbf{x})^T - \sum_{k \in \mathbf{K}} \frac{\partial \mathbf{X}(\tilde{\mathbf{x}})}{\partial x_k} \mathbf{A}_i^k(\mathbf{x})\tilde{\mathbf{x}}(\mathbf{x})$ for $i=1, 2, \dots, p, j=1, 2, \dots, c$ and $d=1, 2, \dots, D$; $\tilde{\mathbf{A}}_i(\mathbf{x}) = \mathbf{T}(\mathbf{x})\mathbf{A}_i(\mathbf{x})$; $\tilde{\mathbf{B}}_i(\mathbf{x}) = \mathbf{T}(\mathbf{x})\mathbf{B}_i(\mathbf{x})$; $\mathbf{T}(\mathbf{x}) \in \mathbb{R}^{N \times n}$ is a polynomial matrix with its (i, j) th entry defined in (20); $\mathbf{A}_i^k(\mathbf{x}) \in \mathbb{R}^N$ and $\mathbf{B}_i^k(\mathbf{x}) \in \mathbb{R}^m, i=1, 2, \dots, p, k=1, 2, \dots, n$, denote the k -th row of $\mathbf{A}_i(\mathbf{x})$ and $\mathbf{B}_i(\mathbf{x})$, respectively; $\underline{h}_{ijd}(\mathbf{x})$ and $\overline{h}_{ijd}(\mathbf{x})$ are the lower and upper bounds of IT2 membership functions $\tilde{h}_{ij}(\mathbf{x})$, respectively, determined in prior and have the property that $0 \leq \underline{h}_{ijd}(\mathbf{x}) \leq \tilde{h}_{ij}(\mathbf{x}) \leq \overline{h}_{ijd}(\mathbf{x}) \leq 1$, for all i, j, d ; similarly, $\underline{\gamma}_{ijd}$ and $\overline{\gamma}_{ijd}$ are the minimum and maximum values of the lower bound $\underline{h}_{ijd}(\mathbf{x})$, respectively, determined in prior and have the property that $0 \leq \underline{\gamma}_{ijd} \leq \underline{h}_{ijd}(\mathbf{x}) \leq \overline{\gamma}_{ijd} \leq 1$ for all i, j, d ; $\underline{\sigma}_{ijd}$ and $\overline{\sigma}_{ijd}$ are the minimum and maximum values of the upper bound $\overline{h}_{ijd}(\mathbf{x})$, respectively, determined in prior and have the property that $0 \leq \underline{\sigma}_{ijd} \leq \overline{h}_{ijd}(\mathbf{x}) \leq \overline{\sigma}_{ijd} \leq 1$ for all i, j, d ; and the polynomial feedback gains are defined as $\mathbf{G}_{jd}(\mathbf{x}) = \mathbf{N}_{jd}(\mathbf{x})\mathbf{X}(\tilde{\mathbf{x}})^{-1}, j = 1, 2, \dots, c$ and $d = 1, 2, \dots, D$.

Remark 3. Since the switching control scheme is employed in Theorem 3, the number of decision variables such as $\mathbf{N}_{jd}(\mathbf{x})$ are increased. Thus the computational demand for finding a feasible solution to stability conditions are increased accordingly. Hence, compared with Theorem 2 under the same number of sub-domains being used, solving a feasible solution to the stability conditions in Theorem 3 will be further increased.

4 Simulation Example

In this section, two examples are provided to verify the stability analysis results by applying the stability conditions in Theorems 1 to 3. The first example is a numerical example which demonstrates that the information of the lower and upper membership functions, and the switching control scheme can offer more relaxed stability analysis results comparatively when they are included in the stability analysis. In the second example, it is to demonstrate that the developed stability conditions can be applied to a nonlinear plant, inverted pendulum, with physical meaning.

4.1 Numerical Example

To demonstrate the effectiveness of the proposed approaches, a nonlinear system represented by an IT2 polynomial fuzzy model is given.

Considering the three-rule IT2 polynomial fuzzy model with $\tilde{\mathbf{x}}(\mathbf{x}) = \mathbf{x} = [x_1 \ x_2]^T$,

$$\mathbf{A}_1(x_1) = \begin{bmatrix} 1.59 - 0.12x_1^2 & 7.29 - 0.25x_1 \\ 0.01 & -0.1 \end{bmatrix},$$

$$\mathbf{A}_2(x_1) = \begin{bmatrix} 0.02 - 0.63x_1^2 & -4.64 + 0.92x_1 \\ 0.35 & -0.21 \end{bmatrix},$$

$$\mathbf{A}_3(x_1) = \begin{bmatrix} -a - 1.12x_1^2 + 0.31x_1 & -4.33 \\ 0 & 0.05 \end{bmatrix},$$

$$\mathbf{B}_1(x_1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$\mathbf{B}_2(x_1) = \begin{bmatrix} 8 \\ 0 \end{bmatrix},$$

$$\mathbf{B}_3(x_1) = \begin{bmatrix} -b + 6 \\ -1 \end{bmatrix}.$$

where a and b are constant parameters to be determined.

The lower and upper membership functions for the IT2 polynomial fuzzy model are chosen as $w_1(x_1) = 1 - 1/(1 + e^{-(x_1+3.5)})$, $w_3(x_1) = 1/(1 + e^{-(x_1-3.5)})$, $\overline{w}_1(x_1) = 1 - 1/(1 + e^{-(x_1+2.5)})$, $\overline{w}_3(x_1) = 1/(1 + e^{-(x_1-2.5)})$, $w_2(x_1) = 1 - \overline{w}_1(x_1) - \overline{w}_3(x_1)$ and $\overline{w}_2(x_1) = 1 - w_1(x_1) - w_3(x_1)$.

Meanwhile, the lower and upper membership functions for the IT2 polynomial fuzzy controller (IT2 switching polynomial fuzzy controller), which are different from those of the IT2 polynomial fuzzy model, are chosen as

$$\underline{m}_1(x_1) = \begin{cases} 1 & \text{for } x_1 < -5.2 \\ \frac{-x_1+4.8}{10} & \text{for } -5.2 \leq x_1 \leq 4.8 \\ 0 & \text{for } x_1 > 4.8 \end{cases},$$

$$\overline{m}_1(x_1) = \begin{cases} 1 & \text{for } x_1 < -4.8 \\ \frac{-x_1+5.2}{10} & \text{for } -4.8 \leq x_1 \leq 5.2 \\ 0 & \text{for } x_1 > 5.2 \end{cases},$$

$$\underline{m}_2(x_1) = 1 - \overline{m}_1(x_1) \text{ and } \overline{m}_2(x_1) = 1 - \underline{m}_1(x_1).$$

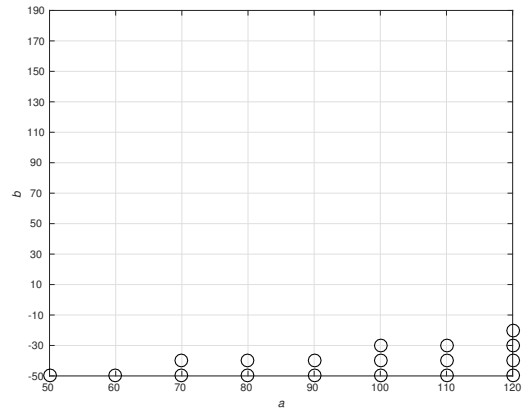


Fig. 1: Feasible region indicated by 'o' for Theorem 1.

By considering different values of a and b in the IT2 polynomial fuzzy model, the influence to feasible regions given by the proposed stability conditions are studied and compared. A larger size of feasible region implies less conservative stability conditions.

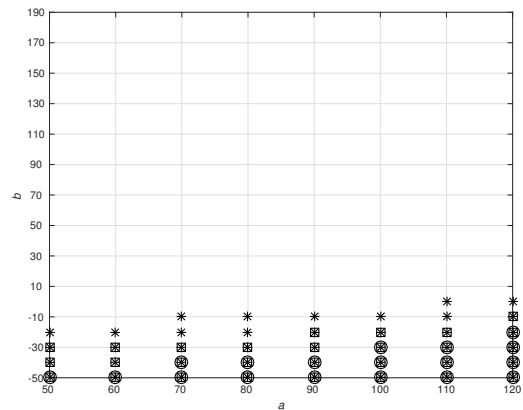


Fig. 2: Feasible region for Theorem 2 with 1 sub-domain, 4 sub-domains and 10 sub-domains. 'o' represents the feasible region for 1 sub-domain; '□' represents the feasible region for 4 sub-domains while '*' represents the feasible region for 10 sub-domains.

We choose the region $50 < a < 120$ and $-50 < b < 190$ with both at the interval of 10, and $\mathbf{X}(\tilde{\mathbf{x}})$ as a constant matrix, and $\mathbf{N}_j(\mathbf{x})$ being a function of monomials $(1, x_1, x_1^2)$, and $\varepsilon_1(\tilde{\mathbf{x}}) = \varepsilon_2(\mathbf{x}) = 0.001$, the feasible region for Theorem 1 is shown in Figure 1. Then the whole operating domain is divided into several sub-domains to compare the feasible regions of Theorem 2 and Theorem 3 with that of Theorem 1. We consider three cases that $D = 1, 4$ and 10 . When $D = 1$, it means that the only sub-domain is the whole operating domain, i.e., $\Phi = \Phi_1 \in (-\infty, +\infty)$. When the operation domain is divided into 4 sub-domains, the sub-domain Φ_d is characterized as $\Phi_1 \in (-\infty - 5]$, $\Phi_2 \in (-5 0]$, $\Phi_3 \in (0 5]$ and $\Phi_4 \in (5 \infty)$, and when it is divided into 10 sub-domains, the sub-domain Φ_d is characterized as $\Phi_1 \in (-\infty - 8]$, $\Phi_2 \in (-8 - 6]$, $\Phi_3 \in (-6 - 4]$, $\Phi_4 \in (-4 - 2]$, $\Phi_5 \in (-2 0]$, $\Phi_6 \in (0 2]$, $\Phi_7 \in (2 4]$, $\Phi_8 \in (4 6]$, $\Phi_9 \in (6 8]$ and $\Phi_{10} \in (8 \infty)$. The boundary informations of the IT2 membership functions in each sub-domain are given in Tables 1 to 4. In this case, setting $\underline{h}_{ijd}(\mathbf{x}) = \underline{\gamma}_{ijd}$ and $\bar{h}_{ijd}(\mathbf{x}) = \bar{\sigma}_{ijd}$, choosing $\underline{\mathbf{R}}_{ijd}(\mathbf{x})$, $\bar{\mathbf{R}}_{ijd}(\mathbf{x})$, $\underline{\mathbf{S}}_{ijd}(\mathbf{x})$, $\bar{\mathbf{S}}_{ijd}(\mathbf{x})$, $\underline{\mathbf{U}}_{ijd}(\mathbf{x})$, $\bar{\mathbf{U}}_{ijd}(\mathbf{x})$, and $\mathbf{X}(\tilde{\mathbf{x}})$ as constant symmetric matrices, $\mathbf{N}_{jd}(\mathbf{x})$ and $\mathbf{N}_j(\mathbf{x})$ being the function of monomials $(1, x_1, x_1^2)$, and $\varepsilon_1(\tilde{\mathbf{x}}) = \varepsilon_2(\mathbf{x}) = 0.001$, the feasible region of Theorem 2 is shown in Figure 2 while that of Theorem 3 is shown in Figure 3 using the same settings.

Remark 4. The membership functions of the IT2 polynomial fuzzy controller are different from those of the IT2 polynomial fuzzy model in this example. Since there are no entries of the row of the input matrices $\mathbf{B}_i(\mathbf{x})$ are zero for all i , $\tilde{\mathbf{x}}$ is chosen to be constant matrix. The predefined scalars $\underline{\gamma}_{ijd}$, $\bar{\gamma}_{ijd}$, $\underline{\sigma}_{ijd}$ and $\bar{\sigma}_{ijd}$, can be computed numerically as the extrema of the upper and lower membership functions in the corresponding sub-domain.

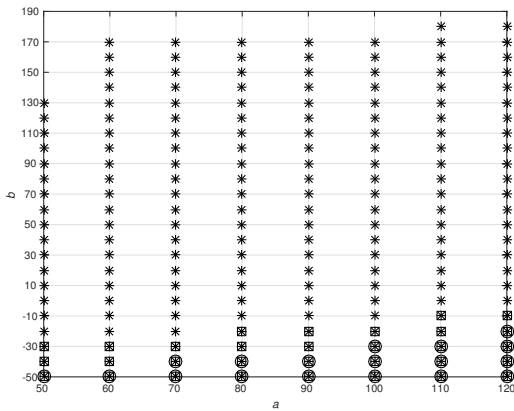


Fig. 3: Feasible region for Theorem 3 with 1 sub-domain, 4 sub-domains and 10 sub-domains. ‘o’ represents the feasible region for 1 sub-domain; ‘□’ represents the feasible region for 4 sub-domains while ‘*’ represents the feasible region for 10 sub-domains.

For Theorem 1, there are 18 feasible points shown in Figure 1, which means that the nonlinear system represented by the IT2 polynomial fuzzy model can be stabilized at the particular a and b accordingly. As for Theorem 2, referring to Figure 2, 18 feasible points for 1 sub-domain, 29 feasible points for 4 sub-domains and 40 feasible points for 10 sub-domains are obtained. For Theorem 3, referring to Figure 3, 18 feasible points are obtained for 1 sub-domain, 31 feasible points for 4 sub-domains and 182 feasible points for 10 sub-domains. Hence, it can be concluded that both Theorem 2 and Theorem 3 could effectively relax conservativeness for the IT2 PFMB control systems. Comparing with Theorem 2, Theorem 3 gets more relaxed results, i.e., the feasible regions for both 4 and 10 sub-domains are larger than that of Theorem 2 when the same boundary informations of the IT2 membership functions are applied.

To verify the analysis results, the phase plots of system states are shown in Figures 4 to 7. For simulation purposes, the IT2 membership functions of the IT2 polynomial fuzzy model are chosen as follows:

$$\tilde{w}_1(x_1) = \frac{\sin(5x_1) + 1}{2}w_1(x_1) + (1 - \frac{\sin(5x_1) + 1}{2})\bar{w}_1(x_1), \quad (42)$$

$$\tilde{w}_2(x_1) = \frac{\cos(5x_1) + 1}{2}w_2(x_1) + (1 - \frac{\cos(5x_1) + 1}{2})\bar{w}_2(x_1), \quad (43)$$

$$\tilde{w}_3(x_1) = 1 - \tilde{w}_1(x_1) - \tilde{w}_2(x_1). \quad (44)$$

When system state x_1 changes, the membership functions of the IT2 polynomial fuzzy model change, which represent the occurrence of uncertainties.

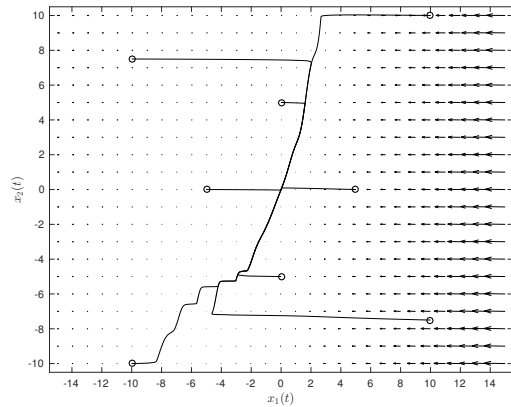


Fig. 4: Phase plot of $x_1(t)$ and $x_2(t)$ for Theorem 2 of 4 sub-domains with $a = 120$ and $b = -10$ where the initial conditions are indicated by ‘o’.

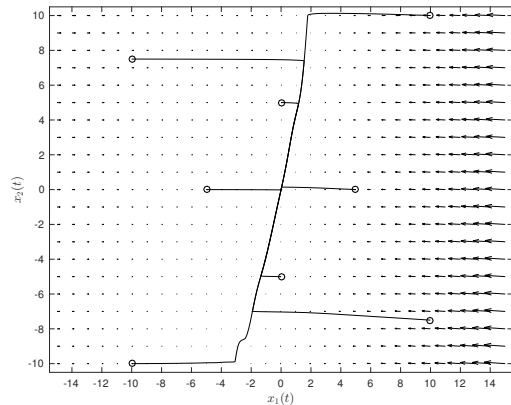


Fig. 5: Phase plot of $x_1(t)$ and $x_2(t)$ for Theorem 2 of 10 sub-domains with $a = 120$ and $b = 0$ where the initial conditions are indicated by ‘o’.

The membership functions of the IT2 polynomial fuzzy controller (IT2 switching polynomial fuzzy controller) are chosen as follows:

$$\tilde{m}_1(x_1) = 0.5m_1(x_1) + 0.5\bar{m}_1(x_1), \quad (45)$$

$$\tilde{m}_2(x_1) = 1 - \tilde{m}_1(x_1). \quad (46)$$

For Theorem 2 with 4 sub-domains, considering $a = 120$ and $b = -10$, the phase plot is shown in Figure 4. The polynomial feedback gains obtained by the SOSTOOLS are as follows:

$$\mathbf{X}(\tilde{\mathbf{x}}) = \begin{bmatrix} 2.2436 \times 10^{-2} & 3.8575 \times 10^{-5} \\ 3.8575 \times 10^{-5} & 4.1107 \times 10^{-4} \end{bmatrix},$$

$$\mathbf{G}_1(x_1) = [-2.6858 \times 10^{-2} x_1^2 + 1.6926 \times 10^{-1} x_1 - 8.5324$$

$$1.0701 \times 10^{-2} x_1^2 - 1.4411 \times 10^{-1} x_1 + 4.8947],$$

$$\mathbf{G}_2(x_1) = [-1.5708 \times 10^{-2} x_1^2 + 1.7171 \times 10^{-1} x_1 - 5.0412$$

$$9.6126 \times 10^{-3} x_1^2 - 1.2095 \times 10^{-1} x_1 + 2.6608].$$

For Theorem 2 with 10 sub-domains, considering $a = 120$ and $b = 0$, the phase plot is shown in Figure 5. The polynomial feedback gains obtained by the SOSTOOLS are as follows:

$$\mathbf{X}(\tilde{\mathbf{x}}) = \begin{bmatrix} 1.9492 \times 10^{-2} & 4.4871 \times 10^{-4} \\ 4.4871 \times 10^{-4} & 1.8900 \times 10^{-3} \end{bmatrix},$$

$$\mathbf{G}_1(x_1) = [-1.0534 \times 10^{-1} x_1^2 + 8.9972 \times 10^{-1} x_1 - 22.1197$$

$$4.6667 \times 10^{-2} x_1^2 - 3.4870 \times 10^{-1} x_1 + 6.3253],$$

$$\mathbf{G}_2(x_1) = [-9.2458 \times 10^{-2} x_1^2 + 8.9192 \times 10^{-2} x_1 - 6.0191$$

$$4.2294 \times 10^{-2} x_1^2 - 7.4132 \times 10^{-2} x_1 + 2.1956].$$

For Theorem 3 with 4 sub-domains, considering $a = 120$ and $b = -10$, the phase plot is shown in Figure 6. The polynomial feedback gains obtained are as follows:

$$\mathbf{X}(\tilde{\mathbf{x}}) = \begin{bmatrix} 141.6031 & 3.1442 \times 10^{-2} \\ 3.1442 \times 10^{-2} & 4.0543 \end{bmatrix},$$

$$\mathbf{G}_{11}(x_1) = [-2.0176 \times 10^{-3} x_1^2 + 1.5396 \times 10^{-1} x_1 - 5.5184$$

$$6.8819 \times 10^{-4} x_1^2 + 4.4443 \times 10^{-2} x_1 - 5.0320],$$

$$\mathbf{G}_{12}(x_1) = [-7.2808 \times 10^{-3} x_1^2 + 3.8638 \times 10^{-2} x_1 - 7.2837$$

$$7.3510 \times 10^{-4} x_1^2 - 1.2434 \times 10^{-2} x_1 + 2.6260],$$

$$\mathbf{G}_{13}(x_1) = [-1.6811 \times 10^{-3} x_1^2 - 3.5488 \times 10^{-2} x_1 - 2.7203$$

$$6.7805 \times 10^{-4} x_1^2 - 1.3293 \times 10^{-3} x_1$$

$$+ 9.1584 \times 10^{-1}],$$

$$\mathbf{G}_{14}(x_1) = [-1.5990 \times 10^{-3} x_1^2 - 5.6217 \times 10^{-3} x_1 + 6.9196$$

$$6.8993 \times 10^{-4} x_1^2 - 2.2696 \times 10^{-4} x_1 + 23.8998],$$

$$\mathbf{G}_{21}(x_1) = [-1.6141 \times 10^{-3} x_1^2 - 1.7223 \times 10^{-3} x_1 + 4.2924$$

$$6.8991 \times 10^{-4} x_1^2 + 1.6437 \times 10^{-3} x_1 - 8.9887],$$

$$\mathbf{G}_{22}(x_1) = [-1.1289 \times 10^{-3} x_1^2 + 1.1062 \times 10^{-1} x_1 - 4.9271$$

$$6.8299 \times 10^{-4} x_1^2 - 2.0180 \times 10^{-2} x_1 + 1.4977],$$

$$\mathbf{G}_{23}(x_1) = [-1.6923 \times 10^{-3} x_1^2 - 3.5024 \times 10^{-2} x_1 - 2.8049$$

$$6.7646 \times 10^{-4} x_1^2 - 1.3881 \times 10^{-3} x_1$$

$$+ 9.6922 \times 10^{-1}],$$

$$\mathbf{G}_{24}(x_1) = [-1.5991 \times 10^{-3} x_1^2 - 8.1618 \times 10^{-3} x_1 + 5.2409$$

$$6.8987 \times 10^{-4} x_1^2 - 1.6830 \times 10^{-4} x_1 + 16.3137].$$

For Theorem 3 with 10 sub-domains, considering $a = 120$ and $b = 180$, the phase plot is shown in Figure 7. The polynomial feedback gains obtained are as follows:

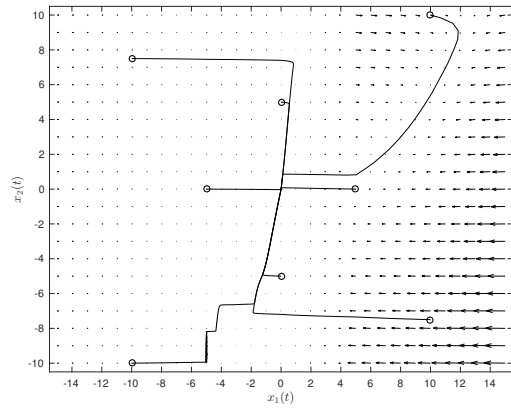


Fig. 6: Phase plot of $x_1(t)$ and $x_2(t)$ for Theorem 3 of 4 sub-domains with $a = 120$ and $b = -10$ where the initial conditions are indicated by 'o'.

$$\mathbf{X}(\tilde{\mathbf{x}}) = \begin{bmatrix} 3.7535 \times 10^{-2} & -2.8585 \times 10^{-4} \\ -2.8585 \times 10^{-4} & 1.3575 \times 10^{-3} \end{bmatrix},$$

$$\mathbf{G}_{11}(x_1) = [2.1481 \times 10^{-3} x_1^2 + 2.0362 \times 10^{-3} x_1 - 1.9462$$

$$2.2390 \times 10^{-3} x_1^2 + 1.4916 \times 10^{-1} x_1 - 7.3793],$$

$$\mathbf{G}_{12}(x_1) = [1.4876 \times 10^{-3} x_1^2 + 2.8200 \times 10^{-3} x_1 - 2.1921$$

$$2.0196 \times 10^{-3} x_1^2 + 4.2479 \times 10^{-2} x_1 - 6.6943],$$

$$\mathbf{G}_{13}(x_1) = [1.9723 \times 10^{-4} x_1^2 + 1.2854 \times 10^{-2} x_1 - 3.7798$$

$$1.5870 \times 10^{-3} x_1^2 - 3.7617 \times 10^{-2} x_1 - 5.5500],$$

$$\mathbf{G}_{14}(x_1) = [8.3476 \times 10^{-4} x_1^2 + 8.4558 \times 10^{-2} x_1 - 6.0365$$

$$2.1028 \times 10^{-3} x_1^2 - 6.9477 \times 10^{-2} x_1 - 6.6323],$$

$$\mathbf{G}_{15}(x_1) = [-2.2271 \times 10^{-5} x_1^2 + 3.5170 \times 10^{-3} x_1$$

$$- 6.7683 \times 10^{-1}$$

$$1.4918 \times 10^{-3} x_1^2 - 1.1906 \times 10^{-2} x_1$$

$$+ 6.8471 \times 10^{-2}],$$

$$\mathbf{G}_{16}(x_1) = [5.6102 \times 10^{-4} x_1^2 - 2.5437 \times 10^{-3} x_1$$

$$- 6.0088 \times 10^{-1}$$

$$1.6146 \times 10^{-3} x_1^2 - 4.9862 \times 10^{-3} x_1$$

$$+ 1.2252 \times 10^{-1}],$$

$$\mathbf{G}_{17}(x_1) = [1.2395 \times 10^{-3} x_1^2 - 1.1151 \times 10^{-3} x_1$$

$$- 5.1500 \times 10^{-1}$$

$$1.8635 \times 10^{-3} x_1^2 - 2.6714 \times 10^{-3} x_1$$

$$+ 1.8443 \times 10^{-1}],$$

$$\mathbf{G}_{18}(x_1) = [2.1142 \times 10^{-3} x_1^2 + 1.1398 \times 10^{-3} x_1$$

$$- 6.2624 \times 10^{-1}$$

$$2.2136 \times 10^{-3} x_1^2 - 9.9151 \times 10^{-4} x_1$$

$$+ 9.8295 \times 10^{-2}],$$

$$\mathbf{G}_{19}(x_1) = [2.4878 \times 10^{-3} x_1^2 + 1.7665 \times 10^{-3} x_1$$

$$- 6.5848 \times 10^{-1}$$

$$2.3735 \times 10^{-3} x_1^2 - 2.3318 \times 10^{-3} x_1$$

$$\begin{aligned}
& + 7.3570 \times 10^{-2}], \\
\mathbf{G}_{1,10}(x_1) = & [2.4877 \times 10^{-3}x_1^2 + 1.7669 \times 10^{-3}x_1 \\
& - 6.6255 \times 10^{-1} \\
& 2.3735 \times 10^{-3}x_1^2 - 2.3161 \times 10^{-3}x_1 \\
& + 7.2495 \times 10^{-2}], \\
\mathbf{G}_{21}(x_1) = & [2.4877 \times 10^{-3}x_1^2 + 1.7666 \times 10^{-3}x_1 \\
& - 6.4268 \times 10^{-1} \\
& 2.3735 \times 10^{-3}x_1^2 - 2.3291 \times 10^{-3}x_1 \\
& + 2.0589 \times 10^{-1}], \\
\mathbf{G}_{22}(x_1) = & [2.4878 \times 10^{-3}x_1^2 + 1.7660 \times 10^{-3}x_1 \\
& - 6.1460 \times 10^{-1} \\
& 2.3735 \times 10^{-3}x_1^2 - 2.3183 \times 10^{-3}x_1 \\
& + 1.7583 \times 10^{-1}], \\
\mathbf{G}_{23}(x_1) = & [2.0439 \times 10^{-3}x_1^2 + 5.8325 \times 10^{-3}x_1 \\
& - 6.6658 \times 10^{-1} \\
& 2.2212 \times 10^{-3}x_1^2 - 3.5744 \times 10^{-2}x_1 - 1.2233], \\
\mathbf{G}_{24}(x_1) = & [-7.3297 \times 10^{-4}x_1^2 + 1.4221 \times 10^{-3}x_1 \\
& - 7.6416 \times 10^{-1} \\
& 1.3597 \times 10^{-3}x_1^2 - 3.5747 \times 10^{-3}x_1 \\
& + 1.3076 \times 10^{-1}], \\
\mathbf{G}_{25}(x_1) = & [5.2529 \times 10^{-4}x_1^2 + 8.5566 \times 10^{-4}x_1 \\
& - 6.6710 \times 10^{-1} \\
& 1.6518 \times 10^{-3}x_1^2 - 9.1199 \times 10^{-3}x_1 \\
& + 8.2080 \times 10^{-2}], \\
\mathbf{G}_{26}(x_1) = & [-1.3436 \times 10^{-4}x_1^2 - 2.8275 \times 10^{-3}x_1 \\
& - 6.1392 \times 10^{-1} \\
& 1.3259 \times 10^{-3}x_1^2 - 3.8814 \times 10^{-3}x_1 \\
& + 1.0074 \times 10^{-1}], \\
\mathbf{G}_{27}(x_1) = & [4.4389 \times 10^{-4}x_1^2 - 2.6004 \times 10^{-3}x_1 \\
& - 1.3704 \times 10^{-1} \\
& 1.5578 \times 10^{-3}x_1^2 - 2.2260 \times 10^{-3}x_1 \\
& + 3.0919 \times 10^{-1}], \\
\mathbf{G}_{28}(x_1) = & [1.3292 \times 10^{-3}x_1^2 - 4.8148 \times 10^{-4}x_1 \\
& - 4.2243 \times 10^{-1} \\
& 1.8901 \times 10^{-3}x_1^2 - 1.2047 \times 10^{-3}x_1 \\
& + 1.9331 \times 10^{-1}], \\
\mathbf{G}_{29}(x_1) = & [2.1913 \times 10^{-3}x_1^2 + 1.3110 \times 10^{-3}x_1 \\
& - 5.9804 \times 10^{-1} \\
& 2.2458 \times 10^{-3}x_1^2 - 3.3326 \times 10^{-4}x_1 \\
& + 1.2055 \times 10^{-1}], \\
\mathbf{G}_{2,10}(x_1) = & [2.4457 \times 10^{-3}x_1^2 + 1.7186 \times 10^{-3}x_1 \\
& - 6.1852 \times 10^{-1} \\
& 2.3555 \times 10^{-3}x_1^2 - 8.4341 \times 10^{-5}x_1
\end{aligned}$$

$$+ 1.1102 \times 10^{-1}].$$

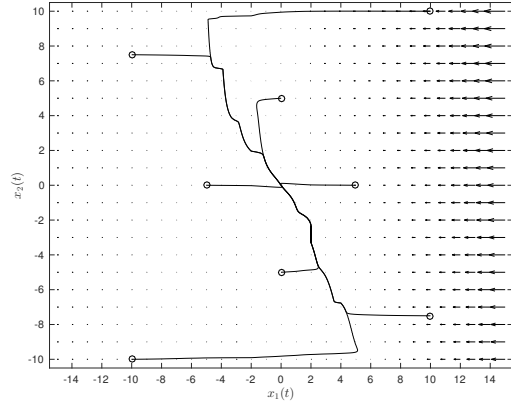


Fig. 7: Phase plot of $x_1(t)$ and $x_2(t)$ for Theorem 3 of 10 subdomains with $a = 120$ and $b = 180$ where the initial conditions are indicated by 'o'.

It can be seen from Figures 4 to 7 that the obtained IT2 polynomial fuzzy controllers are able to stabilize the nonlinear plant subject to uncertainty by driving its system states to the origin.

4.2 Inverted Pendulum

In this section, the stability of an inverted pendulum is investigated to verify the application of the proposed approaches. Consider an inverted pendulum subject to parameter uncertainty as the nonlinear plant. The control task is to apply Theorem 2 and Theorem 3 to find the proper feedback gains to stabilize the inverted pendulum. The dynamic equation for the inverted pendulum [28] is given by

$$\ddot{\theta} = \frac{g \sin(\theta(t)) - am_p S \dot{\theta}(t)^2 \sin(2\theta(t))/2 - a \cos(\theta(t))u(t)}{4S/3 - am_p S \cos^2(\theta(t))}, \quad (47)$$

where $\theta(t)$ is the angular displacement of the inverted pendulum, $g = 9.8 \text{ m/s}^2$, $m_p \in [m_{p_{\min}} \ m_{p_{\max}}] = [2 \ 3] \text{ kg}$ is the mass of the pendulum, $M_c \in [M_{c_{\min}} \ M_{c_{\max}}] = [8 \ 16] \text{ kg}$ is the mass of the cart, $a = \frac{1}{m_p + M_c}$, $2S = 1 \text{ m}$ is the length of the pendulum, and $u(t)$ is the force applied on the cart. In the investigation, m_p and M_c are treated as the parameter uncertainties.

The following 4-rule polynomial fuzzy model is adopted to describe the inverted pendulum:

$$\begin{aligned}
\text{Rule } i : & \text{If } f_1(\mathbf{x}) \text{ is } \tilde{M}_1^i \text{ and } f_2(\mathbf{x}) \text{ is } \tilde{M}_2^i \\
\text{Then } \dot{\mathbf{x}}(t) = & \mathbf{A}_i(\mathbf{x})\dot{\mathbf{x}}(\mathbf{x}) + \mathbf{B}_i(\mathbf{x})\mathbf{u}(t), \\
i = & 1, 2, 3, 4.
\end{aligned} \quad (48)$$

After combining all the fuzzy rules, we have:

$$\dot{\mathbf{x}} = \sum_{i=1}^4 \tilde{w}_i (\mathbf{A}_i(\mathbf{x})\dot{\mathbf{x}}(\mathbf{x}) + \mathbf{B}_i(\mathbf{x})\mathbf{u}(t)), \quad (49)$$

where

$$\begin{aligned}
\dot{\mathbf{x}} = \mathbf{x} = & [x_1 \ x_2]^T = [\theta(t) \ \dot{\theta}(t)]^T, \\
x_1 = & \left[\frac{-5\pi}{12} \ \frac{5\pi}{12} \right], \quad x_2 = [-5 \ 5],
\end{aligned}$$

Table 5 Lower and Upper Membership Functions for the Interval Type-2 Fuzzy Model of the Inverted Pendulum.

Lower and upper membership functions	
$\underline{\mu}_{\tilde{M}_1^1}(f_1(\mathbf{x})) = \underline{\mu}_{\tilde{M}_2^1}(f_1(\mathbf{x}))$	$\underline{\mu}_{\tilde{M}_1^1}(f_2(\mathbf{x})) = \underline{\mu}_{\tilde{M}_3^1}(f_2(\mathbf{x}))$
$= \frac{f_{1\max} - f_1(\mathbf{x})}{f_{1\max} - f_{1\min}}$	$= \frac{f_{2\max} - f_2(\mathbf{x})}{f_{2\max} - f_{2\min}}$
$\bar{\mu}_{\tilde{M}_1^3}(f_1(\mathbf{x})) = \bar{\mu}_{\tilde{M}_1^4}(f_1(\mathbf{x}))$	$\bar{\mu}_{\tilde{M}_1^1}(f_2(\mathbf{x})) = \bar{\mu}_{\tilde{M}_1^4}(f_2(\mathbf{x}))$
$= \frac{f_1(\mathbf{x}) - f_{1\min}}{f_{1\max} - f_{1\min}}$	$= \frac{f_2(\mathbf{x}) - f_{2\min}}{f_{2\max} - f_{2\min}}$
with $x_2(t) = 0, m_p = m_{p\max} = 3\text{kg}$ and $M_c = M_{c\min} = 8\text{kg}$	with $m_p = m_{p\max} = 3\text{kg}$ and $M_c = M_{c\max} = 16\text{kg}$
$\underline{\mu}_{\tilde{M}_1^1}(f_1(\mathbf{x})) = \underline{\mu}_{\tilde{M}_2^2}(f_1(\mathbf{x}))$	$\underline{\mu}_{\tilde{M}_1^1}(f_2(\mathbf{x})) = \underline{\mu}_{\tilde{M}_3^2}(f_2(\mathbf{x}))$
$= \frac{f_{1\max} - f_1(\mathbf{x})}{f_{1\max} - f_{1\min}}$	$= \frac{f_{2\max} - f_2(\mathbf{x})}{f_{2\max} - f_{2\min}}$
$\underline{\mu}_{\tilde{M}_1^3}(f_1(\mathbf{x})) = \underline{\mu}_{\tilde{M}_1^4}(f_2(\mathbf{x}))$	$\underline{\mu}_{\tilde{M}_1^2}(f_2(\mathbf{x})) = \underline{\mu}_{\tilde{M}_1^4}(f_2(\mathbf{x}))$
$= \frac{f_2(\mathbf{x}) - f_{2\min}}{f_{2\max} - f_{2\min}}$	$= \frac{f_2(\mathbf{x}) - f_{2\min}}{f_{2\max} - f_{2\min}}$
with $x_2 = x_{2\max}, m_p = m_{p\max} = 3\text{kg}$ and $M_c = M_{c\min} = 8\text{kg}$	with $m_p = m_{p\min} = 2\text{kg}$ and $M_c = M_{c\min} = 8\text{kg}$

Through a Taylor series based approach [41], the minimum and maximum values of $f_1(\mathbf{x})$ and $f_2(\mathbf{x})$ are as follows:

$$\begin{aligned} f_{1\min} &= -1.8932x_1^2 + 12.0513, \\ f_{1\max} &= -4.3666x_1^2 + 18.4800, \\ f_{2\min} &= -0.0388x_1^4 + 0.1194x_1^2 - 0.1765, \\ f_{2\max} &= -0.0097x_1^4 + 0.0568x_1^2 - 0.0895. \end{aligned}$$

The lower and upper grades of membership are respectively defined as:

$$\begin{aligned} w_i^L(\mathbf{x}) &= \underline{\mu}_{\tilde{M}_1^i}(\mathbf{x}) \times \underline{\mu}_{\tilde{M}_2^i}(\mathbf{x}), \\ w_i^U(\mathbf{x}) &= \bar{\mu}_{\tilde{M}_1^i}(\mathbf{x}) \times \bar{\mu}_{\tilde{M}_2^i}(\mathbf{x}) \end{aligned}$$

for all i .

According to the IT2 PFMB fuzzy model, a two-rule IT2 polynomial fuzzy controller is adopted to stabilize the inverted pendulum.

The following two-rule IT2 polynomial fuzzy controller is adopted to describe the inverted pendulum:

$$\begin{aligned} \text{Rule } j : \text{ If } x_1 \text{ is } \tilde{N}^j \\ \text{Then } u(t) = \mathbf{G}_j \mathbf{x}, \quad j = 1, 2. \end{aligned} \quad (50)$$

After combining of all the fuzzy rules, we have

$$u(t) = \tilde{m}_1(x_1)\mathbf{G}_1\mathbf{x} + \tilde{m}_2(x_1)\mathbf{G}_2\mathbf{x}. \quad (51)$$

where $\tilde{m}_1(x_1)$ and $\tilde{m}_2(x_1)$ are the IT2 membership functions of the polynomial fuzzy controller.

The upper and lower bounds of the membership functions of the fuzzy controller are as follows:

$$\begin{aligned} \bar{m}_1(x_1) &= \begin{cases} 0 & \text{for } x_1 < -\frac{5\pi}{12} \\ \frac{x_1 + 5\pi/12}{5\pi/12} & \text{for } -\frac{5\pi}{12} \leq x_1 \leq 0 \\ \frac{5\pi/12 - x_1}{5\pi/12} & \text{for } 0 \leq x_1 \leq \frac{5\pi}{12} \\ 0 & \text{for } x_1 > \frac{5\pi}{12} \end{cases} \\ m_1(x_1) &= \begin{cases} 0 & \text{for } x_1 < -\frac{5\pi}{12} \\ \frac{0.9(x_1 + 5\pi/12)}{5\pi/12} & \text{for } -\frac{5\pi}{12} \leq x_1 \leq 0 \\ \frac{0.9(5\pi/12 - x_1)}{5\pi/12} & \text{for } 0 \leq x_1 \leq \frac{5\pi}{12} \\ 0 & \text{for } x_1 > \frac{5\pi}{12} \end{cases} \end{aligned}$$

$\bar{m}_2(x_1) = 1 - m_1(x_1)$, $m_2(x_1) = 1 - \bar{m}_1(x_1)$ and $\tilde{m}_1(x_1) + \tilde{m}_2(x_1) = 1$. We choose $\lambda_j(x_1) = \bar{\lambda}_j(x_1) = 0.5, j = 1, 2$.

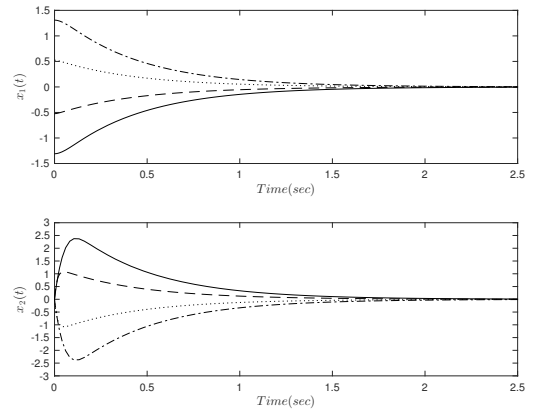


Fig. 8: The top figure is the time responses of $x_1(t)$ and the bottom one is the time responses of $x_2(t)$ for Theorem 2.

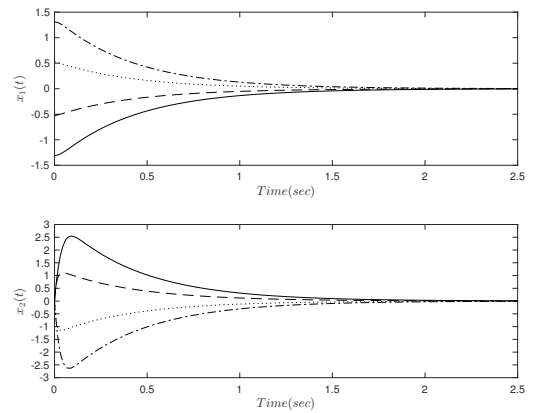


Fig. 9: The top figure is the time responses of $x_1(t)$ and the bottom one is the time responses of $x_2(t)$ for Theorem 3.

During the simulation, for demonstration purposes, we set $m_p = 2.5\text{kg}$ and $m_c = 10\text{kg}$. Based on Theorem 2, we set $\varepsilon_1(\tilde{\mathbf{x}}) = \varepsilon_2(\tilde{\mathbf{x}}) = 0.01$, $\mathbf{X}(\tilde{\mathbf{x}})$ and $\mathbf{N}_j(\mathbf{x})$ as a constant matrix. The operating domain of x_1 is divided into 3 sub-domains, which are characterized as $\Phi_1 = [-\frac{5\pi}{12} -\frac{5\pi}{36})$, $\Phi_2 = [-\frac{5\pi}{36} \frac{5\pi}{36})$ and $\Phi_3 = [\frac{5\pi}{36} \frac{5\pi}{12}]$. The boundary information of the IT2 membership functions are given in Tables 6. The feedback gains are achieved as $\mathbf{G}_1 = [1238.8258 \ 505.4903]$, $\mathbf{G}_2 = [1816.1366 \ 741.3546]$, and $\mathbf{X} = \begin{bmatrix} 0.1586 & -0.3886 \\ -0.3886 & 1.0260 \end{bmatrix}$. The state response for Theorem 2 is shown in Figure 8, where the solid lines are under the initial condition $\mathbf{x}(0) = [-\frac{5\pi}{12} \ 0]$, the dashed lines are under the initial condition $\mathbf{x}(0) = [\frac{5\pi}{6} \ 0]$, the dotted lines are under the initial condition $\mathbf{x}(0) = [-\frac{\pi}{6} \ 0]$ and the dash-dot lines are under the initial condition $\mathbf{x}(0) = [\frac{5\pi}{12} \ 0]$.

Based on Theorem 3, we set $\varepsilon_1(\tilde{\mathbf{x}}) = \varepsilon_2(\tilde{\mathbf{x}}) = 0.01$, $\mathbf{X}(\tilde{\mathbf{x}})$ and $\mathbf{N}_j(\mathbf{x})$ as a constant matrix. The operation domain of x_1 is divided into 2 sub-domains, which are characterized as $\Phi_1 = [-\frac{5\pi}{12} \ 0)$, $\Phi_2 = [0 \ \frac{5\pi}{12}]$. The boundary information of the IT2 membership functions are given in Table 7. The feedback gains are achieved as $\mathbf{G}_{11} = [1244.8920 \ 502.9501]$, $\mathbf{G}_{12} = [1850.2415 \ 747.5784]$, $\mathbf{G}_{21} = [2482.0312 \ 1002.8473]$, $\mathbf{G}_{22} = [3004.2767 \ 1213.8351]$, and $\mathbf{X} = \begin{bmatrix} 0.1910 & -0.4726 \\ -0.4726 & 1.2440 \end{bmatrix}$. The state response for Theorem 3 is shown in Figure 9, where the solid lines are under the initial condition $\mathbf{x}(0) = [-\frac{5\pi}{12} \ 0]$, the dashed lines are under the initial condition $\mathbf{x}(0) = [-\frac{\pi}{6} \ 0]$, the dotted lines are under the initial condition $\mathbf{x}(0) = [\frac{\pi}{6} \ 0]$ and the dash-dot lines are under the initial condition $\mathbf{x}(0) = [\frac{5\pi}{12} \ 0]$.

It can be seen from Figs. 8 and 9 that both IT2 polynomial fuzzy controller and IT2 switching polynomial fuzzy controller designed

Table 6 Boundary informations of IT2 membership functions of inverted pendulum for 3 sub-domains.

$\underline{\gamma}_{ij d} = \underline{h}_{ij d}(\mathbf{x})$	$\bar{\gamma}_{ij d}$	$\underline{\sigma}_{ij d}$	$\bar{h}_{ij d}(\mathbf{x}) = \bar{\sigma}_{ij d}$
$\underline{\gamma}_{111} = 0.0000$	$\bar{\gamma}_{111} = 2.1150 \times 10^{-3}$	$\underline{\sigma}_{111} = 0.0000$	$\bar{\sigma}_{111} = 5.9439 \times 10^{-1}$
$\underline{\gamma}_{121} = 2.6562 \times 10^{-8}$	$\bar{\gamma}_{121} = 6.5540 \times 10^{-3}$	$\underline{\sigma}_{121} = 1.2874 \times 10^{-2}$	$\bar{\sigma}_{121} = 5.5969 \times 10^{-1}$
$\underline{\gamma}_{211} = 0.0000$	$\bar{\gamma}_{211} = 1.2729 \times 10^{-2}$	$\underline{\sigma}_{211} = 0.0000$	$\bar{\sigma}_{211} = 5.9362 \times 10^{-1}$
$\underline{\gamma}_{221} = 0.0000$	$\bar{\gamma}_{221} = 7.7726 \times 10^{-2}$	$\underline{\sigma}_{221} = 3.3282 \times 10^{-1}$	$\bar{\sigma}_{221} = 6.4705 \times 10^{-1}$
$\underline{\gamma}_{311} = 0.0000$	$\bar{\gamma}_{311} = 1.1072 \times 10^{-3}$	$\underline{\sigma}_{311} = 0.0000$	$\bar{\sigma}_{311} = 5.4292 \times 10^{-1}$
$\underline{\gamma}_{321} = 0.0000$	$\bar{\gamma}_{321} = 6.6161 \times 10^{-3}$	$\underline{\sigma}_{321} = 2.8483 \times 10^{-1}$	$\bar{\sigma}_{321} = 4.6539 \times 10^{-1}$
$\underline{\gamma}_{411} = 0.0000$	$\bar{\gamma}_{411} = 1.4038 \times 10^{-2}$	$\underline{\sigma}_{411} = 0.0000$	$\bar{\sigma}_{211} = 5.4222 \times 10^{-1}$
$\underline{\gamma}_{421} = 3.1445 \times 10^{-6}$	$\bar{\gamma}_{421} = 3.9192 \times 10^{-1}$	$\underline{\sigma}_{421} = 3.0400 \times 10^{-1}$	$\bar{\sigma}_{421} = 1.0000$
$\underline{\gamma}_{112} = 3.2541 \times 10^{-10}$	$\bar{\gamma}_{112} = 1.7981 \times 10^{-4}$	$\underline{\sigma}_{112} = 5.9439 \times 10^{-1}$	$\bar{\sigma}_{112} = 8.9899 \times 10^{-1}$
$\underline{\gamma}_{122} = 3.2790 \times 10^{-11}$	$\bar{\gamma}_{122} = 1.0871 \times 10^{-4}$	$\underline{\sigma}_{122} = 7.6601 \times 10^{-4}$	$\bar{\sigma}_{122} = 3.3325 \times 10^{-1}$
$\underline{\gamma}_{212} = 1.8321 \times 10^{-10}$	$\bar{\gamma}_{212} = 1.0178 \times 10^{-4}$	$\underline{\sigma}_{212} = 5.9362 \times 10^{-1}$	$\bar{\sigma}_{212} = 8.9875 \times 10^{-1}$
$\underline{\gamma}_{222} = 1.8462 \times 10^{-11}$	$\bar{\gamma}_{222} = 6.1536 \times 10^{-5}$	$\underline{\sigma}_{222} = 7.6581 \times 10^{-4}$	$\bar{\sigma}_{222} = 3.3282 \times 10^{-1}$
$\underline{\gamma}_{312} = 1.1803 \times 10^{-8}$	$\bar{\gamma}_{312} = 9.1882 \times 10^{-6}$	$\underline{\sigma}_{312} = 5.4292 \times 10^{-1}$	$\bar{\sigma}_{312} = 7.6602 \times 10^{-4}$
$\underline{\gamma}_{322} = 1.2092 \times 10^{-9}$	$\bar{\gamma}_{322} = 5.5551 \times 10^{-6}$	$\underline{\sigma}_{322} = 7.6602 \times 10^{-4}$	$\bar{\sigma}_{322} = 3.0439 \times 10^{-1}$
$\underline{\gamma}_{412} = 6.6407 \times 10^{-9}$	$\bar{\gamma}_{412} = 5.2009 \times 10^{-6}$	$\underline{\sigma}_{412} = 5.4222 \times 10^{-1}$	$\bar{\sigma}_{412} = 8.9877 \times 10^{-1}$
$\underline{\gamma}_{422} = 6.8078 \times 10^{-10}$	$\bar{\gamma}_{422} = 3.1445 \times 10^{-6}$	$\underline{\sigma}_{422} = 7.6582 \times 10^{-4}$	$\bar{\sigma}_{422} = 3.0400 \times 10^{-1}$
$\underline{\gamma}_{113} = 1.1470 \times 10^{-7}$	$\bar{\gamma}_{113} = 2.1148 \times 10^{-3}$	$\underline{\sigma}_{113} = 7.7904 \times 10^{-4}$	$\bar{\sigma}_{113} = 6.0013 \times 10^{-1}$
$\underline{\gamma}_{123} = 1.8678 \times 10^{-5}$	$\bar{\gamma}_{123} = 6.5551 \times 10^{-3}$	$\underline{\sigma}_{123} = 1.4088 \times 10^{-1}$	$\bar{\sigma}_{123} = 5.5968 \times 10^{-1}$
$\underline{\gamma}_{213} = 7.2990 \times 10^{-5}$	$\bar{\gamma}_{213} = 1.2728 \times 10^{-2}$	$\underline{\sigma}_{213} = 2.6024 \times 10^{-3}$	$\bar{\sigma}_{213} = 5.9937 \times 10^{-1}$
$\underline{\gamma}_{223} = 5.3775 \times 10^{-5}$	$\bar{\gamma}_{223} = 7.7733 \times 10^{-2}$	$\underline{\sigma}_{223} = 3.2692 \times 10^{-1}$	$\bar{\sigma}_{223} = 6.4703 \times 10^{-1}$
$\underline{\gamma}_{313} = 3.5381 \times 10^{-6}$	$\bar{\gamma}_{313} = 1.1069 \times 10^{-3}$	$\underline{\sigma}_{313} = 1.6155 \times 10^{-3}$	$\bar{\sigma}_{313} = 5.4971 \times 10^{-1}$
$\underline{\gamma}_{323} = 4.8189 \times 10^{-6}$	$\bar{\gamma}_{323} = 6.6129 \times 10^{-3}$	$\underline{\sigma}_{323} = 2.9213 \times 10^{-1}$	$\bar{\sigma}_{323} = 4.6539 \times 10^{-1}$
$\underline{\gamma}_{413} = 4.4646 \times 10^{-6}$	$\bar{\gamma}_{413} = 1.4032 \times 10^{-2}$	$\underline{\sigma}_{413} = 5.3965 \times 10^{-3}$	$\bar{\sigma}_{413} = 5.4900 \times 10^{-1}$
$\underline{\gamma}_{423} = 2.6381 \times 10^{-6}$	$\bar{\gamma}_{423} = 3.6668 \times 10^{-1}$	$\underline{\sigma}_{423} = 2.9945 \times 10^{-1}$	$\bar{\sigma}_{423} = 9.7587 \times 10^{-1}$

Table 7 Boundary informations of IT2 membership functions of inverted pendulum for 2 sub-domains.

$\underline{\gamma}_{ij d} = \underline{h}_{ij d}(\mathbf{x})$	$\bar{\gamma}_{ij d}$	$\underline{\sigma}_{ij d}$	$\bar{h}_{ij d}(\mathbf{x}) = \bar{\sigma}_{ij d}$
$\underline{\gamma}_{111} = 0.0000$	$\bar{\gamma}_{111} = 2.1150 \times 10^{-3}$	$\underline{\sigma}_{111} = 0.0000$	$\bar{\sigma}_{111} = 8.9349 \times 10^{-1}$
$\underline{\gamma}_{121} = 2.7820 \times 10^{-9}$	$\bar{\gamma}_{121} = 6.5540 \times 10^{-3}$	$\underline{\sigma}_{121} = 6.8707 \times 10^{-3}$	$\bar{\sigma}_{121} = 5.5969 \times 10^{-1}$
$\underline{\gamma}_{211} = 0.0000$	$\bar{\gamma}_{211} = 1.2729 \times 10^{-2}$	$\underline{\sigma}_{211} = 0.0000$	$\bar{\sigma}_{211} = 8.9326 \times 10^{-1}$
$\underline{\gamma}_{221} = 0.0000$	$\bar{\gamma}_{221} = 7.7726 \times 10^{-2}$	$\underline{\sigma}_{221} = 6.8689 \times 10^{-3}$	$\bar{\sigma}_{221} = 6.4705 \times 10^{-1}$
$\underline{\gamma}_{311} = 0.0000$	$\bar{\gamma}_{311} = 1.1072 \times 10^{-3}$	$\underline{\sigma}_{311} = 0.0000$	$\bar{\sigma}_{311} = 8.9347 \times 10^{-1}$
$\underline{\gamma}_{321} = 0.0000$	$\bar{\gamma}_{321} = 6.6161 \times 10^{-3}$	$\underline{\sigma}_{321} = 6.8705 \times 10^{-3}$	$\bar{\sigma}_{321} = 4.6539 \times 10^{-1}$
$\underline{\gamma}_{411} = 0.0000$	$\bar{\gamma}_{411} = 1.4038 \times 10^{-2}$	$\underline{\sigma}_{411} = 0.0000$	$\bar{\sigma}_{211} = 8.9324 \times 10^{-1}$
$\underline{\gamma}_{421} = 7.1798 \times 10^{-10}$	$\bar{\gamma}_{421} = 3.9192 \times 10^{-1}$	$\underline{\sigma}_{421} = 6.8687 \times 10^{-3}$	$\bar{\sigma}_{421} = 1.0000$
$\underline{\gamma}_{112} = 3.2541 \times 10^{-10}$	$\bar{\gamma}_{112} = 2.1148 \times 10^{-3}$	$\underline{\sigma}_{112} = 7.7904 \times 10^{-4}$	$\bar{\sigma}_{112} = 8.9899 \times 10^{-1}$
$\underline{\gamma}_{122} = 3.2790 \times 10^{-11}$	$\bar{\gamma}_{122} = 6.5551 \times 10^{-3}$	$\underline{\sigma}_{122} = 7.6601 \times 10^{-4}$	$\bar{\sigma}_{122} = 5.5968 \times 10^{-1}$
$\underline{\gamma}_{212} = 1.8321 \times 10^{-10}$	$\bar{\gamma}_{212} = 1.2728 \times 10^{-2}$	$\underline{\sigma}_{212} = 2.6024 \times 10^{-3}$	$\bar{\sigma}_{212} = 8.9875 \times 10^{-1}$
$\underline{\gamma}_{222} = 1.8462 \times 10^{-11}$	$\bar{\gamma}_{222} = 7.7733 \times 10^{-2}$	$\underline{\sigma}_{222} = 7.6581 \times 10^{-4}$	$\bar{\sigma}_{222} = 6.4703 \times 10^{-1}$
$\underline{\gamma}_{312} = 1.1803 \times 10^{-8}$	$\bar{\gamma}_{312} = 1.1069 \times 10^{-3}$	$\underline{\sigma}_{312} = 1.6155 \times 10^{-3}$	$\bar{\sigma}_{312} = 8.9901 \times 10^{-1}$
$\underline{\gamma}_{322} = 1.2092 \times 10^{-9}$	$\bar{\gamma}_{322} = 6.6129 \times 10^{-3}$	$\underline{\sigma}_{322} = 7.6602 \times 10^{-4}$	$\bar{\sigma}_{322} = 4.6539 \times 10^{-1}$
$\underline{\gamma}_{412} = 6.6407 \times 10^{-9}$	$\bar{\gamma}_{412} = 1.4032 \times 10^{-2}$	$\underline{\sigma}_{412} = 5.3965 \times 10^{-3}$	$\bar{\sigma}_{412} = 8.9877 \times 10^{-1}$
$\underline{\gamma}_{422} = 6.8078 \times 10^{-10}$	$\bar{\gamma}_{422} = 3.6668 \times 10^{-1}$	$\underline{\sigma}_{422} = 7.6582 \times 10^{-4}$	$\bar{\sigma}_{422} = 9.7587 \times 10^{-1}$

through Theorems 2 and 3, respectively, can stabilize the inverted pendulum. However, as illustrated in Section 4.1, Theorem 3 provide more relaxed stability conditions thanks to the switching control scheme.

5 Conclusion

In this paper, the stability analysis of IT2 PFMB control systems is considered and the stability conditions are obtained in terms of SOS. By applying IT2 fuzzy sets into the polynomial fuzzy model, uncertainties can be captured by the lower and upper membership functions. In addition, the boundary informations of both membership functions are considered in the stability analysis. With such information, slack matrices are introduced to relax conservativeness. Based on the Lyapunov stability theory, MFI, MFD and MFD switching stability conditions, in terms of SOS are derived to achieve a stable IT2 PFMB control system. Both of MFD and MFD switching stability conditions are more relaxed compared with MFI stability conditions, while the MFD switching stability conditions

are more effective than the MFD stability conditions. Numerical examples have been given and simulation results have been shown to verify the analysis results.

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