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An application of the method of moments to volatility estimation using daily high, low, opening and closing prices

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Abstract

We use the expectation of the range of an arithmetic Brownian motion and the method of moments on the daily high, low, opening and closing prices to estimate the volatility of the stock price. The daily price jump at the opening is considered to be the result of the unobserved evolution of an after-hours virtual trading day. The annualized volatility is used to calculate Black-Scholes prices for European options, and a trading strategy is devised to profit when these prices differ flagrantly from the market prices.

Key words: Range-based volatility estimation, method of moments, daily high, low, opening and closing prices, density and expectation of the range of an arithmetic Brownian motion.

AMS subject classifications: 91G20, 60J65, 62F10, 62G05, 62P05

JEL classification: G12, G13, C13, C46.

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1 Introduction

This article is a modified version of what has been studied in the Ph.D. thesis of Koné (1996). It concerns the application of the method of moments to range-based volatility estimation using daily high, low, opening and closing stock prices. Aiming to estimate volatility and not to measure it, we assume a Black-Scholes framework with constant volatility and use daily data to achieve it (see Rogers and Zhou (2008) for further motivation for this choice).

Incidental to this is the derivation of the density and expectation of the range of an arithmetic Brownian motion. Subsequent to this thesis, portions of it have been studied for different purposes (see, for instance, Sutrick et al (1997) for the use of the density of the range of an arithmetic Brownian motion in the *do-nothing option*, or Magdon-Ismail et al (2000, 2004) for the use of the expectation of the range of an arithmetic Brownian motion in different contexts). In particular, expressing the density of the range in the context of Sutrick et al (1997) corrects their expression.

The literature on range-based volatility estimation includes classic work by Garman and Klass (1980), Parkinson (1980), Rogers and Satchell (1991) and Rogers et al (1994), whose estimators are reviewed in Yang and Zhang (2000). Of these, the latter paper is most related to the current one because it considers after-hours price jumps in addition to drift. However, the methods presented here are different², and perhaps more practical (see the remarks of Chan and Lien (2003) on the empirical availability of certain parameters in Yang and Zhang (2000)).

Starting from the joint density of the running maximum and the current value of an arithmetic Brownian motion, the density of their difference (referred to as half-range) is obtained, and its expectation computed. This allows the computation of the expectation of the full range (defined as maximum minus minimum), which will then be used in the method of moments for intra-day volatility estimation.

After-hours arrival of information results in price jumps at the opening, and we model this as a virtual trading day which is unobservable, but which, when succeeding the trading day, gives on average the complete statistical representation of one day.

Black-Scholes option prices are computed using the parameter estimates, and when they differ the most from the observed market prices a profit is made by an appropriately devised trading strategy.

The paper is organized as follows. In Section 2 we derive the expectation and the density function of an arithmetic Brownian motion. In Section 3 the method of moments is used to estimate the parameters of the stock price based on daily high, low, opening and closing data. The estimated parameters are then used in Section 4 to price European options on the stock, which are then compared to market prices to identify instances of flagrant differences. The effect of the mispricing is estimated by computing the profit to be made in these opportunities. We conclude in Section 5 with some comments on the efficiency of the method of moments in volatility estimation.

²see Remarks 3.1 and 3.2

2 The range of an arithmetic Brownian motion: expectation and density

Let $\{\Omega, \mathcal{F}, P\}$ be a probability space endowed with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$, and let $\{W_t\}_{t \geq 0}$ be a one-dimensional standard Brownian motion adapted to $\{\mathcal{F}_t\}_{t \geq 0}$. For $t \geq 0$ let X_t denote a standard arithmetic Brownian motion with drift μ and volatility $\sigma > 0$:

$$X_t = \mu t + \sigma W_t, \quad X_0 = 0, \quad (2.1)$$

and let $\{M_t\}_{t \geq 0}$, $\{m_t\}_{t \geq 0}$ and $\{R_t\}_{t \geq 0}$ denote its running maximum, minimum, and range, respectively:

$$M_t := \sup_{0 \leq s \leq t} X_s, \quad m_t = \inf_{0 \leq s \leq t} X_s, \quad R_t := M_t - m_t. \quad (2.2)$$

First we derive the expectation $E[R_t]$ of the range of the arithmetic Brownian motion X_t . This is achieved by computing the density and expectation of the half-range $M_t - X_t$ from the joint density of X_t and M_t .

Lemma 2.1. *The joint density function of an arithmetic Brownian motion and its running maximum can be expressed as:*

$$P(X_t \in da, M_t \in db) = \frac{2(2b-a)}{\sqrt{2\pi t^3} \sigma^3} \exp \left\{ -\frac{(2b-a)^2}{2t\sigma^2} + \frac{\mu}{\sigma^2} a - \frac{1}{2} \frac{\mu^2}{\sigma^2} t \right\} da db. \quad (2.3)$$

Proof. The proof is standard. Using the martingale $Z_t(Y) = \exp\{W_t \mu / \sigma - t \mu^2 / (2\sigma^2)\}$, Girsanov's change of measure defines a new probability measure \tilde{P} for any measurable set A by $\tilde{P}(A) = E[Z_t(Y) 1_A]$. Theorem 3.2.2 of Karatzas and Shreve (1988) with $Y_t = \mu / \sigma$ and $N_t = \sigma W_t$ gives that $\tilde{N}_t = \sigma W_t - \mu t$ is a local martingale. The process $\tilde{W}_t = W_t - t \mu / \sigma$ is a Brownian motion under the new probability measure \tilde{P} . Equivalently, $\sigma W_t = \mu t + \sigma \tilde{W}_t$ is a Brownian motion with drift under \tilde{P} , and we can write:

$$\begin{aligned} P(X_t \leq a, M_t \leq b) &= \tilde{P}(\sigma W_t \leq a, \sigma \bar{M}_t \leq b) \\ &= \int_{-\infty}^a \exp \left(\frac{\mu}{\sigma^2} x - \frac{1}{2} \frac{\mu^2}{\sigma^2} t \right) P(\sigma W_t \in dx, \sigma \bar{M}_t \leq b). \end{aligned} \quad (2.4)$$

An application of the reflection principle gives:

$$\begin{aligned} P(\sigma W_t \leq a, \sigma \bar{M}_t \leq b) &= P\left(W_t \leq \frac{a}{\sigma}\right) - P\left(W_t \leq \frac{a}{\sigma}, \sigma \bar{M}_t > b\right) \\ &= P\left(W_t \leq \frac{a}{\sigma}\right) - P\left(W_t > \frac{2b-a}{\sigma}\right) \\ &= \Phi\left(\frac{a}{\sigma\sqrt{t}}\right) - 1 + \Phi\left(\frac{2b-a}{\sigma\sqrt{t}}\right), \end{aligned}$$

where we denote by $\phi(\cdot)$ and $\Phi(\cdot)$ the standard normal density and cumulative distribution functions, respectively.

Differentiating the formula above with respect to a gives:

$$P(\sigma W_t \in da, \sigma \bar{M}_t \leq b) = \frac{1}{\sigma\sqrt{t}} \left(\phi\left(\frac{a}{\sigma\sqrt{t}}\right) - \phi\left(\frac{2b-a}{\sigma\sqrt{t}}\right) \right). \quad (2.5)$$

Replacing (2.5) in (2.4) and differentiating first with respect to a gives:

$$P(X_t \in da, M_t \leq b) = \frac{1}{\sigma\sqrt{t}} \exp\left(\frac{\mu}{\sigma^2}a - \frac{1}{2}\frac{\mu^2}{\sigma^2}t\right) \left(\phi\left(\frac{a}{\sigma\sqrt{t}}\right) - \phi\left(\frac{2b-a}{\sigma\sqrt{t}}\right) \right) da,$$

and differentiating then with respect to b gives:

$$\begin{aligned} P(X_t \in da, M_t \in db) &= \frac{1}{\sigma\sqrt{t}} \exp\left(\frac{\mu}{\sigma^2}a - \frac{1}{2}\frac{\mu^2}{\sigma^2}t\right) \left(-\frac{2}{\sigma\sqrt{t}}\right) \phi'\left(\frac{2b-a}{\sigma\sqrt{t}}\right) da db \\ &= \left(-\frac{2}{t\sigma^2}\right) \exp\left(\frac{\mu}{\sigma^2}a - \frac{1}{2}\frac{\mu^2}{\sigma^2}t\right) \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{2b-a}{\sigma\sqrt{t}}\right)^2\right) \left(-\frac{1}{2}\right) 2\frac{2b-a}{\sigma\sqrt{t}} da db \\ &= \frac{2(2b-a)}{\sqrt{2\pi t^3}\sigma^3} \exp\left\{-\frac{(2b-a)^2}{2t\sigma^2} + \frac{\mu}{\sigma^2}a - \frac{1}{2}\frac{\mu^2}{\sigma^2}t\right\} da db. \end{aligned}$$

The joint density $f_{X_t, M_t}(a, b)$ is given by the term multiplying $da db$ above. \square

Remark 2.1. *This is one of those results that seemed to be always at hand (it can be obtained from equation (1.8.8) of Harrison (1985)), but never derived. Note the typo in Yang and Zhang (2000), whose expression (B1) has a plus for the first fraction in the exponential. For $\sigma = 1$ this was used in Example E5 of Karatzas and Shreve (1998) in relationship to Clark's formula to obtain explicitly the hedging portfolio.*

The density of the half-range $M_t - X_t$ can be obtained using a standard two-dimensional transformation of the above joint density.

Lemma 2.2. *The density of the half-range $M_t - X_t$ is given by:*

$$f_{M_t - X_t}(c) = 2\frac{\mu}{\sigma^2} \Phi\left(\frac{\mu t - c}{\sigma\sqrt{t}}\right) \exp\left(-2\frac{\mu}{\sigma^2}c\right) + \frac{2}{\sigma\sqrt{2t\pi}} \exp\left\{-\frac{(\mu t + c)^2}{2t\sigma^2}\right\}. \quad (2.6)$$

Proof. For $Y_1 = X_1 + X_2$ and $Y_2 = X_1 - X_2$ the joint density is $f_{Y_1, Y_2}(y_1, y_2) = \frac{1}{2}f_{X_1, X_2}\left(\frac{y_1 + y_2}{2}, \frac{y_1 - y_2}{2}\right)$. Taking $X_1 = M_t$ and $X_2 = X_t$ and using Lemma 2.1 gives:

$$\begin{aligned} f_{Y_1, Y_2}(y_1, y_2) &= \frac{1}{2}f_{M_t, X_t}\left(\frac{y_1 + y_2}{2}, \frac{y_1 - y_2}{2}\right) = \frac{1}{2}f_{X_t, M_t}\left(\frac{y_1 - y_2}{2}, \frac{y_1 + y_2}{2}\right) \\ &= \frac{1}{2} \frac{2y_1 + 2y_2 - y_1 + y_2}{\sqrt{2\pi t^3}\sigma^3} \exp\left\{-\frac{(2y_1 + 2y_2 - y_1 + y_2)^2}{8t\sigma^2} + \frac{\mu}{\sigma^2} \frac{y_1 - y_2}{2} - \frac{1}{2}\frac{\mu^2}{\sigma^2}t\right\} \\ &= \frac{1}{2} \frac{y_1 + 3y_2}{\sqrt{2\pi t^3}\sigma^3} \exp\left\{-\frac{(y_1 + 3y_2)^2}{8t\sigma^2} + \frac{\mu}{\sigma^2} \frac{y_1 - y_2}{2} - \frac{1}{2}\frac{\mu^2}{\sigma^2}t\right\} \\ &= \frac{1}{2} \frac{y_1 + 3y_2}{\sqrt{2\pi t^3}\sigma^3} \exp\left\{-\frac{1}{2} \frac{(y_1 + 3y_2 - 2\mu t)^2}{4t\sigma^2} - 2\frac{\mu}{\sigma^2}y_2\right\} \end{aligned}$$

Note that $M_t \geq 0$ implies $Y_1 \geq -Y_2$, thus the marginal density of $Y_2 = M_t - X_t$ is:

$$f_{Y_2}(y_2) = \int_{-y_2}^{\infty} f_{Y_1, Y_2}(y_1, y_2) dy_1.$$

A change of variable $z = (y_1 + 3y_2 - 2\mu t)/(2\sigma\sqrt{t})$ gives:

$$z > z_0 := \frac{y_2 - \mu t}{\sigma\sqrt{t}}, \quad dy_1 = 2\sigma\sqrt{t} dz,$$

therefore:

$$\begin{aligned} f_{Y_2}(y_2) &= \int_{z_0}^{\infty} \frac{1}{2} \frac{2z\sigma\sqrt{t} + 2\mu t}{\sqrt{2\pi t^3} \sigma^3} \exp \left\{ -\frac{1}{2} z^2 - 2\frac{\mu}{\sigma^2} y_2 \right\} 2\sigma\sqrt{t} dz \\ &= \frac{2}{\sigma\sqrt{t}} \int_{z_0}^{\infty} z \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz \exp \left\{ -2\frac{\mu}{\sigma^2} y_2 \right\} + 2\frac{\mu}{\sigma^2} \int_{z_0}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz \exp \left\{ -2\frac{\mu}{\sigma^2} y_2 \right\} \\ &= \frac{2}{\sigma\sqrt{t}} \int_{z_0}^{\infty} -\left(\frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \right)' dz \exp \left\{ -2\frac{\mu}{\sigma^2} y_2 \right\} + 2\frac{\mu}{\sigma^2} \left(1 - \Phi(z_0) \right) \exp \left\{ -2\frac{\mu}{\sigma^2} y_2 \right\} \\ &= \frac{2}{\sigma\sqrt{t}} \frac{1}{\sqrt{2\pi}} e^{-\frac{z_0^2}{2}} \exp \left\{ -2\frac{\mu}{\sigma^2} y_2 \right\} + 2\frac{\mu}{\sigma^2} \Phi(-z_0) \exp \left\{ -2\frac{\mu}{\sigma^2} y_2 \right\}. \end{aligned}$$

This can be rewritten as:

$$\begin{aligned} P(M_t - X_t \in dc) &= \left(2\frac{\mu}{\sigma^2} \Phi \left(\frac{\mu t - c}{\sigma\sqrt{t}} \right) \exp \left(-2\frac{\mu}{\sigma^2} c \right) \right. \\ &\quad \left. + \frac{2}{\sigma\sqrt{2t\pi}} \exp \left\{ -\frac{(\mu t - c)^2}{2t\sigma^2} \right\} \exp \left(-2\frac{\mu}{\sigma^2} c \right) \right) dc, \quad (2.7) \end{aligned}$$

and the result follows. \square

Proposition 2.1. *The expectation of the half-range is given by:*

$$\begin{aligned} E(M_t - X_t) &= \frac{\sigma^2}{2\mu} \Phi \left(\frac{\mu}{\sigma} \sqrt{t} \right) - \left(\mu t + \frac{\sigma^2}{2\mu} \right) \left(1 - \Phi \left(\frac{\mu}{\sigma} \sqrt{t} \right) \right) \\ &\quad + \frac{\sigma\sqrt{t}}{\sqrt{2\pi}} \exp \left(-\frac{t\mu^2}{2\sigma^2} \right). \quad (2.8) \end{aligned}$$

Proof. A simple calculation yields:

$$\begin{aligned}
 E(M_t - X_t) &= \int_0^\infty c \frac{2\mu}{\sigma^2} \Phi\left(\frac{\mu t - c}{\sigma\sqrt{t}}\right) \exp\left(-2\frac{\mu}{\sigma^2}c\right) dc \\
 &\quad + \int_0^\infty c \frac{2}{\sigma\sqrt{2t\pi}} \exp\left\{-\frac{(\mu t + c)^2}{2t\sigma^2}\right\} dc \\
 &= \int_0^\infty \left\{-(c + \frac{\sigma^2}{2\mu}) \exp\left(-2\frac{\mu}{\sigma^2}c\right)\right\}' \Phi\left(\frac{\mu t - c}{\sigma\sqrt{t}}\right) dc \\
 &\quad + \int_0^\infty c \frac{2}{\sigma\sqrt{2t\pi}} \exp\left\{-\frac{(\mu t + c)^2}{2t\sigma^2}\right\} dc \\
 &= \frac{\sigma^2}{2\mu} \Phi\left(\frac{\mu}{\sigma}\sqrt{t}\right) + \frac{1}{\sigma\sqrt{t}} \frac{1}{\sqrt{2\pi}} \int_0^\infty \left(-c - \frac{2\sigma^2}{\mu} + 2c\right) \exp\left\{-\frac{(\mu t + c)^2}{2t\sigma^2}\right\} dc.
 \end{aligned}$$

A change of variable $c = z\sigma\sqrt{t} - \mu t$ gives the result. \square

Consider now $E[X_t - m_t]$. For each path of the Brownian motion X_t with drift μ consider a symmetric path of a Brownian motion \tilde{X}_t having drift $-\mu$. Then $X_t - m_t = -(\tilde{X}_t - \tilde{M}_t)$ and $E[X_t - m_t]$ can be calculated using the equation (2.8) with μ replaced by $-\mu$. Whereas the formula for the expectation of the range follows.

Theorem 2.1. *The expectation of the range of the arithmetic Brownian motion X_t defined in (2.1) is given by:*

$$E[R_t] = \left(\mu t + \frac{\sigma^2}{\mu}\right) \left(1 - 2\Phi\left(-\sqrt{t}\frac{\mu}{\sigma}\right)\right) + 2\frac{\sigma\sqrt{t}}{\sqrt{2\pi}} \exp\left(-\frac{t\mu^2}{2\sigma^2}\right). \quad (2.9)$$

Let us denote this expected range function by $ER(\mu, \sigma, t)$. On closer inspection this can be further simplified as a function of just two quantities:

$$E[R_t] = ER(\mu, \sigma, t) = h\left(\frac{\mu t}{\sigma\sqrt{t}}, \frac{\sigma^2}{\mu}\right), \quad (2.10)$$

where the function h is defined by:

$$h(x, y) := \left\{ (x^2 + 1)(2\Phi(x) - 1) + \frac{2x}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \right\} y. \quad (2.11)$$

Note that $ER(\mu, \sigma, t) = ER(\mu t, \sigma\sqrt{t}, 1)$ as it should (the range over a time interval $(0, t)$ of an arithmetic Brownian motion with parameters μ and σ is the same as that over $(0, 1)$ when the parameters change to μt and $\sigma\sqrt{t}$).

In the remainder of this section we derive the density of the range R_t of the arithmetic Brownian motion X_t . This is achieved by the use of the joint density of the minimum and the maximum of X_t , a result with its own merit, that we could not find

published prior to Koné (1996) (Borodin and Salminen (1996, 1.15.4) published in the same year the joint cumulative distribution function only in terms of some definite integrals).

A version of this result is used in Sutrick et al (1997) for the same purpose, but there it seems to have incorporated an error.

To obtain the joint density $F(a, b)$ of the maximum and the minimum we start with a lemma.

Lemma 2.3. *We can write:*

$$F(a, b) = \int_a^b h(a, b, x) \exp\left(\frac{\mu}{\sigma^2}x - \frac{1}{2}\frac{\mu^2}{\sigma^2}t\right) dx, \quad (2.12)$$

where

$$\begin{aligned} h(a, b, x) &= h_1(a, b, x) - h_2(a, b, x), \quad (2.13) \\ h_1(a, b, x) &= \sum_{k=-\infty}^{\infty} \frac{2k(2k-2)}{\sigma^3 t \sqrt{2\pi t}} \left[1 - \frac{[2k(b-a) - 2b + x]^2}{t\sigma^2} \right] \\ &\quad \times \exp\left(-\frac{[2k(b-a) - 2b + x]^2}{2t\sigma^2}\right), \\ h_2(a, b, x) &= \sum_{k=-\infty}^{\infty} \frac{4k^2}{\sigma^3 t \sqrt{2\pi t}} \left[1 - \frac{[2k(b-a) - x]^2}{t\sigma^2} \right] \exp\left(-\frac{[2k(b-a) - x]^2}{2t\sigma^2}\right). \end{aligned}$$

Proof. Using the change of measure of the proof of Lemma 2.1 and Girsanov's theorem we have:

$$P(a < m_t < M_t < b) = \int_a^b p_t(x; a, b) \exp\left(\frac{\mu}{\sigma^2}x - \frac{1}{2}\frac{\mu^2}{\sigma^2}t\right) dx, \quad (2.14)$$

which gives $F(a, b)$ via:

$$F(a, b) = -\frac{\partial^2 P(a < m_t < M_t < b)}{\partial a \partial b}. \quad (2.15)$$

Leibniz rule of differentiation:

$$\frac{\partial}{\partial z} \int_{a(z)}^{b(z)} f(x, z) dx = \int_{a(z)}^{b(z)} \frac{\partial f(x, z)}{\partial z} dx + f(b(z), z) \frac{\partial b}{\partial z} - f(a(z), z) \frac{\partial a}{\partial z} \quad (2.16)$$

gives the partial derivative of (2.36) wrt b :

$$\begin{aligned} \int_a^b \frac{1}{\sigma\sqrt{t}} \sum_{k=-\infty}^{\infty} \left[\frac{\partial \phi}{\partial b} \left(\frac{2k(b-a) - x}{\sigma\sqrt{t}} \right) - \frac{\partial \phi}{\partial b} \left(\frac{2k(b-a) - 2b + x}{\sigma\sqrt{t}} \right) \right] \\ \times \exp\left(\frac{\mu}{\sigma^2}x - \frac{1}{2}\frac{\mu^2}{\sigma^2}t\right) dx. \quad (2.17) \end{aligned}$$

Differentiating wrt a this last equation gives:

$$F(a, b) = - \int_a^b \frac{1}{\sigma\sqrt{t}} \sum_{k=-\infty}^{\infty} \left[\frac{\partial^2 \phi}{\partial a \partial b} \left(\frac{2k(b-a) - x}{\sigma\sqrt{t}} \right) - \frac{\partial^2 \phi}{\partial a \partial b} \left(\frac{2k(b-a) - 2b + x}{\sigma\sqrt{t}} \right) \right] \times \exp \left(\frac{\mu}{\sigma^2} x - \frac{1}{2} \frac{\mu^2}{\sigma^2} t \right) dx. \quad (2.18)$$

Direct calculation gives:

$$\begin{aligned} \frac{\partial^2 \phi}{\partial a \partial b} \left(\frac{2k(b-a) - x}{\sigma\sqrt{t}} \right) &= \frac{\partial}{\partial a} \left[\frac{(-2k)[2k(b-a) - x]}{\sqrt{2\pi}t\sigma^2} \exp \left(- \frac{[2k(b-a) - x]^2}{2t\sigma^2} \right) \right] \\ &= \frac{4k^2}{\sqrt{2\pi}t\sigma^2} \exp \left(- \frac{[2k(b-a) - x]^2}{2t\sigma^2} \right) \left(1 - \frac{[2k(b-a) - x]^2}{t\sigma^2} \right), \end{aligned} \quad (2.19)$$

$$\begin{aligned} \frac{\partial^2 \phi}{\partial a \partial b} \left(\frac{2k(b-a) - 2b + x}{\sigma\sqrt{t}} \right) &= \frac{\partial}{\partial a} \left[\frac{-(2k-2)[2k(b-a) - 2b + x]}{\sqrt{2\pi}t\sigma^2} \exp \left(- \frac{[2k(b-a) - 2b + x]^2}{2t\sigma^2} \right) \right] \\ &= \frac{4k(k-1)}{\sqrt{2\pi}t\sigma^2} \exp \left(- \frac{[2k(b-a) - 2b + x]^2}{2t\sigma^2} \right) \left(1 - \frac{[2k(b-a) - 2b + x]^2}{t\sigma^2} \right). \end{aligned} \quad (2.20)$$

Substituting (2.19)-(2.20) in (2.18) gives the result. \square

Proposition 2.2. *The joint density function $F(a, b)$ of M_t and m_t can be represented as:*

$$\begin{aligned} F(a, b) &= F_1(a, b) - F_2(a, b) - F_3(a, b) + F_4(a, b) \\ &\quad - F_5(a, b) + F_6(a, b) + F_7(a, b) - F_8(a, b), \end{aligned} \quad (2.21)$$

with

$$\begin{aligned} F_1(a, b) &= \sum_{k=-\infty}^{\infty} \frac{4k(k-1)}{t\sigma^3\sqrt{2\pi}t} [(2k-1)b - 2ka + \mu t] \\ &\quad \times \exp \left\{ - \frac{\mu}{\sigma^2} [2(k-1)b - 2ka] - \frac{[(2k-1)b - 2ka - \mu t]^2}{2t\sigma^2} \right\}, \end{aligned} \quad (2.22)$$

$$\begin{aligned} F_2(a, b) &= \sum_{k=-\infty}^{\infty} \frac{4k(k-1)}{t\sigma^3\sqrt{2\pi}t} [2(k-1)b - (2k-1)a + \mu t] \\ &\quad \times \exp \left\{ - \frac{\mu}{\sigma^2} [2(k-1)b - 2ka] - \frac{[2(k-1)b - (2k-1)a - \mu t]^2}{2t\sigma^2} \right\}, \end{aligned} \quad (2.23)$$

$$F_3(a, b) = \sum_{k=-\infty}^{\infty} \frac{4k(k-1)\mu^2}{2\sigma^4} \exp \left\{ -\frac{\mu}{\sigma^2} [2(k-1)b - 2ka] \right\} \\ \times \operatorname{erf} \left(\frac{(2k-1)b - 2ka - \mu t}{\sigma\sqrt{2t}} \right), \quad (2.24)$$

$$F_4(a, b) = \sum_{k=-\infty}^{\infty} \frac{4k(k-1)\mu^2}{2\sigma^4} \exp \left\{ -\frac{\mu}{\sigma^2} [2(k-1)b - 2ka] \right\} \\ \times \operatorname{erf} \left(\frac{2(k-1)b - (2k-1)a - \mu t}{\sigma\sqrt{2t}} \right), \quad (2.25)$$

$$F_5(a, b) = \sum_{k=-\infty}^{\infty} \frac{4k^2}{t\sigma^3\sqrt{2\pi t}} [(2k+1)b - 2ka + \mu t] \\ \times \exp \left\{ -\frac{\mu}{\sigma^2} 2k(b-a) - \frac{[(2k+1)b - 2ka - \mu t]^2}{2t\sigma^2} \right\}, \quad (2.26)$$

$$F_6(a, b) = \sum_{k=-\infty}^{\infty} \frac{4k^2}{t\sigma^3\sqrt{2\pi t}} [2kb - (2k-1)a + \mu t] \\ \times \exp \left\{ -\frac{\mu}{\sigma^2} 2k(b-a) - \frac{[2kb - (2k-1)a - \mu t]^2}{2t\sigma^2} \right\}, \quad (2.27)$$

$$F_7(a, b) = \sum_{k=-\infty}^{\infty} \frac{4k^2\mu^2}{2\sigma^4} \exp \left\{ -\frac{\mu}{\sigma^2} 2k(b-a) \right\} \operatorname{erf} \left(\frac{(2k+1)b - 2ka - \mu t}{\sigma\sqrt{2t}} \right), \quad (2.28)$$

$$F_8(a, b) = \sum_{k=-\infty}^{\infty} \frac{4k^2\mu^2}{2\sigma^4} \exp \left\{ -\frac{\mu}{\sigma^2} 2k(b-a) \right\} \operatorname{erf} \left(\frac{2kb - (2k-1)a - \mu t}{\sigma\sqrt{2t}} \right), \quad (2.29)$$

where

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt. \quad (2.30)$$

Proof. From Lemma 2.3 we write $F(a, b)$ as the difference of two terms, which we denote I_1 and I_2 :

$$I_1 := \int_a^b h_1(a, b, x) \exp \left(\frac{\mu}{\sigma^2} x - \frac{1}{2} \frac{\mu^2}{\sigma^2} t \right) dx, \\ I_2 := \int_a^b h_2(a, b, x) \exp \left(\frac{\mu}{\sigma^2} x - \frac{1}{2} \frac{\mu^2}{\sigma^2} t \right) dx. \quad (2.31)$$

In each I_1 and I_2 we combine the exponents and then use, respectively, a change of variable:

$$z = \frac{x + 2k(b - a) - 2b - \mu t}{\sigma\sqrt{t}}, \quad z = \frac{x + 2k(b - a) - \mu t}{\sigma\sqrt{t}}, \quad (2.32)$$

followed by an integration by parts for $\int z^2 \exp(-z^2/2)dz$ and replacement of $\Phi(x)$ by $\operatorname{erf}(x)$ via $\Phi(x) = 0.5 \operatorname{erf}(x/\sqrt{2}) + 0.5$. The resulting eight terms are then denoted $F_i(a, b)$, $i = 1, \dots, 8$. \square

We use the above expression to derive the density of the range. To make it suitable for comparison with that obtained by Sutrick et al (1997) in their Proposition 1, we change $k \rightarrow k + 1$ in the summations of F_1, F_2, F_3 and F_4 .

Proposition 2.3. *The density function $f_{R_t}(r)$ for the range of an arithmetic Brownian motion can be written as:*

$$f_{R_t}(r) = \frac{1}{\sigma\sqrt{t}} \sum_{k=-\infty}^{\infty} 4k^2 I(k) + \frac{1}{\sigma\sqrt{t}} \sum_{k=-\infty}^{\infty} 4k(k+1) J(k), \quad (2.33)$$

where

$$\begin{aligned} I(k) &= e^{-\frac{2\mu kr}{\sigma^2}} (1 + c^2) (\phi(K_1 - c) - 2\phi(K_0 - c) + \phi(K_{-1} - c)) \\ &+ e^{-\frac{2\mu kr}{\sigma^2}} [(c^2 K_1 - 2c - c^3) \Phi(K_1 - c) - 2(c^2 K_0 - 2c - c^3) \Phi(K_0 - c) \\ &+ (c^2 K_{-1} - 2c - c^3) \Phi(K_{-1} - c)], \end{aligned} \quad (2.34)$$

and

$$\begin{aligned} J(k) &= e^{\frac{2\mu kr}{\sigma^2}} (\phi(K_1 + c) - \phi(K_0 + c)) - e^{\frac{2\mu(k+1)r}{\sigma^2}} (\phi(K_2 + c) - \phi(K_1 + c)) \\ &+ e^{-\frac{2\mu kr}{\sigma^2}} \left(-\frac{c}{2} \Phi(K_1 - c) + \frac{c}{2} \Phi(K_0 - c) \right) \\ &- e^{-\frac{2\mu(k+1)r}{\sigma^2}} \left(-\frac{c}{2} \Phi(K_2 - c) + \frac{c}{2} \Phi(K_1 - c) \right) \\ &+ e^{\frac{2\mu kr}{\sigma^2}} (\Phi(K_1 + c) - \Phi(K_0 + c)) \\ &- e^{\frac{2\mu(k+1)r}{\sigma^2}} (\Phi(K_2 + c) - \Phi(K_1 + c)), \end{aligned} \quad (2.35)$$

with

$$K_2 = \frac{(2k+2)r}{\sigma\sqrt{t}}, \quad K_1 = \frac{(2k+1)r}{\sigma\sqrt{t}}, \quad K_0 = \frac{2kr}{\sigma\sqrt{t}}, \quad K_{-1} = \frac{(2k-1)r}{\sigma\sqrt{t}}, \quad c = \frac{\mu\sqrt{t}}{\sigma}.$$

Proof. After replacing k by $k+1$ in F_i , $i = 1, \dots, 4$ of Proposition 2.2, a two-dimensional transformation $a = u - v, b = u$, gives, via Jacobian, the density of the range and the running maximum. Its marginal density is the one we seek:

$$f(r) = \int_0^r F(u - r, u) du.$$

Applying a change of variable and integration by parts gives the result. \square

Remark 2.2. *This result corrects that of Sutrick et al (1997) where there appears to be a mistake in the computations.*

Remark 2.3. *The probabilistic starting point for both Koné (1996) and Sutrick et al (1997) is $p_t(x; a, b) dx := P(a < m_t < M_t < b, x \leq X_t < x + dx | X_0 = 0)$. The former uses a result of Feller (1951) that can be traced to Lévy (1948):*

$$p_t(x; a, b) = \frac{1}{\sigma\sqrt{t}} \sum_{k=-\infty}^{\infty} \left[\phi\left(\frac{2k(b-a) - x}{\sigma\sqrt{t}}\right) - \phi\left(\frac{2k(b-a) - 2b + x}{\sigma\sqrt{t}}\right) \right], \quad (2.36)$$

while the latter uses a result of Billingsley (1968):

$$p_t(x; a, b) = \frac{1}{\sigma\sqrt{t}} \sum_{k=-\infty}^{\infty} \left[\phi\left(\frac{x + 2k(b-a)}{\sigma\sqrt{t}}\right) - \phi\left(\frac{2b - x + 2k(b-a)}{\sigma\sqrt{t}}\right) \right]. \quad (2.37)$$

The probabilistic results (2.36) and (2.37) are in fact equivalent, as one can be obtained from the other by appropriately replacing the summation index k with $-k$ and $\phi(x)$ with $\phi(-x)$.

3 The method of moments applied to volatility estimation using daily high, low, opening and closing prices

In this section we apply Theorem 2.1 to the estimation of the drift and volatility parameters of the stock price from market data on high, low, opening and closing prices.

Definition 3.1. *i) A trading day is the period elapsed between the opening and the closing bells of a calendar day.*

ii) A virtual trading day is the after-hours period beginning from the closing of one trading day and ending at the opening of the next trading day.

iii) A one-day period consists of one trading day followed by one virtual trading day.

We assume that the stock price S_t has the usual geometric Brownian motion dynamics:

$$\frac{dS_t}{S_t} = \mu_s dt + \sigma dW_t, \quad t \geq 0. \quad (3.1)$$

Then the log-stock price $\log S_t$ is the arithmetic Brownian motion X_t defined in (2.1) with drift coefficient

$$\mu = \mu_s - \frac{\sigma^2}{2}. \quad (3.2)$$

Note that μ_s is the *one-day period* drift of the stock price S_t , while μ is the similar drift of the log-price $X_t = \log S_t$; the volatility parameter σ is the same for both S_t and X_t .

The market data used for parameter estimation is as follows: for each one-period day $i \in \{1, 2, \dots, n\}$ we denote by S_{i-1} the opening price and by H_i and L_i the intra-day high and low prices, respectively (i.e. the high and low are observed only during the trading day, and not the virtual trading day - see Figure 1).

The after-hours arrival of information in the market determines a jump between the closing price of one trading day and the opening price of the next day. We model this jump by letting the same geometric Brownian motion S_t have an unobserved evolution during a virtual trading day. The length of this virtual trading day is assumed to be, on average, a fraction f of the unit length of the one-day period.

Remark 3.1. *This assumption follows Garman and Klass (1980) and Yang and Zhang (2000), except that they assume the after-hours trading day precedes the actual trading day. They call it the opening jump (from C_{i-1} to O_i), and assume it is modeled by a Poisson process.*

Thus for $i \in \{1, 2, \dots, n\}$ we have (see Figure 1):

$$\text{OPEN}(i) = O_i = S_{i-1}, \quad \text{CLOSE}(i) = C_i = S_{i-f}, \quad \text{HIGH}(i) = H_i, \quad \text{LOW}(i) = L_i, \quad (3.3)$$

where:

$$H_i = \sup_{t \in [i-1, i-f]} S_t, \quad L_i = \inf_{t \in [i-1, i-f]} S_t. \quad (3.4)$$

The evolution of the price during the trading period $i-1 \leq t < i-f$ is given by:

$$\log S_t = \log O_i + \mu(t-i+1) + \sigma(W_t - W_{t-i+1}), \quad (3.5)$$

and during the after-hours virtual trading period $i-f \leq t < i$ by:

$$\log S_t = \log C_i + \mu(t-i+f) + \sigma(W_t - W_{i-f}). \quad (3.6)$$

Taking expectation in (3.5) when $t \nearrow (i-f)$ and in (3.6) when $t \nearrow i$ gives:

$$E \left[\log \frac{C_i}{O_i} \right] = E \left[\log \frac{S_{i-f}}{S_{i-1}} \right] = \mu(1-f), \quad (3.7)$$

$$E \left[\log \frac{O_{i+1}}{C_i} \right] = E \left[\log \frac{S_i}{S_{i-f}} \right] = \mu f. \quad (3.8)$$

Using $W_t - W_s$ identically distributed to W_{t-s} , the trading day and virtual trading day variances are obtained, respectively, as:

$$\begin{aligned} \text{VAR} \left[\log \frac{C_i}{O_i} \right] &= \text{VAR} \left[\log \frac{S_{i-f}}{S_{i-1}} \right] = \sigma^2(1-f), \\ \text{VAR} \left[\log \frac{O_{i+1}}{C_i} \right] &= \text{VAR} \left[\log \frac{S_i}{S_{i-f}} \right] = \sigma^2 f. \end{aligned} \quad (3.9)$$

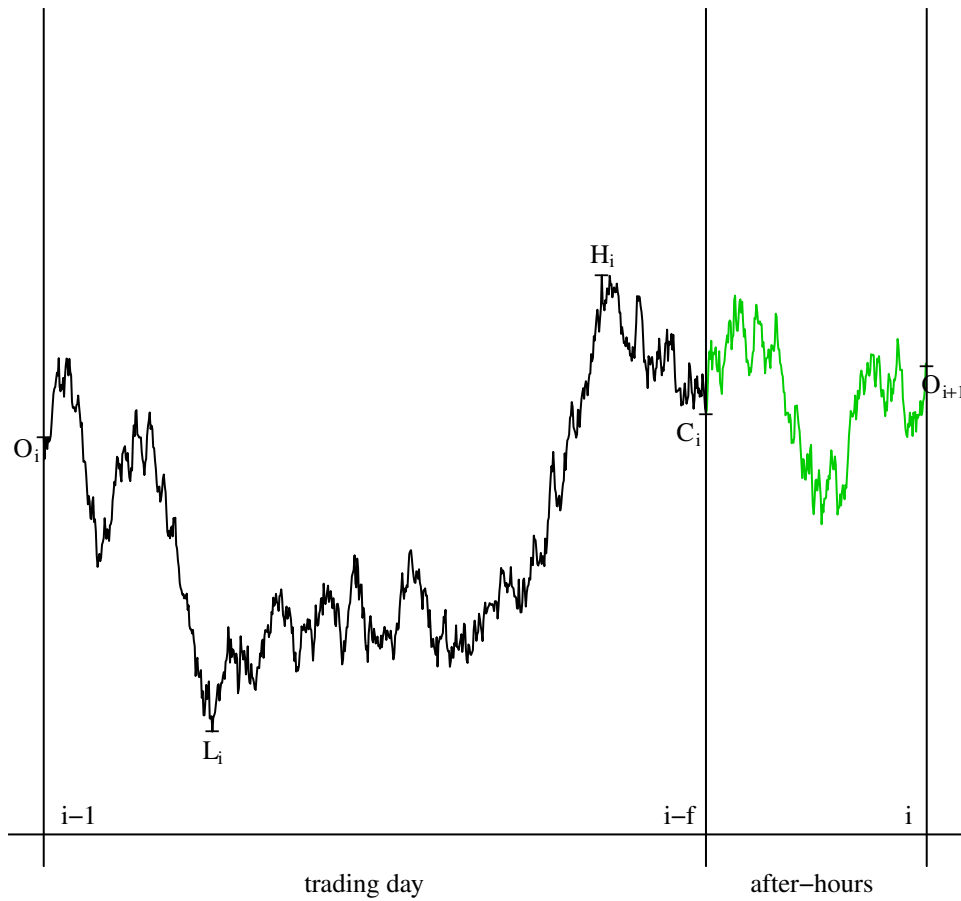


Figure 1: A one-day period consisting of a trading day and an after-hours period

Thus, we can write heuristically:

$$\sigma^2 = \text{VAR} \left[\log \frac{C_i}{O_i} \right] + \text{VAR} \left[\log \frac{O_{i+1}}{C_i} \right] = \text{VAR}(\text{trading day}) + \text{VAR}(\text{after hours}).$$

To estimate the variance over the trading day we use the method of moments. The range $R_{1-f} = \log H_1 - \log L_1$ of the arithmetic Brownian motion $X_t = \log S_t$ over the trading day $[0, 1 - f]$ was obtained in equation (2.10):

$$E(R_{1-f}) = ER(\mu, \sigma, 1 - f) = ER(\mu(1 - f), \sigma\sqrt{1 - f}, 1). \quad (3.10)$$

In (3.10) we estimate $E(R_{1-f})$ using the daily range data:

$$k_1 := \frac{1}{n} \sum_{i=1}^n \log \frac{H_i}{L_i}. \quad (3.11)$$

and $\mu(1 - f)$ by (see (3.7)):

$$k_2 := \frac{1}{n} \sum_{i=1}^n \log \frac{C_i}{O_i}. \quad (3.12)$$

This leads to the following equation to be solved for x , the estimate of $\sigma\sqrt{1-f}$:

$$k_1 = h\left(\frac{k_2}{x}, \frac{x^2}{k_2}\right). \quad (3.13)$$

The squared of this solution gives an estimate $V_i = x^2$ of the variance (volatility squared) corresponding to the trading day part of a one-day period.

For the after-hours part of the one-day period we have two choices: V_0 (centered approach) used in Yang and Zhang (2000), or V'_0 (non-centered) used in Garman and Klass (1980). Using the former (i.e. the sample standard variance V_0), we obtain the estimate for the variance of the entire one-day period as:

$$V_Z := V_0 + V_i, \quad (3.14)$$

or, in annualized form, as:

$$\sigma_a^2 := 252V_Z. \quad (3.15)$$

Denoting by V_C the sample variance of $\log(C_i/O_i)$ used in their estimator by Yang and Zhang (2000) :

$$V_{YZ} = V_0 + kV_C + (1 - k)V_{RS},$$

where V_{RS} is the estimator of Rogers and Satchell (1991) and Rogers, Satchell and Yoon (1994) and k is a constant, we note the following.

Remark 3.2. *i) The term V_i replaces the linear combination of V_C and V_{RS} used by Yang and Zhang (2000) for the intra-day trading period, and it does not need estimating the value of k that achieves minimum variance.*

ii) Our estimator is a true range-based estimator (log-range to be precise since it uses $\log(H_i/L_i)$), unlike that of Yang and Zhang (2000).

iii) Our estimator V_Z is independent of both the drift and the weight f of the after-hours information.

Example 3.1. *Consider the market data on the high, low, opening and closing prices for the IBM stock for the period from May 26, 2010 to June 18, 2010. For each of these days we consider the historical 3-month³ estimates of k_1 and k_2 , and we solve the corresponding equation (3.13).*

The solution is our estimate of the volatility $\sigma\sqrt{1-f}$ corresponding to the trading day, and we present it in annualized form (i.e. multiplied by $\sqrt{252}$) in Figure 2. We compare our estimate of the volatility corresponding to a one-day period with the one of Yang and Zhang (2000). On June 18, 2010 they are $\sigma_a = 0.2781$ (see (3.15)) and 0.2982 (annualized volatility corresponding to V_{YZ}).

³For parameter estimation Hull (2006, p. 287) recommends using historical data of 90 to 180 days.

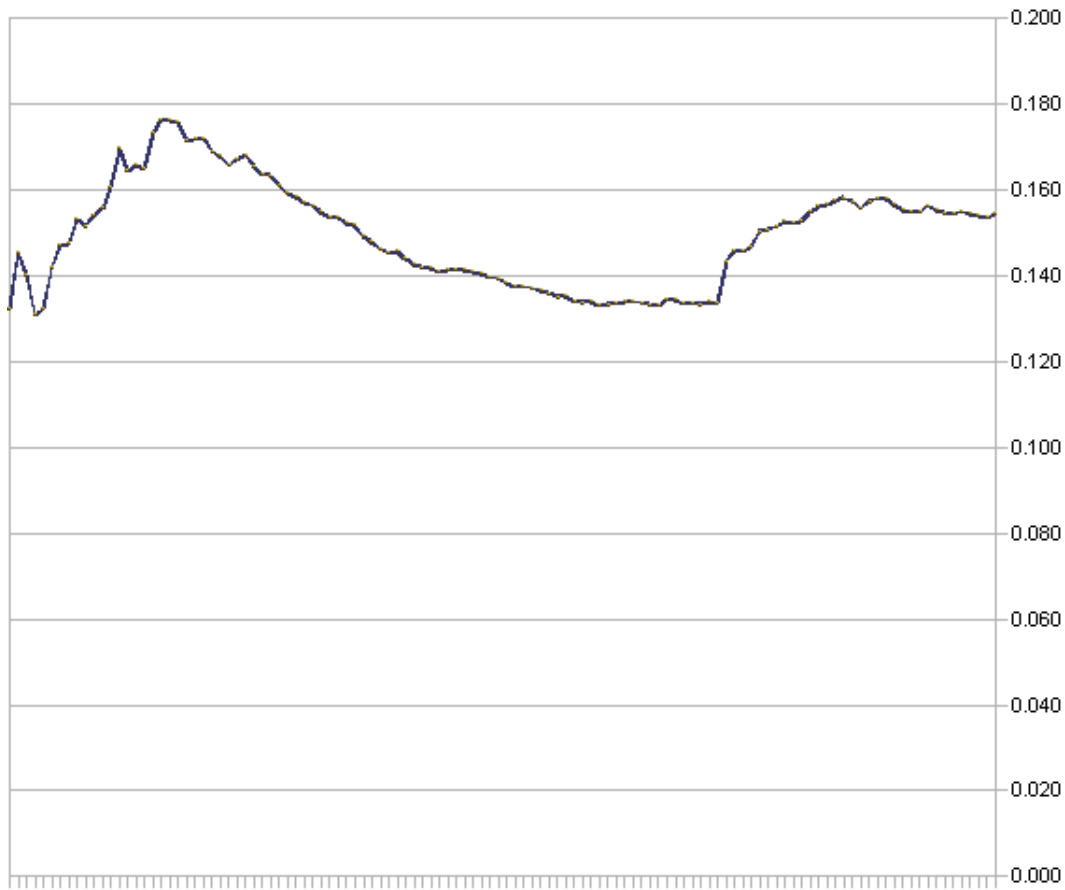


Figure 2: Estimated intra-day IBM volatility - May 26 to June 18, 2010

4 European options: mispricing opportunities

We use the resulting annualized volatility to compute the Black-Scholes prices of European options on the stock. We then seek those instances when the computed prices differ the most from the market prices, and devise trading strategies to take advantage of the price difference.

We now use the volatility parameter estimated above to price European call options using the Black-Scholes formula:

$$C_t = S_t \Phi(d_1) - K e^{-r(T-t)} \Phi(d_2), \quad (4.16)$$

with

$$d_1 = \frac{\log(S_t/K) + (r + \sigma_a^2/2)(T-t)}{\sigma_a \sqrt{T-t}},$$

$$d_2 = d_1 - \sigma_a \sqrt{T-t}.$$

We devise a trading strategy to take advantage of the information differential between our estimated prices and market prices. For simplicity we trade only in European

call options, and assume that at expiry there is a payment equal to the payoff so that no actual trading occurs in the underlying stock (naked trading).

Having assumed a constant volatility there is no volatility smile and no stochastic volatility⁴, so we restrict our analysis to European call options whose strike prices are relatively close to the stock price at the beginning of the period (preferably in the money), and whose expiry dates are up to three months (the parameters can be re-estimated later in view of new data).

Example 4.1. *Consider the market prices for the European call options on IBM for the period May 26, 2010 to June 18, 2010 with expiry dates June 18 and July 16, and strike prices $K \in \{115, 120, 125, 130\}$ (the stock price on May 26 was 125.91). We compare these market prices with the Black-Scholes prices calculated using (4.16). Here the inputs are the stock price, the volatility estimated in Example 3.1, and r the value of the 1-month US Treasury bill yield for the previous day (online Treasury data⁵*

Since the intra-day volatility of Example 3.1 that we use in Black-Scholes formula does not include the effect of the after-hours evolution, we compensate by allowing our prices to differ by up to 10% from the bid-ask spread. Thus, we trade when our estimated call price falls outside the interval $(0.9 \times \text{bid-price}, 1.1 \times \text{ask-price})$.

There are two cases. If our price is lower, then we short-sell the option at the bid price and wait for the first day when the estimated price is no longer lower to buy back the option at the then ask price. If it expires and the call option is exercised then we buy the stock in the market and deliver it.

If our price is higher, then we buy the option at ask price and wait for the first day when the price is no longer higher to sell it at the then bid price. If it reaches expiry date, then we exercise it.

This trading strategy is summarized in Algorithm 1 for t between May 26, 2010 and June 18, 2010 for European call options expiring at close June 18, 2010. The data is retrieved once a day, except on expiration date when it is retrieved several times a day (this can be implemented as an algorithmic trading strategy and deployed continuously without much effort, especially by those interested in technical trading).

Remark 4.1. *This strategy results in an overall loss of 0.68 (see Table 1). This is due mostly to one large loss induced by a large sudden move in the stock price on June 10 (127.3 versus 123.9 the day before). That is because we use yesterday's intra-day volatility to trade in today's world.*

Over a time horizon longer than a month the strategy can absorb such shocks in the stock prices, provided they are sparse. Alternatively, one can implement an additional stopping rule when the change in the stock price exceeds a pre-determined margin.

A similar behaviour is exhibited when applying the same trading strategy to the European call option expiring at close July 16, 2010, but as the expiry date is longer than a couple of months the limitations of the assumptions of the model become apparent.

⁴Alternative approaches like stochastic volatility or econometric models (ARMA, GARCH etc) are not discussed here.

⁵<http://www.treasury.gov/resource-center/data-chart-center/interest-rates/Pages/default.aspx>

t	K	$\hat{C}(t)$	(bid,ask)	trade	t	(bid,ask)	trade	profit
May 26	130	0.90	(1.16,1.17)	sell	May 27	(0.96,0.99)	buy	0.17
May 28	130	0.57	(0.75,0.78)	sell	Jun 2	(0.67,0.70)	buy	0.05
Jun 7	125	1.92	(2.20,2.23)	sell	Jun 8	(1.21,1.23)	buy	0.97
Jun 7	130	0.29	(0.42,0.44)	sell	Jun 8	(0.15,0.17)	buy	0.25
Jun 8	120	3.69	(4.15,4.20)	sell	Jun 9	(4.60,4.75)	buy	(0.6)
Jun 9	125	1.09	(1.30,1.38)	sell	Jun 10	(3.05,3.15)	buy	(1.77)
Jun 17	130	1.09	(0.90,0.94)	buy	Jun 19	(1.00,1.05)	sell	0.06
Jun 18 ^a	130	0.60	(0.48,0.51)	buy	Jun 18 ^b	(0.63,0.69)	sell	0.12
Jun 18 ^c	130	0.18	(0.21,0.25)	sell	Jun 18 ^d	$S_t=130.14$	ex ^e	0.07

^a at 12:27pm ^b at 13:36pm ^c at 15:58pm ^d at 16:00pm ^e if exercised

Table 1: Trading in the call option expiring June 18, 2010
(left: open a position, right: close position)

5 Conclusions

We have used the method of moments to estimate the volatility of the stock price and used this to identify arbitrage opportunities in the market of European options. As a by-product we have derived the density and expectation of the range of an arithmetic Brownian motion.

In comparison to the estimate of Yang and Zhang (2000), our volatility estimate takes advantage of the actual range of the Brownian motion and perhaps does not overestimate as much. It is most useful for short expiration dates and for strike prices that are not far out. We believe it is an efficient alternative that can be easily computed and has a practical implementation. These traits recommend it to the attention of practioners in the field.

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Algorithm 1: Trading strategy for a mispricing opportunity found at time t

```

input : Parameters  $t, \sigma_a, r, S_t$ , bid( $t$ ) and ask( $t$ ) (European call prices)
output: Profit of trading strategy

1 profit  $\leftarrow$  0; // initialize
2  $T \leftarrow$  June 18, 2010; // expiry date
3 compute  $\sigma_a$ ; // Example 3.1
4 for  $K \leftarrow$  115 to 130 do
5    $\hat{C}(t) \leftarrow$  BlackScholesCall( $t, T, K, \sigma_a, r, S_t$ ); // (4.16)
6   if  $\hat{C} < 0.9 \times \text{bid}(t)$  then
7     profit  $\leftarrow$  profit + bid( $t$ ); // short-sell call
8     while  $t < T$  and  $\hat{C}(t) < 0.9 \times \text{bid}(t)$  do
9        $t \leftarrow t+1$ ; // wait 1 day
10      compute  $\sigma_a$ ; // Example 3.1
11       $\hat{C}(t) \leftarrow$  BlackScholesCall( $t, T, K, \sigma_a, r, S_t$ ); // (4.16)
12    end
13    if  $t < T$  then
14      profit  $\leftarrow$  profit - ask( $t$ ); // buy back call
15    else
16      if call is exercised then
17        profit  $\leftarrow$  profit - ( $S_t - K$ );
18        // buy stock and deliver for  $K$ 
19      end
20    end
21  else
22    if  $\hat{C} > 1.1 \times \text{ask}(t)$  then
23      profit  $\leftarrow$  profit - ask( $t$ ); // buy call
24      while  $t < T$  and  $\hat{C}(t) > 1.1 \times \text{ask}(t)$  do
25         $t \leftarrow t+1$ ; // wait 1 day
26        compute  $\sigma_a$ ; // Example 3.1
27         $\hat{C}(t) \leftarrow$  BlackScholesCall( $t, T, K, \sigma_a, r, S_t$ );
28      end
29      if  $t < T$  then
30        profit  $\leftarrow$  profit + bid( $t$ ); // sell call
31      else
32        profit  $\leftarrow$  profit + ( $S_t - K$ ); // exercise call
33      end
34    end
35 end
36 return profit;

```
