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An application of the method of moments to range-based volatility estimation using daily opening, high, low and closing (OHLC) prices

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Abstract

We use the expectation of the range of an arithmetic Brownian motion and the method of moments on the daily high, low, opening and closing prices to estimate the volatility of the stock price. This novel theoretical approach results in an estimator that is genuinely range-based on daily opening, high, low and closing data, unlike current estimators in the literature. The daily price jump at the opening is considered to be the result of the unobserved evolution of an after-hours virtual trading day. In comparison to existing drift-independent estimator we find that our estimator is actually more efficient when using a smaller number of data points, while for a larger number of points the efficiency of our estimator stays within 99% of the existing one. A toy example that uses this method to take advantage of mispricing opportunities in the options market illustrates potential applications of this method to algorithmic trading.

Key words: Range-based volatility estimation, method of moments, daily high, low, opening and closing prices, density and expectation of the range of an arithmetic Brownian motion, algorithmic trading.

AMS subject classifications: 91G20, 60J65, 62F10, 62G05, 62P05

JEL classification: G12, G13, C13, C46.

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1 Introduction

We present a novel theoretical approach to the problem of volatility estimation that is based on what was studied in the Ph.D. thesis of Koné (1996). Aiming to estimate volatility and not to measure it, we assume a Black-Scholes framework with constant volatility and use daily data to achieve it (see Rogers and Zhou (2008) for further motivation for this choice).

After deriving an expression for the expectation of the range of an arithmetic Brownian motion, daily opening-high-low-closing data (OHLC) is used via the method of moments for intra-day volatility estimation by means of an implicit equation.

After-hours arrival of information results in price jumps at the opening relative to previous closing, and we model this as a virtual/unobservable trading day that, in conjunction with the intra-day, gives a statistical representation of one trading day (from one opening bell to the next opening bell, or from one closing bell to the next closing bell). This after-hours information is incorporated in the final form of the volatility estimator.

As pointed out by Yang and Zhang (2000), the literature on volatility estimation (or variance estimation, defined as volatility squared) assumes either that security price has no drift, resulting in overestimation, or that there is no price jump at opening (i.e. closing price is the same as next day's opening price), resulting in underestimation. In this regard the variance estimator V_{GK} of Garman and Klass (1980) assumes no drift, V_P of Parkinson (1980) uses only high-low data, while V_{RS} of Rogers and Satchell (1991) and Rogers et al (1994) uses OHLC data but has no opening jumps. Yang and Zhang (2000) caution that simply adding to V_{RS} a term V_0 to account for the opening jump does not result in an estimator with minimum variance, and proposed instead the estimator:

$$V_{YZ} = V_0 + kV_C + (1 - k)V_{RS}, \quad (1)$$

where V_0 is the sample variance of $\ln \frac{O_{i+1}}{C_i}$, V_C is the sample variance of $\ln \frac{C_i}{O_i}$, and

$$V_{RS} = \frac{1}{n} \sum_{i=1}^n \left(\ln \frac{H_i}{O_i} \left(\ln \frac{H_i}{O_i} - \ln \frac{C_i}{O_i} \right) + \ln \frac{L_i}{O_i} \left(\ln \frac{L_i}{O_i} - \ln \frac{C_i}{O_i} \right) \right). \quad (2)$$

The opening price for day i is denoted by O_i , the high H_i , the low L_i and the closing price by C_i . The estimator V_{YZ} is drift independent, and the constant k is chosen so that V_{YZ} is MVUE (minimum variance unbiased estimator) for fixed number of data points.

In contrast to (1), our proposed estimator is:

$$V_Z := V_0 + V_i, \quad (3)$$

with V_i being the squared of the solution of an implicit equation that depends on the range data $\ln \frac{H_i}{L_i}$ and on $\ln \frac{C_i}{O_i}$.

The immediate difference between (3) and (1) is that we use true range data $\ln \frac{H_i}{L_i}$ instead of the normalized highs and lows $\ln \frac{H_i}{O_i}$ and $\ln \frac{L_i}{O_i}$. We show that this range data captures the volatility features as well as the normalized highs and lows do, despite providing fewer measurements (for further comparison see Remarks 3.1 and 3.2). Our method is perhaps more practical (see the remarks of Chan and Lien (2003) on the empirical availability of certain parameters in V_{YZ}).

Due to the implicit nature of our estimator, we cannot derive explicit results on the MVUE property of V_Z , so we resort instead to Monte-Carlo simulation and look at alternative measures: for unbiasedness we look at the proportion of times our estimator is closer to the true value and also compare their mean absolute error, and for minimum variance we look at the variances and efficiency. We find that our estimator is closer to the real value than the unbiased estimator V_{YZ} in 40 to 60 percent of scenarios (unbiased), and that its efficiency is higher for small number of data points, while for large number of data points the efficiency doesn't drop by more than 1 percent.

Next a simple implementation of this method of volatility estimation that takes advantage of mispricing opportunities for European options illustrates potential applications to algorithmic trading.

Incidental to this is the derivation of the density and expectation of the range of an arithmetic Brownian motion. Subsequent to Koné (1996), portions of it have been studied for different purposes (see, for instance, Sutrick et al (1997) for the use of the density of the range of an arithmetic Brownian motion in the *do-nothing option*, or Magdon-Ismail et al (2000, 2004) for the use of the expectation of the range of an arithmetic Brownian motion in different contexts). In particular, writing the density of the range in the context of Sutrick et al (1997) allows us to correct their expression.

The paper is organized as follows. In Section 2 we derive the expectation and the density function of an arithmetic Brownian motion. In Section 3 the method of moments is used to estimate the parameters of the stock price based on daily OHLC data. In Section 4 the Monte-Carlo simulation is used to compare the statistical properties of V_Z and V_{YZ} . The example of Section 5 uses the estimated volatility to price European options on a stock, which are then compared to market prices to identify profit opportunities. We conclude in Section 6 with a summary of the results.

2 The range of an arithmetic Brownian motion: expectation and density

Let $\{\Omega, \mathcal{F}, P\}$ be a probability space endowed with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$, and let $\{W_t\}_{t \geq 0}$ be a one-dimensional standard Brownian motion adapted to $\{\mathcal{F}_t\}_{t \geq 0}$. For $t \geq 0$ let X_t denote a standard arithmetic Brownian motion with drift μ and volatility $\sigma > 0$:

$$X_t = \mu t + \sigma W_t, \quad X_0 = 0, \quad (4)$$

and let $\{M_t\}_{t \geq 0}$, $\{m_t\}_{t \geq 0}$ and $\{R_t\}_{t \geq 0}$ denote its running maximum, minimum, and range, respectively:

$$M_t := \sup_{0 \leq s \leq t} X_s, \quad m_t = \inf_{0 \leq s \leq t} X_s, \quad R_t := M_t - m_t. \quad (5)$$

First we derive the expectation $E[R_t]$ of the range of the arithmetic Brownian motion X_t . This is achieved by computing the density and expectation of the half-range $M_t - X_t$ from the joint density of X_t and M_t .

Lemma 2.1. *The joint density function of an arithmetic Brownian motion and its running maximum can be expressed as:*

$$P(X_t \in da, M_t \in db) = \frac{2(2b-a)}{\sqrt{2\pi t^3} \sigma^3} \exp \left\{ -\frac{(2b-a)^2}{2t\sigma^2} + \frac{\mu}{\sigma^2} a - \frac{1}{2} \frac{\mu^2}{\sigma^2} t \right\} da db. \quad (6)$$

Proof. The proof is standard. Using the martingale $Z_t(Y) = \exp\{W_t\mu/\sigma - t\mu^2/(2\sigma^2)\}$, Girsanov's change of measure defines a new probability measure \tilde{P} for any measurable set A by $\tilde{P}(A) = E[Z_t(Y)1_A]$. Theorem 3.2.2 of Karatzas and Shreve (1988) with $Y_t = \mu/\sigma$ and $N_t = \sigma W_t$ gives that $\tilde{N}_t = \sigma W_t - \mu t$ is a local martingale. The process $\tilde{W}_t = W_t - t\mu/\sigma$ is a Brownian motion under the new probability measure \tilde{P} . Equivalently, $\sigma W_t = \mu t + \sigma \tilde{W}_t$ is a Brownian motion with drift under \tilde{P} , and we can write:

$$\begin{aligned} P(X_t \leq a, M_t \leq b) &= \tilde{P}(\sigma W_t \leq a, \sigma \bar{M}_t \leq b) \\ &= \int_{-\infty}^a \exp\left(\frac{\mu}{\sigma^2} x - \frac{1}{2} \frac{\mu^2}{\sigma^2} t\right) P(\sigma W_t \in dx, \sigma \bar{M}_t \leq b). \end{aligned} \quad (7)$$

An application of the reflection principle gives:

$$\begin{aligned} P(\sigma W_t \leq a, \sigma \bar{M}_t \leq b) &= P\left(W_t \leq \frac{a}{\sigma}\right) - P\left(W_t \leq \frac{a}{\sigma}, \sigma \bar{M}_t > b\right) \\ &= P\left(W_t \leq \frac{a}{\sigma}\right) - P\left(W_t > \frac{2b-a}{\sigma}\right) \\ &= \Phi\left(\frac{a}{\sigma\sqrt{t}}\right) - 1 + \Phi\left(\frac{2b-a}{\sigma\sqrt{t}}\right), \end{aligned}$$

where we denote by $\phi(\cdot)$ and $\Phi(\cdot)$ the standard normal density and cumulative distribution functions, respectively.

Differentiating the formula above with respect to a gives:

$$P(\sigma W_t \in da, \sigma \bar{M}_t \leq b) = \frac{1}{\sigma\sqrt{t}} \left(\phi\left(\frac{a}{\sigma\sqrt{t}}\right) - \phi\left(\frac{2b-a}{\sigma\sqrt{t}}\right) \right). \quad (8)$$

Replacing (8) in (7) and differentiating first with respect to a gives:

$$P(X_t \in da, M_t \leq b) = \frac{1}{\sigma\sqrt{t}} \exp\left(\frac{\mu}{\sigma^2} a - \frac{1}{2} \frac{\mu^2}{\sigma^2} t\right) \left(\phi\left(\frac{a}{\sigma\sqrt{t}}\right) - \phi\left(\frac{2b-a}{\sigma\sqrt{t}}\right) \right) da,$$

and differentiating then with respect to b gives:

$$\begin{aligned} P(X_t \in da, M_t \in db) &= \frac{1}{\sigma\sqrt{t}} \exp\left(\frac{\mu}{\sigma^2}a - \frac{1}{2}\frac{\mu^2}{\sigma^2}t\right) \left(-\frac{2}{\sigma\sqrt{t}}\right) \phi'\left(\frac{2b-a}{\sigma\sqrt{t}}\right) da db \\ &= \left(-\frac{2}{t\sigma^2}\right) \exp\left(\frac{\mu}{\sigma^2}a - \frac{1}{2}\frac{\mu^2}{\sigma^2}t\right) \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{2b-a}{\sigma\sqrt{t}}\right)^2\right) \left(-\frac{1}{2}\right) 2\frac{2b-a}{\sigma\sqrt{t}} da db \\ &= \frac{2(2b-a)}{\sqrt{2\pi t^3}\sigma^3} \exp\left\{-\frac{(2b-a)^2}{2t\sigma^2} + \frac{\mu}{\sigma^2}a - \frac{1}{2}\frac{\mu^2}{\sigma^2}t\right\} da db. \end{aligned}$$

The joint density $f_{X_t, M_t}(a, b)$ is given by the term multiplying $da db$ above. \square

Remark 2.1. *This is one of those results that seemed to be always at hand (it can be obtained from equation (1.8.8) of Harrison (1985)), but never derived. Note the typo in Yang and Zhang (2000), whose expression (B1) has a plus for the first fraction in the exponential. For $\sigma = 1$ this was used in Example E5 of Karatzas and Shreve (1998) in relationship to Clark's formula to obtain explicitly the hedging portfolio.*

The density of the half-range $M_t - X_t$ can be obtained using a standard two-dimensional transformation of the above joint density.

Lemma 2.2. *The density of the half-range $M_t - X_t$ is given by:*

$$f_{M_t - X_t}(c) = 2\frac{\mu}{\sigma^2} \Phi\left(\frac{\mu t - c}{\sigma\sqrt{t}}\right) \exp\left(-2\frac{\mu}{\sigma^2}c\right) + \frac{2}{\sigma\sqrt{2t\pi}} \exp\left\{-\frac{(\mu t + c)^2}{2t\sigma^2}\right\}. \quad (9)$$

Proof. For $Y_1 = X_1 + X_2$ and $Y_2 = X_1 - X_2$ the joint density is $f_{Y_1, Y_2}(y_1, y_2) = \frac{1}{2}f_{X_1, X_2}\left(\frac{y_1+y_2}{2}, \frac{y_1-y_2}{2}\right)$. Taking $X_1 = M_t$ and $X_2 = X_t$ and using Lemma 2.1 gives:

$$\begin{aligned} f_{Y_1, Y_2}(y_1, y_2) &= \frac{1}{2}f_{M_t, X_t}\left(\frac{y_1+y_2}{2}, \frac{y_1-y_2}{2}\right) = \frac{1}{2}f_{X_t, M_t}\left(\frac{y_1-y_2}{2}, \frac{y_1+y_2}{2}\right) \\ &= \frac{1}{2} \frac{2y_1 + 2y_2 - y_1 + y_2}{\sqrt{2\pi t^3}\sigma^3} \exp\left\{-\frac{(2y_1 + 2y_2 - y_1 + y_2)^2}{8t\sigma^2} + \frac{\mu}{\sigma^2} \frac{y_1 - y_2}{2} - \frac{1}{2}\frac{\mu^2}{\sigma^2}t\right\} \\ &= \frac{1}{2} \frac{y_1 + 3y_2}{\sqrt{2\pi t^3}\sigma^3} \exp\left\{-\frac{(y_1 + 3y_2)^2}{8t\sigma^2} + \frac{\mu}{\sigma^2} \frac{y_1 - y_2}{2} - \frac{1}{2}\frac{\mu^2}{\sigma^2}t\right\} \\ &= \frac{1}{2} \frac{y_1 + 3y_2}{\sqrt{2\pi t^3}\sigma^3} \exp\left\{-\frac{1}{2} \frac{(y_1 + 3y_2 - 2\mu t)^2}{4t\sigma^2} - 2\frac{\mu}{\sigma^2}y_2\right\} \end{aligned}$$

Note that $M_t \geq 0$ implies $Y_1 \geq -Y_2$, thus the marginal density of $Y_2 = M_t - X_t$ is:

$$f_{Y_2}(y_2) = \int_{-y_2}^{\infty} f_{Y_1, Y_2}(y_1, y_2) dy_1.$$

A change of variable $z = (y_1 + 3y_2 - 2\mu t)/(2\sigma\sqrt{t})$ gives:

$$z > z_0 := \frac{y_2 - \mu t}{\sigma\sqrt{t}}, \quad dy_1 = 2\sigma\sqrt{t}dz,$$

therefore:

$$\begin{aligned}
f_{Y_2}(y_2) &= \int_{z_0}^{\infty} \frac{1}{2} \frac{2z\sigma\sqrt{t} + 2\mu t}{\sqrt{2\pi t^3}\sigma^3} \exp\left\{-\frac{1}{2}z^2 - 2\frac{\mu}{\sigma^2}y_2\right\} 2\sigma\sqrt{t} dz \\
&= \frac{2}{\sigma\sqrt{t}} \int_{z_0}^{\infty} z \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz \exp\left\{-2\frac{\mu}{\sigma^2}y_2\right\} + 2\frac{\mu}{\sigma^2} \int_{z_0}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz \exp\left\{-2\frac{\mu}{\sigma^2}y_2\right\} \\
&= \frac{2}{\sigma\sqrt{t}} \int_{z_0}^{\infty} -\left(\frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}\right)' dz \exp\left\{-2\frac{\mu}{\sigma^2}y_2\right\} + 2\frac{\mu}{\sigma^2} \left(1 - \Phi(z_0)\right) \exp\left\{-2\frac{\mu}{\sigma^2}y_2\right\} \\
&= \frac{2}{\sigma\sqrt{t}} \frac{1}{\sqrt{2\pi}} e^{-\frac{z_0^2}{2}} \exp\left\{-2\frac{\mu}{\sigma^2}y_2\right\} + 2\frac{\mu}{\sigma^2} \Phi(-z_0) \exp\left\{-2\frac{\mu}{\sigma^2}y_2\right\}.
\end{aligned}$$

This can be rewritten as:

$$\begin{aligned}
P(M_t - X_t \in dc) &= \left(2\frac{\mu}{\sigma^2} \Phi\left(\frac{\mu t - c}{\sigma\sqrt{t}}\right) \exp\left(-2\frac{\mu}{\sigma^2}c\right) \right. \\
&\quad \left. + \frac{2}{\sigma\sqrt{2t\pi}} \exp\left\{-\frac{(\mu t - c)^2}{2t\sigma^2}\right\} \exp\left(-2\frac{\mu}{\sigma^2}c\right) \right) dc, \quad (10)
\end{aligned}$$

and the result follows. \square

Proposition 2.1. *The expectation of the half-range is given by:*

$$\begin{aligned}
E(M_t - X_t) &= \frac{\sigma^2}{2\mu} \Phi\left(\frac{\mu}{\sigma}\sqrt{t}\right) - \left(\mu t + \frac{\sigma^2}{2\mu}\right) \left(1 - \Phi\left(\frac{\mu}{\sigma}\sqrt{t}\right)\right) \\
&\quad + \frac{\sigma\sqrt{t}}{\sqrt{2\pi}} \exp\left(-\frac{t\mu^2}{2\sigma^2}\right). \quad (11)
\end{aligned}$$

Proof. A simple calculation yields:

$$\begin{aligned}
E(M_t - X_t) &= \int_0^{\infty} c \frac{2\mu}{\sigma^2} \Phi\left(\frac{\mu t - c}{\sigma\sqrt{t}}\right) \exp\left(-2\frac{\mu}{\sigma^2}c\right) dc \\
&\quad + \int_0^{\infty} c \frac{2}{\sigma\sqrt{2t\pi}} \exp\left\{-\frac{(\mu t + c)^2}{2t\sigma^2}\right\} dc \\
&= \int_0^{\infty} \left\{ -\left(c + \frac{\sigma^2}{2\mu}\right) \exp\left(-2\frac{\mu}{\sigma^2}c\right) \right\}' \Phi\left(\frac{\mu t - c}{\sigma\sqrt{t}}\right) dc \\
&\quad + \int_0^{\infty} c \frac{2}{\sigma\sqrt{2t\pi}} \exp\left\{-\frac{(\mu t + c)^2}{2t\sigma^2}\right\} dc \\
&= \frac{\sigma^2}{2\mu} \Phi\left(\frac{\mu}{\sigma}\sqrt{t}\right) + \frac{1}{\sigma\sqrt{t}} \frac{1}{\sqrt{2\pi}} \int_0^{\infty} \left(-c - \frac{2\sigma^2}{\mu} + 2c\right) \exp\left\{-\frac{(\mu t + c)^2}{2t\sigma^2}\right\} dc.
\end{aligned}$$

A change of variable $c = z\sigma\sqrt{t} - \mu t$ gives the result. \square

Consider now $E[X_t - m_t]$. For each path of the Brownian motion X_t with drift μ consider a symmetric path of a Brownian motion \tilde{X}_t having drift $-\mu$. Then $X_t - m_t = -(\tilde{X}_t - \tilde{M}_t)$ and $E[X_t - m_t]$ can be calculated using the equation (11) with μ replaced by $-\mu$. Whereas the formula for the expectation of the range follows.

Theorem 2.1. *The expectation of the range of the arithmetic Brownian motion X_t defined in (4) is given by:*

$$E[R_t] = \left(\mu t + \frac{\sigma^2}{\mu} \right) \left(1 - 2\Phi \left(-\sqrt{t} \frac{\mu}{\sigma} \right) \right) + 2 \frac{\sigma \sqrt{t}}{\sqrt{2\pi}} \exp \left(-\frac{t\mu^2}{2\sigma^2} \right). \quad (12)$$

Let us denote this expected range function by $ER(\mu, \sigma, t)$. On closer inspection this can be further simplified as a function of just two quantities:

$$E[R_t] = ER(\mu, \sigma, t) = h \left(\frac{\mu t}{\sigma \sqrt{t}}, \frac{\sigma^2}{\mu} \right), \quad (13)$$

where the function h is defined by:

$$h(x, y) := \left\{ (x^2 + 1)(2\Phi(x) - 1) + \frac{2x}{\sqrt{2\pi}} \exp \left(-\frac{x^2}{2} \right) \right\} y. \quad (14)$$

Note that $ER(\mu, \sigma, t) = ER(\mu t, \sigma \sqrt{t}, 1)$ as it should (the range over a time interval $(0, t)$ of an arithmetic Brownian motion with parameters μ and σ is the same as that over $(0, 1)$ when the parameters change to μt and $\sigma \sqrt{t}$).

In the remainder of this section we derive the density of the range R_t of the arithmetic Brownian motion X_t . This is achieved by the use of the joint density of the minimum and the maximum of X_t , a result with its own merit, that we could not find published prior to Koné (1996) (Borodin and Salminen(1996, 1.15.4) published in the same year the joint cumulative distribution function only in terms of some definite integrals).

A version of this result is used in Sutrick et al (1997) for the same purpose, but there it seems to have incorporated an error.

To obtain the joint density $F(a, b)$ of the maximum and the minimum we start with a lemma.

Lemma 2.3. *We can write:*

$$F(a, b) = \int_a^b h(a, b, x) \exp \left(\frac{\mu}{\sigma^2} x - \frac{1}{2} \frac{\mu^2}{\sigma^2} t \right) dx, \quad (15)$$

where

$$\begin{aligned}
h(a, b, x) &= h_1(a, b, x) - h_2(a, b, x), \tag{16} \\
h_1(a, b, x) &= \sum_{k=-\infty}^{\infty} \frac{2k(2k-2)}{\sigma^3 t \sqrt{2\pi t}} \left[1 - \frac{[2k(b-a) - 2b + x]^2}{t\sigma^2} \right] \\
&\quad \times \exp\left(-\frac{[2k(b-a) - 2b + x]^2}{2t\sigma^2}\right), \\
h_2(a, b, x) &= \sum_{k=-\infty}^{\infty} \frac{4k^2}{\sigma^3 t \sqrt{2\pi t}} \left[1 - \frac{[2k(b-a) - x]^2}{t\sigma^2} \right] \exp\left(-\frac{[2k(b-a) - x]^2}{2t\sigma^2}\right).
\end{aligned}$$

Proof. Using the change of measure of the proof of Lemma 2.1 and Girsanov's theorem we have:

$$P(a < m_t < M_t < b) = \int_a^b p_t(x; a, b) \exp\left(\frac{\mu}{\sigma^2}x - \frac{1}{2}\frac{\mu^2}{\sigma^2}t\right) dx, \tag{17}$$

which gives $F(a, b)$ via:

$$F(a, b) = -\frac{\partial^2 P(a < m_t < M_t < b)}{\partial a \partial b}. \tag{18}$$

Leibniz rule of differentiation:

$$\frac{\partial}{\partial z} \int_{a(z)}^{b(z)} f(x, z) dx = \int_{a(z)}^{b(z)} \frac{\partial f(x, z)}{\partial z} dx + f(b(z), z) \frac{\partial b}{\partial z} - f(a(z), z) \frac{\partial a}{\partial z} \tag{19}$$

gives the partial derivative of (39) wrt b :

$$\begin{aligned}
&\int_a^b \frac{1}{\sigma\sqrt{t}} \sum_{k=-\infty}^{\infty} \left[\frac{\partial \phi}{\partial b} \left(\frac{2k(b-a) - x}{\sigma\sqrt{t}} \right) - \frac{\partial \phi}{\partial b} \left(\frac{2k(b-a) - 2b + x}{\sigma\sqrt{t}} \right) \right] \\
&\quad \times \exp\left(\frac{\mu}{\sigma^2}x - \frac{1}{2}\frac{\mu^2}{\sigma^2}t\right) dx. \tag{20}
\end{aligned}$$

Differentiating wrt a this last equation gives:

$$\begin{aligned}
F(a, b) &= -\int_a^b \frac{1}{\sigma\sqrt{t}} \sum_{k=-\infty}^{\infty} \left[\frac{\partial^2 \phi}{\partial a \partial b} \left(\frac{2k(b-a) - x}{\sigma\sqrt{t}} \right) - \frac{\partial^2 \phi}{\partial a \partial b} \left(\frac{2k(b-a) - 2b + x}{\sigma\sqrt{t}} \right) \right] \\
&\quad \times \exp\left(\frac{\mu}{\sigma^2}x - \frac{1}{2}\frac{\mu^2}{\sigma^2}t\right) dx. \tag{21}
\end{aligned}$$

Direct calculation gives:

$$\begin{aligned}
\frac{\partial^2 \phi}{\partial a \partial b} \left(\frac{2k(b-a) - x}{\sigma\sqrt{t}} \right) &= \frac{\partial}{\partial a} \left[\frac{(-2k)[2k(b-a) - x]}{\sqrt{2\pi t}\sigma^2} \exp\left(-\frac{[2k(b-a) - x]^2}{2t\sigma^2}\right) \right] \\
&= \frac{4k^2}{\sqrt{2\pi t}\sigma^2} \exp\left(-\frac{[2k(b-a) - x]^2}{2t\sigma^2}\right) \left(1 - \frac{[2k(b-a) - x]^2}{t\sigma^2}\right), \tag{22}
\end{aligned}$$

$$\begin{aligned}
& \frac{\partial^2 \phi}{\partial a \partial b} \left(\frac{2k(b-a) - 2b + x}{\sigma \sqrt{t}} \right) \\
&= \frac{\partial}{\partial a} \left[\frac{-(2k-2)[2k(b-a) - 2b + x]}{\sqrt{2\pi}t\sigma^2} \exp \left(-\frac{[2k(b-a) - 2b + x]^2}{2t\sigma^2} \right) \right] \\
&= \frac{4k(k-1)}{\sqrt{2\pi}t\sigma^2} \exp \left(-\frac{[2k(b-a) - 2b + x]^2}{2t\sigma^2} \right) \left(1 - \frac{[2k(b-a) - 2b + x]^2}{t\sigma^2} \right). \quad (23)
\end{aligned}$$

Substituting (22)-(23) in (21) gives the result. \square

Proposition 2.2. *The joint density function $F(a, b)$ of M_t and m_t can be represented as:*

$$\begin{aligned}
F(a, b) &= F_1(a, b) - F_2(a, b) - F_3(a, b) + F_4(a, b) \\
&\quad - F_5(a, b) + F_6(a, b) + F_7(a, b) - F_8(a, b), \quad (24)
\end{aligned}$$

with

$$\begin{aligned}
F_1(a, b) &= \sum_{k=-\infty}^{\infty} \frac{4k(k-1)}{t\sigma^3\sqrt{2\pi}t} [(2k-1)b - 2ka + \mu t] \\
&\quad \times \exp \left\{ -\frac{\mu}{\sigma^2} [2(k-1)b - 2ka] - \frac{[(2k-1)b - 2ka - \mu t]^2}{2t\sigma^2} \right\}, \quad (25)
\end{aligned}$$

$$\begin{aligned}
F_2(a, b) &= \sum_{k=-\infty}^{\infty} \frac{4k(k-1)}{t\sigma^3\sqrt{2\pi}t} [2(k-1)b - (2k-1)a + \mu t] \\
&\quad \times \exp \left\{ -\frac{\mu}{\sigma^2} [2(k-1)b - 2ka] - \frac{[2(k-1)b - (2k-1)a - \mu t]^2}{2t\sigma^2} \right\}, \quad (26)
\end{aligned}$$

$$\begin{aligned}
F_3(a, b) &= \sum_{k=-\infty}^{\infty} \frac{4k(k-1)\mu^2}{2\sigma^4} \exp \left\{ -\frac{\mu}{\sigma^2} [2(k-1)b - 2ka] \right\} \\
&\quad \times \operatorname{erf} \left(\frac{(2k-1)b - 2ka - \mu t}{\sigma \sqrt{2t}} \right), \quad (27)
\end{aligned}$$

$$\begin{aligned}
F_4(a, b) &= \sum_{k=-\infty}^{\infty} \frac{4k(k-1)\mu^2}{2\sigma^4} \exp \left\{ -\frac{\mu}{\sigma^2} [2(k-1)b - 2ka] \right\} \\
&\quad \times \operatorname{erf} \left(\frac{2(k-1)b - (2k-1)a - \mu t}{\sigma \sqrt{2t}} \right), \quad (28)
\end{aligned}$$

$$F_5(a, b) = \sum_{k=-\infty}^{\infty} \frac{4k^2}{t\sigma^3\sqrt{2\pi t}} [(2k+1)b - 2ka + \mu t] \times \exp \left\{ -\frac{\mu}{\sigma^2} 2k(b-a) - \frac{[(2k+1)b - 2ka - \mu t]^2}{2t\sigma^2} \right\}, \quad (29)$$

$$F_6(a, b) = \sum_{k=-\infty}^{\infty} \frac{4k^2}{t\sigma^3\sqrt{2\pi t}} [2kb - (2k-1)a + \mu t] \times \exp \left\{ -\frac{\mu}{\sigma^2} 2k(b-a) - \frac{[2kb - (2k-1)a - \mu t]^2}{2t\sigma^2} \right\}, \quad (30)$$

$$F_7(a, b) = \sum_{k=-\infty}^{\infty} \frac{4k^2\mu^2}{2\sigma^4} \exp \left\{ -\frac{\mu}{\sigma^2} 2k(b-a) \right\} \operatorname{erf} \left(\frac{(2k+1)b - 2ka - \mu t}{\sigma\sqrt{2t}} \right), \quad (31)$$

$$F_8(a, b) = \sum_{k=-\infty}^{\infty} \frac{4k^2\mu^2}{2\sigma^4} \exp \left\{ -\frac{\mu}{\sigma^2} 2k(b-a) \right\} \operatorname{erf} \left(\frac{2kb - (2k-1)a - \mu t}{\sigma\sqrt{2t}} \right), \quad (32)$$

where

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt. \quad (33)$$

Proof. From Lemma 2.3 we write $F(a, b)$ as the difference of two terms, which we denote I_1 and I_2 :

$$\begin{aligned} I_1 &:= \int_a^b h_1(a, b, x) \exp \left(\frac{\mu}{\sigma^2} x - \frac{1}{2} \frac{\mu^2}{\sigma^2} t \right) dx, \\ I_2 &:= \int_a^b h_2(a, b, x) \exp \left(\frac{\mu}{\sigma^2} x - \frac{1}{2} \frac{\mu^2}{\sigma^2} t \right) dx. \end{aligned} \quad (34)$$

In each I_1 and I_2 we combine the exponents and then use, respectively, a change of variable:

$$z = \frac{x + 2k(b-a) - 2b - \mu t}{\sigma\sqrt{t}}, \quad z = \frac{x + 2k(b-a) - \mu t}{\sigma\sqrt{t}}, \quad (35)$$

followed by an integration by parts for $\int z^2 \exp(-z^2/2) dz$ and replacement of $\Phi(x)$ by $\operatorname{erf}(x)$ via $\Phi(x) = 0.5 \operatorname{erf}(x/\sqrt{2}) + 0.5$. The resulting eight terms are then denoted $F_i(a, b)$, $i = 1, \dots, 8$. \square

We use the above expression to derive the density of the range. To make it suitable for comparison with that obtained by Sutrick et al (1997) in their Proposition 1, we change $k \rightarrow k+1$ in the summations of F_1, F_2, F_3 and F_4 .

Proposition 2.3. *The density function $f_{R_t}(r)$ for the range of an arithmetic Brownian motion can be written as:*

$$f_{R_t}(r) = \frac{1}{\sigma\sqrt{t}} \sum_{k=-\infty}^{\infty} 4k^2 I(k) + \frac{1}{\sigma\sqrt{t}} \sum_{k=-\infty}^{\infty} 4k(k+1) J(k), \quad (36)$$

where

$$\begin{aligned} I(k) &= e^{-\frac{2\mu kr}{\sigma^2}} (1+c^2)(\phi(K_1-c) - 2\phi(K_0-c) + \phi(K_{-1}-c)) \\ &+ e^{-\frac{2\mu kr}{\sigma^2}} [(c^2 K_1 - 2c - c^3)\Phi(K_1-c) - 2(c^2 K_0 - 2c - c^3)\Phi(K_0-c) \\ &+ (c^2 K_{-1} - 2c - c^3)\Phi(K_{-1}-c)], \end{aligned} \quad (37)$$

and

$$\begin{aligned} J(k) &= e^{\frac{2\mu kr}{\sigma^2}} (\phi(K_1+c) - \phi(K_0+c)) - e^{\frac{2\mu(k+1)r}{\sigma^2}} (\phi(K_2+c) - \phi(K_1+c)) \\ &+ e^{-\frac{2\mu kr}{\sigma^2}} \left(-\frac{c}{2}\Phi(K_1-c) + \frac{c}{2}\Phi(K_0-c) \right) \\ &- e^{-\frac{2\mu(k+1)r}{\sigma^2}} \left(-\frac{c}{2}\Phi(K_2-c) + \frac{c}{2}\Phi(K_1-c) \right) \\ &+ e^{\frac{2\mu kr}{\sigma^2}} (\Phi(K_1+c) - \Phi(K_0+c)) \\ &- e^{\frac{2\mu(k+1)r}{\sigma^2}} (\Phi(K_2+c) - \Phi(K_1+c)), \end{aligned} \quad (38)$$

with

$$K_2 = \frac{(2k+2)r}{\sigma\sqrt{t}}, \quad K_1 = \frac{(2k+1)r}{\sigma\sqrt{t}}, \quad K_0 = \frac{2kr}{\sigma\sqrt{t}}, \quad K_{-1} = \frac{(2k-1)r}{\sigma\sqrt{t}}, \quad c = \frac{\mu\sqrt{t}}{\sigma}.$$

Proof. After replacing k by $k+1$ in $F_i, i = 1, \dots, 4$ of Proposition 2.2, a two-dimensional transformation $a = u - v, b = u$, gives, via Jacobian, the density of the range and the running maximum. Its marginal density is the one we seek:

$$f(r) = \int_0^r F(u-r, u) du.$$

Applying a change of variable and integration by parts gives the result. \square

Remark 2.2. *This result corrects that of Sutrick et al (1997) where there appears to be a mistake in the computations (see the term $J(k)$).*

Remark 2.3. *The probabilistic starting point for both Koné (1996) and Sutrick et al (1997) is*

$p_t(x; a, b) dx := P(a < m_t < M_t < b, x \leq X_t < x + dx | X_0 = 0)$. *The former uses a result of Feller (1951) that can be traced to Lévy (1948):*

$$p_t(x; a, b) = \frac{1}{\sigma\sqrt{t}} \sum_{k=-\infty}^{\infty} \left[\phi\left(\frac{2k(b-a)-x}{\sigma\sqrt{t}}\right) - \phi\left(\frac{2k(b-a)-2b+x}{\sigma\sqrt{t}}\right) \right], \quad (39)$$

while the latter uses a result of Billingsley (1968):

$$p_t(x; a, b) = \frac{1}{\sigma\sqrt{t}} \sum_{k=-\infty}^{\infty} \left[\phi\left(\frac{x + 2k(b-a)}{\sigma\sqrt{t}}\right) - \phi\left(\frac{2b-x + 2k(b-a)}{\sigma\sqrt{t}}\right) \right]. \quad (40)$$

The probabilistic results (39) and (40) are in fact equivalent, as one can be obtained from the other by appropriately replacing the summation index k with $-k$ and $\phi(x)$ with $\phi(-x)$.

3 The method of moments applied to volatility estimation using daily OHLC prices

In this section we apply Theorem 2.1 to the estimation of the volatility of the stock price from market data on OHLC prices, data which is readily available, even freely (see, for instance, Yahoo Finance).

Definition 3.1. *i) A trading day is the intra-day period elapsed between the opening and the closing bells of a business calendar day.*

ii) A virtual trading day is the after-hours period elapsed between the closing bell of one trading day and the opening bell of the next trading day.

iii) A one-day period consists of one trading day followed by one virtual trading day.

We assume that the stock price S_t has the usual geometric Brownian motion dynamics:

$$\frac{dS_t}{S_t} = \mu_s dt + \sigma dW_t, \quad t \geq 0. \quad (41)$$

Then the log-stock price $\log S_t$ is the arithmetic Brownian motion X_t defined in (4) with drift coefficient

$$\mu = \mu_s - \frac{\sigma^2}{2}. \quad (42)$$

Note that μ_s is the *one-day period* drift of the stock price S_t , while μ is the similar drift of the log-price $X_t = \log S_t$; the volatility parameter σ is the same for both S_t and X_t .

The market data used for parameter estimation is as follows: for each one-period day $i \in \{1, 2, \dots, n\}$ we denote by S_{i-1} the opening price and by H_i and L_i the intra-day high and low prices, respectively (i.e. the high and low are observed only during the trading day, and not the virtual trading day - see Figure 1).

The after-hours arrival of information in the market determines a jump between the closing price of one trading day and the opening price of the next day. We model this jump by letting the same geometric Brownian motion S_t have an unobserved evolution during a virtual trading day. The length of this virtual trading day is assumed to be, on average, a fraction f of the unit length of the one-day period.

Remark 3.1. *This assumption follows Garman and Klass (1980) and Yang and Zhang (2000), except that they assume the after-hours trading day precedes the actual trading day. They call it the opening jump (from C_{i-1} to O_i), and assume it is modeled by a Poisson process.*

Thus for $i \in \{1, 2, \dots, n\}$ we have (see Figure 1):

$$\text{OPEN}(i) = O_i = S_{i-1}, \text{ CLOSE}(i) = C_i = S_{i-f}, \text{ HIGH}(i) = H_i, \text{ LOW}(i) = L_i, \tag{43}$$

where:

$$H_i = \sup_{t \in [i-1, i-f]} S_t, \quad L_i = \inf_{t \in [i-1, i-f]} S_t. \tag{44}$$

The evolution of the price during the trading period $i - 1 \leq t < i - f$ is given by:

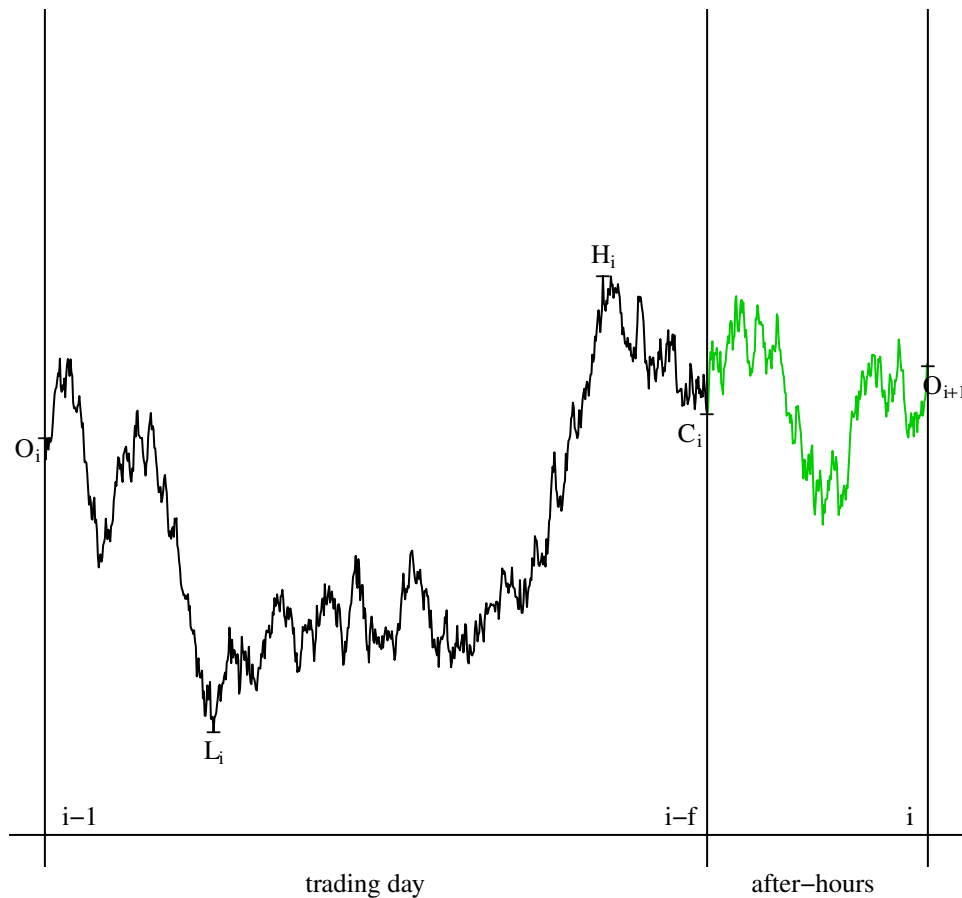


Figure 1: A one-day period consisting of a trading day and an after-hours period

$$\log S_t = \log O_i + \mu (t - i + 1) + \sigma(W_t - W_{t-i+1}), \tag{45}$$

and during the after-hours virtual trading period $i - f \leq t < i$ by:

$$\log S_t = \log C_i + \mu(t - i + f) + \sigma(W_t - W_{i-f}). \quad (46)$$

Taking expectation in (45) when $t \nearrow (i - f)$ and in (46) when $t \nearrow i$ gives:

$$E \left[\log \frac{C_i}{O_i} \right] = E \left[\log \frac{S_{i-f}}{S_{i-1}} \right] = \mu (1 - f), \quad (47)$$

$$E \left[\log \frac{O_{i+1}}{C_i} \right] = E \left[\log \frac{S_i}{S_{i-f}} \right] = \mu f. \quad (48)$$

Using $W_t - W_s$ identically distributed to W_{t-s} , the trading day and virtual trading day variances are obtained, respectively, as:

$$\begin{aligned} \text{VAR} \left[\log \frac{C_i}{O_i} \right] &= \text{VAR} \left[\log \frac{S_{i-f}}{S_{i-1}} \right] = \sigma^2 (1 - f), \\ \text{VAR} \left[\log \frac{O_{i+1}}{C_i} \right] &= \text{VAR} \left[\log \frac{S_i}{S_{i-f}} \right] = \sigma^2 f. \end{aligned} \quad (49)$$

Thus, we can write heuristically:

$$\sigma^2 = \text{VAR} \left[\log \frac{C_i}{O_i} \right] + \text{VAR} \left[\log \frac{O_{i+1}}{C_i} \right] = \text{VAR}(\text{trading day}) + \text{VAR}(\text{after hours}).$$

To estimate the variance over the trading day we use the method of moments. The range $R_{1-f} = \log H_1 - \log L_1$ of the arithmetic Brownian motion $X_t = \log S_t$ over the trading day $[0, 1 - f]$ was obtained in equation (13):

$$E(R_{1-f}) = ER(\mu, \sigma, 1 - f) = ER(\mu(1 - f), \sigma\sqrt{1 - f}, 1). \quad (50)$$

In (50) we estimate the left hand side $E(R_{1-f})$ using the daily range data:

$$k_1 := \frac{1}{n} \sum_{i=1}^n \log \frac{H_i}{L_i}. \quad (51)$$

and $\mu(1 - f)$ by (see (47)):

$$k_2 := \frac{1}{n} \sum_{i=1}^n \log \frac{C_i}{O_i}. \quad (52)$$

This leads to the following equation to be solved for x , the estimate of $\sigma\sqrt{1 - f}$:

$$k_1 = h \left(\frac{k_2}{x}, \frac{x^2}{k_2} \right). \quad (53)$$

The squared of this solution gives an estimate

$$V_i = x^2 \quad (54)$$

of the variance (volatility squared) corresponding to the trading day portion of a one-day period.

For the after-hours portion of the one-day period we have two choices: V_0 (centered approach) used in Yang and Zhang (2000), or V_0' (non-centered) used in Garman and Klass (1980). Using the former (i.e. the sample variance V_0 of $\ln(O_{i+1}/C_i)$), we obtain the estimate for the variance of the entire one-day period as given in equation (3), or, in annualised form, as:

$$\sigma_a^2 := 252V_Z. \quad (55)$$

Remark 3.2. *i) The term V_i replaces the linear combination of V_C and V_{RS} used by Yang and Zhang (2000) for the intra-day trading period, and it does not need estimating the value of k that achieves minimum variance.*

ii) Our estimator is a true range-based estimator (log-range to be precise since it uses $\log(H_i/L_i)$), unlike that of Yang and Zhang (2000).

iii) Our estimator V_Z is independent of both the drift and the weight f of the after-hours information.

4 Comparison results based on Monte Carlo simulation

In the following we assume the stock price follows a geometric Brownian motion with constant volatility as in (41) (except now we assume both drift and volatility are annualised), and compare the volatility estimator obtained in (3), (55) with V_{YZ} of (1).

Consider a partition Π of the time axis with a constant mesh size of dt days (or $dt_a = dt/252$ years) consisting of n_t time points: $\Pi = \{k \times dt_a, 1 \leq k \leq n_t\}$. At each point of the partition Π we compute the stock price in terms of the i.i.d. standard normal random variables $\{Z_k, 2 \leq k \leq n_t\}$:

$$\begin{aligned} S_{1 \times dt_a} &= S_0, \\ S_{k \times dt_a} &= S_{(k-1) \times dt_a} \exp \left\{ \left(\mu_s - \frac{1}{2} \sigma^2 \right) (dt_a) + \sigma \sqrt{dt_a} Z_k \right\}, 2 \leq k \leq n_t. \end{aligned} \quad (56)$$

The time horizon is $hor = dt \times n_t$ days, or $hor/252 = dt_a \times n_t$ years. The number of time points allocated to one day is $1/dt$ (day one prices are $\{S_{k \times dt_a}, 1 \leq k \leq 1/dt\}$), of which the first $(1-f)/dt$ form the trading day (intra-day) and the last f/dt points form the virtual trading day (after-hours) as illustrated in Figure 1.

The simulated opening, high, low and closing stock prices (OHLC) are obtained for each day $i \in \{1, \dots, hor\}$ in terms of the left time point $l(i) = i/dt + 1$ and of the right time point $r(i) = (i + 1 - f)/dt$ as:

$$O_i = S_{l(i)dt_a}, \quad H_i = \max_{t \in \{l(i)dt_a, \dots, r(i)dt_a\}} S_t, \quad L_i = \min_{t \in \{l(i)dt_a, \dots, r(i)dt_a\}} S_t, \quad C_i = S_{r(i)dt_a},$$

with an opening jump occurring from $C_{i-1} = S_{r(i-1)dt_a}$ to $O_i = S_{l(i)dt_a}$ for $i > 1$.

Example 4.1. For the stock price modeled by (41) let $\mu_s = 0.015$, $\sigma = 0.2$ (annualised), $S_0 = 100$, and assume $f = 0.25$, $dt = 0.005$ and $n_t = 50000$, so $hor = 250$ trading days. We compute the volatility estimate $\sigma_{YZ} = \sqrt{V_{YZ}}$ using

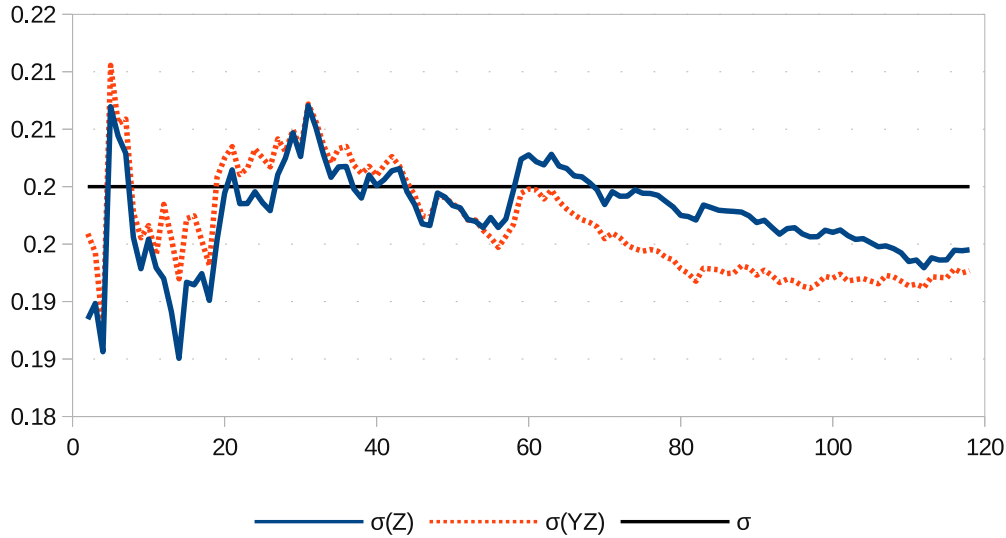


Figure 2: Comparison of the two volatility estimates and true value vs number of data points

(1) and $\sigma_Z = \sqrt{V_Z}$ using (3) for different values of n ranging from 2 to 118^3 . We plot in Figure 2 the term structure of our (annualised) volatility estimates for one scenario of simulated stock prices.

A remarkable feature of Figure 2 is that two estimation methods that use different range data (absolute range vs range relative to the opening and closing prices) result in strikingly similar term structure patterns.

We now repeat the simulation of Example 4.1 for $n_s = 5000$ times and compute the two estimates in each scenario.

Comparing the efficiency of the two estimators, or obtaining an MVUE (minimum variance unbiased estimate) result is not possible due to the implicit nature of equation (53) that needs to be solved. So we use alternative measures instead; we first compute the proportion of scenarios where σ_Z is closer than σ_{YZ} to the true value $\sigma = 0.2$ (i.e. $|\sigma_Z - \sigma| < |\sigma_{YZ} - \sigma|$). This proportion (empirical probability) is plotted in Figure 3 as a function of the number n of data points used in the estimation.

Figure 3 shows that with only two-data points the estimate σ_Z is closer than σ_{YZ} to the true σ in 39 percent of the scenarios. As we increase the number of data points we notice that at $n = 34$ we reach parity, while for $n \geq 37$ the estimate σ_Z is more often

³We omit the dependence on n in the notation, but we recall (51),(52),(2), and (V_0)

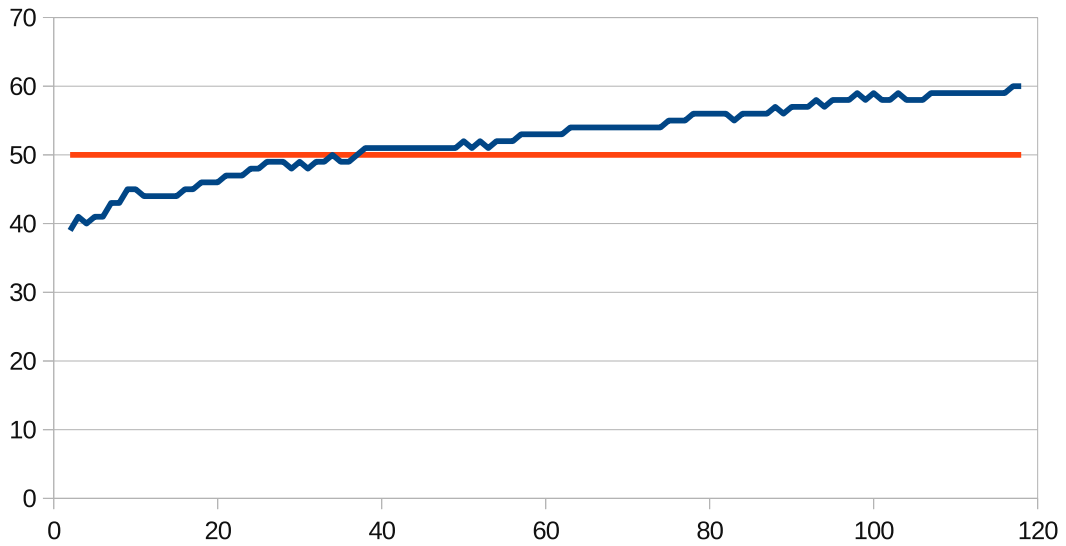


Figure 3: Percentage of scenarios where σ_Z is better than σ_{YZ} vs number of data points

(> 50%) more accurate than σ_{YZ} . This observation is required due to the potential concern that our estimator may be biased. In fact it behaves remarkably similar to σ_{YZ} which was shown to be unbiased.

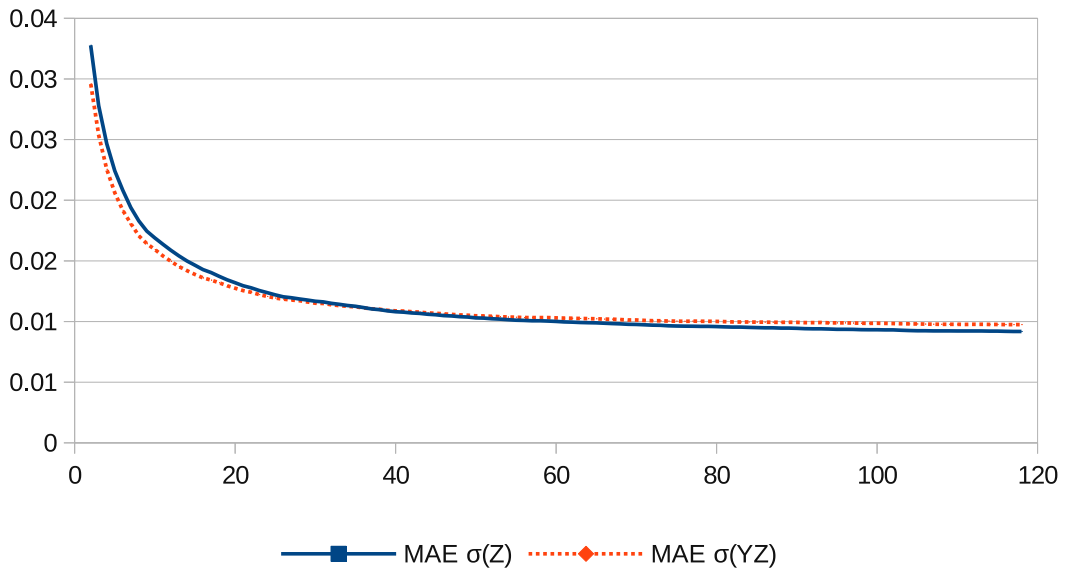


Figure 4: MAE for each estimate vs number of data points

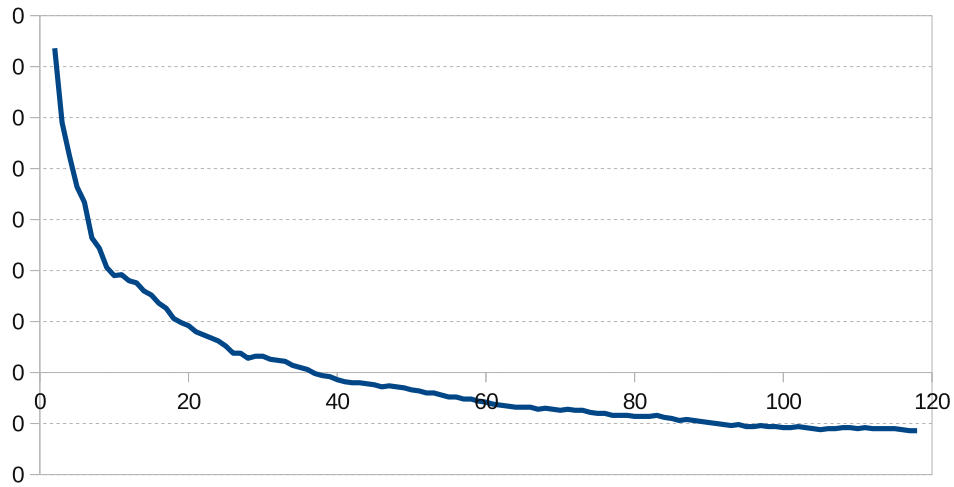


Figure 5: Difference in MAE between σ_Z and σ_{YZ} vs number of data points

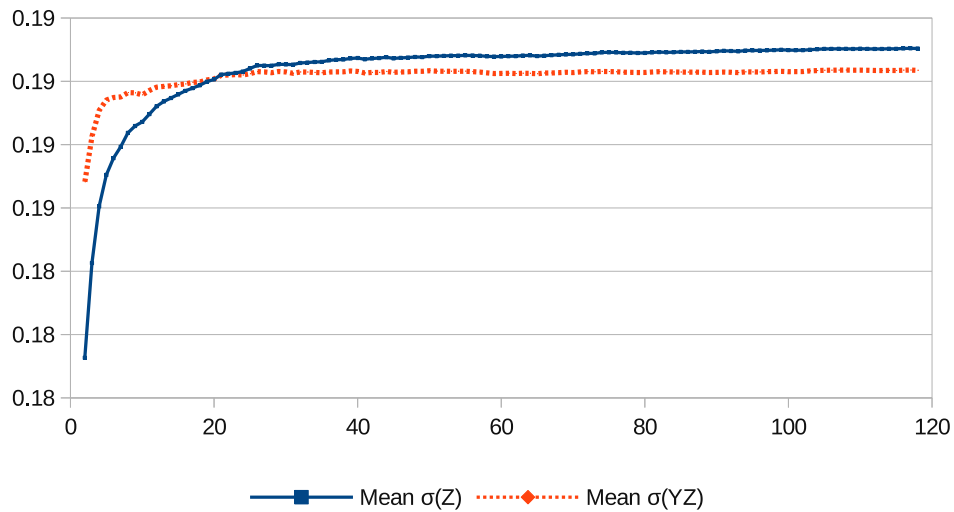


Figure 6: Average estimates σ_Z and σ_{YZ} vs number of data points

In Figure 4 we compare the average L^1 -error (Mean Absolute Error=MAE) of each estimate; we can notice that for $n \geq 55$ the mean absolute error for each of the two estimates tends to stabilize around a minimum value.

In Figure 5 we plot the MAE difference of the two estimators. In this case the MAE of σ_Z is smaller than that of σ_{YZ} for $n \geq 37$.

We plot the average of each volatility estimate over the number of scenarios in

Figure 6. We notice that for $n \geq 21$ the estimate σ_Z is, on average, closer to the true value $\sigma = 0.2$.

Remark 4.1. *In Yang and Zhang (2000) it is stated that the range of the fraction f is $[0.18, 0.3]$, with a typical value of 0.25. For this reason we have considered not only the case $f = 0.25$, but also $f = 0$, $f = 0.18$ and $f = 0.3$, but the plots were similar to those already presented. Even in the driftless case ($\mu = 0$) the values $f = 0$, $f = 0.18$, $f = 0.25$ and $f = 0.3$ resulted in findings that were qualitatively similar.*

Remark 4.2. *Based on these insights a good trade-off seems to be to use $n = 55$ data points in our volatility estimation.*

Following Garman and Klass (1980) we define the efficiency of our estimator with respect to V_{YZ} as:

$$Eff := \frac{\text{VAR}(V_{YZ})}{\text{VAR}(V_Z)}. \quad (57)$$

Since our estimator V_Z is based on the implicit equation (53), an explicit formula for Eff cannot be found. Instead, we resort to simulation to obtain a numerical approximation.

In Figure 7 we plot the efficiency of σ_Z wrt σ_{YZ} vs number of data points. We note that for a smaller number of data points our estimator is actually more efficient, while for a larger number of points the efficiency of our estimator stays within 99% of that of σ_{YZ} .

We conclude that in terms of unbiasedness and minimum variance, respectively, our estimator is:

- closer than σ_{YZ} to the real value σ in 40-60 percent of scenarios (Figure 3)
- with efficiency that doesn't drop by more than 1% (Figure 7).

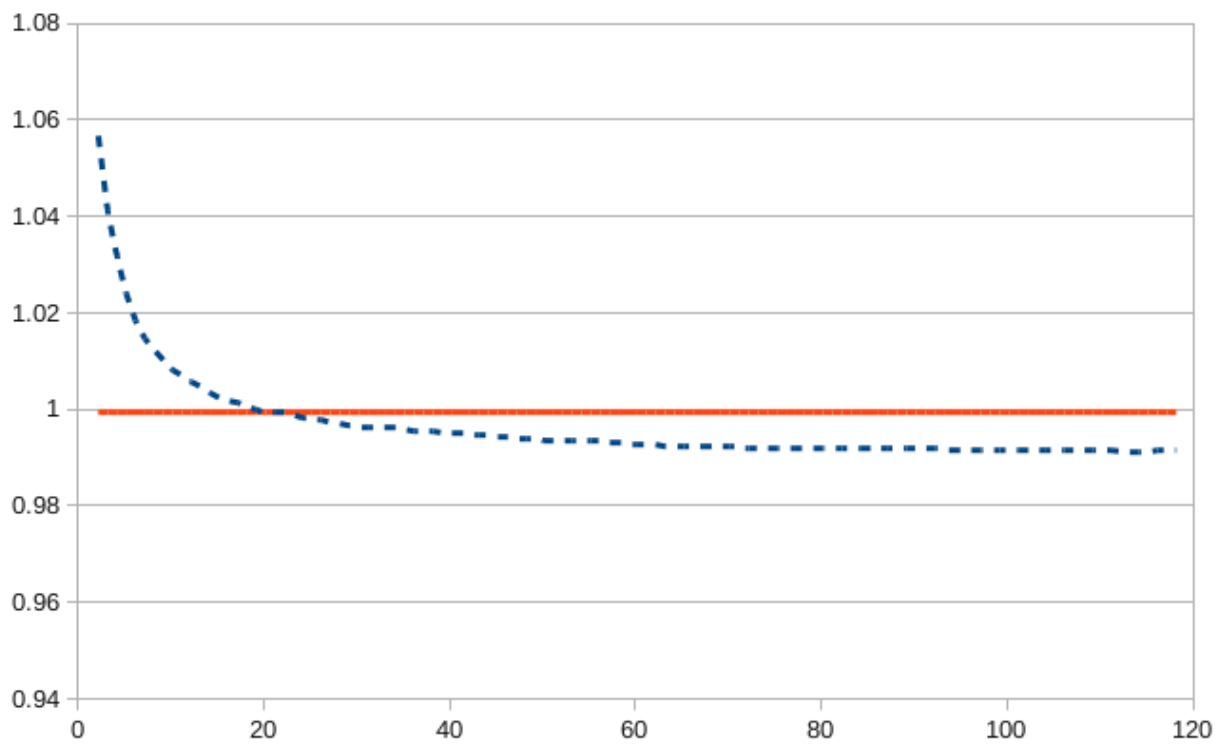


Figure 7: Efficiency of σ_Z wrt σ_{YZ} vs number of data points

5 Algorithmic implementation for mispricing opportunities with European call options

In this section we present a toy example of how this estimator could be used in an algorithmic trading context. Of course, a more careful analysis is required in devising such algorithms in practice.

First we solve the implicit equation (53) for a specific market data set, and then, after accounting for the after-hours effect, we use the resulting value to estimate Black-Scholes call option prices. A trading strategy takes advantage of mispricing opportunities that arise (i.e. of the difference between the estimated option prices and actual market prices).

Example 5.1. Consider the market data on the opening, high, low and closing (OHLC) prices for the IBM stock for the period from May 26, 2010 to June 18, 2010. For each of these days we consider the historical 3-month⁴ estimates of k_1 and k_2 ($n = 90$), and we solve the corresponding equation (53). We present in Figure 8 the solution in annualised form, after also including the effect of the after-hours movement (i.e. using (55) and annualised (1)).

We compare our estimate of the volatility corresponding to a one-day period with the one of Yang and Zhang (2000). On June 18, 2010 they are $\sigma_a = 0.2726$ (see (55)) and 0.2985 (annualised volatility corresponding to V_{YZ}).

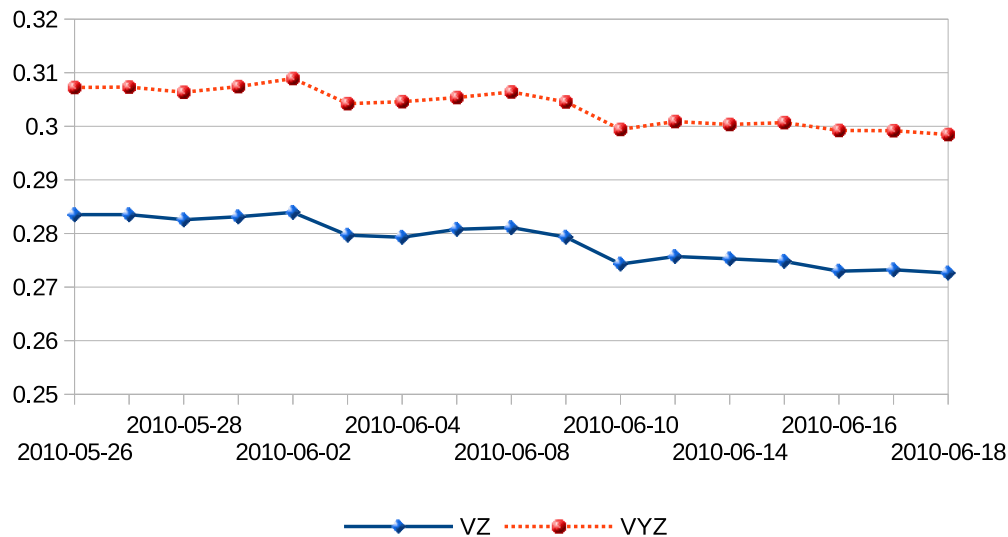


Figure 8: Estimated intra-day IBM volatility - May 26 to June 18, 2010

⁴For parameter estimation Hull (2006, p. 287) recommends using historical data of 90 to 180 days.

The resulting annualised volatility is used to compute the Black-Scholes prices of European call options on the stock:

$$C_t = S_t \Phi(d_1) - K e^{-r(T-t)} \Phi(d_2), \quad (58)$$

with

$$d_1 = \frac{\log(S_t/K) + (r + \sigma_a^2/2)(T-t)}{\sigma_a \sqrt{T-t}},$$

$$d_2 = d_1 - \sigma_a \sqrt{T-t}.$$

We then seek instances when the estimated prices differ the most from the market prices, and devise a trading strategy to take advantage of the price difference. For simplicity we trade only in European call options, and assume that at expiry there is a payment equal to the payoff so that no actual trading occurs in the underlying stock (naked trading). The purpose is to illustrate how easy the estimation can be incorporated in a trading algorithm.

Having assumed a constant volatility there is no volatility smile and no stochastic volatility⁵, so we restrict our analysis to European call options whose strike prices are relatively close to the stock price at the beginning of the period (preferably in the money), and whose expiry dates are up to three months (the parameters can be re-estimated later in view of new data).

Example 5.2. Consider the market prices for the European call options on IBM from May 26, 2010 to June 18, 2010, with expiries June 18 and July 16, and strikes $K \in \{115, 120, 125, 130\}$ (the stock price on May 26 was 125.91). We compare these market prices with the Black-Scholes prices calculated using (58). Here the inputs are the stock price, the volatility estimated in Example 5.1, and r the value of the 1-month US Treasury bill yield for the previous day (from online Treasury data⁶

We allow prices to differ by up to 10% from the bid-ask spread. Thus, we trade when our estimated call price falls outside the interval $(0.9 \times \text{bid}, 1.1 \times \text{ask})$.

There are two cases. If our price is lower, then we short-sell the option at the bid price and wait for the first day when the estimated price is no longer lower to buy back the option at the then ask price. If it expires and the call option is exercised then we buy the stock in the market and deliver it.

If our price is higher, then we buy the option at ask price and wait for the first day when the price is no longer higher to sell it at the then bid price. If it reaches expiry date, then we exercise it.

This trading strategy is summarized in Algorithm 1 for t between May 26, 2010 and June 18, 2010 for European call options expiring at close June 18, 2010. The data

⁵Alternative approaches like stochastic volatility or econometric models (ARMA, GARCH etc) are not discussed here.

⁶<http://www.treasury.gov/resource-center/data-chart-center/interest-rates/Pages/default.aspx>

is retrieved once a day, except on expiration date when it is retrieved several times a day (this can be implemented as an algorithmic trading strategy and deployed continuously without much effort, especially by those interested in technical trading).

t	K	$\hat{C}(t)$	(bid,ask)	trade	t	(bid,ask)	trade	profit
May 26	130	0.90	(1.16,1.17)	sell	May 27	(0.96,0.99)	buy	0.17
May 28	130	0.57	(0.75,0.78)	sell	Jun 2	(0.67,0.70)	buy	0.05
Jun 7	125	1.92	(2.20,2.23)	sell	Jun 8	(1.21,1.23)	buy	0.97
Jun 7	130	0.29	(0.42,0.44)	sell	Jun 8	(0.15,0.17)	buy	0.25
Jun 8	120	3.69	(4.15,4.20)	sell	Jun 9	(4.60,4.75)	buy	(0.6)
Jun 9	125	1.09	(1.30,1.38)	sell	Jun 10	(3.05,3.15)	buy	(1.77)
Jun 17	130	1.09	(0.90,0.94)	buy	Jun 19	(1.00,1.05)	sell	0.06
Jun 18 ^a	130	0.60	(0.48,0.51)	buy	Jun 18 ^b	(0.63,0.69)	sell	0.12
Jun 18 ^c	130	0.18	(0.21,0.25)	sell	Jun 18 ^d	$S_t=130.14$	ex ^e	0.07

^a at 12:27 ^b at 13:36 ^c at 15:58 ^d at 16:00 ^e if exercised

Table 1: Trading in the call option expiring June 18, 2010 (left: open a position, right: close position)

Remark 5.1. *According to Table 1 this strategy results in an overall loss of 0.68, due mostly to one large loss on June 10 (we use yesterday's volatility to trade in today's world).*

Over a time horizon longer than a month the strategy can absorb such shocks in the stock prices, provided they are sparse. Alternatively, one can implement an additional stopping rule when the change in the stock price exceeds a pre-determined margin.

A similar behaviour is exhibited when applying the same trading strategy to the European call option expiring at close July 16, 2010, but as the expiry date is longer than a couple of months the limitations of the assumptions of the model become apparent. However, the ease of implementation of this estimator into algorithmic trading is clear.

6 Conclusions

We have used the method of moments to estimate the volatility of the stock price as a solution of an implicit equation and an additional term due to the after-hours effect. In terms of unbiasedness and minimum variance we looked at alternative measures due to the implicit nature of the equation: for a small number of data points our estimator is more efficient than that of Yang and Zhang (2000), while for larger number of data points its efficiency drops by no more than 1%. A simple example illustrated how this can be implemented in an algorithm to profit from mispricing opportunities.

As a by-product we derived the density and expectation of the range of an arithmetic Brownian motion.

In comparison to the estimator of Yang and Zhang (2000), our volatility estimator takes advantage of the actual range of the Brownian motion. It is most useful for short expiration dates and for strike prices that are not far out. We found that it is an efficient alternative that can be easily computed and has a fast practical implementation. These traits recommend it to the attention of practioners in the field.

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Algorithm 1: A trading strategy implementing new estimator

```

input : Parameters  $t, \sigma_a, r, S_t$ , bid(t) and ask(t) (European call prices)
output: Profit of trading strategy

1 profit  $\leftarrow$  0; // initialize
2  $T \leftarrow$  June 18, 2010; // expiry date
3 compute  $\sigma_a$ ; // Example 5.1
4 for  $K \leftarrow$  115 to 130 do
5    $\hat{C}(t) \leftarrow$  BlackScholesCall( $t, T, K, \sigma_a, r, S_t$ ); // (58)
6   if  $\hat{C} < 0.9 \times \text{bid}(t)$  then
7     profit  $\leftarrow$  profit + bid(t); // short-sell call
8     while  $t < T$  and  $\hat{C}(t) < 0.9 \times \text{bid}(t)$  do
9        $t \leftarrow t+1$ ; // wait 1 day
10      compute  $\sigma_a$ ; // Example 5.1
11       $\hat{C}(t) \leftarrow$  BlackScholesCall( $t, T, K, \sigma_a, r, S_t$ ); // (58)
12    end
13    if  $t < T$  then
14      profit  $\leftarrow$  profit - ask(t); // buy back call
15    else
16      if call is exercised then
17        profit  $\leftarrow$  profit - ( $S_t - K$ );
18        // buy stock and deliver for K
19      end
20    end
21  else
22    if  $\hat{C} > 1.1 \times \text{ask}(t)$  then
23      profit  $\leftarrow$  profit - ask(t); // buy call
24      while  $t < T$  and  $\hat{C}(t) > 1.1 \times \text{ask}(t)$  do
25         $t \leftarrow t+1$ ; // wait 1 day
26        compute  $\sigma_a$ ; // Example 5.1
27         $\hat{C}(t) \leftarrow$  BlackScholesCall( $t, T, K, \sigma_a, r, S_t$ );
28      end
29      if  $t < T$  then
30        profit  $\leftarrow$  profit + bid(t); // sell call
31      else
32        profit  $\leftarrow$  profit + ( $S_t - K$ ); // exercise call
33      end
34    end
35  end
36 return profit;

```
