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Stability Analysis of Positive Polynomial Fuzzy-Model-Based Control Systems with Time Delay under Imperfect Premise Matching

Xiaomiao Li^{*,1}, Hak Keung Lam^{*,2}, Ge Song^{*,3}, Fucui Liu^{*,4}

Abstract—This paper deals with the stability and positivity analysis of polynomial-fuzzy-model-based (PFMB) control systems with time delay, which is formed by a polynomial fuzzy model and a polynomial fuzzy controller connected in a closed loop, under imperfect premise matching. To improve the design and realization flexibility, the polynomial fuzzy model and the polynomial fuzzy controller are allowed to have their own set of premise membership functions. A sum-of-squares (SOS)-based stability analysis approach using the Lyapunov stability theory is employed to investigate the positivity and stability of the PFMB control systems and synthesize the polynomial fuzzy controller. In order to relax the stability results, we propose two methods: 1) membership functions are considered as symbolic variables in the stability analysis, 2) the property of the membership functions and the boundary information of the membership functions are considered in the stability analysis. A simulation example is given to illustrate the effectiveness of the proposed approach.

Index Terms—Polynomial fuzzy-model-based (PFMB) control systems, Time Delay, Stability analysis, Mismatched premise membership functions, Sum of squares (SOS).

I. INTRODUCTION

IN many biological and engineered systems, the dynamics is always affected by time delay. Some work [1]–[5] have shown that time delay is the factor causes the instability and poor performance in nonlinear systems. In the real world, there exists a particular type of practical systems called positive systems whose states variables are confined to be positive (at least nonnegative) for any nonnegative initial condition [6]–[9]. For the stability issue of positive system with time delay, [10], [11] presented the stability conditions based on popular quadratic Lyapunov–Krasovskii functional (QLKF) [10], [11] for both continuous and discrete-time systems. However, the results are very conservative without considering the nature of positive systems. Given full consideration to the property of positive systems, more work applied the linear co-positive Lyapunov function (LCLF) [12]–[14] instead of the QLKF [10], [11] to investigate the stability analysis. The approach of LCLF needs to obey the following Lyapunov principles:

the stability of a positive system can be guaranteed if there is a function $V(t)$ that is positive definite and its derivative $\dot{V}(t)$ is negative definite when the states are in the positive orthant. Compared with QLKF, LCLF considers the nature of positivity in the stability analysis. Additionally, the stability conditions obtained based on LCLF are not affected by the values of time delays [13]–[17]. This motivates us to adopt the LCLF to investigate the stability analysis of nonlinear systems, and the results obtained are less conservative.

The Takagi-Sugeno (T-S) fuzzy model [5], [18]–[20] provides an effective and systematic way to represent a nonlinear system. As an important concept of fuzzy-model-based (FMB) control, the T-S fuzzy model which connects with the T-S fuzzy controller is considered as the most classic FMB control. It has been shown that if a set of linear-matrix-inequalities (LMIs) can be solved, the FMB control system is asymptotically stable [21]. This approach has been applied widely in a variety of nonlinear systems. Hence, the T-S FMB control system has been applied in some works [12]–[14] for stability issues due to its efficiency and maturity. The work [12]–[14] investigated the stability of continuous/discrete-time positive T-S fuzzy systems based on the LCLF. Whereas, the limitations among those papers are that they just considered the basic stability of closed-loop control systems, without the information from membership functions. Therefore the conditions derived were very conservative.

The parallel-distribution-compensation (PDC) design concept was proposed in [22], [23] that the fuzzy controller shares the same premise membership functions and the same number of rules as those of the T-S fuzzy model. The advantage of PDC is that stability analysis is done using the characteristics of membership functions and thus the stability conditions obtained are more relaxed. However, it constrains the design flexibility of fuzzy controller. When the membership functions of the fuzzy model are complicated and/or the number of fuzzy rules is large, the cost of controller implementation will be increased. In order to alleviate this issue, [24]–[28] proposed that the fuzzy model and the fuzzy controller do not need to share the same premise membership functions and/or number of rules. Compared with the PDC design, although it makes the stability analysis more difficult and lead to potentially conservative results, the design flexibility and robustness are improved. Inspired by these work, we investigate the stability of positive system with time delay under imperfect premise matching that the premise membership functions are not required to be the same for the fuzzy model and fuzzy

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controller. As pointed out in [24]–[28], the information from membership functions plays a key role in the stability analysis under imperfect premise matching. To the best knowledge of the authors, these information is capable of approaching the stability analysis more easily and avoiding the potentially conservative stability analysis result. However, in order to avoid the difficulty caused by imperfect premise matching, most current work [12]–[15], [29] applied PDC design concept to the positive systems with time delay.

Recently, some work [30]–[34] have extended the T-S fuzzy model to polynomial fuzzy model whose sub-systems are represented by polynomial variables. Compared with T-S fuzzy model, the polynomial fuzzy model possesses the capability of describing a wider class of nonlinear systems, because sub-systems are able to contain an enormous amount of information compared with those of T-S fuzzy model. However, LMI-based approach cannot investigate the stability analysis as the LMI solver cannot deal with polynomial terms. To deal with this issue, some work employed SOS-based approach to obtain the SOS-based stability conditions [24], [25] to guarantee the stability of the polynomial fuzzy-model-based (PFMB) control systems. In order to relax the stability analysis results, [25]–[27] employed the information or property from membership functions for stability analysis. [24], [28] considered the membership functions as symbolic variables to the stability conditions.

However, up to now, we can only find two literatures [16], [17] studying the problems of the stability and positivity of polynomial fuzzy-model-based (PFMB) control systems with time delay. The achievement in [16] obtained the basic open-loop stability conditions which is relaxed by considering the information from the membership functions. [17] extended the stability and positivity analysis from open-loop systems with time delay to the real PFMB control system with time delay. [17] investigated the stability and positivity analysis of PFMB control system with time delay under imperfect premise matching. For the case of imperfect premise matching in [17], the author setted that the fuzzy model and the fuzzy controller do not need to share the same premise membership functions or number of rules. Then, in order to deal with the difficulty of stability analysis and achieve potentially less conservative results from the mismatched premise membership functions, the authors proposed approximated membership functions representing the original ones. Through considering the relationship between approximated membership functions and the original ones, the information of membership functions was brought into the stability conditions, and the stability conditions were relaxed.

In this paper, we propose another method to deal with the difficulty resulting from the mismatched premise membership functions to the stability/stabilization analysis and relax the stability conditions by considering the membership function as symbolic variables in the stability analysis. **In order to obtain more relaxed stability conditions**, the property and the boundary information of membership function are considered and imposed on the stability analysis through some Positivstellensatz multipliers. To reduce the computational demand on finding a feasible solution, the number of symbolic variables

is reduced through using the relationship of membership functions between polynomial fuzzy model and polynomial fuzzy controller, i.e., use the membership functions of polynomial fuzzy controller to represent those of the polynomial fuzzy model. However, it is required that the number of rules of the premise membership functions for both the polynomial fuzzy model and the polynomial fuzzy controller are the same.

In view of the issues mentioned above, The main contributions of this paper lies in the following aspects 1) we tend to investigate the stability and positivity of PFMB control system with time delay under imperfect premise matching. In this way, the membership functions for the polynomial fuzzy controller can be allowed to choose freely to improve its design flexibility. 2) Inspired by [24], [28], we consider the membership functions as symbolic variables in the stability analysis which makes the stability analysis easy and allows the feedback gains to be a function of membership functions to improve its feedback compensation capability. Furthermore, the property and the boundary information of membership functions are considered and imposed on the stability analysis through some Positivstellensatz multipliers, resulting in relaxed stability conditions which are then verified by a simulation example.

II. NOTATIONS AND PRELIMINARIES

A. Notation

Throughout this paper, the following notations are adopted. A monomial in $\mathbf{x}(t) = [x_1(t), x_2(t), \dots, x_n(t)]$ is a function of the form $x_1^{d_1}(t), x_2^{d_2}(t), \dots, x_n^{d_n}(t)$, where d_i , $i \in \{1, 2, \dots, n\}$ is a nonnegative integer. The degree of a monomial is defined as $d = \sum_{i=1}^n d_i$. $\mathbf{p}(\mathbf{x}(t))$ is a polynomial if it can be expressed as a finite linear combination of monomials with real coefficients. $\mathbf{p}(\mathbf{x}(t)) = \sum_{j=1}^m \mathbf{q}_j(\mathbf{x}(t))^2$ indicates the polynomial $\mathbf{p}(\mathbf{x}(t))$ is an SOS implying $\mathbf{p}(\mathbf{x}(t)) \geq 0$ where $\mathbf{q}_j(\mathbf{x}(t))$ is a polynomial and m is a non-zero positive integer. $\mathbf{Z} \succ 0$, $\mathbf{Z} \succeq 0$, $\mathbf{Z} \prec 0$ and $\mathbf{Z} \preceq 0$ mean that all the elements of the matrix \mathbf{Z} are positive, semipositive, negative and seminegative, respectively. A matrix \mathbf{A} is called a Metzler matrix, if its off diagonal elements are all non-negative [15], [35]. It means if $\mathbf{A} = (a_{rs})$, $r, s = \{1, 2, \dots, n\}$ where a_{rs} denotes the (r, s) -th element of \mathbf{A} , \mathbf{A} is Metzler if $a_{rs} \geq 0$ for all $r \neq s$. \mathbf{R}^T stands for the transpose of a real matrix \mathbf{R} . $\underline{n} = \{1, 2, \dots, n\}$ where $n \in \mathbb{Z}^+$ denotes the order of fuzzy model. $\underline{p} = \{1, 2, \dots, p\}$ where $p \in \mathbb{Z}^+$ is the rule number of fuzzy model. We define the normalized grade of membership function of fuzzy model and fuzzy controller as $\mathbf{w}(\mathbf{x}(t)) = [w_1(\mathbf{x}(t)), w_2(\mathbf{x}(t)), \dots, w_i(\mathbf{x}(t))]$, $i \in \underline{p}$ and $\mathbf{m}(\mathbf{x}(t)) = [m_1(\mathbf{x}(t)), m_2(\mathbf{x}(t)), \dots, m_j(\mathbf{x}(t))]$, $j \in \underline{p}$ respectively. With the third-party MATLAB toolbox SOSTOOLS, SOS programmes can be solved efficiently [36].

B. Polynomial Fuzzy Models With Time Delay

The dynamical behaviour of the nonlinear plant can be represented by polynomial fuzzy model. We define the i^{th}

rule of the polynomial fuzzy model with time delay for the nonlinear plant as follows:

Rule $i, i \in \underline{p}$:

IF $f_1(\mathbf{x}(t))$ is M_1^i AND \dots AND $f_\Psi(\mathbf{x}(t))$ is M_Ψ^i

THEN $\dot{\mathbf{x}}(t) = \mathbf{A}_{i0}(\mathbf{x}(t))\mathbf{x}(t)$

$$+ \sum_{l=1}^d \mathbf{A}_{il}(\mathbf{x}(t - \tau_l))\mathbf{x}(t - \tau_l) + \mathbf{B}_i(\mathbf{x}(t))\mathbf{u}(t) \quad (1)$$

$$\mathbf{x}(t) = \phi(t), \quad t \in [-\tau, 0], \tau = \max_{l \in \{1, 2, \dots, d\}} \{\tau_l\} \quad (2)$$

where $M_l^i, l = \{1, 2, \dots, \Psi\}$ is the fuzzy set in rule i corresponding to premise variable $f_l(\mathbf{x}(t)), l = \{1, 2, \dots, \Psi\}$; Ψ is a positive integer; $\phi(t)$ is the vector valued initial function; $\mathbf{A}_{i0}(\mathbf{x}(t)) \in \mathbb{R}^{n \times n}$, $\mathbf{A}_{il}(\mathbf{x}(t - \tau_l)) \in \mathbb{R}^{n \times n}$ and $\mathbf{B}_i(\mathbf{x}(t)) \in \mathbb{R}^{n \times m}$ are polynomial system, time delay and input matrices respectively; $\mathbf{x}(t) \in \mathbb{R}^n$ and $\mathbf{u}(t) \in \mathbb{R}^m$ are state vector and control input vector respectively; $\tau_l, l \in \underline{d} = \{1, 2, \dots, d\}$, is the constant time delay, p is the total number of rules of the polynomial fuzzy model.

The system dynamics is described by :

$$\dot{\mathbf{x}}(t) = \sum_{i=1}^p w_i(\mathbf{x}(t)) \left(\mathbf{A}_{i0}(\mathbf{x}(t))\mathbf{x}(t) + \sum_{l=1}^d \mathbf{A}_{il}(\mathbf{x}(t - \tau_l))\mathbf{x}(t - \tau_l) + \mathbf{B}_i(\mathbf{x}(t))\mathbf{u}(t) \right) \quad (3)$$

where $w_i(\mathbf{x}(t)) = \frac{\prod_{l=1}^{\Psi} \mu_{M_l^i}(f_l(\mathbf{x}(t)))}{\sum_{k=1}^p \prod_{l=1}^{\Psi} \mu_{M_l^k}(f_l(\mathbf{x}(t)))} \geq 0$ is the normalized grade of membership which satisfies $\sum_{i=1}^p w_i(\mathbf{x}(t)) = 1$, and $\mu_{M_l^i}(f_l(\mathbf{x}(t)))$ is the grade of membership corresponding to the fuzzy term M_l^i .

Definition 1 ([15]): A system is said to be positive if the initial condition $\phi(\cdot) \succeq 0$ holds and the corresponding trajectory $\mathbf{x}(t) \succeq 0$ for all $t \succeq 0$ is satisfied.

Lemma 1 ([15]): A system with time delay is said to be controlled positive if the system and time delay matrices satisfy the conditions that $\mathbf{A}_{i0}(\mathbf{x}(t))$ is a Metzler matrix and $\mathbf{A}_{il}(\mathbf{x}(t - \tau_l)) \succeq 0$ when $\mathbf{u}(t) = \mathbf{0}$.

Assumption 1: Assuming that $\mathbf{A}_{il}(\mathbf{x}(t - \tau_l)) \succeq 0$ is satisfied by the polynomial fuzzy model (3), otherwise, the system is not a positive system or cannot be controlled positive.

C. Polynomial Fuzzy Controller

A polynomial fuzzy controller is employed to control the nonlinear system represented by the polynomial fuzzy model (3) such that the closed-loop system is positive and stable. We define the j^{th} rule of the polynomial fuzzy controller under imperfect premise matching as follows:

Rule $j, j \in \underline{p}$:

IF $g_1(\mathbf{x}(t))$ is N_1^j AND \dots AND $g_\Omega(\mathbf{x}(t))$ is M_Ω^j

THEN $\mathbf{u}(t) = \mathbf{G}_j(\mathbf{x}(t), \mathbf{m}(\mathbf{x}(t)))\mathbf{x}(t)$ (4)

where N_l^j is the fuzzy set in rule j corresponding to the premise variable $g_l(\mathbf{x}(t)), l \in \{1, 2, \dots, \Omega\}$, and Ω is a positive integer; $\mathbf{G}_j(\mathbf{x}(t), \mathbf{m}(\mathbf{x}(t))) \in \mathbb{R}^{m \times n}$, $j \in$

\underline{p} , is the polynomial feedback gain to be determined.

$m_j(\mathbf{x}(t)) = \frac{\prod_{i=1}^{\Omega} \mu_{N_l^j}(g_l(\mathbf{x}(t)))}{\sum_{k=1}^p \prod_{i=1}^{\Omega} \mu_{N_l^k}(g_l(\mathbf{x}(t)))} \geq 0$ is the normalized grade of membership which satisfies $\sum_{j=1}^p m_j(\mathbf{x}(t)) = 1$ and $\mu_{N_l^j}(g_l(\mathbf{x}(t)))$ is the grade of membership corresponding to the fuzzy term N_l^j .

The polynomial fuzzy controller is described as follows:

$$\mathbf{u}(t) = \sum_{j=1}^p m_j(\mathbf{x}(t)) \mathbf{G}_j(\mathbf{x}(t), \mathbf{m}(\mathbf{x}(t)))\mathbf{x}(t). \quad (5)$$

Remark 1: Under the imperfect premise matching, the membership function $m_j(\mathbf{x}(t))$ of the polynomial fuzzy controller can be chosen $m_j(\mathbf{x}(t)) \neq w_j(\mathbf{x}(t))$ for any $j \in \underline{p}$ is allowed.

Remark 2: Under the imperfect premise matching, the polynomial fuzzy controller $\mathbf{G}_j(\mathbf{x}(t), \mathbf{m}(\mathbf{x}(t)))$ can be chosen as $\mathbf{G}_j(\mathbf{x}(t))$ which is the function of $\mathbf{x}(t)$ to reduce the proposed polynomial fuzzy controller with traditional feedback gains.

Remark 3: The proposed polynomial feedback gains $\mathbf{G}_j(\mathbf{x}(t), \mathbf{m}(\mathbf{x}(t)))$ are function of both $\mathbf{x}(t)$ and $\mathbf{m}(\mathbf{x}(t))$. When using SOSTOOLS [36], both $\mathbf{x}(t)$ and $\mathbf{m}(\mathbf{x}(t))$ are considered as symbolic variables so that the feedback gains will be automatically determined.

III. STABILITY ANALYSIS

In the following, for brevity, the time t associated with the variables is dropped for the situation without ambiguity, $\mathbf{x}(t)$ and $\mathbf{u}(t)$ are denoted as \mathbf{x} and \mathbf{u} , respectively. Furthermore, $\mathbf{w}(\mathbf{x}(t))$ and $\mathbf{m}(\mathbf{x}(t))$ are denoted as \mathbf{w} and \mathbf{m} , respectively.

A PFMB control system with time delay is formed by the polynomial fuzzy model (3) and polynomial fuzzy controller (5) as follows:

$$\dot{\mathbf{x}} = \sum_{i=1}^p \sum_{j=1}^p w_i m_j \left((\mathbf{A}_{i0}(\mathbf{x}) + \mathbf{B}_i(\mathbf{x})\mathbf{G}_j(\mathbf{x}, \mathbf{m}))\mathbf{x} + \sum_{l=1}^d \mathbf{A}_{il}(\mathbf{x}(t - \tau_l))\mathbf{x}(t - \tau_l) \right). \quad (6)$$

Lemma 2 ([15], [37]): The system (6) is controlled positive for $\phi(\cdot) > 0$ if $\mathbf{A}_{i0}(\mathbf{x}) + \mathbf{B}_i(\mathbf{x})\mathbf{G}_j(\mathbf{x}, \mathbf{m})$ is a Metzler matrix and $\mathbf{A}_{il}(\mathbf{x}(t - \tau_l)) \succeq 0$.

A. Basic Stability Analysis

Subject to Assumption 1, we choose the following polynomial Lyapunov functional candidate to investigate the system stability of (6):

$$V(t) = \mathbf{x}^T(t)\lambda + \sum_{l=1}^d \left(\int_{t-\tau_l}^t \mathbf{x}(i)^T \mathbf{A}_l(\mathbf{x}(i))\lambda di \right) \quad (7)$$

where $\lambda = [\lambda_1, \lambda_2, \dots, \lambda_n]^T \succ 0$; $\mathbf{A}_l(\mathbf{x}(t - \tau_l)) = (a_{rs}^l(\mathbf{x}(t - \tau_l))) \succeq 0$ whose element $a_{rs}^{il}(\mathbf{x}(t - \tau_l))$ is defined as $a_{rs}^{il}(\mathbf{x}(t - \tau_l)) = \max_{i \in \underline{p}} \{a_{rs}^{il}(\mathbf{x}(t - \tau_l))\}$, $l \in \underline{d} = \{1, 2, \dots, d\}$, where $a_{rs}^l(\mathbf{x}(t - \tau_l))$ and $a_{rs}^{il}(\mathbf{x}(t - \tau_l))$ are the (r, s) -th element of the matrices $\mathbf{A}_l(\mathbf{x}(t - \tau_l))$ and $\mathbf{A}_{il}(\mathbf{x}(t - \tau_l))$, respectively.

Remark 4: A system whose matrices are transposed is called dual system of another system. As the equivalence of stability between two systems under duality, some work [13], [14], [38] had investigated useful properties of positive systems which makes the stability analysis easier.

The dual system of (6) as follows can be used to investigate the stability:

$$\begin{aligned} \dot{\mathbf{x}} = & \sum_{i=1}^p \sum_{j=1}^p w_i m_j ((\mathbf{A}_{i0}(\mathbf{x}) + \mathbf{B}_i(\mathbf{x}) \mathbf{G}_j(\mathbf{x}, \mathbf{m}))^T \mathbf{x} \\ & + \sum_{l=1}^d \mathbf{A}_{il}^T(\mathbf{x}(t - \tau_l)) \mathbf{x}(t - \tau_l)). \end{aligned} \quad (8)$$

Before we conduct the stability analysis, an assumption that the PFMB control system is positive needs to be satisfied. Specially, the positivity issue of the PFMB control system is discussed according to Lemma 2. To facilitate the stability analysis, the feedback gains are chosen as $\mathbf{G}_j(\mathbf{x}, \mathbf{m}) = \left[\frac{\mathbf{y}_1^j(\mathbf{x}, \mathbf{m})}{\lambda_1}, \frac{\mathbf{y}_2^j(\mathbf{x}, \mathbf{m})}{\lambda_2}, \dots, \frac{\mathbf{y}_n^j(\mathbf{x}, \mathbf{m})}{\lambda_n} \right]$ where $\mathbf{y}_1^j(\mathbf{x}, \mathbf{m}), \mathbf{y}_2^j(\mathbf{x}, \mathbf{m}), \dots, \mathbf{y}_n^j(\mathbf{x}, \mathbf{m}) \in \mathfrak{R}^m$ for $j \in \underline{p}$ are to be determined.

Referring to Lemma 2, the positivity conditions for the PFMB control system are obtained in the following theorem.

Theorem 1: The PFMB control system (6) or dual system (8) with the initial condition $\phi(\cdot) \succcurlyeq 0$ can be controlled positive if there exist $\lambda \in \mathfrak{R}^n$ and $\mathbf{y}_k^j(\mathbf{x}, \mathbf{m}) \in \mathfrak{R}^m$ for $j \in \underline{p}$ and $k \in \underline{n}$ such that the following SOS-based conditions are satisfied:

$$a_{rs}^i(\mathbf{x})\lambda_s + \mathbf{b}_r^i(\mathbf{x})\mathbf{y}_s^j(\mathbf{x}, \mathbf{m}) \text{ is SOS}, i, j \in \underline{p}; r \neq s \in \underline{n} \quad (9)$$

$$a_{rs}^{il}(\mathbf{x}(t - \tau_l)) \text{ is SOS}, i, j \in \underline{p}; r, s \in \underline{n}; l \in \underline{d} \quad (10)$$

where $\lambda = [\lambda_1, \lambda_2, \dots, \lambda_n]^T \succ 0$, $\mathbf{A}_i(\mathbf{x}) = (a_{rs}^i(\mathbf{x}))$, $\mathbf{A}_{il}(\mathbf{x}(t - \tau_l)) = (a_{rs}^{il}(\mathbf{x}(t - \tau_l)))$ and $\mathbf{B}_i(\mathbf{x}) = [\mathbf{b}_1^i(\mathbf{x})^T, \mathbf{b}_2^i(\mathbf{x})^T, \dots, \mathbf{b}_n^i(\mathbf{x})^T]^T$, $i \in \underline{p}$, $r, s \in \underline{n}$.

Remark 5: The SOS-based conditions in Theorem 1 are to make sure that the PFMB control system is positive. The condition (9) is to make sure that the matrix $\mathbf{A}_i(\mathbf{x}) + \mathbf{B}_i(\mathbf{x})\mathbf{G}_j(\mathbf{x}, \mathbf{m})$ is Metzler for all i and j which can be achieved by satisfying $a_{rs}^i(\mathbf{x})\lambda_s + \mathbf{b}_r^i(\mathbf{x})\mathbf{y}_s^j(\mathbf{x}, \mathbf{m}) \geq 0$ for all $r \neq s$. Referring to Assumption 1, the condition (10) is to make sure that all elements in the matrix $\mathbf{A}_{il}(\mathbf{x}(t - \tau_l))$ is non-negative, i.e., $a_{rs}^{il}(\mathbf{x}(t - \tau_l)) \geq 0$ for all i, l, r and s .

In the following, we obtain the SOS-based stability conditions to guarantee the system stability and facilitate the control synthesis. From (7) and (8), we have:

$$\begin{aligned} \dot{V}(t) = & \dot{\mathbf{x}}^T \lambda + \mathbf{x}^T \sum_{l=1}^d \mathbf{A}_l(\mathbf{x}) \lambda \\ & - \mathbf{x}^T(t - \tau_l) \sum_{l=1}^d \mathbf{A}_l(\mathbf{x}(t - \tau_l)) \lambda \\ \leq & \sum_{i=1}^p \sum_{j=1}^p w_i m_j \mathbf{x}^T ((\mathbf{A}_{i0}(\mathbf{x}) + \sum_{l=1}^d \mathbf{A}_l(\mathbf{x})) \lambda \\ & + \mathbf{B}_i(\mathbf{x}) \sum_{s=1}^n \mathbf{y}_s^j(\mathbf{x}, \mathbf{m})) \end{aligned}$$

$$= \sum_{i=1}^p \sum_{j=1}^p w_i m_j \mathbf{x}^T \mathbf{Q}_{ij}(\mathbf{x}, \mathbf{m}) \quad (11)$$

where

$$\begin{aligned} \mathbf{Q}_{ij}(\mathbf{x}, \mathbf{m}) = & ((\mathbf{A}_{i0}(\mathbf{x}) + \sum_{l=1}^d \mathbf{A}_l(\mathbf{x})) \lambda + \mathbf{B}_i(\mathbf{x}) \sum_{s=1}^n \mathbf{y}_s^j(\mathbf{x}, \mathbf{m})) \\ = & [q_1^{ij}(\mathbf{x}, \mathbf{m}), q_2^{ij}(\mathbf{x}, \mathbf{m}), \dots, q_n^{ij}(\mathbf{x}, \mathbf{m})]^T. \end{aligned} \quad (12)$$

It can be seen from (11) that based on the Lyapunov stability theory, $V(t) > 0$ and $\dot{V}(t) < 0$ (excluding $\mathbf{x}(t) = 0$) can guarantee stability of the dual system (8). The stability conditions are summarized in the following theorem.

Theorem 2: The PFMB control system (6) is positive and asymptotically stable if there exist $\lambda \in \mathfrak{R}^n$ and $\mathbf{y}_k^j(\mathbf{x}) \in \mathfrak{R}^m$ for $j \in \underline{p}$, $k \in \underline{n}$ such that Theorem 1 and the following SOS-based conditions are satisfied:

$$\lambda_k - \varepsilon_1 \text{ is SOS}, k \in \underline{n} \quad (13)$$

$$-(q_k^{ij}(\mathbf{x}, \mathbf{m}) + \varepsilon_2(\mathbf{x}, \mathbf{m})) \text{ is SOS}, i \in \underline{p}, j \in \underline{p}, k \in \underline{n} \quad (14)$$

where $\varepsilon_1 > 0$ is a predefined scalar and $\varepsilon_2(\mathbf{x}, \mathbf{m}) > 0$ is a predefined scalar polynomial; $q_k^{ij}(\mathbf{x}, \mathbf{m})$ is defined in (12); the feedback gains and the other variables are defined in Theorem 1. λ is a decision variable and obtained by the SOS approach by satisfying Theorems 1 and 2.

Remark 6: The PFMB control system (6) is guaranteed to be asymptotically stable and positive if the stability and positive conditions in Theorems 1 and 2 are satisfied as equivalence of stability between the PFMB control system with time delay (6) and its dual system (8).

Remark 7: We can obtain the basic SOS-based stability conditions under imperfect premise matching provided by Theorems 1 and 2 to synthesize the polynomial fuzzy controller with the consideration of the positivity and stability of the PFMB control system with time delay. However, the stability conditions are conservative as information and properties of the membership functions w_i and m_j are not considered.

Remark 8: In the development of the stability conditions, we need to ensure that the system states are positive, i.e., $x_k \geq 0$ for $k \in \underline{n}$. When using convex programming software to find a feasible solution, in order to impose the constraint $x_k \geq 0$, the technique of variable transformation [17] is employed which simply turns x_k to x_k^2 in this paper.

In the following, to relax the stability conditions, we consider the membership function as symbolic variables and carry the property of membership functions into stability conditions. From (11), the stability of the system can be guaranteed by satisfying all elements of $\sum_{i=1}^p \sum_{j=1}^p w_i m_j \mathbf{Q}_{ij}(\mathbf{x}, \mathbf{m}) \prec 0$ for all i and j . In this condition, there are three kinds of symbolic variables which need to be considered when using SOSTOOLS, i.e., \mathbf{w} , \mathbf{m} and \mathbf{x} .

The more number of symbolic variables we have, the more likely to cause SOSTOOLS reaching its limits easier especially when higher degrees of monomials are employed in the SOS program. To reduce the computational demand for the solution search process, the number of symbolic

variables can be reduced by applying the both two methods as following: 1) considering the property of the membership functions $m_p = 1 - \sum_{j=1}^{p-1} m_j$; 2) getting rid of the symbolic variables w_i , we consider the relationship between w_i and m_j as follows:

$$w_i = m_i - s_i + \epsilon_i(\mathbf{x}) \quad (15)$$

where s_i is a constant scalar to be determined such that $\epsilon_i(\mathbf{x}) > 0$. Because of $\sum_{i=1}^p (w_i - m_i) = \sum_{i=1}^p (-s_i + \epsilon_i(\mathbf{x})) = 0$, we have $\sum_{i=1}^p \sum_{j=1}^p (-s_i + \epsilon_i(\mathbf{x})) m_j \mathbf{W}_j(\mathbf{x}, \mathbf{m}) = 0$, where $\mathbf{W}_j(\mathbf{x}, \mathbf{m})$ is an arbitrary polynomial vector.

From (11) and (15), we can obtain:

$$\begin{aligned} \dot{V}(t) &\leq \sum_{i=1}^p \sum_{j=1}^p w_i m_j \mathbf{x}^T(t) \mathbf{Q}_{ij}(\mathbf{x}, \mathbf{m}) \\ &= \sum_{i=1}^p \sum_{j=1}^p m_i m_j \mathbf{x}^T \mathbf{Q}_{ij}(\mathbf{x}, \mathbf{m}) \\ &+ \sum_{i=1}^p \sum_{j=1}^p (-s_i + \epsilon_i(\mathbf{x})) m_j \mathbf{x}^T (\mathbf{Q}_{ij}(\mathbf{x}, \mathbf{m}) + \mathbf{W}_j(\mathbf{x}, \mathbf{m})) \\ &= \sum_{i=1}^p \sum_{j=1}^p (m_i - s_i) m_j \mathbf{x}^T \mathbf{Q}_{ij}(\mathbf{x}, \mathbf{m}) \\ &- \sum_{i=1}^p \sum_{j=1}^p s_i m_j \mathbf{x}^T \mathbf{W}_j(\mathbf{x}, \mathbf{m}) \\ &+ \sum_{i=1}^p \sum_{j=1}^p \epsilon_i(\mathbf{x}) m_j \mathbf{x}^T (\mathbf{Q}_{ij}(\mathbf{x}, \mathbf{m}) + \mathbf{W}_j(\mathbf{x}, \mathbf{m})). \end{aligned} \quad (16)$$

In the following, we propose a stability analysis approach to relax the basic SOS-based stability conditions by considering property of membership functions. To incorporate property of the membership functions, we have $\sum_{j=1}^p m_j - 1 = 0$ and $m_i(1 - m_j) \geq 0$ for all i and j such that the following equalities and inequalities hold:

$$\left(\sum_{j=1}^p m_j - 1 \right) \mathbf{M}(\mathbf{x}, \mathbf{m}) = 0 \quad (17)$$

$$\sum_{i=1}^p \sum_{j=1}^p m_i (1 - m_j) \mathbf{H}_{ij}(\mathbf{x}, \mathbf{m}) \succcurlyeq 0 \quad (18)$$

$$\sum_{i=1}^p \left(\sum_{j=1}^p m_j - 1 \right) \epsilon_i(\mathbf{x}) \mathbf{T}(\mathbf{x}, \mathbf{m}) = 0 \quad (19)$$

$$\sum_{i=1}^p \sum_{j=1}^p m_j (1 - m_j) \epsilon_i(\mathbf{x}) \mathbf{U}_{ij}(\mathbf{x}, \mathbf{m}) \succcurlyeq 0 \quad (20)$$

where $\mathbf{M}(\mathbf{x}, \mathbf{m}) \in \mathbb{R}^n$ and $\mathbf{T}(\mathbf{x}, \mathbf{m}) \in \mathbb{R}^n$ are arbitrary polynomial matrices, $0 \preccurlyeq \mathbf{H}_{ij}(\mathbf{x}, \mathbf{m}) \in \mathbb{R}^n$ and $0 \preccurlyeq \mathbf{U}_{ij}(\mathbf{x}, \mathbf{m}) \in \mathbb{R}^n$ are polynomial matrices. And we define $\mathbf{M}(\mathbf{x}, \mathbf{m}) = [m_1(\mathbf{x}, \mathbf{m}), m_2(\mathbf{x}, \mathbf{m}), \dots, m_n(\mathbf{x}, \mathbf{m})]^T$, $\mathbf{T}(\mathbf{x}, \mathbf{m}) = [t_1(\mathbf{x}, \mathbf{m}), t_2(\mathbf{x}, \mathbf{m}), \dots, t_n(\mathbf{x}, \mathbf{m})]^T$,

$$\begin{aligned} \mathbf{H}_{ij}(\mathbf{x}, \mathbf{m}) &= [h_1^{ij}(\mathbf{x}, \mathbf{m}), h_2^{ij}(\mathbf{x}, \mathbf{m}), \dots, h_n^{ij}(\mathbf{x}, \mathbf{m})]^T \\ \text{and } \mathbf{U}_{ij}(\mathbf{x}, \mathbf{m}) &= [u_1^{ij}(\mathbf{x}, \mathbf{m}), u_2^{ij}(\mathbf{x}, \mathbf{m}), \dots, u_n^{ij}(\mathbf{x}, \mathbf{m})]^T. \end{aligned}$$

From (16) to (20), we have

$$\begin{aligned} \dot{V}(t) &\leq \mathbf{x}^T \left(\sum_{i=1}^p \sum_{j=1}^p (m_i - s_i) m_j \mathbf{Q}_{ij}(\mathbf{x}) \right. \\ &- \sum_{i=1}^p \sum_{j=1}^p s_i m_j \mathbf{W}_j(\mathbf{x}, \mathbf{m}) + \left(\sum_{i=1}^p m_i - 1 \right) \mathbf{M}(\mathbf{x}, \mathbf{m}) \\ &+ \sum_{i=1}^p \sum_{j=1}^p m_i (1 - m_j) \mathbf{H}_{ij}(\mathbf{x}, \mathbf{m}) \left. \right) \\ &+ \mathbf{x}^T \sum_{i=1}^p \epsilon_i(\mathbf{x}) \left(\sum_{j=1}^p m_j (\mathbf{Q}_{ij}(\mathbf{x}, \mathbf{m}) + \mathbf{W}_j(\mathbf{x}, \mathbf{m})) \right. \\ &+ \left. \left(\sum_{j=1}^p m_j - 1 \right) \mathbf{T}(\mathbf{x}, \mathbf{m}) + \sum_{j=1}^p m_j (1 - m_j) \mathbf{U}_{ij}(\mathbf{x}, \mathbf{m}) \right). \end{aligned} \quad (21)$$

Based on the Lyapunov stability theory and from (21), the stability conditions which can guarantee stability of the dual system (8) are summarized in the following theorem.

Theorem 3: The PFMB control system (6) is positive and asymptotically stable if there exist polynomial matrices $\lambda \in \mathbb{R}^n$, $\mathbf{y}_k^j(\mathbf{x}) \in \mathbb{R}^m$ for $j \in \underline{p}$, $k \in \underline{n}$, $\mathbf{W}_j(\mathbf{x}, \mathbf{m}) \in \mathbb{R}^n$, $\mathbf{M}(\mathbf{x}, \mathbf{m}) \in \mathbb{R}^n$, $\mathbf{T}(\mathbf{x}, \mathbf{m}) \in \mathbb{R}^n$ are arbitrary polynomial matrices, $\mathbf{H}_{ij}(\mathbf{x}, \mathbf{m}) \in \mathbb{R}^n$, $\mathbf{U}_{ij}(\mathbf{x}, \mathbf{m}) \in \mathbb{R}^n$, such that Theorem 1 and the following SOS-based conditions are satisfied: $\lambda_k - \epsilon_1$ is SOS, $k \in \underline{n}$; $h_k^{ij}(\mathbf{x}, \mathbf{m})$ is SOS, $k \in \underline{n}$, $i \in \underline{p}$, $j \in \underline{p}$; $u_k^{ij}(\mathbf{x}, \mathbf{m})$ is SOS, $k \in \underline{n}$, $i \in \underline{p}$, $j \in \underline{p}$;

$$\begin{aligned} &- \left(\sum_{i=1}^p \sum_{j=1}^p (m_i - s_i) m_j q_k^{ij}(\mathbf{x}, \mathbf{m}) - \sum_{i=1}^p \sum_{j=1}^p s_i m_j w_k^j(\mathbf{x}, \mathbf{m}) \right. \\ &+ \left. \left(\sum_{j=1}^p m_j - 1 \right) m_k(\mathbf{x}, \mathbf{m}) + \sum_{i=1}^p \sum_{j=1}^p m_i (1 - m_j) h_k^{ij}(\mathbf{x}, \mathbf{m}) \right. \\ &+ \left. \epsilon_2(\mathbf{x}, \mathbf{m}) \right) \text{ is SOS, } k \in \underline{n}, i \in \underline{p}, j \in \underline{p}; \end{aligned} \quad (22)$$

$$\begin{aligned} &- \left(\sum_{j=1}^p m_j (q_k^{ij}(\mathbf{x}, \mathbf{m}) + w_k^j(\mathbf{x}, \mathbf{m})) \right. \\ &+ \left. \left(\sum_{j=1}^p m_j - 1 \right) t_k(\mathbf{x}, \mathbf{m}) + \sum_{j=1}^p m_j (1 - m_j) u_k^{ij}(\mathbf{x}, \mathbf{m}) \right. \\ &+ \left. \epsilon_3(\mathbf{x}, \mathbf{m}) \right) \text{ is SOS, } k \in \underline{n}, i \in \underline{p}, j \in \underline{p} \end{aligned} \quad (23)$$

where $\epsilon_1 > 0$ is a predefined scalar, $\epsilon_2(\mathbf{x}, \mathbf{m}) > 0$ and $\epsilon_3(\mathbf{x}, \mathbf{m}) > 0$ are predefined scalar polynomials; q_k^{ij} is defined in (12); the feedback gains and the other variables are defined in Theorem 1. λ is a decision variable and obtained by the SOS approach by satisfying Theorems 1 and 3. In order to impose the constraint $x_k \geq 0$, the technique of variable transformation is employed which simply turns x_k to x_k^2 , $k \in \underline{n}$.

In the following, in order to further relax the SOS-based stability conditions under imperfect premise matching obtained

in Theorems 3, we consider the upper and lower boundaries of membership functions. In this way, the shape information of the membership functions is considered in the stability conditions, resulting in more relaxed results. Defining \bar{w}_i and \underline{w}_i as the upper and lower bounds of w_i , respectively, the following inequalities hold:

$$\sum_{j=1}^p (\bar{m}_j - m_j) \mathbf{S}_j(\mathbf{x}, \mathbf{m}) \succcurlyeq 0 \quad (24)$$

$$\sum_{j=1}^p (m_j - \underline{m}_j) \mathbf{Z}_j(\mathbf{x}, \mathbf{m}) \succcurlyeq 0 \quad (25)$$

$$\sum_{i=1}^p \sum_{j=1}^p (\bar{m}_j - m_j) \epsilon_i(\mathbf{x}) \mathbf{D}_j(\mathbf{x}, \mathbf{m}) \succcurlyeq 0 \quad (26)$$

$$\sum_{i=1}^p \sum_{j=1}^p (m_j - \underline{m}_j) \epsilon_i(\mathbf{x}) \mathbf{V}_j(\mathbf{x}, \mathbf{m}) \succcurlyeq 0 \quad (27)$$

where $0 \preccurlyeq \mathbf{S}_j(\mathbf{x}, \mathbf{m}) \in \mathbb{R}^n$, $0 \preccurlyeq \mathbf{Z}_j(\mathbf{x}, \mathbf{m}) \in \mathbb{R}^n$, $0 \preccurlyeq \mathbf{D}_j(\mathbf{x}, \mathbf{m}) \in \mathbb{R}^n$, $0 \preccurlyeq \mathbf{V}_j(\mathbf{x}, \mathbf{m}) \in \mathbb{R}^n$ are polynomial vectors. And we define

$$\mathbf{S}_j(\mathbf{x}, \mathbf{m}) = \begin{bmatrix} s_1^j(\mathbf{x}, \mathbf{m}), s_2^j(\mathbf{x}, \mathbf{m}), \dots, s_n^j(\mathbf{x}, \mathbf{m}) \end{bmatrix}^T,$$

$$\mathbf{Z}_j(\mathbf{x}, \mathbf{m}) = \begin{bmatrix} z_1^j(\mathbf{x}, \mathbf{m}), z_2^j(\mathbf{x}, \mathbf{m}), \dots, z_n^j(\mathbf{x}, \mathbf{m}) \end{bmatrix}^T,$$

$$\mathbf{D}_j(\mathbf{x}, \mathbf{m}) = \begin{bmatrix} d_1^j(\mathbf{x}, \mathbf{m}), d_2^j(\mathbf{x}, \mathbf{m}), \dots, d_n^j(\mathbf{x}, \mathbf{m}) \end{bmatrix}^T \text{ and}$$

$$\mathbf{V}_j(\mathbf{x}, \mathbf{m}) = \begin{bmatrix} v_1^j(\mathbf{x}, \mathbf{m}), v_2^j(\mathbf{x}, \mathbf{m}), \dots, v_n^j(\mathbf{x}, \mathbf{m}) \end{bmatrix}^T.$$

From (21) to (27), we have

$$\begin{aligned} \dot{V}(t) &\leq \mathbf{x}^T \left(\sum_{i=1}^p \sum_{j=1}^p (m_i - s_i) m_j \mathbf{Q}_{ij}(\mathbf{x}, \mathbf{m}) \right. \\ &\quad - \sum_{i=1}^p \sum_{j=1}^p s_i m_j \mathbf{W}_j(\mathbf{x}, \mathbf{m}) + \left(\sum_{i=1}^p m_i - 1 \right) \mathbf{M}(\mathbf{x}, \mathbf{m}) \\ &\quad + \sum_{i=1}^p \sum_{j=1}^p m_i (1 - m_j) \mathbf{H}_{ij}(\mathbf{x}, \mathbf{m}) \\ &\quad + \sum_{j=1}^p (\bar{m}_j - m_j) \mathbf{S}_j(\mathbf{x}, \mathbf{m}) + \sum_{j=1}^p (m_j - \underline{m}_j) \mathbf{Z}_j(\mathbf{x}, \mathbf{m}) \Big) \\ &\quad + \mathbf{x}^T \sum_{i=1}^p \epsilon_i(\mathbf{x}) \left(\sum_{j=1}^p m_j (\mathbf{Q}_{ij}(\mathbf{x}, \mathbf{m}) + \mathbf{W}_j(\mathbf{x}, \mathbf{m})) \right. \\ &\quad + \left(\sum_{j=1}^p m_j - 1 \right) \mathbf{T}(\mathbf{x}, \mathbf{m}) + \sum_{j=1}^p m_j (1 - m_j) \mathbf{U}_{ij}(\mathbf{x}, \mathbf{m}) \\ &\quad \left. + \sum_{j=1}^p (\bar{m}_j - m_j) \mathbf{D}_j(\mathbf{x}, \mathbf{m}) + \sum_{j=1}^p (m_j - \underline{m}_j) \mathbf{V}_j(\mathbf{x}, \mathbf{m}) \right). \end{aligned} \quad (28)$$

Based on the Lyapunov stability theory and from (28), the PFMB control system (6) is positive subject to Theorem 1 and asymptotically stable under imperfect premise matching are summarised in the following theorem.

Theorem 4: The PFMB control system (6) is positive and

asymptotically stable if there exist polynomial vectors $\lambda \in \mathbb{R}^n$, $\mathbf{y}_k^j(\mathbf{x}) \in \mathbb{R}^m$ for $j \in \underline{p}$, $k \in \underline{n}$, arbitrary polynomial vectors. $\mathbf{W}_j(\mathbf{x}, \mathbf{m}) \in \mathbb{R}^n$, $\mathbf{M}(\mathbf{x}, \mathbf{m}) \in \mathbb{R}^n$, $\mathbf{T}(\mathbf{x}, \mathbf{m}) \in \mathbb{R}^n$, polynomial matrices, $\mathbf{H}_{ij}(\mathbf{x}, \mathbf{m}) \in \mathbb{R}^n$, $\mathbf{U}_{ij}(\mathbf{x}, \mathbf{m}) \in \mathbb{R}^n$, $\mathbf{S}_j(\mathbf{x}, \mathbf{m}) \in \mathbb{R}^n$, $\mathbf{Z}_j(\mathbf{x}, \mathbf{m}) \in \mathbb{R}^n$, $\mathbf{D}_j(\mathbf{x}, \mathbf{m}) \in \mathbb{R}^n$ and $\mathbf{V}_j(\mathbf{x}, \mathbf{m}) \in \mathbb{R}^n$ such that Theorem 1 and the following SOS-based conditions are satisfied: $\lambda_k - \epsilon_1$ is SOS, $k \in \underline{n}$; $h_k^{ij}(\mathbf{x}, \mathbf{m})$ is SOS, $k \in \underline{n}$, $i \in \underline{p}$, $j \in \underline{p}$; $u_k^{ij}(\mathbf{x}, \mathbf{m})$ is SOS, $k \in \underline{n}$, $i \in \underline{p}$, $j \in \underline{p}$; $s_k^j(\mathbf{x}, \mathbf{m})$ is SOS, $k \in \underline{n}$, $i \in \underline{p}$, $j \in \underline{p}$; $z_k^j(\mathbf{x}, \mathbf{m})$ is SOS, $k \in \underline{n}$, $i \in \underline{p}$, $j \in \underline{p}$; $d_k^j(\mathbf{x}, \mathbf{m})$ is SOS, $k \in \underline{n}$, $i \in \underline{p}$, $j \in \underline{p}$;

$$\begin{aligned} &- \left(\sum_{i=1}^p \sum_{j=1}^p (m_i - s_i) m_j q_k^{ij}(\mathbf{x}, \mathbf{m}) - \sum_{i=1}^p \sum_{j=1}^p s_i m_j w_k^j(\mathbf{x}, \mathbf{m}) \right. \\ &\quad + \left(\sum_{j=1}^p m_j - 1 \right) m_k(\mathbf{x}, \mathbf{m}) + \sum_{i=1}^p \sum_{j=1}^p m_i (1 - m_j) h_k^{ij}(\mathbf{x}, \mathbf{m}) \\ &\quad + \sum_{j=1}^p (\bar{m}_j - m_j) s_k^j(\mathbf{x}, \mathbf{w}) + \sum_{j=1}^p (m_j - \underline{m}_j) z_k^j(\mathbf{x}, \mathbf{m}) \\ &\quad \left. + \epsilon_2(\mathbf{x}, \mathbf{m}) \right) \text{ is SOS, } k \in \underline{n}, i \in \underline{p}, j \in \underline{p} \end{aligned} \quad (29)$$

$$\begin{aligned} &- \left(\sum_{j=1}^p m_j (q_k^{ij}(\mathbf{x}, \mathbf{m}) + w_k^j(\mathbf{x}, \mathbf{m})) \right. \\ &\quad + \left(\sum_{j=1}^p m_j - 1 \right) t_k(\mathbf{x}, \mathbf{m}) + \sum_{j=1}^p m_j (1 - m_j) u_k^{ij}(\mathbf{x}, \mathbf{m}) \\ &\quad + \sum_{j=1}^p (\bar{m}_j - m_j) d_k^j(\mathbf{x}, \mathbf{m}) + \sum_{j=1}^p (m_j - \underline{m}_j) v_k^j(\mathbf{x}, \mathbf{m}) \\ &\quad \left. + \epsilon_3(\mathbf{x}, \mathbf{m}) \right) \text{ is SOS, } k \in \underline{n}, i \in \underline{p}, j \in \underline{p} \end{aligned} \quad (30)$$

where $\epsilon_1 > 0$ is a predefined scalar, $\epsilon_2(\mathbf{x}, \mathbf{m}) > 0$ and $\epsilon_3(\mathbf{x}, \mathbf{m}) > 0$ are predefined scalar polynomials; q_k^{ij} is defined in (12); the feedback gains and the other variables are defined in Theorem 1. λ is a decision variable and obtained by the SOS approach by satisfying Theorems 1 and 4.

Remark 9: In the above, we only provide the stability and positivity conditions where the system performance is not considered. Readers are referred to [39] for performance realization.

IV. SIMULATION EXAMPLE

In this section, a simulation example is given to demonstrate the effectiveness of the proposed SOS-based stability conditions. A polynomial fuzzy model with time delay is considered with the following system and input matrices under three fuzzy rules: $\mathbf{x} = [x_1 \ x_2]^T$,

$$\mathbf{A}_{10}(x_1) = \begin{bmatrix} 1.59 - 0.12x_1^2 & -7.29 - 0.25x_1 \\ 0.01 & -0.10 \end{bmatrix},$$

$$\mathbf{A}_{20}(x_1) = \begin{bmatrix} 0.02 - 0.63x_1^2 & -4.64 + 0.92x_1 \\ 0.35 & -0.21 \end{bmatrix},$$

$$\mathbf{A}_{30}(x_1) = \begin{bmatrix} -a - 1.12x_1^2 + 0.31x_1 & -4.33 \\ 0 & 0.05 \end{bmatrix},$$

$$\mathbf{B}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{B}_2 = \begin{bmatrix} 8 \\ 0 \end{bmatrix}, \mathbf{B}_3 = \begin{bmatrix} -b + 6 \\ -1 \end{bmatrix},$$

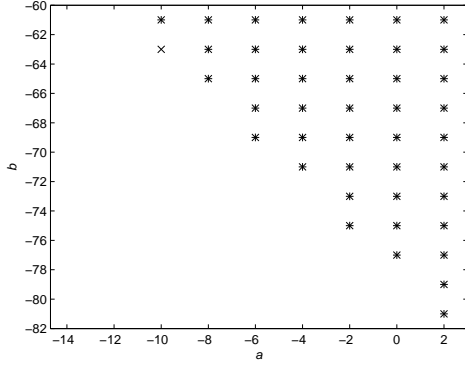


Fig. 1. stability regions given by Theorem 2 with $\mathbf{y}_k^j(\mathbf{x})$ as polynomials of degrees 0 to 2 in (x_1) (indicated by the symbols “+”) and (x_1, m_1, m_3) (indicated by the “x”).

$$\mathbf{A}_{11}(x_1) = \mathbf{A}_{21}(x_1) = \mathbf{A}_{31}(x_1) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

$$\mathbf{A}_{12} = \mathbf{A}_{22} = \mathbf{A}_{32} = \begin{bmatrix} 0.01 & 0 \\ 0 & 0 \end{bmatrix},$$

where the parameters a and b are constant scalars to be determined. The membership functions of polynomial fuzzy model are chosen as: $w_1(x_1) = 1 - \frac{1}{(1+e^{-(x_1-3)})/0.7}$, $w_2(x_1) = 1 - w_1(x_1) - w_3(x_1)$, $w_3(x_1) = \frac{1}{(1+e^{-(x_1-7)})/0.7}$.

Under the imperfect premise matching design, a three-rule fuzzy controller is employed to guarantee the positivity and stability of the system. The membership functions of the fuzzy controller are chosen as follows: $m_1(x_1) = 1$ for $x_1 < 0.8$; $= \frac{-x_1+5.2}{4.4}$ for $0.8 \leq x_1 \leq 5.2$; $= 0$ for $x_1 > 5.2$; $m_2(x_1) = 1 - m_1(x_1) - m_3(x_1)$; $m_3(x_1) = 1$ for $x_1 < 4.8$; $= \frac{x_1-4.8}{4.4}$ for $4.8 \leq x_1 \leq 9.2$; $= 0$ for $x_1 > 9.2$.

Based on (15), we found numerically that $s_1 = s_3 = 0.0795$, $s_2 = 0.0803$. In order to demonstrate the effectiveness of the approach we propose in this paper, the stability regions for the PFMB control system with time delay obtained by the stability conditions of Theorems 2 to 4 are compared. The constant parameters a and b are chosen in the range of $-14 \leq a \leq 2$ and $-81 \leq b \leq -61$ with the interval of 2.

Choosing $\varepsilon_1 = \varepsilon_2(\mathbf{x}) = 0.0010$, the stability conditions in Theorem 2 are applied to determine the stability regions which is shown in Fig. 1 where the stability region is indicated by “+” with $\mathbf{y}_k^j(\mathbf{x}, \mathbf{m})$ of degrees 0 to 2 for (x_1) , “x” is $\mathbf{y}_k^j(\mathbf{x}, \mathbf{m})$ with degrees 0 to 2 for (x_1, m_1, m_3) .

The SOS-based stability conditions in Theorem 3 are employed to check the system positivity and stability. With the above settings, choose $\varepsilon_1 = \varepsilon_2(\mathbf{x}, \mathbf{m}) = \varepsilon_3(\mathbf{x}, \mathbf{m}) = 0.0010$, $\mathbf{M}(\mathbf{x}, \mathbf{m})$ and $\mathbf{T}(\mathbf{x}, \mathbf{m})$ as polynomials of degree of 0 to 2 in (x_1, m_1, m_3) , $\mathbf{H}_{ij}(\mathbf{x}, \mathbf{m})$ and $\mathbf{U}_{ij}(\mathbf{x}, \mathbf{m})$ as polynomials of $(1, x_1^2)$. The stability regions are shown in Fig. 2 where the stability region indicated by “+” is with $\mathbf{y}_k^j(\mathbf{x}, \mathbf{m})$ as polynomials of degrees 0 to 2 in (x_1) , “x” with $\mathbf{y}_k^j(\mathbf{x}, \mathbf{m})$ as polynomials of degrees 0 to 2 in (x_1, m_1, m_3) .

The SOS-based stability conditions in Theorem 4 are employed to check the positivity and stability of PFMB control system with time delay. With the above settings, we have the upper bounds $\bar{m}_1 = \bar{m}_3 = 1$ and $\bar{m}_2 = 0.9091$, the lower

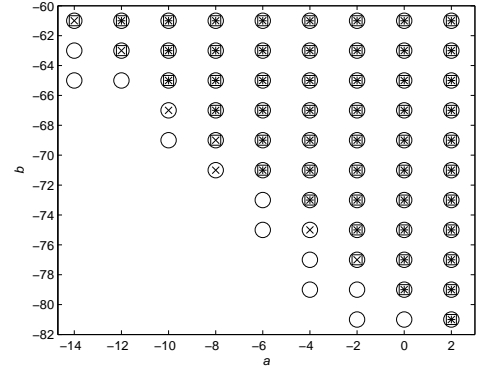


Fig. 2. stability regions given by Theorem 3 with $\mathbf{y}_k^j(\mathbf{x})$ as polynomials of degrees 0 to 2 in (x_1) (indicated by the symbols “+”) and (x_1, m_1, m_3) (indicated by the symbols “x”), given by Theorem 4 with $\mathbf{y}_k^j(\mathbf{x})$ as polynomials of degrees 0 to 2 in (x_1) (indicated by the symbols “□”) and (x_1, m_1, m_3) (indicated by the symbols “o”). Both $\mathbf{H}_{ij}(\mathbf{x}, \mathbf{m})$ and $\mathbf{U}_{ij}(\mathbf{x}, \mathbf{m})$ are polynomials of $(1, x_1^2)$.

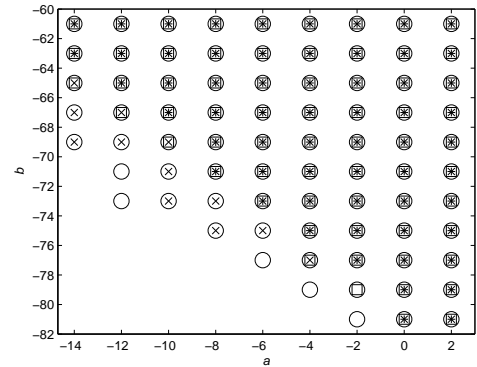


Fig. 3. stability regions given by Theorem 3 with $\mathbf{y}_k^j(\mathbf{x})$ as polynomials of degrees 0 to 2 in (x_1) (indicated by the symbols “+”) and (x_1, m_1, m_3) (indicated by the symbols “x”), given by Theorem 4 with $\mathbf{y}_k^j(\mathbf{x})$ as polynomials of degrees 0 to 2 in (x_1) (indicated by the symbols “□”) and (x_1, m_1, m_3) (indicated by the symbols “o”). Both $\mathbf{H}_{ij}(\mathbf{x}, \mathbf{m})$ and $\mathbf{U}_{ij}(\mathbf{x}, \mathbf{m})$ are polynomials of degree of 0 to 2 in (x_1, m_1, m_3) .

bounds $\bar{m}_1 = \bar{m}_2 = \bar{m}_3 = 0$, and choose $\mathbf{S}_j(\mathbf{x}, \mathbf{m})$, $\mathbf{D}_j(\mathbf{x}, \mathbf{m})$, $\mathbf{Z}_j(\mathbf{x}, \mathbf{m})$ and $\mathbf{V}_j(\mathbf{x}, \mathbf{m})$ as polynomials of degree of 0 to 2 in (x_1, m_1, m_3) . The stability regions are shown in Fig. 2 where the stability region indicated by “□” is with $\mathbf{y}_k^j(\mathbf{x}, \mathbf{m})$ as polynomials of degrees 0 to 2 in (x_1) , “o” with $\mathbf{y}_k^j(\mathbf{x}, \mathbf{m})$ as polynomials of degrees 0 to 2 in (x_1, m_1, m_3) .

For comparison purposes, keeping the same above settings in Fig. 2 but choosing $\mathbf{H}_{ij}(\mathbf{x}, \mathbf{m})$ and $\mathbf{U}_{ij}(\mathbf{x}, \mathbf{m})$ as polynomials of degree of 0 to 2 in (x_1, m_1, m_3) . The stability regions given by Theorem 3 are shown in Fig. 3 where the stability region is indicated by “+” with $\mathbf{y}_k^j(\mathbf{x}, \mathbf{m})$ of degrees 0 to 2 in (x_1) , “x” with $\mathbf{y}_k^j(\mathbf{x}, \mathbf{m})$ as polynomials of degrees 0 to 2 in (x_1, m_1, m_3) . The stability regions given by Theorem 4 are shown in Fig. 3 where the stability region indicated by “□” is with $\mathbf{y}_k^j(\mathbf{x}, \mathbf{m})$ as polynomials of degrees 0 to 2 in (x_1) , “o” with $\mathbf{y}_k^j(\mathbf{x}, \mathbf{m})$ as polynomials of degrees 0 to 2 in (x_1, m_1, m_3) .

As can be seen that from Fig. 1, Fig. 2 or Fig. 3 individually,

the polynomial fuzzy controller considering more variables always offers a larger stability regions. Comparing stability regions in Fig. 1 obtained by the basic SOS-based stability conditions in Theorem 2 with the stability regions in Fig. 2 or Fig. 3 whose stability conditions all consider m_1, m_2 and m_3 as symbolic variables in the stability conditions, it can be seen that Theorem 3 or 4 offers larger stability regions which demonstrates the effectiveness of m_1, m_2 and m_3 being considered in the stability conditions. This is because symbolic variables are able to carry information of membership functions (i.e., positivity of the grades of membership) to the stability analysis and to facilitate the completing square process in the SOS-based analysis. Therefore, more relaxed stability conditions are obtained.

Comparing the stability regions in Fig. 2 or Fig. 3 individually, it can be seen that the stability regions obtained by Theorem 4 considering the upper and lower bounds of membership functions are larger than in Theorem 3 without considering these information because the symbolic variables are only able to carry limited information of membership functions in Theorem 3. When the property of membership functions, i.e., the constraint on the sum of membership grades and the boundary information of membership grades which gives an idea on the shape of membership functions, are considered in the stability analysis in Theorem 4, we can obtain more relaxed stability conditions as the stability analysis is carried out with richer information being considered.

Comparing the stability regions in Fig. 2 (where $\mathbf{H}_{ij}(\mathbf{x}, \mathbf{m})$ and $\mathbf{U}_{ij}(\mathbf{x}, \mathbf{m})$ are function of monomials $(1, x_1^2)$) and Fig. 3 (where $\mathbf{H}_{ij}(\mathbf{x}, \mathbf{m})$ and $\mathbf{U}_{ij}(\mathbf{x}, \mathbf{m})$ are polynomials of degree of 0 to 2 in (x_1, m_1, m_3)), it can be seen that the polynomial matrices of $\mathbf{H}_{ij}(\mathbf{x}, \mathbf{m})$ and $\mathbf{U}_{ij}(\mathbf{x}, \mathbf{m})$ with higher degrees of monomials are able to produce larger stability regions.

To verify the obtained stability and positivity regions we acquire, the simulation for the phase plots of the PFMB control system states are conducted. Referring to Fig. 1, we choose $a = -10; b = -61$ and $a = -10; b = -63$ (stability regions indicated by “+” and “×”, respectively). The phase plots are shown in Fig. 4. The top left and top right phase plots are $a = -10; b = -61$ (stability regions indicated by “+”) with time delay $\tau_1 = \tau_2 = 1$ and $\tau_1 = 50, \tau_2 = 50$, respectively. The bottom left and bottom right phase plots are $a = -10; b = -63$ (stability region indicated by “×”) with time delay $\tau_1 = \tau_2 = 1$ and $\tau_1 = 50, \tau_2 = 50$, respectively.

Referring to Fig. 2, we choose $a = -12; b = -61, a = -14; b = -61, a = -14; b = -61$ and $a = -14; b = -65$ (stability region indicated by “+”, “×”, “□” and “o”, respectively). The phase plots are shown in Fig. 5 with time delay $\tau_1 = \tau_2 = 1$ and Fig. 6 with time delay $\tau_1 = 50, \tau_2 = 50$.

Referring to Fig. 3, we choose $a = -14; b = -61, a = -14; b = -69, a = -14; b = -69$ and $a = -12; b = -71$ (stability regions indicated by “+”, “×”, “□” and “o”, respectively). The phase plots are shown in Fig. 7 and 8. The phase plots are shown in Fig. 7 with time delay $\tau_1 = \tau_2 = 1$ and Fig. 8 with time delay $\tau_1 = 50, \tau_2 = 50$.

As seen from Fig. 4 to Fig. 8, the obtained polynomial fuzzy controller designed is able to drive all system states always to be positive and the origin. Furthermore, the stability can be

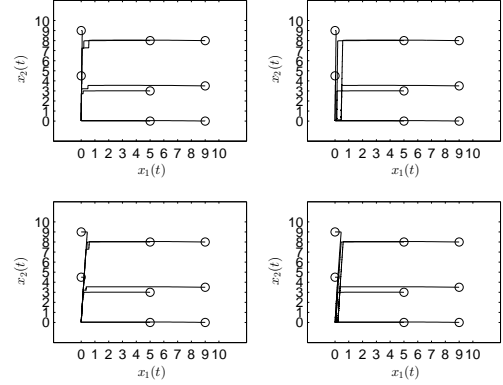


Fig. 4. The top left and top right phase plots are $a = -10; b = -61$ (stability regions indicated by “+”) with time delay $\tau_1 = \tau_2 = 1$ and $\tau_1 = 50, \tau_2 = 50$, respectively. The bottom left and bottom right phase plots are $a = -10; b = -63$ (stability region indicated by “×”) with time delay $\tau_1 = \tau_2 = 1$ and $\tau_1 = 50, \tau_2 = 50$ respectively all with referring to Fig. 1

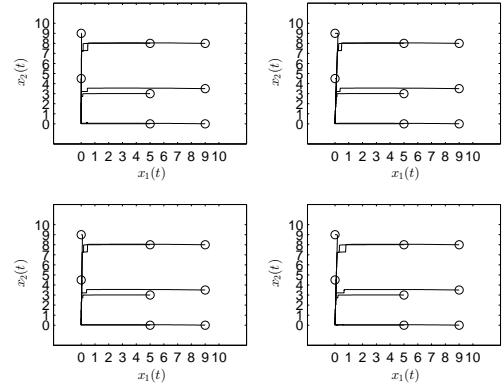


Fig. 5. The top left phase plots are $a = -12; b = -61$ (stability regions indicated by the symbols “+”), the top right phase plots are $a = -14; b = -61$ (stability region indicated by the symbols “×”). The bottom left phase plots are $a = -14; b = -61$ (stability regions indicated by the symbols “□”), the bottom right phase plots are $a = -14; b = -65$ (stability region indicated by the symbols “o”) all with referring to Fig. 2. And time delay $\tau_1 = \tau_2 = 1$.

guaranteed regardless of the values of time delays. The reason is that the stability conditions we obtain from Theorems 2 to 4 are delay independent. But different values of time delay will affect the dynamic response.

V. CONCLUSION

The positivity and stability analysis of positive polynomial-fuzzy-model-based control systems with time delay, which is formed by a polynomial fuzzy model and a polynomial fuzzy controller connected in a closed loop, under imperfect premise matching, has been investigated using SOS approach based on the Lyapunov stability theory. An improved SOS-based stability analysis approach has been proposed, which not only considers membership functions as symbolic variables in the stability analysis, but also considers the property of the membership functions and the boundary information of the membership functions in the stability analysis. A simulation

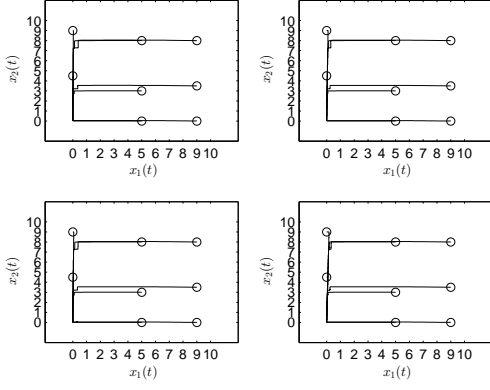


Fig. 6. The top left phase plots are $a = -12; b = -61$ (stability regions indicated by the symbols “+”), the top right phase plots are $a = -14; b = -61$ (stability region indicated by the symbols “x”). The bottom left phase plots are $a = -14; b = -61$ (stability regions indicated by the symbols “□”), the bottom right phase plots are $a = -14; b = -65$ (stability region indicated by the symbols “o”) referring to Fig. 2. And time delay $\tau_1 = \tau_2 = 50$.

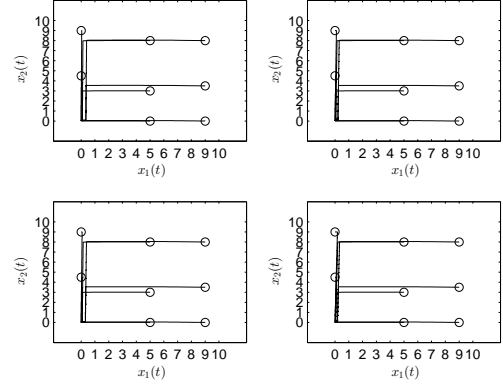


Fig. 8. The top left phase plots are $a = -14; b = -61$ (stability regions indicated by the symbols “+”), the top right phase plots are $a = -14; b = -69$ (stability region indicated by the symbols “x”). The bottom left phase plots are $a = -14; b = -69$ (stability regions indicated by the symbols “□”), the bottom right phase plots are $a = -12; b = -71$ (stability region indicated by the symbols “o”) all referring to Fig. 3. And time delay $\tau_1 = \tau_2 = 50$.

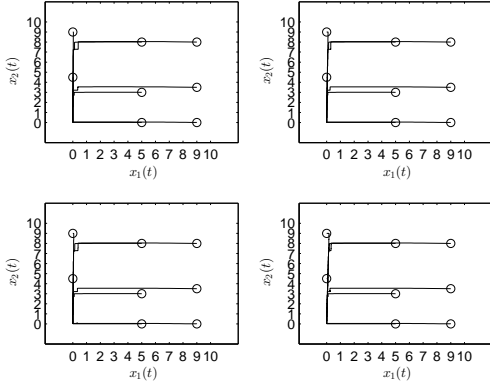


Fig. 7. The top left phase plots are $a = -14; b = -61$ (stability regions indicated by the symbols “+”), the top right phase plots are $a = -14; b = -69$ (stability region indicated by the symbols “x”). The bottom left phase plots are $a = -14; b = -69$ (stability regions indicated by the symbols “□”), the bottom right phase plots are $a = -12; b = -71$ (stability region indicated by the symbols “o”) all referring to Fig. 3. And time delay $\tau_1 = \tau_2 = 1$.

example has been given to show the merits of the proposed method. Furthermore, in the future, the proposed method in this paper can be further relaxed by considering more system information such as the information of premise variables and so on. In the mean time, this method can be applied to design various kinds of fuzzy controllers such as output-feedback controller and observer-based fuzzy controller and so on aiming for different kinds of nonlinear systems.

VI. APPENDIX

The feedback gains corresponding to $a = -10; b = -61$ (stability region indicated by “+”) in Fig. 1 are obtained as $\mathbf{G}_1(x_1) = [-22.5846x_1^2 - 2.3597x_1 - 93.2369, 1.9102x_1^2 + 0.1994x_1 + 7.9358]$, $\mathbf{G}_2(x_1) = [-22.5846x_1^2 - 2.3597x_1 - 93.2369, 1.9102x_1^2 + 0.1994x_1 + 7.9358]$, $\mathbf{G}_3(x_1) = [-22.5846x_1^2 - 2.3597x_1 - 93.2369, 1.9102x_1^2 + 0.1994x_1 + 7.9358]$.

The feedback gains corresponding to $a = -10; b = -63$ (stability region indicated by “x”) in Fig. 1 are obtained as $\mathbf{G}_1(x_1, m_1, m_3) = [-83.3706m_1^2 - 83.3706m_3^2 - 83.3852x_1^2 - 0.1063 \times 10^{-12}m_1x_1 - 0.5207 \times 10^{-13}m_1 - 0.1007 \times 10^{-12}m_3x_1 - 0.2040 \times 10^{-15}m_3 + 0.2015 \times 10^{-13}m_1m_3 - 1.2984x_1 - 127.9484, 6.6728m_1^2 + 6.6728m_3^2 + 6.6740x_1^2 - 0.8512 \times 10^{-14}m_1x_1 - 0.4167 \times 10^{-14}m_1 + 0.1007 \times 10^{-13}m_3x_1 - 0.2040 \times 10^{-16}m_3 + 0.1613 \times 10^{-13}m_1m_3 + 0.1039x_1 + 10.2908]$, $\mathbf{G}_2(x_1, m_1, m_3) = [-83.3706m_1^2 - 83.3706m_3^2 - 83.3852x_1^2 - 0.1819 \times 10^{-11}m_1x_1 + 0.3547 \times 10^{-12}m_1 + 0.7249 \times 10^{-12}m_3x_1 - 0.1481 \times 10^{-11}m_3 - 0.1456 \times 10^{-11}m_1m_3 - 1.2984x_1 - 127.9484, 6.6728m_1^2 + 6.6728m_3^2 + 6.6740x_1^2 + 0.1456 \times 10^{-12}m_1x_1 - 0.2839 \times 10^{-13}m_1 - 0.1277 \times 10^{-13}m_3x_1 + 0.1007 \times 10^{-14}m_3 + 0.3102 \times 10^{-12}m_1m_3 + 0.1039x_1 + 10.2908]$, $\mathbf{G}_3(x_1, m_1, m_3) = [-83.3706m_1^2 - 83.3706m_3^2 - 83.3852x_1^2 - 0.1750 \times 10^{-12}m_1x_1 - 0.1618 \times 10^{-12}m_1 + 0.7249 \times 10^{-12}m_3x_1 - 0.1481 \times 10^{-12}m_3 - 0.1166 \times 10^{-11}m_1m_3 + 1.2984x_1 - 127.9484, 6.6728m_1^2 + 6.6728m_3^2 + 6.6740x_1^2 - 0.1400 \times 10^{-13}m_1x_1 - 0.1295 \times 10^{-13}m_1 - 0.5802 \times 10^{-13}m_3x_1 + 0.1185 \times 10^{-14}m_3 + 0.1166 \times 10^{-12}m_1m_3 + 0.1039x_1 + 10.2908]$.

The feedback gains corresponding to $a = -12; b = -61$ (stability region indicated by “+”) in Fig. 2 are obtained as $\mathbf{G}_1(x_1) = [-11.2193x_1^2 - 3.0262x_1 - 99.8543, 0.8388x_1^2 + 0.2237x_1 + 7.4865]$, $\mathbf{G}_2(x_1) = [-11.2127x_1^2 - 3.2268x_1 - 100.9379, 0.8381x_1^2 + 0.2234x_1 + 7.4858]$, $\mathbf{G}_3(x_1) = [-11.2161x_1^2 - 2.9988x_1 - 99.5609, 0.8383x_1^2 + 0.2237x_1 + 7.4862]$.

The feedback gains corresponding to $a = -14; b = -61$ (stability regions indicated by “x”) in Fig. 2 are obtained as $\mathbf{G}_1(x_1, m_1, m_3) = [-361.6651m_1^2 - 397.7504m_3^2 - 37.8012x_1^2 - 0.3358m_1x_1 + 429.7925m_1 - 0.4970m_3x_1 + 135.6141m_3 - 29.7473m_1m_3 - 1.8764x_1 - 219.4902, 2.1466m_1^2 + 2.0352m_3^2 + 2.6553x_1^2 + 0.0126m_1x_1 + 0.3954m_1 - 0.0037m_3x_1 + 0.1044m_3 + 0.0537m_1m_3 + 0.1678x_1 + 8.1332]$, $\mathbf{G}_2(x_1, m_1, m_3) = [-360.1127m_1^2 -$

$360.2003m_3^2 - 37.7800x_1^2 - 0.4701m_1x_1 + 103.2877m_1 - 0.4935m_3x_1 + 102.9673m_3 - 66.3263m_1m_3 - 2.6521x_1 - 115.8577, 2.0375m_1^2 + 2.0376m_3^2 + 2.6527x_1^2 - 0.0031m_1x_1 + 0.0728m_1 - 0.0032m_3x_1 + 0.0719m_3 - 0.0036m_1m_3 + 0.1699x_1 + 8.0928], \mathbf{G}_3(x_1, m_1, m_3) = [-397.4775m_1^2 - 361.7657m_3^2 - 37.7894x_1^2 - 0.4508m_1x_1 + 136.1019m_1 - 0.3586m_3x_1 + 429.5434m_3 + 29.5629m_1m_3 - 1.8508x_1 - 219.1143, 2.0352m_1^2 + 2.1476m_3^2 + 2.6534x_1^2 - 0.0037m_1x_1 + 0.1048m_1 - 0.0127m_3x_1 + 0.3956m_3 - 0.0540m_1m_3 + 0.1677x_1 + 8.1333].$

The feedback gains corresponding to $a = -14; b = -61$ (stability region indicated by “□”) in Fig. 2 are obtained as $\mathbf{G}_1(x_1) = [-48.6515x_1^2 - 2.2975x_1 - 120.2110, 3.3723x_1^2 + 0.1610x_1 + 8.3485], \mathbf{G}_2(x_1) = [-48.6272x_1^2 - 2.5605x_1 - 121.5924, 3.3697x_1^2 + 0.1595x_1 + 8.3471], \mathbf{G}_3(x_1) = [-48.6064x_1^2 - 2.3245x_1 - 119.9645, 3.3682x_1^2 + 0.1609x_1 + 8.3622].$

The feedback gains corresponding to $a = -14; b = -65$ (stability region indicated by “o”) in Fig. 2 are obtained as $\mathbf{G}_1(x_1, m_1, m_3) = [-125.4687m_1^2 - 142.6125m_3^2 - 9.9651x_1^2 - 0.3832m_1x_1 + 139.4900m_1 - 0.4135m_3x_1 - 28.9491m_3 + 2.8521m_1m_3 - 1.3584x_1 - 86.1288, 0.8950m_1^2 + 0.8802m_3^2 + 1.1580x_1^2 - 0.0061m_1x_1 + 0.0946m_1 - 0.0017m_3x_1 + 0.0227m_3 + 0.0046m_1m_3 + 0.2112x_1 + 7.5052], \mathbf{G}_2(x_1, m_1, m_3) = [-125.3564m_1^2 - 211.3135m_3^2 - 10.0227x_1^2 - 0.4246m_1x_1 + 22.2521m_1 - 0.4158m_3x_1 + 22.0893m_3 - 14.1475m_1m_3 - 1.9473x_1 - 63.9553, 0.8817m_1^2 + 0.8817m_3^2 + 1.1540x_1^2 + 0.0014m_1x_1 + 0.0175m_1 + 0.0014m_3x_1 + 0.0168m_3 - 0.0025m_1m_3 + 0.2118x_1 + 7.4836], \mathbf{G}_3(x_1, m_1, m_3) = [-142.4581m_1^2 - 125.7550m_3^2 - 10.1090x_1^2 - 0.4202m_1x_1 + 29.4880m_1 - 0.3764m_3x_1 + 139.6261m_3 + 2.7750m_1m_3 - 1.3651x_1 - 86.1990, 0.8802m_1^2 + 0.8959m_3^2 + 1.1499x_1^2 - 0.0017m_1x_1 + 0.0217m_1 - 0.0060m_3x_1 + 0.0884m_3 + 0.0049m_1m_3 + 0.2115x_1 + 7.5077].$

The feedback gains of $a = -14; b = -61$ (stability regions indicated by “+”) referring to Fig. 3 are obtained as $\mathbf{G}_1(x_1) = [-26.3303x_1^2 - 3.4227x_1 - 126.3813, 1.5977x_1^2 + 0.2026x_1 + 7.7274], \mathbf{G}_2(x_1) = [-26.3282x_1^2 - 4.0256x_1 - 128.1270, 1.5969x_1^2 + 0.2008x_1 + 7.7287], \mathbf{G}_3(x_1) = [-26.3045x_1^2 - 3.3473x_1 - 126.5332, 1.5961x_1^2 + 0.2028x_1 + 7.7240].$

The feedback gains corresponding to $a = -14; b = -69$ (stability regions indicated by “x”) in Fig. 3 are obtained as $\mathbf{G}_1(x_1, m_1, m_3) = [-496.7636m_1^2 - 545.6796m_3^2 - 40.8410x_1^2 - 0.8049m_1x_1 + 598.0971m_1 - 0.7558m_3x_1 + 186.0998m_3 - 44.1162m_1m_3 - 3.4351x_1 - 311.19445, 1.4750m_1^2 + 1.4250m_3^2 + 1.8652x_1^2 - 0.0117m_1x_1 + 0.1732m_1 - 0.0033m_3x_1 + 0.0434m_3 + 0.0219m_1m_3 + 0.1912x_1 + 7.8690], \mathbf{G}_2(x_1, m_1, m_3) = [-496.6929m_1^2 - 507.5867m_3^2 - 40.8234x_1^2 - 0.9187m_1x_1 - 140.4803m_1 - 0.2332m_3x_1 + 168.8466m_3 - 94.7016m_1m_3 - 5.3135x_1 - 190.2168, 1.4261m_1^2 + 1.4264m_3^2 + 1.8641x_1^2 - 0.0029m_1x_1 + 0.0246m_1 - 0.0026m_3x_1 + 0.0304m_3 - 0.0025m_1m_3 + 0.1940x_1 + 7.8478], \mathbf{G}_3(x_1, m_1, m_3) = [-545.3966m_1^2 - 483.3233m_3^2 - 40.7880x_1^2 - 0.9552m_1x_1 + 177.2751m_1 - 0.8934m_3x_1 - 572.9140m_3 + 36.7453m_1m_3 - 3.0724x_1 - 297.1138, 1.4249m_1^2 + 1.4784m_3^2 + 1.8625x_1^2 + 0.0032m_1x_1 + 0.0439m_1 - 0.0111m_3x_1 + 0.1770m_3 - 0.0228m_1m_3 +$

$0.1921x_1 + 7.8703].$

The feedback gains corresponding to $a = -14; b = -69$ (stability regions indicated by “□”) in Fig. 3 are obtained as $\mathbf{G}_1(x_1) = [-50.5851x_1^2 - 3.0418x_1 - 140.3193, 2.9396x_1^2 + 0.1693x_1 + 8.2198], \mathbf{G}_2(x_1) = [-50.5727x_1^2 - 3.6755x_1 - 141.9099, 2.9381x_1^2 + 0.1652x_1 + 8.2196], \mathbf{G}_3(x_1) = [-50.5260x_1^2 - 2.9244x_1 - 140.4469, 2.9359x_1^2 + 0.1698x_1 + 8.2088].$

The feedback gains corresponding to $a = -12; b = -71$ (stability regions indicated by “o”) in Fig. 3 are obtained as $\mathbf{G}_1(x_1, m_1, m_3) = [-626.1074m_1^2 - 670.5284m_3^2 - 58.9988x_1^2 - 0.6542m_1x_1 + 758.0385m_1 - 0.6593m_3x_1 - 290.4993m_3 + 147.1005m_1m_3 - 3.2659x_1 - 383.9601, 2.3709m_1^2 + 2.1635m_3^2 + 2.7998x_1^2 - 0.0083m_1x_1 + 0.8135m_1 - 0.0023m_3x_1 + 0.2336m_3 + 0.1102m_1m_3 + 0.1821x_1 + 8.7733], \mathbf{G}_2(x_1, m_1, m_3) = [-622.9477m_1^2 - 631.7142m_3^2 - 58.9802x_1^2 - 0.6894m_1x_1 - 199.7583m_1 - 0.0859m_3x_1 - 220.4510m_3 - 191.5773m_1m_3 - 4.8630x_1 - 197.3297, 2.1589m_1^2 + 2.1614m_3^2 + 2.7986x_1^2 - 0.0030m_1x_1 - 0.1476m_1 - 0.0026m_3x_1 - 0.1665m_3 - 0.0036m_1m_3 + 0.1785x_1 + 8.5842], \mathbf{G}_3(x_1, m_1, m_3) = [-666.0368m_1^2 - 612.5704m_3^2 - 58.9368x_1^2 - 0.7835m_1x_1 + 276.7703m_1 - 0.8169m_3x_1 + 728.4554m_3 - 133.3529m_1m_3 - 2.8835x_1 - 367.9759, 2.1635m_1^2 + 2.3779m_3^2 + 2.7965x_1^2 - 0.0022m_1x_1 - 0.2377m_1 - 0.8397m_3x_1 - 0.8397m_3 - 0.1121m_1m_3 + 0.1828x_1 + 8.7924].$

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