MODULARITY LIFTING RESULTS IN PARALLEL WEIGHT ONE AND APPLICATIONS TO THE ARTIN CONJECTURE: THE TAMELY RAMIFIED CASE

PAYMAN L KASSAEI, SHU SASAKI, YICHAO TIAN

Abstract. We extend the modularity lifting result of [17] to allow Galois representations with some ramification at \( p \). We also prove modularity mod 2 and 5 of certain Galois representations. We use these results to prove many new cases of the strong Artin conjecture over totally real fields in which 5 is unramified.

1. Introduction

The seminal work of Buzzard and Taylor [4] beautifully combined methods of Wiles and Taylor (à la Diamond [9]) with a geometric analysis of overconvergent \( p \)-adic modular forms to prove a modularity lifting result for geometric representations of \( \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \) which are split and unramified at \( p \). Combined with the works of Shepherd-Barron-Taylor, and Dickinson, these ideas came together in [3], where a program laid out by R. Taylor came to fruition in proving many cases of the Artin conjecture over \( \mathbb{Q} \). Later, Buzzard generalized the results of [4], allowing the Galois representation to have ramification at \( p \), and these results were used by Taylor to prove more cases of the Artin conjecture.

In [17], the first-named author proved a generalization of the main result of Buzzard and Taylor to representations of \( \text{Gal}(\bar{\mathbb{Q}}/F) \), where \( F \) is a totally real field in which \( p \) is unramified. At the heart of the argument in [17] lies the proof that a collection of weight one specializations of Hida families are all classical Hilbert modular forms. The proof closely employs the geometry of the Hilbert modular variety \( \mathfrak{Y}_{\text{rig}} \) of level \( \Gamma_1(N) \cap \Gamma_0(p) \) (studied in [13]) to provide analytic continuation of these overconvergent Hilbert modular forms to a big enough region \( \mathfrak{Y}_{\text{rig}} \cap \mathfrak{X}_{\text{rig}} \subset \mathfrak{Y}_{\text{rig}} \), which, crucially, contains a region saturated with respect to the forgetful map

\[
\pi : \mathfrak{Y}_{\text{rig}} \to \mathfrak{X}_{\text{rig}}
\]

to the Hilbert modular variety \( \mathfrak{X}_{\text{rig}} \) of level \( \Gamma_1(N) \). It is, then, proved that certain linear combinations of the collection of forms at hand descend under the map \( \pi \) to a region \( \mathfrak{X}_{\text{rig}} \cap \mathfrak{X}_{\text{rig}} \supseteq \mathfrak{X}_{\text{rig}} \), over which a rigid-analytic Koecher principle can be applied to extend the descended forms to the entire \( \mathfrak{X}_{\text{rig}} \).

In this paper, we build on the methods of [17] and generalize the result, allowing some ramification at \( p \) for the Galois representation. More precisely, we prove:

**Theorem 1.1.** Let \( p > 2 \) be a prime number, and \( F \) a totally real field in which \( p \) is unramified. For any prime ideal \( \mathfrak{p} | p \), let \( D_{\mathfrak{p}} \) denote a decomposition group of \( \text{Gal}(\bar{\mathbb{Q}}/L) \) at \( \mathfrak{p} \),

\[
\text{Gal}(\bar{\mathbb{Q}}/L) 
\]
and $I_p$ the inertia subgroup. Let $\rho: \text{Gal}(\overline{\mathbb{Q}}/F) \to \text{GL}_2(\mathcal{O})$ be a continuous representation, where $\mathcal{O}$ is the ring of integers in a finite extension of $\mathbb{Q}_p$, and $m$ its maximal ideal. Assume

- $\rho$ is unramified outside a finite set of primes,
- For every prime $p|p$, we have
  \[ \rho|_{I_p} \cong \alpha_p \oplus \beta_p, \]
  where $\alpha_p, \beta_p: D_p \to \mathcal{O}^\times$ are characters distinct modulo $m$, and $\alpha_p(I_p)$ and $\beta_p(I_p)$ are finite, and $\alpha_p/\beta_p$ is tamely ramified,
- $\mathcal{P} := \left( \rho \mod m \right)$ is ordinarily modular, i.e., there exists a classical Hilbert modular form $g$ of parallel weight 2 such that $\rho \equiv \rho_g(\mod m)$ and $\rho_g$ is potentially ordinary and potentially Barsotti-Tate at every prime of $F$ dividing $p$,
- $\mathcal{P}$ is absolutely irreducible when restricted to $\text{Gal}(\overline{\mathbb{Q}}/F(\zeta_p))$.

Then, $\rho$ is isomorphic to $\rho_f$, the Galois representation associated to a Hilbert modular eigenform $f$ of weight $(1,1,\cdots,1)$ and level $\Gamma_1(Np)$, for some integer $N$ prime to $p$.

We first produce a collection of overconvergent Hilbert modular forms of parallel weight one whose associated Galois representations are isomorphic to various twists of $\rho$, and then prove that they are classical. These modular forms live over $\tilde{\mathcal{X}}$, the Hilbert modular variety of level $\Gamma_1(Np)$. What makes the descent process possible in [17] is the fact that the Galois representation $\rho$ is assumed unramified at $p$. Since here we are not making that assumption, it is no longer possible to prove the classicality of the forms by descending them to the conveniently big region $X_{\text{rig}}^{[p]} \leq 1$. Instead, we proceed as follows. We first extend the forms to a large admissible open $W \subset \tilde{\mathcal{X}}$ applying analytic continuation results obtained in [17], as well as through a gluing process as in [2]. We then prove that $W$ is large enough, so that a combination of multiple applications of the rigid analytic Koecher principle, and a certain principle of degree increasing under the $U_p$ operator will allow us to extend any finite slope $U_p$-eigenform from $W$ to the entire $\tilde{\mathcal{X}}$. This requires a refined understanding of the special fibre of $\tilde{\mathcal{Y}}_{\text{rig}}$, which we study using the results of [13].

In the second part of the paper, we apply our results to prove certain cases of the strong Artin conjecture over totally real fields. We first prove modularity of certain Galois representations mod 2 and 5. Combining these results with Theorem [11], we prove many new cases of the strong Artin conjecture over the totally real field $F$. When $p$ is splits in $F$, the following is a result of the second-named author.

**Theorem 1.2.** Let $F$ be a totally real field in which 5 is unramified. Let

\[ \rho: \text{Gal}(\overline{\mathbb{Q}}/F) \to \text{GL}_2(\mathbb{C}) \]

be a totally odd and continuous representation satisfying the following conditions:

- $\rho$ has the projective image $A_5$.
- For every place $p$ above 5, the projective image of the decomposition group $D_p$ at $p$ has order 2. Furthermore, the quadratic extension of $F_p$ fixed by the kernel of $D_p \hookrightarrow G_F^{\text{proj}} \cong A_5$ is not $F_p(\sqrt{5})$.

Then, there exists a holomorphic Hilbert cuspidal eigenform $f$ of weight 1 such that $\rho$ arises from $f$ in the sense of Rogawski-Tunnell, and the Artin $L$-function $L(\rho,s)$ is entire.
While the article [17] was under review for publication, V. Pilloni sent the first-named author a preprint where results similar to those in [17] were presented, allowing ramification $e \leq p - 1$ for the totally real field at primes dividing $p$. As we show in this article, the method in [17] is amenable to generalization to cases where the Galois representation is not assumed unramified at $p$. We are working on extending the results of this paper to the case of wild ramification. Also, in a joint work with V. Pilloni, we hope to generalize the results allowing ramification at $p$, simultaneously, for the totally real field and the Galois representation.

1.3. Notation. Let $p$ be a prime number. Let $L/\mathbb{Q}$ be a totally real field of degree $g > 1$ in which $p$ is unramified, $\mathcal{O}_L$ its ring of integers, and $\mathfrak{d}_L$ its different ideal. For a prime ideal $\mathfrak{p}$ of $\mathcal{O}_L$ dividing $p$, let $\kappa_\mathfrak{p} = \mathcal{O}_L/\mathfrak{p}$, a finite field of order $p^h$. Let $\kappa$ be a finite field containing an isomorphic copy of all $\kappa_\mathfrak{p}$ which is generated by their images. We identify $\kappa_\mathfrak{p}$ with a subfield of $\kappa$ once and for all. Let $\mathbb{Q}_\kappa$ be the fraction field of $W(\kappa)$. We fix embeddings $\mathbb{Q}_\kappa \subset \mathbb{Q}_p^{ur} \subset \mathbb{C}_p$. Let $v_p$ the $p$-adic valuation on $\mathbb{C}_p$.

Let $S = \{\mathfrak{p}|p\}$ be the set of prime ideals of $\mathcal{O}_L$ dividing $p$. Let $\mathbb{B} = \text{Emb}(L, \mathbb{Q}_\kappa) = \prod_{\mathfrak{p} \in S} \mathbb{B}_\mathfrak{p}$, where $\mathbb{B}_\mathfrak{p} = \{\beta \in \mathbb{B} : \beta^{-1}(pW(\kappa)) = \mathfrak{p}\}$, for every prime ideal $\mathfrak{p}$ dividing $p$. Let $\sigma$ denote the Frobenius automorphism of $\mathbb{Q}_\kappa$, lifting $x \mapsto x^p$ modulo $p$. It acts on $\mathbb{B}$ via $\beta \mapsto \sigma \circ \beta$, and transitively on each $\mathbb{B}_\mathfrak{p}$. For $S \subseteq \mathbb{B}$, we let $S^c = \mathbb{B} - S$, and $\ell(S) = \{\sigma^{-1} \circ \beta : \beta \in S\}$, the left shift of $S$. Similarly, define the right shift of $S$, denoted $r(S)$. We denote by $|S|$ the cardinality of $S$. The decomposition

$$\mathcal{O}_L \otimes \mathbb{Z} W(\kappa) = \bigoplus_{\beta \in \mathbb{B}} W(\kappa)_\beta,$$

where $W(\kappa)_\beta$ is $W(\kappa)$ with the $\mathcal{O}_L$-action given by $\beta$, induces a decomposition,

$$M = \bigoplus_{\beta \in \mathbb{B}} M_\beta,$$

on any $\mathcal{O}_L \otimes \mathbb{Z} W(\kappa)$-module $M$.

Let $N \geq 4$ be an integer prime to $p$. Throughout the paper, we will fix a finite extension $K$ of $\mathbb{Q}_\kappa$ with residue field $\kappa_K$, and ring of integers $\mathcal{O}_K$.

We denote by $X$ the Hilbert modular scheme of level $\Gamma_1(N)$ over $\text{Spec}(\mathcal{O}_K)$, and $Y$ the Hilbert modular scheme over $\text{Spec}(\mathcal{O}_K)$ of level $\Gamma_1(N) \cap \Gamma_0(p)$. The base extensions of $X, Y$ to $\text{Spec}(K)$ are denoted, respectively, $X_K, Y_K$. The connected components of $X$ and $Y$ are both in natural bijection with the set of strict ideal class group $c_l^+$. For a representative $(a, a^+)$ of $c_l^+$, we denote the corresponding connected component of $X$ (respectively, $Y$) by $X_a$ (respectively, $Y_a$). We apply the same convention to other variants of $X, Y$ appearing in this paper. Let $\mathcal{X}, \mathcal{Y}$ be, respectively, the special fibres of $X$ and $Y$, $\mathfrak{X}$ and $\mathfrak{Y}$, the formal completions of $X$ and $Y$ along their special fibres, and $\mathfrak{X}_{\text{rig}}$ and $\mathfrak{Y}_{\text{rig}}$, the associated rigid analytic generic fibres over $K$. By abuse of notation, we always denote by $\pi$ the natural forgetful projection from level $\Gamma_1(N) \cap \Gamma_0(p)$ to $\Gamma_1(N)$ in various settings, e.g. $\pi : Y \to X$, $\pi : \mathcal{Y} \to \mathcal{X}$, $\pi : \mathfrak{Y}_{\text{rig}} \to \mathfrak{X}_{\text{rig}}$. 
Let $A^\text{univ}$ be the universal abelian scheme over $X$, and $e : X \to A^\text{univ}$ be the unit section. We put $\omega = e^*\Omega^1_{A^\text{univ}/X}$. This is a locally free $(\mathcal{O}_X \otimes_{\mathbb{Z}} \mathcal{O}_L)$-module of rank 1, so that we have a canonical decomposition $\omega = \bigoplus_{\beta \in \mathbb{B}} \omega_\beta$ according to the natural action of $\mathcal{O}_L$ on $\omega$. For each $k = (k_\beta)_{\beta \in \mathbb{B}} \in \mathbb{Z}^\mathbb{B}$, we put

$$\omega^k = \bigotimes_{\beta \in \mathbb{B}} \omega_\beta^{\otimes k_\beta}.$$  

We still denote by $\omega$, $\omega_\beta$ and $\omega^k$ the pull-back to $Y$ via $\pi$ or the corresponding sheaves in the formal or rigid analytic settings.

We choose toroidal compactifications $\tilde{X}$ and $\tilde{Y}$ based on a common fixed choice of rational polyhedral cone decompositions. We still denote by $\pi$ the natural map $\tilde{Y} \to \tilde{X}$. There is a semi-abelian scheme $\tilde{A}$ over $\tilde{X}$ extending the universal abelian scheme $A^\text{univ}$. We denote by $\tilde{X}$ and $\tilde{Y}$ the corresponding formal completions along the special fibres, by $\tilde{X}_\text{rig} = \tilde{X}^\text{an}$ and $\tilde{Y}_\text{rig} = \tilde{Y}^\text{an}$ the associated rigid analytic spaces. For any $k \in \mathbb{Z}^\mathbb{B}$, the line bundle $\omega^k$ on $X$ (resp. on $Y$) extends to a line bundle over $\tilde{X}$ (resp. $\tilde{Y}$), still denoted by $\omega^k$, by using the semi-abelian scheme $\tilde{A}$ over $\tilde{X}$ (resp. $\tilde{Y}$).

Let $\tilde{X}_\text{rig}$ be the ordinary locus of $X_\text{rig}$, i.e., the quasi-compact admissible open subdomain of $X_\text{rig}$, where the rigid semi-abelian scheme $\tilde{A}^\text{an}$ has good ordinary reduction or specializes to cusps, and $X_\text{ord} = \tilde{X}_\text{rig} \cap X_\text{rig}$. Similarly, let $\tilde{Y}_\text{rig}$ be the locus of $\pi^{-1}(X_\text{ord})$ where the finite flat subgroup $H$ is of multiplicative type, and $\tilde{Y}_\text{ord} \subset \tilde{Y}_\text{rig}$ be the union of $\tilde{Y}_\text{ord}$ together with the locus of $\tilde{Y}_\text{rig}$ with reduction to unramified cups. Let $\tilde{Y}_\text{ord} = \tilde{Y}_\text{rig} \cap \tilde{Y}_\text{rig}$.

## 2. Preliminaries on the Geometry of $\tilde{Y}$

In this section, we prove some results on the geometry of the special fibre of $Y$, which will be useful in the analytic continuation process of later sections.

### 2.1. Atkin-Lehner automorphisms

Let $S$ be an $\mathcal{O}_K$-scheme, and consider an $S$-valued point $Q$ of $Y$ corresponding to $(A, H) = (A, i_N, \lambda, H)$. We have a canonical decomposition $H = \bigcap_{q \in S} H[q]$. For any $p \in S$, we put

$$w_p(Q) = (A/H[p], H'),$$  

where $A/H[p]$ denotes the quotient of $A$ by $H[p]$ along with its induce PEL data (as defined in [13] §2.1), and $H' = \bigcap_q H'[q]$, with $H'[p] = A[p]/H[p]$ and $H'[q] = H[q]$, under the identification $(A/H[p])[q] \cong A[q]$ for $q \neq p$. The automorphisms $w_p$ for $p \in S$ commute with each other. We put $w_T = \prod_{p \in T} w_p$ for any subset $T \subset S$, and $w = w_S$. In view of

$$w^2(Q) = \left( \prod_{p \in S} w_p^2 \right)(Q) = (A, \lambda, pi_N, H),$$

we see that each $w_p$ is automorphism on $Y$, and we call it $p$-th Atkin-Lehner automorphism. The automorphisms $w_p$ extend naturally to the chosen toroidal compactification $\tilde{Y}$ We still denote by $w_p$ the automorphisms induced on $\tilde{Y}_K$, $\tilde{\mathcal{O}}$, $\tilde{\mathcal{Y}}$, etc.
2.2. Partial degrees and valuations. We recall the notions of partial degrees introduced in [20], and partial valuations defined in [13], and discuss the relationship between the two notions. Let $Q = (\mathcal{A}, H)$ be a rigid point of $\mathfrak{Y}_{\rig}$ defined over a finite extension $K'/K$.

Case 1. $A$ has good reduction over $\mathcal{O}_{K'}$. Then $A$ and $H$ can be defined over $\mathcal{O}_{K'}$. Let $\omega_H$ be the module of invariant differential 1-forms of $H$. We have a canonical decomposition $\omega_H = \bigoplus_{\beta \in \mathcal{B}} \omega_{H,\beta}$, where each $\omega_{H,\beta}$ is a torsion $\mathcal{O}_{K'}$-module generated by one element. So there exists $a_\beta \in \mathcal{O}_{K'}$ such that $\omega_{H,\beta} \simeq \mathcal{O}_{K'}/(a_\beta)$. We put

$$\deg_{\beta}(Q) = \deg_{\beta}(H) = v_p(a_\beta) \in \mathbb{Q} \cap [0, 1].$$

Case 2. $A$ has semi-stable reduction over $\mathcal{O}_{K'}$. Then we have a canonical decomposition $H \simeq \prod_{p \in \mathcal{S}} H[p]$ of group schemes over $K'$. Recall that $A[p]$ has a unique maximal $\mathcal{O}_{L'}$-subgroup, denoted by $A[p]^\mu$, which extends to a group scheme over $\mathcal{O}_{K'}$ of multiplicative type. We put, for all $\beta \in \mathcal{B}$,

$$\deg_{\beta}(Q) = \begin{cases} 1 & \text{if } H[p] = A[p]^\mu, \\ 0 & \text{otherwise}. \end{cases}$$

We get therefore a parametrization of $\mathfrak{Y}_{\rig}$ by the partial degrees [13, 20, 11]:

$$\deg = (\deg_{\beta})_{\beta \in \mathcal{B}} : \mathfrak{Y}_{\rig} \to [0, 1]^\mathcal{B}.$$ 

For an ideal $t$ of $\mathcal{O}_L$ dividing $(p)$, we put $t^* = (p)/t$, $\mathcal{B}_t = \prod_{p \mid t} \mathcal{B}_p$. Then we have $\mathcal{B} = \mathcal{B}_t \coprod \mathcal{B}_{t^*}$ and a decomposition of rigid analytic spaces

$$\mathfrak{Y}_{\rig} - \mathfrak{Y}_{\rig} = \prod_{(p) \subset \mathcal{O}_L} (\mathfrak{Y}_{\rig} - \mathfrak{Y}_{\rig})_{t},$$

where $(\mathfrak{Y}_{\rig} - \mathfrak{Y}_{\rig})_t$ consists of points $Q \in \mathfrak{Y}_{\rig} - \mathfrak{Y}_{\rig}$ with $\deg_{\beta}(Q) = 1$ for $\beta \in \mathcal{B}_t$ and $\deg_{\beta}(Q) = 0$ for $\beta \in \mathcal{B}_{t^*}$.

Using the additivity of partial degrees [20, 3.6], we have

$$(2.2.1) \quad \deg_{\beta}(w_p(Q)) = \begin{cases} 1 - \deg_{\beta}(Q) & \text{if } \beta \in \mathcal{B}_p \\ \deg_{\beta}(Q) & \text{if } \beta \not\in \mathcal{B}_p, \end{cases}$$

for any $Q \in \mathfrak{Y}_{\rig}$ and $p \in \mathcal{S}$.

**Definition 2.3.** Let $Q \in \mathfrak{Y}_{\rig}$. For any $p \in \mathcal{S}$, we define $\deg_{p}(Q) = \sum_{\beta \in \mathcal{B}_p} \deg_{\beta}(Q)$. If $\mathcal{I}$ is a multiset of intervals indexed by $\mathcal{S}$, $\mathcal{I} = \{I_p \subset [0, f_p] : p \in \mathcal{S}\}$, and $\mathcal{V} \subset \mathfrak{Y}_{\rig}$, we define an admissible open of $\mathfrak{Y}_{\rig}$

$$\mathcal{V}_{\mathcal{I}} = \{Q \in \mathcal{V} : \deg_{p}(Q) \in I_p, \forall p \in \mathcal{S}\}.$$ 

Also, for an interval $I \subset [0, f]$, define $\mathfrak{Y}_{\rig} I = \{Q \in \mathfrak{Y}_{\rig} : \deg(Q) = \sum_{p \in \mathcal{S}} \deg_{p}(Q) \in I\}$.

We now discuss the relation to partial valuations $\nu_{\beta}(Q)$ defined in [13, §4.2], using the mod-$p$ geometry of $\mathfrak{Y}_{\rig}$ The valuations and partial degrees are related as follows

$$\nu_{\beta}(Q) = 1 - \deg_{\beta}(Q).$$
Remark 2.4. In this paper, we have decided to use the partial degrees which are more intrinsically defined in terms of the subgroup $H$. Since we refer often to the results of [13], and [17], the reader should be mindful of the slight change in the notation. In those references, intervals are formed using partial valuations rather than partial degrees. The two notions are simply related by interchanging $I$ with $I^w$ obtained by replacing any interval $I_p = [a, b]$, with $I_p^{w} := [f_p - b, f_p - a]$ (similarly, for open, half-open intervals).

2.5. Mod-$p$ geometry. Let $x = (x_\beta) \in \{0, 1\}^B$ be a vertex point of $[0,1]^B$. Define

$$\varphi_x = \{ \beta \in B : x_{r-1,\beta} = 1 \}$$

$$\eta_x = \{ \beta \in B : x_\beta = 0 \}$$

Let $W_x = W_{\varphi_x, \eta_x}$ be a stratum in the stratification $\{W_{\varphi, \eta}\}$ studied in [13]. Then, by [13], 4.3.1], we have

$$\deg^{-1}(x) = \begin{cases} \text{sp}^{-1}(W_x) \cup (\{2\}_{\text{rig}} - \{2\}_{\text{rig}}) \text{t} & \text{if } x = x_t \text{ for some } t(p); \\ \text{sp}^{-1}(W_x) & \text{otherwise.} \end{cases}$$

Here, $\text{sp} : \{2\}_{\text{rig}} \to \mathbb{Y}$ is the specialization map, and $x_t$ is the vertex point whose $\beta$-th component is 1 if $\beta \in \mathbb{B}_t$, and 0 otherwise. By [13, 2.5.2], the map $x \mapsto W_x$ establishes a one-to-one correspondence between $\{0,1\}^B$ and the set of codimension 0 strata in the stratification $\{W_{\varphi, \eta}\}$.

Definition 2.6. Let $p \in S$. Let $x$ be a vertex of the cube $[0,1]^B$. We say $x$ (or $W_x$) is

1. $p$-ordinary, if for all $\beta \in \mathbb{B}_p$, we have $x_\beta = 1$, or for all $\beta \in \mathbb{B}_p$, we have $x_\beta = 0$;
2. of type 1 at $p$, if one of the following three equivalent conditions is satisfied:
   (a) the cardinality of $\varphi_x \cap \eta_x \cap \mathbb{B}_p$ is 1;
   (b) the cardinality of $r(\eta_x) \cap \ell(\varphi_x) \cap \mathbb{B}_p$ is 1;
   (c) there exists a $\beta_p \in \mathbb{B}_p$, and an integer $0 \leq m_p - 1 \leq f_p - 1$, such that we have $\varphi_x \cap \mathbb{B}_p = \{\sigma^j \circ \beta_p : 1 \leq j \leq m_p \}$ (Note that we always have $\eta_x = \ell(\varphi_x)^c$).

The vertex points of type 1 are stable under the involution $x \mapsto w_p(x)$, where the $\beta$-component of $w_p(x)$ is $1 - x_\beta$ for any $\beta \in \mathbb{B}_p$, and $x_\beta$ otherwise. We have $w_p(W_x) = W_{w_p(x)}$.

Definition 2.7. For any vertex $x$, and $S \subset S$, let $W_x^{sp,S}$ be the closed subscheme of $W_x$ defined as

$$W_x^{sp,S} = \{Q \in W_x : \tau(\pi(Q)) \supset r(\eta_x) \cap \ell(\varphi_x) \cap \mathbb{B}_p, \forall p \in S \}.$$ 

If $S = \{p\}$ (respectively, $S = \{p, q\}$), we write $W_x^{sp,p}$ (respectively, $W_x^{sp,pq}$) for $W_x^{sp,S}$.

Proposition 2.8. Let $p \in S$. Let $x$ be a vertex point, and $S \subset S$. Then, every irreducible component of $W_x^{sp,S}$ has codimension $\sum_{p \in S} |r(\eta_x) \cap \ell(\varphi_x) \cap \mathbb{B}_p|$.

Proof. By [13, 2.6.16], we have

$$\pi(W_x) = \bigcup_{\tau \in \mathcal{T}_x} W_{\tau},$$
where $T_{\underline{z}} = \{ \tau \subseteq B : \varphi_{\underline{z}} \cap \eta_{\underline{z}} \subseteq \tau \subseteq (\varphi_{\underline{z}} \cap \eta_{\underline{z}}) \cup (r(\eta_{\underline{z}}) \cap \ell(\varphi_{\underline{z}}))\}$. It follows from definition that

$$W_{\underline{z}}^{sp,S} = \pi^{-1}( \bigcup_{\tau \in T_{\underline{z}} \cap W_{\underline{z}}} W_{\tau} \cap W_{\underline{z}}),$$

where, $T_{\underline{z}} \cap W_{\underline{z}} = \{ \tau \in T_{\underline{z}} : \tau \cap B_p = (\varphi_{\underline{z}} \cap \eta_{\underline{z}} \cap B_p) \cup (r(\eta_{\underline{z}}) \cap \ell(\varphi_{\underline{z}}) \cap B_p), \forall p \in S \}$. Let $\tau_0 \in T_{\underline{z}} \cap W_{\underline{z}}$ be defined as

$$\tau_0 = \bigcap_{\tau \in T_{\underline{z}} \cap W_{\underline{z}}} \tau.$$

Then $W_{\underline{z}}$ is Zariski dense in $\pi(W_{\underline{z}}^{sp,S})$, and it is equidimensional of dimension $g - |\tau_0|$ by results of [12].

For each point $\tau \in \pi(W_{\underline{z}})$, the fibre $\pi^{-1}(\tau) \cap W_{\underline{z}}$ is irreducible of dimension $g - |\varphi_{\underline{z}} \cup \eta_{\underline{z}}|$ by loc. cit. Hence, $\pi^{-1}(W_{\tau_0})$ is Zariski dense in $W_{\underline{z}}^{sp,S}$, and every irreducible component $D$ of $W_{\underline{z}}^{sp,S}$ is the Zariski closure (in $W_{\underline{z}}^{sp,S}$) of a set of the form $\pi^{-1}(C) \cap W_{\underline{z}}$, where $C$ is an irreducible component of $W_{\tau_0}$. It also follows that the codimension of $D$ in $W_{\underline{z}}$ is the same as that of $C$ in $\pi(W_{\underline{z}})$. This codimension equals $\sum_{p \in S} |r(\eta_{\underline{z}}) \cap \ell(\varphi_{\underline{z}}) \cap B_p|$. □

The following is a corollary of Lemma 2.8.

**Corollary 2.9.** Let $\underline{z}$ be a vertex. Let $p \neq q$ be elements of $S$.

1. If $\underline{z}$ is neither $p$-ordinary, nor of type 1 at $p$, then $W_{\underline{z}}^{sp,p}$ is a closed subset of $W_{\underline{z}}$ of codimension $\geq 2$.

2. If $\underline{z}$ is neither $p$-ordinary, nor $q$-ordinary, then $W_{\underline{z}}^{sp,pq}$ is a closed subset of $W_{\underline{z}}$ of codimension $\geq 2$.

**Definition 2.10.** For a vertex $\underline{z}$, and $p \in S$, we define an open subset of $W_{\underline{z}}$

$$W_{\underline{z}}^{[\tau_p] \leq 1} = \{ \overline{Q} \in W_{\underline{z}} : |\tau(\overline{Q})| \cap B_p \leq 1 \}.$$

We define $W_{\underline{z}}^{[\tau_p] > 1}$ to be the complement of $W_{\underline{z}}^{[\tau_p] \leq 1}$ in $W_{\underline{z}}$.

**Proposition 2.11.** Let $p \in S$, and $\underline{z}$ be a vertex which is of type 1 at $p$, and $q$-ordinary for all $q \in S \setminus \{p\}$. Then,

$$W_{\underline{z}}^{[\tau_p] \leq 1} \cup w_p^{-1}(W_{\underline{z}}^{[\tau_p] \leq 1})$$

is an open subset of $W_{\underline{z}}$ whose complement in $W_{\underline{z}}$ has codimension $\geq 2$.

**Proof.** The complement of $W_{\underline{z}}^{[\tau_p] \leq 1} \cup w_p^{-1}(W_{\underline{z}}^{[\tau_p] \leq 1})$ in $W_{\underline{z}}$ equals

$$W_{\underline{z}}^{[\tau_p] > 1} \cap w_p^{-1}(W_{\underline{z}}^{[\tau_p] > 1}).$$

It is enough to show that the closed subschemes $W_{\underline{z}}^{[\tau_p] > 1}$ and $W_{\underline{z}}^{[\tau_p] > 1}$ are divisors, and that no irreducible component of $W_{\underline{z}}^{[\tau_p] > 1}$ coincides with an irreducible component of $w_p^{-1}(W_{\underline{z}}^{[\tau_p] > 1})$. By assumption, there is an integer $1 \leq m_p \leq f_p - 1$, and $\beta_p \in B_p$, such that $\varphi_{\underline{z}} \cap B_p = \{ \sigma \circ \beta_p : 1 \leq j \leq m_p \}$, and $\eta_{\underline{z}} \cap B_p = \{ \sigma \circ \beta_p : m_p \leq f_p - 1 \}$. Hence, we
have \( \varphi_{\mathcal{A}} \cap \eta_{\mathcal{A}} \cap \mathbb{B}_p = \{ \sigma^{mp} \circ \beta_p \} \), and \( \tau(\eta_{\mathcal{A}}) \cap \ell(\varphi_{\mathcal{A}}) \cap \mathbb{B}_p = \{ \beta_p \} \). Since \( \mathcal{A} \) is \( q \)-ordinary for all \( q \in S - \{ p \} \), we have either \( \varphi_{\mathcal{A}} \cap \mathbb{B}_q = \emptyset \) or \( \eta_{\mathcal{A}} \cap \mathbb{B}_q = \emptyset \). Hence, we get \( \varphi_{\mathcal{A}} \cap \eta_{\mathcal{A}} = \{ \sigma^{mp} \circ \beta_p \} \), and similarly \( \tau(\eta_{\mathcal{A}}) \cap \ell(\varphi_{\mathcal{A}}) = \{ \beta_p \} \). Using the notation of \([13] \ 2.6.16\), we have
\[
\pi(W_{\mathcal{A}}) = \pi(W_{\varphi, \eta}) = W_{\{ \sigma^{mp} \circ \beta_p \}} \cup W_{\{ \sigma^{mp} \circ \beta_p, \beta_p \}}.
\]

It follows that we have
\[
W_{\mathcal{A}}^{[\tau_p]} = \pi^{-1}(W_{\{ \beta_p, \sigma^{mp} \circ \beta_p \}}) \cap W_{\mathcal{A}} = \pi^{-1}(W_{\{ \beta_p \}}) \cap W_{\mathcal{A}}.
\]
In particular, \( W_{\mathcal{A}}^{[\tau_p]} \) is the closed subscheme of \( W_{\mathcal{A}} \) defined by \( \pi^*(h_{\beta_p}) = 0 \), where \( h_{\beta_p} \) denotes the \( \beta_p \)-th partial Hasse invariant on \( \mathcal{X} \). By \([13] \ 2.6.7\), for any \( \mathcal{T} \in \pi(W_{\mathcal{A}}) \), the fibre \( \pi^{-1}(\mathcal{T}) \cap W_{\mathcal{A}} \) is equi-dimensional of dimension \( g - \ell(\mathcal{T}) \). Since the Oort stratum \( W_{\{ \beta_p, \sigma^{mp} \circ \beta_p \}} \) is equi-dimensional of dimension \( g - 2 \), we see that \( W_{\mathcal{A}}^{[\tau_p]} \) is a divisor in \( W_{\mathcal{A}} \). Similarly, \( w_p^{-1}(W_{\mathcal{A}}^{[\tau_p]}) \) is a divisor in \( W_{\mathcal{A}} \), defined by the vanishing of \( w_p^* \pi^*(h_{\sigma^{mp} \circ \beta_q}) \).

Let \( Z_{\mathcal{A}} \) be the closure of \( W_{\mathcal{A}} \) in \( \mathcal{Y} \). In the notation of \([13] \ 2.5.1\), we have
\[
Z_{\mathcal{A}} = Z_{\varphi, \eta} \cup \bigcup_{(\varphi, \eta) \not\supseteq (\varphi_{\mathcal{A}}, \eta_{\mathcal{A}})} W_{\varphi, \eta},
\]
where \( (\varphi, \eta) \) runs through the admissible pairs of subsets of \( \mathbb{B} \) (in the sense of \([13] \ 2.3.1\)), such that \( \varphi \supseteq \varphi_{\mathcal{A}} \) and \( \eta \supseteq \eta_{\mathcal{A}} \). It is clear that the Zariski closures of \( W_{\mathcal{A}}^{[\tau_p]} \) and \( w_p^{-1}(W_{\mathcal{A}}^{[\tau_p]}) \) in \( Z_{\mathcal{A}} \) are still, respectively, defined by the equations \( \pi^*(h_{\beta_p}) = 0 \) and \( w_p^* \pi^*(h_{\sigma^{mp} \circ \beta_q}) = 0 \).

Lemma 2.12. Let notation be as in Proposition 2.11. The closure in \( Z_{\mathcal{A}} \) of each irreducible component of \( W_{\mathcal{A}}^{[\tau_p]} \) passes through a point in \( W_{3, \mathcal{B}} \).

We assume this Lemma for a moment, and finish the proof of 2.11 as follows. Let \( \mathcal{Q} \) be a point in \( W_{3, \mathcal{B}} \). By Stamm’s theorem \([35]\), we have an isomorphism
\[
\widehat{\mathcal{O}}_{\mathcal{Y}, \mathcal{Q}} \cong \kappa_K[[z_{\beta}, y_{\beta} : \beta \in \mathcal{B}]]/(x_{\beta}y_{\beta})_{\beta \in \mathcal{B}},
\]
where \( x_{\beta}, y_{\beta} \) are some canonically defined local parameters. It follows from \([13] \ 2.5.2\)(4) that
\[
\widehat{\mathcal{O}}_{Z_{\mathcal{A}}, \mathcal{Q}} = \kappa_K[[z_{\beta} : \beta \in \mathcal{B}}],
\]
where each \( z_{\beta} \) is either \( x_{\beta} \) or \( y_{\beta} \), and \( z_{\beta_p} = x_{\beta_p}, z_{\sigma^{-1}0_{\beta_p}} = y_{\sigma^{-1}0_{\beta_p}}, z_{\sigma^{mp}0_{\beta_p}} = x_{\sigma^{mp}0_{\beta_p}}, z_{\sigma^{mp}^{-1}0_{\beta_p}} = y_{\sigma^{mp}^{-1}0_{\beta_p}} \).

By Lemma 2.12 above, to complete the proof of 2.11 we just need to show that the closed subscheme defined by \( \pi^*(h_{\beta_p}) \) and \( w_p^* \pi^*(h_{\sigma^{mp} \circ \beta_q}) \) in \( \text{Spec}(\widehat{\mathcal{O}}_{Z_{\mathcal{A}}, \mathcal{Q}}) \) have no common irreducible components. The Key Lemma 2.8.1 in \([13]\) implies that
\[
\pi^*(h_{\beta_p}) = ux_{\beta_p} + vy_{\sigma^{-1}0_{\beta_p}} \quad \text{and} \quad \pi^*(h_{\sigma^{mp} \circ \beta_q}) = u'x_{\sigma^{mp} \circ \beta_p} + v'y_{\sigma^{mp}^{-1}0_{\beta_p}},
\]
in the local ring $\hat{O}_{Γ,Γ}$, where $u, v, u', v'$ are units in $\hat{O}_{Γ,Γ}$. Since $w_p^*$ exchanges $x_β$ and $y_β$, for all $β \in \mathbb{B}_p$ (this can be proved as in [13, 2.7.2]), we find that

$$w_p^*(π^*(h_σm_0p_0β)) = w_p^*(u'j_σm_0p_0β) + w_p^*(v')x_σm_0p_1β.$$

It is now immediate to see that $π^*(h_β)$ and $w_p^*(π^*(h_σm_0p_0β))$ cut out two irreducible and distinct divisors in $\text{Spec}(\hat{O}_{Z,Γ})$. Now it remains to prove Lemma 2.12.

□

Proof of 2.12. Let $C$ be an irreducible component of $W_β^{[1]}$, and $\overline{C}$ be its closure in $Z_\hat{\mathbb{A}}$. Since $Z_\hat{\mathbb{A}}$ is smooth [13, 2.5.2](4), there exists a unique irreducible component $D$ of $Z_\hat{\mathbb{A}}$ containing $\overline{C}$. By [13, 2.6.4](1), $π(D)$ is an irreducible component of

$$π(Z_\hat{\mathbb{A}}) = π(φ_{\hat{\mathbb{A}},η_\hat{\mathbb{A}}}) = \bigcup_{τ ≥ \{σm_0p_0β\}} W_τ.$$

Since $π$ is proper, $π(\overline{C})$ is an irreducible component of $\bigcup_{τ ≥ \{β_p,σm_0p_0β\}} W_τ$. In particular, $π(\overline{C})$ intersects $W_β$. By [13, 2.6.16], if a stratum $W_ϕ,η \subset Z_{φ_{\hat{\mathbb{A}},η_\hat{\mathbb{A}}}}$ meets $π^{-1}(W_β)$, we must have $φ = η, φ ≥ φ_\hat{\mathbb{A}}$ and $η ≥ η_\hat{\mathbb{A}}$. Therefore, we have $π^{-1}(W_β) \cap Z_{φ_{\hat{\mathbb{A}},η_\hat{\mathbb{A}}}} \subset Z_{β,β-β_p}$, and hence

$$E := \overline{C} \cap Z_{β-β_p,β-β_p}$$

is nonempty. Since $\overline{C}$ is a divisor, and $Z_{β,β-β_p}$ is equi-dimensional of dimension 2 [13, 2.5.2](2), $E$ has dimension at least 1. By [13, 2.3.4], the type of any point in $Z_{β-β_p}$ always contains $β-β_p$. But the type of any point on $\overline{C}$ also contains $\{β_p\}$, so we have

$$E \subset π^{-1}(W_β) \cap Z_{β-β_p,β-β_p} = π^{-1}(W_β) \cap (W_{β-β_p,β-β_p} \cup W_{β,β})$$

where the last equality follows from [13, 2.3.4]. Now, assume the lemma doesn’t hold, i.e., $\overline{C} \cap W_{β,β} = \emptyset$. Then, we must have

$$E \subset π^{-1}(W_β) \cap W_{β-β_p,β-β_p}.$$

But this is impossible, as the left side is a projective variety of dimension at least 1, and the right side is a union of finitely many irreducible components each homeomorphic to $\mathbb{A}^1$, by [13, 2.6.12]. This proves Lemma 2.12.

□

2.13. The locus $|τ| ≤ 1$. Define an open subset of $\tilde{X}$ as follows:

$$\tilde{X}^{[1]} := (\tilde{X} \setminus \bigcup_{|τ \cap β_p| > 1, β_p} W_τ).$$

We define $\tilde{X}_\text{rig}^{[1]} := sp^{-1}(\tilde{X}^{[1]}_\text{rig})$ and $\tilde{Y}^{[1]}_\text{rig} = π^{-1}(\tilde{X}_\text{rig}^{[1]}).$ These are admissible open subsets of $\tilde{X}_\text{rig}$ and $\tilde{Y}_\text{rig}$, respectively.
3. Analytic continuation

3.1. Notation. Recall our fixed choice of \( K \), a finite extension of \( \mathbb{Q}_\kappa \). We assume that \( K \) contains the \( p \)-th roots of unity, and let \( \zeta \) be a fixed primitive \( p \)-th root of unity in \( K \). Let \( Z_K \) be the Hilbert modular variety of level \( \Gamma_1(Np) \) over \( \text{Spec}(K) \), i.e., the scheme that represents the functor attaching to a \( K \)-scheme \( S \), the set of isomorphic classes of the 4-tuples, \((A, \lambda, i_N, P)\), where

- \( A \) is an abelian scheme of dimension \( g \) over \( S \) with real multiplication by \( \mathcal{O}_L \);
- \( \lambda : A \to A^\vee \) is a prime-to-\( p \) polarization compatible with the action of \( \mathcal{O}_L \);
- \( i_N : \mu_N \otimes \mathbb{Z} \cdot \mathcal{O}_L^{-1} \hookrightarrow A[N] \) is an \( \mathcal{O}_L \)-equivariant closed immersion of group schemes;
- \( P : S \to A[p] \) is a section of the finite flat group scheme \( A[p] \) of order \( p \), such that the \( \mathcal{O}_L \)-subgroup generated by \( P \) is a free \((\mathcal{O}_L/p)\)-module of rank 1.

We will often use the abbreviation \( \underline{A} = (A, \lambda, i_N) \), and denote a \( S \)-valued point of \( Z_K \) simply by \((\underline{A}, P)\). Let \( H = (P) \) be the \( \mathcal{O}_L \)-subgroup of \( A[p] \) generated by \( P \). We have a canonical decomposition \( H = \prod_{\mathfrak{p} \in \mathcal{S}} H[p] \) corresponding to the decomposition \( \mathcal{O}_L/p = \prod_{\mathfrak{p} \in \mathcal{S}} \mathcal{O}_L/p \). We denote by \( P_{\mathfrak{p}} \) the image of \( P \) under the natural projection \( H \to H[p] \). Then \( P_{\mathfrak{p}} \) is a generator of \( H[p] \) as \( \mathcal{O}_L \)-module, and we have \( P = \prod_{\mathfrak{p} \in \mathcal{S}} P_{\mathfrak{p}} \). Let

\[
\alpha : Z_K \to Y_K
\]

be the map given by \((\underline{A}, P) \mapsto (\underline{A}, (P))\). The map \( \alpha \) is finite étale of degree \( \prod_{\mathfrak{p} \in \mathcal{S}} (p^{f_{\mathfrak{p}}} - 1) \). We choose the same rational polyhedral cone decomposition as that of \( Y \) to construct a toroidal compactification \( \tilde{Z}_K \) of \( Z_K \). The morphism \( \alpha \) extends to a finite flat map \( \alpha : \tilde{Z}_K \to \tilde{Y}_K \).

We will use \( \mathfrak{F} \) as a shorthand notation for \( \tilde{Z}_K \), the rigid analytification of \( \tilde{Z}_K \). We will continue to denote by \( \alpha : \mathfrak{F} \to \tilde{Y}_K = \mathfrak{Y}_\text{an} = \mathfrak{Y}_\text{rig} \), the analytification of \( \alpha : \tilde{Z}_K \to \tilde{Y}_K \). We set \( \mathfrak{F}_\text{ord} = \alpha^{-1}(\mathfrak{Y}_\text{rig}) \).

3.2. Automorphisms \( w_\mathfrak{p} \). Given the fixed choice of \( \zeta \in \mu_p(K) \), we define an automorphism \( w_\mathfrak{p} \) on \( Z_K \) for \( \mathfrak{p} \in \mathcal{S} \) similar to \([21]\) as follows. Let \( x = (\underline{A}, P) \) be a point of \( Z_K \) with values in a \( K \)-scheme \( S \). For each \( \mathfrak{p} \in \mathcal{S} \), the prime-to-\( p \) polarization \( \lambda \) induces a perfect Weil pairing

\[
\langle \cdot, \cdot \rangle_\mathfrak{p} : A[p] \times A[p] \to \mu_p.
\]

Let \( \phi_\mathfrak{p} : A \to A' = A/H[p] \) be the canonical isogeny, and \( \phi'_\mathfrak{p} : A' \to A/A[p] \) be the canonical isogeny with kernel \( A[p]/H[p] \). Since \( H[p] \) is (automatically) isotropic, the Weil pairing \( \langle \cdot, \cdot \rangle_\mathfrak{p} \) induces a perfect duality pairing

\[
\langle \cdot, \cdot \rangle_{\phi_\mathfrak{p}} : \text{Ker}(\phi_\mathfrak{p}) \times \text{Ker}(\phi'_\mathfrak{p}) \to \mu_p.
\]

We define \( w_\mathfrak{p}(x) = (A', Q) = (A', \lambda', i_N', Q) \), where \( \lambda' \) and \( i_N' \) are respectively the induced polarization and \( \Gamma_1(N) \)-structure on \( A' \), and \( Q = \prod_{\mathfrak{q} \in \mathcal{S}} Q_{\mathfrak{q}} \in A'[p](S) \) is given as follows: For \( \mathfrak{q} \neq \mathfrak{p} \), we put

\[
Q_{\mathfrak{q}} = P_{\mathfrak{q}} \in A'[\mathfrak{q}](S)
\]

and \( Q_{\mathfrak{p}} \) is the unique point of \( \text{Ker}(\phi'_\mathfrak{p})(S) \) such that \( (P_{\mathfrak{p}}, Q_{\mathfrak{p}})_{\phi_\mathfrak{p}} = \zeta \).
We see easily that \( w_p \) and \( w_q \) commute for \( p, q \in S \), so that \( w_T = \prod_{p \in T} w_p \) is well-defined for any subset \( T \subset S \). We put \( w = w_\emptyset \). Note that \( w^2(A, \lambda, i_N, P) = (A, \lambda, pi_N, -P) \), since 
\[
\langle P_p, Q_p \rangle_{\phi_p} \langle Q_p, P_p \rangle_{\phi_p} = 1
\]
for any \( p \in S \). Since a certain power of \( w^2 \) is the identity, each \( w_p \) is an automorphism. The automorphism \( w_p \) extends to the toroidal compactification \( \tilde{Z}_K \). Via the natural projection \( \alpha : \tilde{Z}_K \to \tilde{Y}_K \), the automorphism \( w_p \) is compatible with the Atkin-Lehner automorphism \( w_p \) on \( \tilde{Y}_K \) defined in \( \ref{2.1} \).

### 3.3. Geometric Hilbert modular forms.

For \( k \in \mathbb{Z}^d \), the space of geometric Hilbert modular forms of level \( \Gamma_1(Np) \) and weight \( \underline{k} \) is defined as 
\[
\mathcal{M}_k(\Gamma_1(Np); K) = H^0(Z_K, \omega_{\underline{k}}).
\]
As usual, the line bundle \( \omega_{\underline{k}} \) extends to \( \tilde{Z} \), and the classical Koecher principle \[ \ref{2.1} \text{ 4.9} \] implies that \( \mathcal{M}_k(\Gamma_1(Np); K) = H^0(\tilde{Z}_K, \omega_{\underline{k}}) \). We will denote the subspace of cusp forms by \( \mathcal{S}_k(\Gamma_1(Np); K) \). The space of overconvergent Hilbert modular forms of level \( \Gamma_1(Np) \) and weight \( \underline{k} \) over \( K \) is 
\[
\mathcal{M}_{k,\underline{k}}^1(\Gamma_1(Np); K) = \lim_{\mathcal{V} \uparrow \mathcal{G}^\text{ord}} H^0(V, \omega_{\underline{k}}),
\]
where \( \mathcal{V} \) runs through the strict neighborhoods of \( \mathcal{G}^\text{ord} \) in \( \mathcal{G} \). We will denote the subspace of cusp forms by \( \mathcal{S}_{k,\underline{k}}^1(\Gamma_1(Np); K) \).

We have a natural injection \( \mathcal{M}_{k,\underline{k}}(\Gamma_1(Np); K) \hookrightarrow \mathcal{M}_{k,\underline{k}}^1(\Gamma_1(Np); K) \) sending cusp forms to cusp forms. Both the source and target of this injection are equipped with an action of Hecke operators, and the action is compatible with the Hecke action. The overconvergent modular forms in the image of \( \mathcal{M}_{k,\underline{k}}(\Gamma_1(Np); K) \) are called classical.

We also define the space of geometric Hilbert modular forms of level \( \Gamma_1(N) \cap \Gamma_0(p) \) and weight \( \underline{k} \) 
\[
\mathcal{M}_{\underline{k}}(\Gamma_1(N) \cap \Gamma_0(p); K) = H^0(\mathcal{Z}_K^\text{rig}, \omega_{\underline{k}}),
\]
with the subspace of cusp forms denoted by \( \mathcal{S}_{\underline{k}}(\Gamma_1(N) \cap \Gamma_0(p); K) \). Similarly, one can define the space of overconvergent Hilbert modular forms of level \( \Gamma_1(N) \cap \Gamma_0(p) \) and weight \( \underline{k} \) as 
\[
\mathcal{M}_{\underline{k}}^1(\Gamma_1(N) \cap \Gamma_0(p); K) = \lim_{\mathcal{V} \uparrow \mathcal{G}^\text{rig}\text{ord}} H^0(V, \omega_{\underline{k}}),
\]
where \( \mathcal{V} \) runs through the strict neighborhoods of \( \mathcal{G}^\text{rig}\text{ord} \) in \( \mathcal{G}^\text{rig} \). The subspace of cusp forms is denoted by \( \mathcal{S}_{\underline{k}}^1(\Gamma_1(N) \cap \Gamma_0(p); K) \).

**Remark 3.4.** Geometric Hilbert modular forms of are not exactly automorphic forms for \( \text{Res}_{L/Q} \text{GL}_2 \). In level \( \Gamma_1(N) \), these automorphic forms can be identified as the space of geometric Hilbert modular forms which are invariant under the natural action of \( \mathcal{O}_L^{\times, +} / \mathcal{O}_L^{\times, +} \) (induced by a natural action on \( \mathcal{O} = (\mathcal{A}, \lambda, i_N) \) via the polarization \( \lambda \)). Here, \( \mathcal{O}_L^{\times, +} \) denotes the group of totally positive units of \( L \), and \( \mathcal{O}_L^{\times, +} \) the group of units congruent to 1 mod \( N \). See \[ \ref{20} \text{ Remark 1.11.8} \] for more details.
In the second part of this paper, where applications to the Artin conjecture are discussed, we will be working with spaces of automorphic forms for $\text{Res}_{L/Q}\text{GL}_2,Q$. The results of the first part of the paper will be applied via the above identification.

3.5. The operators $U_p, w_p$. We will describe the action of the $U_p$-operators, for $p \in S$. We need to define some auxiliary modular varieties as follows. For any $p \in S$, let $Z^K_p$ be the scheme which classifies triples $(A, P, D)$ over $K$-schemes, where $(A, P)$ is classified by $Z^K$, and $D$ is a finite-flat $p$-torsion subgroup of $A$ of rank $p^{f_p}$ such that $(P) \cap D = 0$. We let $\tilde{Z}^p$ denote the toroidal compactification of $Z^K_p$, obtained using our fixed choice of collection of cone decompositions. We use the same notation to denote the associated rigid analytic variety. There are two finite flat morphisms

$\pi_{1,p}, \pi_{2,p} : \tilde{Z}^p \to \tilde{3}$,

defined by $\pi_{1,p}(A, P, D) = (A, P)$ and $\pi_{2,p}(A, P, D) = (A/D, \overline{P})$ on the non-cuspidal part, where $\overline{P}$ is the image of $P$ in $A/D$.

We have the same setup over $Y$, i.e., for every $p \in S$, a (rigid analytic) variety $\tilde{Y}^p_{\text{rig}}$ defined over $K$, and two finite flat morphisms

$\pi_{1,p}, \pi_{2,p} : \tilde{Y}^p_{\text{rig}} \to \tilde{3}_{\text{rig}}$,

defined similarly. See [17, §4] for details.

We define $\gamma : \tilde{3}^p \to \tilde{Y}^p_{\text{rig}}$ to be the morphism which sends $(A, P, D) \to (A, (P), D)$. We have, for $i = 1, 2$,

$\gamma \circ \pi_{i,p} = \pi_{i, p} \circ \alpha$.

Recall the standard construction of overconvergent correspondences explained in Definition 2.19 of [18]: If $p \in S$, and we have $\pi_{1,p}^{-1}(\mathcal{V}_1) \subset \pi_{2,p}^{-1}(\mathcal{V}_2)$ for admissible opens $\mathcal{V}_1, \mathcal{V}_2$ of $\tilde{3}$, we define an operator

$U_p : \omega^k(\mathcal{U}_2) \to \omega^k(\mathcal{U}_1)$,

via the formula

$U_p(f) = (1/p^{f_p})(\pi_{1,p})_*(\text{res}(\text{pr}^*\pi_{2,p}^*(f)))$,

where $\text{res}$ is restriction from $\pi_{2,p}^{-1}(\mathcal{V}_2)$ to $\pi_{1,p}^{-1}(\mathcal{V}_1)$, and $\text{pr}^* : \pi_{2,p}^*\omega^k \to \pi_{1,p}^*\omega^k$ is a morphism of sheaves on $\tilde{3}^p$, which at $(A, P, D)$ is induced by $\text{pr}^* : \Omega_{A/D} \to \Omega_A$ coming from the natural projection $\text{pr} : A \to A/D$.

For $0 < r < 1$, let $\tilde{3}[1 - r, 1] = \alpha^{-1}(\tilde{Y}^p_{\text{rig}}[1 - r, 1])$ (Definition 2.3). Then, as $r \in Q$ goes to $0$, the $\tilde{3}[1 - r, 1]$ form a basis of strict neighborhoods of $\tilde{3}$, and considering Remark 2.4 for $r$ close to $0$, we have

$\pi_{1, p}^{-1}(\tilde{Y}^p_{\text{rig}}[1 - r, 1]) \subset \pi_{2, p}^{-1}(\tilde{Y}^p_{\text{rig}}[1 - r, 1])$.

Applying Equation 3.5.2 we get $\pi_{1, p}^{-1}(\tilde{3}[1 - r, 1]) \subset \pi_{2, p}^{-1}(\tilde{3}[1 - r, 1])$.

Definition 3.6. Let $p \in S$. The $U_p$ operator on $\mathcal{M}_k(\Gamma_1(Np); K)$ is obtained as above, using the above fact that for $r \in Q$ close to $0$, we have

$\pi_{1, p}^{-1}(\tilde{3}[1 - r, 1]) \subset \pi_{2, p}^{-1}(\tilde{3}[1 - r, 1])$. 

The $U_p$ operators commute with each other and their composition is called $U_p$.

**Definition 3.7.** We also define, for any $U \subset \tilde{\mathcal{Z}}$, and any $p|p$,

$$w_p : \omega(U) \to \omega(w_p^{-1}(U)),$$

by $w_p(f) = \text{pr}^*w_p^*(f)$. We can similarly define

$$w : \omega(U) \to \omega(w(U)).$$

The $w_p$ operators commute with each other and their composition equals $w$.

3.8. More auxiliary modular varieties. Recall our fixed choice of $p$-th root of unity $\zeta$ in $K$. For any $p \in S$, let $Z^p_K$ be the scheme which classifies triples $(A, P, Q_p)$ over $K$-schemes, where $(A, P)$ is classified by $Z_K$, $Q_p$ is a generator of $A[p]$ different from $P_p$, and $<P_p, Q_p> = \zeta$. Let $\tilde{\mathcal{Z}}^p$ denote the toroidal compactification of $Z^p_K$, obtained using our fixed choice of collection of cone decompositions, with the same notation used for the associated rigid analytic variety. For any $j \in \mathcal{O}/p$, there is a finite flat morphism

$$\pi_{2, p,j} : \tilde{\mathcal{Z}}^p \to \tilde{\mathcal{Z}}^p,$$

defined by $\pi_{2, p,j}(A, P, Q_p) = (A/(jP_p + Q_p), \mathcal{P})$ on the non-cuspidal part, where $\mathcal{P}$ denotes the image of $P$ in $A/(jP_p + Q_p)$.

We also define

$$\pi_{1, p} : \tilde{\mathcal{Z}}^p \to \tilde{\mathcal{Z}},$$

via $\pi_{1, p}(A, P, Q_p) = (A, P)$.

3.9. The statement of the Main theorem. Before stating the main theorem, we set up some notation. Recall the admissible open $\tilde{\mathcal{Y}} | \tau | \leq 1$ of $\tilde{\mathcal{Y}}_\text{rig}$ from §2.13. We define

$$\tilde{\mathcal{Y}} | | \tau | \leq 1^w := \alpha^{-1}(\tilde{\mathcal{Y}} | | \tau | \leq 1).$$

We now define some admissible opens in $\tilde{\mathcal{Y}}$ defined by degree constraints.

**Definition 3.10.** For any multiset of intervals $\mathcal{I}$ as in Definition 2.3, we define

$$\tilde{\mathcal{Y}}^\mathcal{I} = \alpha^{-1}(\tilde{\mathcal{Y}}_\text{rig}^\mathcal{I}).$$

For any admissible open subset $\mathcal{W} \subset \tilde{\mathcal{Y}}$, we define $\mathcal{W}^\mathcal{I} = \mathcal{W} \cap \tilde{\mathcal{Y}}^\mathcal{I}$.

Define a map $w_T : \mathbb{R}^S \to \mathbb{R}^S$ by

$$w_T((x_p)_{p \in S}) = (y_p)_{p \in S},$$

where $y_p = x_p$ if $p \notin T$, and $y_p = f_p - x_p$ if $p \in T$. For a multiset of intervals $\mathcal{L} = \{I_p : p \in S\}$, define $I^\mathcal{L} = w_T(I)$. Since $w_p$ on $\tilde{\mathcal{Y}}$ is compatible with $w_p$ on $\tilde{\mathcal{Y}}_\text{rig}$ via $\alpha$, Equation 2.2.1 implies the following result.

**Proposition 3.11.** Let $T \subset \mathcal{S}$, and let $\mathcal{L}$ be a multiset of intervals as in Definition 2.3. Then, $w_T^{-1}(\tilde{\mathcal{Y}}^\mathcal{I}) = \mathcal{W}^{I^\mathcal{L}}$.
Definition 3.12. Let \( \mathfrak{p} \in S \). We define \( \mathcal{I}_\mathfrak{p} \) to be the interval \( (f_\mathfrak{p} - 1, f_\mathfrak{p}] \) if \( f_\mathfrak{p} > 1 \), and \( (1/(p + 1), 1] \) if \( f_\mathfrak{p} = 1 \). We define \( \mathcal{I}_\mathfrak{p}^* \) to be the interval \( (\sum_{i=1}^{f_\mathfrak{p} - 1} 1/p^i, f_\mathfrak{p}] \) if \( f_\mathfrak{p} > 1 \), and \( (0, 1] \) if \( f_\mathfrak{p} = 1 \). We also define \( \mathcal{I}_\mathfrak{p}^{**} \) to be the interval \( (\sum_{i=1}^{f_\mathfrak{p} - 1} 1/p^i, 1) \) if \( f_\mathfrak{p} > 1 \), and \( (0, 1) \) if \( f_\mathfrak{p} = 1 \). We define \( \mathcal{I} \) (respectively, \( \mathcal{I}', \mathcal{I}^* \)) to be the multiset of intervals whose component at \( \mathfrak{p} \) is \( \mathcal{I}_\mathfrak{p} \) (respectively, \( \mathcal{I}_\mathfrak{p}', \mathcal{I}_\mathfrak{p}^* \)).

We now state the main Theorem of this section. Assume we are given a collection of nonzero elements of \( K \), \( \{a_\mathfrak{p}, b_\mathfrak{p} : \mathfrak{p} \in S\} \), such that \( a_\mathfrak{p} \neq b_\mathfrak{p} \) for all \( \mathfrak{p} \in S \).

Theorem 3.13. Consider a collection of elements of \( \mathcal{M}_L^1(\Gamma_1(Np); K) \) of parallel weight \( k \),
\[
\{f_T : T \subset S\},
\]
which are all Hecke eigenforms of prime-to-\( p \) nebentypus \( \chi_N \) (independent of \( T \)). Consider, also, a collection of characters \( \{\chi_T : T \subset S\} \) of \( C_{L,Np} \), the strict ray class group of \( L \) modulo \( Np\infty \), satisfying:

- each \( f_T \) is normalized and has \( p \)-nebentypus \( \chi_T/\chi_T \);
- for all pairs \( T, T' \subset S \), we have \( \chi_T(l)c(f_T, l) = \chi_{T'}(l)c(f_{T'}, l) \), for any prime ideal \( l \) not dividing \( pN \);
- \( c(f_T, l) = 0 \), for all \( l|N \);
- \( c(f_T, \mathfrak{p}) \) is \( a_\mathfrak{p} \) if \( \mathfrak{p} \in T \), and \( b_\mathfrak{p} \) if \( \mathfrak{p} \notin T \);
- \( a_\mathfrak{p} b_\mathfrak{p} = (N_{L/Q}\mathfrak{p})^{k-1}\chi_N(\mathfrak{p}) \).

Then, all \( f_T \)'s are classical.

The rest of this section will be devoted to the proof of this Theorem, which will follow directly from Theorems 3.24 and 3.27.

3.14. The first step of analytic continuation. Let \( \mathcal{U}_1 \subset \mathcal{U}_2 \) be open subsets of \( \mathfrak{Z} \), and \( f \) a \( U_\mathfrak{p} \)-eigenform of nonzero eigenvalue defined on \( \mathcal{U}_1 \). By Buzzard’s method of analytic continuation [2], to extend \( f \) from \( \mathcal{U}_1 \) to \( \mathcal{U}_2 \) it is enough to show that
\[
\pi_{1,\mathfrak{p}}^{-1}(\mathcal{U}_1) \subset \pi_{2,\mathfrak{p}}^{-1}(\mathcal{U}_2)
\]
on \( \mathfrak{Z}_p \) (See Proposition 3.1 of [18] for details). Now assume \( \mathcal{U}_1 = \alpha^{-1}(\mathcal{V}_1) \) and \( \mathcal{U}_2 = \alpha^{-1}(\mathcal{V}_2) \) for admissible open subsets of \( \mathfrak{Z}_{\text{rig}} \). Then, by Equation 3.3.2 the above inclusion is equivalent to
\[
\pi_{1,\mathfrak{p}}^{-1}(\mathcal{V}_1) \subset \pi_{2,\mathfrak{p}}^{-1}(\mathcal{V}_2)
\]

This allows us to immediately extend the analytic continuation results in [17] to this case. Applying this principle to the proof of [17, 5.2], we obtain:

Lemma 3.15. Any \( f \in \mathcal{M}_L^1(\Gamma_1(Np)) \) with a nonzero \( U_\mathfrak{p} \)-eigenvalue can be analytically extended to \( \mathfrak{Z}_{|t| \leq 1}^p \).

We can now prove the following general analytic continuation for overconvergent Hilbert modular forms, essentially as in [17].
Proposition 3.16. Let $q \in \mathcal{S}$. Let $f$ be an overconvergent Hilbert modular form of weight $\kappa$, which is a $U_q$ eigenform with a nonzero eigenvalue $a_q$. Let $I'$ be a multiset of intervals such that $I'_q = \mathbb{I}'$. Assume $f$ is defined on $\tilde{3}|r| \leq 1 I'$. Then, $f$ can be analytically extended to $\tilde{3}|r| \leq 1 L$, where $L$ is such that $I_p = I_p'$ for $p \neq q$, and $I_q = \mathbb{I}_q$.

Proof. Apply the above principle to the proof of Propositions 5.7 (and 7.6) of [17].

Proposition 3.15 combined with Proposition 3.16 imply the following result.

Corollary 3.17. Let $f \in \mathcal{M}_k^\dagger(\Gamma_1(Np); K)$ have a nonzero $U_p$-eigenvalue. Then, $f$ extends analytically to $\tilde{3}|r| \leq 1 \mathbb{I}$.  

3.18. Gluing forms. We begin by proving a few results that will be used in the gluing process.

Lemma 3.19. Let $p \in \mathcal{S}$. Let $L$ be a multiset of intervals as in Definition 2.3 such that $L_p = \mathbb{I}_p$. For all $j \in \mathcal{O}_L/p$, we have the following inclusion of open subsets of $\tilde{3}L$:

$$\pi_{1,p}^{-1}(\tilde{3}|r| \leq 1 L) \subset \pi_{2,p,j}^{-1}(\tilde{3}|r| \leq 1 L).$$

Proof. When $p$ is inert in $\mathcal{O}_L$, this follows from [17, Remark 5.8] after unravelling definitions. Keep in mind the slight change of notation from [17] explained in the Remark 2.4.

The exact same proof works in the general case, noting that all the maps involved keep partial degrees away from $p$ unchanged (See Equation 2.2.1).

The following lemma is crucial for the gluing process and was essentially proved in [17].

Lemma 3.20. Let $p \in \mathcal{S}$. Let $L^*$ be a multiset of intervals as in Definition 2.3 such that $L^*_p = \mathbb{I}_p^*$.

1. If $(A, P) \in \tilde{3}|r| \leq 1 L^*$, then for any nonzero point $Q_p$ of $A[p]$, we have

$$(A, P^p \times Q_p) \in \tilde{3}|r| \leq 1 L^*.$$  

2. For all $j \in \mathcal{O}_L/p$, we have the following inclusion of $\tilde{3}L^*$:

$$\pi_{1,p}^{-1}(\tilde{3}|r| \leq 1 L^*) \subset \pi_{2,p,j}^{-1}(\tilde{3}|r| \leq 1 L^*).$$

Proof. The first statement follows directly from [17, Lemmas 5.9, 7.8] (keeping in mind Remark 2.4). The second statement follows from the first statement combined with Lemma 3.19.

Let $\iota : \tilde{3}L \to \tilde{3}L$ be the map $(A, P, Q_p) \to (A, P^p \times Q_p, -P_p)$.

Lemma 3.21. Let $p \in \mathcal{S}$. Let $L^*$ be a multiset of intervals as in Definition 2.3 such that $L^*_p = \mathbb{I}_p^*$. Let $h, h', h''$ be three sections of $\omega^\perp$ on $\tilde{3}|r| \leq 1 L^*$, where $h$ has a nontrivial character $\psi_p$ at $p$. Assume that

$$(3.21.1) \quad \pi_{1,p}^*(h) = \text{pr}^*\pi_{2,p,0}^*(h') + (\iota^{-1})^*\pi_{1,p}^*(h''),$$

on $\pi_{1,p}^{-1}(\tilde{3}|r| \leq 1 L^*)$, Then $h = 0$. 


Proof. First, we point out that part (1) of Lemma 3.20 implies that
\[
\pi_{1,p}^{-1}(\tilde{3}^{[r]\leq 1}I^r) \subset \pi_{1,p}^{-1} \circ \iota(\tilde{3}^{[r]\leq 1}I^r).
\]
Also, we have \(\pi_{1,p}^{-1}(\tilde{3}^{[r]\leq 1}I^r) \subset \pi_{2,p,0}^{-1}(\tilde{3}^{[r]\leq 1}I^r)\) by part (2) of the same Lemma. In particular, the given expression is defined on \(\pi_{1,p}^{-1}(\tilde{3}^{[r]\leq 1}I^r)\).

We show that even though \(h\) is not defined over the entire non-ordinary locus, an argument similar to the one given at the end of the proof of Theorem 10.1 of [2] can be repeated, essentially, because of part (1) of Lemma 3.20. The Lemma implies that if \((A, P) \in \tilde{3}^{[r]\leq 1}I^r\), and \((A, P, Q_p) \in \tilde{3}^{L^p}\), then for any \(j \in O_L/p\), the point \((A, P^p \times (P_p + jQ_p), Q_p)\) belongs to \(\pi_{1,p}^{-1}(\tilde{3}^{[r]\leq 1}I^r)\), and, hence, \(h\) can be applied at it.

Let \((A, P) \in \tilde{3}^{[r]\leq 1}I^r\) and choose a point \((A, P, Q_p) \in \tilde{3}^{L^p}\). Writing Equation 3.21 at \((A, P^p \times (P_p + jQ_p), Q_p)\) for all \(j \in (O_L/p)^\times\), we obtain
\[
h(A, P) = h(A, P^p \times (P_p + jQ_p)).
\]
Replacing \(Q_p\) with \(Q_p - j^*P_p\) (where \(jj^* = 1\) in \((O_L/p)^\times\)) in this equation, we obtain
\[
h(A, P) = h(A, P^p \times jQ_p),
\]
for all \(j \in (O_L/p)^\times\). In particular, for all \(j \in (O_L/p)^\times\), we have
\[
h(A, P^p \times Q_p) = h(A, P^p \times jQ_p) = \psi_p(j)h(A, P^p \times Q_p).
\]
Assume \(h \neq 0\). Then, we can find \(P, Q_p\) as above such that \(h(A, P^p \times Q_p) \neq 0\). It would then follow that \(\psi_p(j) = 1\) for all \(j \in (O_L/p)^\times\), which would contradict the assumption.

We need the following connectivity result, essentially proved in [17].

Lemma 3.22. Let \(p \in S\). Let \(I\) be a multiset of intervals as in Definition 2.3 such that \(I_p = \eta_p\). Let \(I' \subseteq I\) be such that \(I'_p = \eta'_p\). Then, \(\tilde{3}^{[r]\leq 1}I^{r}\) intersects every connected component of \(\tilde{3}^{[r]\leq 1}I \cap w_p^{-1}(\tilde{3}^{[r]\leq 1}I)\).

Proof. The same statement with \(\tilde{3}^{[r]\leq 1}\) replaced with \(\tilde{2} \tilde{3}^{[r]\leq 1}\) follows from exactly the same proof as in [17] Lemma 6.6. We have
\[
\tilde{3}^{[r]\leq 1}I \cap w_p^{-1}(\tilde{3}^{[r]\leq 1}I) = \alpha^{-1}(\tilde{2} \tilde{3}^{[r]\leq 1}I \cap w_p^{-1}(\tilde{2} \tilde{3}^{[r]\leq 1}I)),
\]
which implies that every connected component of \(\tilde{3}^{[r]\leq 1}I \cap w_p^{-1}(\tilde{3}^{[r]\leq 1}I)\) maps surjectively to a connected component of \(\tilde{2} \tilde{3}^{[r]\leq 1}I \cap w_p^{-1}(\tilde{2} \tilde{3}^{[r]\leq 1}I)\) under the finite flat map \(\alpha\). The result now follows since \(\tilde{3}^{[r]\leq 1}I' = \alpha^{-1}(\tilde{2} \tilde{3}^{[r]\leq 1}I')\).

For any \(S \subseteq S\), define \(\mathcal{W}_S = \bigcup_{T \subseteq S} w_T^{-1}(\tilde{3}^{[r]\leq 1}I)\).

Lemma 3.23. Assume \(p \notin S\). Then, \(\tilde{3}^{[r]\leq 1}I^r\) (See Definition 3.12) intersects every connected component of \(\mathcal{W}_S \cap w_p^{-1}(\mathcal{W}_S)\).
Proof. For any $q \in S$, and any point $\overline{q} \in \overline{Y}$, we have
\[
\tau(\pi(q(\overline{q}))) \cap (\mathbb{B} - \mathbb{B}_q) = \tau(\pi(\overline{q})) \cap (\mathbb{B} - \mathbb{B}_q).
\]
Using this fact, calculation of partial degrees under $w_T$’s (Proposition 3.11, and Lemma 7.4), it is straightforward to see that for all $T_1, T_2 \subseteq S$, we have
\[
w_{T_1}^{-1}(\tilde{3}^{[0,1]}_T) \cap w_{T_2 \cup \{p\}}^{-1}(\tilde{3}^{[0,1]}_T) \subseteq w_{T_1 \cap T_2}^{-1}(\tilde{3}^{[0,1]}_T) \cap w_{T_1 \cap T_2 \cup \{p\}}^{-1}(\tilde{3}^{[0,1]}_T).
\]
This implies that
\[
W_S \cap w_p^{-1}(W_S) = \bigcup_{T \subseteq S} w_T^{-1}(\tilde{3}^{[0,1]}_T) \cap w_{T \cup \{p\}}^{-1}(\tilde{3}^{[0,1]}_T).
\]
Hence, to prove the claim, it is enough to show that for any $T \subseteq S$, the region $\tilde{3}^{[0,1]}_T$ intersects every connected component of the region $w_T^{-1}(\tilde{3}^{[0,1]}_T) \cap w_{T \cup \{p\}}^{-1}(\tilde{3}^{[0,1]}_T)$. Since $w_T$ is an isomorphism, this is equivalent to showing that every connected component of the region $\tilde{3}^{[0,1]}_T \cap w_p^{-1}(\tilde{3}^{[0,1]}_T)$ intersects $w_T(\tilde{3}^{[0,1]}_T)$. This follows from Lemma 3.22 noting that $w_T(\tilde{3}^{[0,1]}_T)$ is of the form $\tilde{3}^{[0,1]}_I'$, with $I' \subseteq \mathbb{I}$, and such that $I'_p = \mathbb{I}_p^*$, using the assumption that $p \not\in T$, and [17 Lemma 7.4].

Let $\mathfrak{x}$ be a vertex of the cube $[0,1]^3$, as in §2. We say that a region $\mathcal{U} \subseteq \text{sp}^{-1}(W_{\mathfrak{x}})$ is of negligible complement, if $\mathcal{U} = \text{sp}^{-1}(C)$, where $W_{\mathfrak{x}} - C \subset W_{\mathfrak{x}}$ has codimension at least 2. To prove the classicality of $f_T$’s, as in Theorem 3.13, we proceed in two steps. In the first step, below, we show that given the assumptions of Theorem 3.13, we can extend all $f_T$’s to a large admissible open $\mathcal{W}$, whose image under $\alpha$ contains a region of negligible complement inside every $\text{sp}^{-1}(W_{\mathfrak{x}})$ (using results of §2). In the second step, 3.26, we show that any finite slope overconvergent Hilbert eigenform defined on $\mathcal{W}$, automatically, extends to the entire $\tilde{3}$.

Now, assume we are in the situation of Theorem 3.13.

Theorem 3.24. Consider a collection $\{f_T : T \subseteq S\}$ of elements of $\mathcal{M}^1_{\tilde{3}}(\Gamma_1(Np); K)$, as in Theorem 3.13. Then, each $f_T$ extends analytically from $\tilde{3}^{[0,1]}_T$ to
\[
\mathcal{W} = \bigcup_{T \subseteq S} w_T^{-1}(\tilde{3}^{[0,1]}_T).
\]
Proof. We prove the claim by induction. Let $S \subseteq S$, and $p \not\in S$. Recall
\[
\mathcal{W}_S = \bigcup_{T \subseteq S} w_T^{-1}(\tilde{3}^{[0,1]}_T),
\]
defined before Lemma 3.22. Given $S \subseteq S$, we assume that for all $T \subseteq S$, $f_T$ is defined over $\mathcal{W}_S$. We prove that for all $T \subseteq S$, $f_T$ can be extended to $\mathcal{W}_{S \cup \{p\}}$. This would prove the claim as $\mathcal{W}_{\emptyset} = \tilde{3}^{[0,1]}_1$ and $\mathcal{W}_S = \mathcal{W}$.

In the following, we will show that our analytic continuation results are enough to enable us to carry out an argument as in the proof [2 Theorem 10.1]. Consider a cusp of $\tilde{3}$
\[
Ta_0 = ((\mathbb{G}_m \otimes \mathbb{D}_S^{-1})/q^{a-1}, [\zeta]_p),
\]
where underline indicates the inclusion of standard PEL structure away from \( p \). Here, \([\zeta]\) denotes an \( O_L\)-generator of \((G_m \otimes \mathcal{O}_L^{-1})[p]\), and \([\zeta]_p\) the \( p\)-component of \([\zeta]\). We assume that \([\zeta]\) is chosen such that for all \( p \in \mathcal{S} \), under \((G_m \otimes \mathcal{O}_L^{-1})_R \cong \text{Spec}(R[\mathbf{X}^\lambda : \lambda \in \mathcal{O}_L])\), we have \([\zeta]_p(\mathbf{X}) = \zeta\), our fixed choice of \( p\)-th root of unity. Let \( \eta^\zeta \) denote a generator of the sheaf \( \omega^\zeta \) on the base of \( T_a \). Let \( p \not\in T \subset \mathcal{S} \), and write

\[
\begin{align*}
    f_T(T_a) &= \sum_{\xi \in (a^{-1})^+} a_{\xi(a)} q^\xi \eta^{\zeta}, \\
    f_{T \cup \{p\}}(T_a) &= \sum_{\xi \in (a^{-1})^+} b_{\xi(a)} q^\xi \eta^{\zeta},
\end{align*}
\]

and assume that they are both normalized. Choose \( c \in p^{-1}a^{-1} - a^{-1} \). Let

\[
t : p^{-1}a^{-1} \to p^{-1}a^{-1}/a^{-1} \to O_L/p
\]

be the induced map. Let \( \chi_p : (O/p)^\times \to \mathbb{C}_p^\times \) be the character of \( f_T \) at \( p \), which we can assume to be nontrivial (otherwise, the result follows from Main theorem of [17]). Then, the character at \( p \) of \( f_{T \cup \{p\}} \) equals \( \chi_p^{-1} \), by assumptions. Our assumptions on \( f_T \) and \( f_{T \cup \{p\}} \) (as in Theorem 3.13), also imply that

- \( a_{\xi(a)} = \chi_p(t(\xi)) b_{\xi(a)} \) for all \( \xi \not\in p(a^{-1})^+ \),
- \( a_{\xi(pa)} = a_p a_{\xi(a)} \), \( b_{\xi(pa)} = b_{\xi(a)} \) for all \( \xi \in (a^{-1})^+ \).

Let \( T_a^0 = (T_a, [\zeta], p, q) \). By definitions, we have:

\[
\pi_{1,p}^*(f_{T \cup \{p\}})(T_a^0) = f_{T \cup \{p\}}(T_a, [\zeta]) = \sum_{\xi \in (a^{-1})^+} a_{\xi(a)} q^\xi \eta^{\zeta}. 
\]

For any \( j \in O_L/p \), we can write

\[
\begin{align*}
    \pi_{2,p,j}^*(f_T)(T_a^0) &= \text{pr}^* f(T_a^0) \langle q^\alpha \zeta^j \rangle, [\zeta]_p \\
    &= f_T(T_a, [\zeta]), [\zeta]_p \\
    &= \sum_{\xi \in p^{-1}(a^{-1})^+} a_{\xi(pa)} q^{j(\xi)} \eta^{\zeta}. 
\end{align*}
\]

Similarly,

\[
\pi_{2,p,0}^*(f_{T \cup \{p\}})(T_a^0) = \sum_{\xi \in p^{-1}(a^{-1})^+} b_{\xi(pa)} q^\xi \eta^{\zeta}. 
\]

**Lemma 3.25.** **We have the following equality over** \( \pi_{1,p}^{-1}(3 [\mathbf{I}]^{\mathbf{I}}) \subset 3^\mathbf{p} \).

\[
\sum_{j \in (O_L/p)^\times} \chi_p(j) \pi_{2,p,j}^*(f_T) = W(\chi_p)(\pi_{2,p,0}^*(f_{T \cup \{p\}}) - \pi_{1,p}^*(f_{T \cup \{p\}})).
\]

**Proof.** We first show the two sides have the same \( q \)-expansions, and then appeal to a connectivity result proven in [17 Lemma 6.3] to end the proof.

Using the above calculations, it follows easily that the two sides have the same \( q \)-expansion at the cusp \( T_a^0 \). To prove the lemma, it is enough to show that every connected component of \( \pi_{1,p}^{-1}(3 [\mathbf{I}]^{\mathbf{I}}) \subset 3^\mathbf{p} \) contains such a cusp.
Let \( \mathfrak{K} \to \mathfrak{K}_{\text{rig}} \) denote the map \((A, P, Q) \mapsto (A, (P))\). Then, by Definition 2.3, we have \( \pi_1^{-1}(\mathfrak{K}[|\leq 1]) = \alpha^{-1}(\mathfrak{K}_{\text{rig}}[|\leq 1]) \). Since \( \alpha \) is finite flat, it follows that every connected component of \( \pi_1^{-1}(\mathfrak{K}[|\leq 1]) \) maps surjectively to a connected component of \( \mathfrak{K}_{\text{rig}}[|\leq 1] \). The claim now follows from the fact, shown in the proof of [17] Prop. 6.8, that every connected component of \( \mathfrak{K}_{\text{rig}}[|\leq 1] \) contains a cusp of the form \((Ta_a, [c]_p)\).

We continue with the proof of Theorem 3.24. Let \( T \subset S \) be such that \( p \not\in T \). By the induction assumption, \( f_{T \cup \{p\}} \) can be extended to \( W_S \), and \( w_p(f_T) \) can be extended to \( u_p^{-1}(W_S) \). It follows that

\[
\tilde{h} := N_{L/Q}(p)a_p f_{T \cup \{p\}} - W(\chi_p) w_p(f_T)
\]

is defined over \( W_S \cap u_p^{-1}(W_S) \), and has (nontrivial) character \( \chi_p \) at \( p \). If we show that \( \tilde{h} = 0 \) on this intersection, then the two sections \( W(\chi_p)(N_{L/Q}(p)a_p)^{-1} u_p(f_T) \) and \( f_{T \cup \{p\}} \) would glue together to form an extension of \( f_{T \cup \{p\}} \) to \( W_{S \cup \{p\}} = W_S \cup u_p^{-1}(W_S) \). Similarly, it would follow that \( f_T \) extends to \( W_{S \cup \{p\}} \). Varying \( T \subset S \), this would prove the induction step.

Let \( h'' = w^2(f_T) - N_{L/Q}(p)^{k-1} \chi_p(-1) \chi_N(p) f_T \). By Proposition 3.11, \( h'' \) is defined over \( \mathfrak{K}[|\leq 1]) \).

**Sublemma:** There are constants \( C_1, C_2 \) such that

\[
\pi_1^*(h) = C_1 \pi_2^*(h) + C_2 (\iota^{-1})^* \pi_1^*(h'')
\]

on \( \mathfrak{K}[|\leq 1]) \).

**Proof.** Let \((A, P) \in \mathfrak{K}[|\leq 1]) \), and \((A, P, Q_p) \in \mathfrak{K}^\circ\). To avoid clutter, in the following calculation, we simplify the notation as: \( f := f_{T \cup \{p\}}, g := f_T \). We will also suppress the away-from-\( p \) data, by using \( A := (A, P_p) \). We write

\[
N_{L/Q}(p)a_p f_{A, Q_p} - \text{pr}^* f_{A/(P_p), Q_p} = \sum_{j \in (O_L/p)^\times} \text{pr}^* f_{A/(jP_p + Q_p, Q_p)}
\]

\[
= \chi_p(-1) \sum_{j \in (O_L/p)^\times} \chi_p(j) \text{pr}^* f_{A/(jP_p + Q_p, Q_p)}
\]

\[
= \chi_p(-1) \sum_{j \in (O_L/p)^\times} \chi_p(j) \pi_2^*(f)(A_p, Q_p)
\]

\[
= \chi_p(-1) W(\chi_p)(\text{pr}^* g(A/(Q_p), Q_p) - b_p g(A_p, P_p)),
\]

where, in the first equality, we have used the fact that \( U_p(f_{T \cup \{p\}}) = a_p f_{T \cup \{p\}} \), and, in the last equality, we have used Lemma 3.25. We must note that by Lemmas 3.19, 3.20, all the terms in the above calculation are well-defined.

To prove the desired equality, we first apply \( \iota^* \), and unravel the definitions to see that the equality follows directly from the above calculation.

\[\square\]
Using Lemma 3.21, we deduce that \( h = 0 \) on \( \tilde{Z} |_{\tau \leq 1} \). By Lemma 3.23, this implies the vanishing of \( h \) on the bigger domain \( W_{S} \cap w_{p}^{-1}(W_{S}) \), as desired. This ends the proof of Theorem 3.24.

\[ \square \]

3.26. The second step of analytic continuation. The aim of this section is to show that the overconvergent modular forms obtained in previous sections can be, further, analytically continued to the entire Hilbert modular variety; i.e., they are classical.

**Theorem 3.27.** Let \( f \) be an overconvergent Hilbert modular form of level \( \Gamma_{1}(N_{p}) \) and weight \( k \), which is an eigenforms for the \( U_{p} \)-operator with a non-zero eigenvalue \( \lambda_{p} \) for all \( p \in S \). Assume that \( f \) extends analytically to \( W = \bigcup_{T \subseteq S} w_{p}^{-1}(\tilde{Z} |_{\tau \leq 1}) \).

Then, \( f \) is classical.

Using Theorems 3.24 and 3.27, one immediately deduces Theorem 3.13. In the following, we prove Theorem 3.27.

3.28. Special loci. Let \( P \in X_{\text{rig}} \) be a rigid point. One can associate to \( P \) a vector \( (\nu_{\beta}(P) : \beta \in B) \in [0, 1]^{B} \), where each \( \nu_{\beta}(P) \) is the truncated \( p \)-adic valuation of a lift of the \( \beta \)-partial Hasse invariant \( h_{\beta} \) applied at the reduction of \( P \). Saying that \( \nu_{\beta}(P) \geq r \) is equivalent to saying that any lift of the \( \beta \)-partial Hasse invariant at the reduction of \( P \) is divisible by \( p^{r} \). For more details, see [13, §4.2] and [41, 4.6.1] where \( (\nu_{\beta}(P))_{\beta} \) are called, respectively, *valuations* of \( P \) and *partial Hodge heights* of \( P \).

We recall the following degree increasing principle introduced by Pilloni in [25] using Fargues’ theory of degree functions. Its variant in the Hilbert case was considered in [41, 26, 17], and stated as follows:

**Proposition 3.29** ([41] Prop. 4.16). Let \( Q = (A, H) \) be a rigid point of \( Y_{\text{rig}} \) (resp. of \( Z \)) with values in a finite extension \( K'/K \). We let, for any \( p \in S \),

\[ U_{p}(Q) := \pi_{2,p}(\pi_{1,p}^{-1}([Q])), \]

the image of \( Q \) under the Hecke correspondence \( U_{p} \). Here, \( \pi_{1,p} \) and \( \pi_{2,p} \) are defined as in 3.5.1.

(1) For any \( Q' \in U_{p}(Q) \), we have \( \deg_{p}(Q') \geq \deg_{p}(Q) \).

(2) Assume there exists a \( Q' \in U_{p}(Q) \) such that \( \deg_{p}(Q') = \deg_{p}(Q) \). Then, we have \( \deg_{\beta} \in \{0, 1\} \), for all \( \beta \in B_{p} \). Moreover, defining

\[ \eta_{p}(Q) := \{ \beta \in B_{p} : \deg_{\beta}(Q) = 0 \}, \]

we will be in one of the two following cases:

(a) We have \( \eta_{p}(Q) = \emptyset \) or \( S \). In this case, \( Q \) is \( p \)-ordinary, i.e. the \( p \)-divisible group \( A[p^{\infty}] \) is an extension of an étale \( p \)-divisible group by a multiplicative one.
(b) We have $\eta_p(Q) \not\in \{0, S\}$. In this case, if we let

$$\varphi_p(Q) := \{\beta \in \mathbb{B}_p : \deg_{\sigma^{-1}o\beta}(Q) = 1\},$$

then $\nu_\beta(\pi(Q)) = 1$ for all $\beta \in (\varphi_p(Q) \cap \eta_p(Q)) \cup (r(\eta_p(Q)) \cap \ell(\varphi_p(Q)))$.

This proposition motivates the following definition of the special locus in $\mathcal{Y}_{\text{rig}}$. There will be several references to Definition 3.30 in what follows.

**Definition 3.30.** Let $\overline{x}$ be a vertex point of the cube $[0, 1]^{\mathbb{B}}$.

1. For $p \in S$, we define $V_{\overline{x}}^{sp,p} \subset \deg^{-1}(\overline{x})$, called the special locus at $p$ of $\deg^{-1}(\overline{x}) \subset \mathcal{Y}_{\text{rig}}$, as follows:
   a. if $\overline{x}$ is $p$-ordinary, we put $V_{\overline{x}}^{sp,p} = \deg^{-1}(\overline{x})$;
   b. otherwise, we define

$$V_{\overline{x}}^{sp,p} = \{Q \in \deg^{-1}(\overline{x}) : \nu_\beta(Q) = 1 \land \beta \in (\varphi_\overline{x} \cap \eta_\overline{x}) \cup (r(\eta_\overline{x}) \cap \ell(\varphi_\overline{x})) \cap \mathbb{B}_p\}.$$

This is a quasi-compact admissible open subset of $\deg^{-1}(\overline{x}) = sp^{-1}(W_{\overline{x}})$. In particular, $sp^{-1}(W_{\overline{x}})$ (Definition 2.7) is a strict neighborhood of $V_{\overline{x}}^{sp,p}$ in $\deg^{-1}(\overline{x})$.

2. In general, for an ideal $t$ of $O_L$ dividing $(p)$, we define the special locus at $t$ of $\deg^{-1}(\overline{x})$ as

$$V_{\overline{x}}^{sp,t} = \bigcap_{p \mid t} V_{\overline{x}}^{sp,p}.$$

For any subset of points $S \subset \mathcal{Y}_{\text{rig}}$, and $p \in S$, let $U_p(S) = \cup_{Q \in S} U_p(Q)$. For any ideal $t \subset O_L$ dividing $(p)$, and $Q \in \mathcal{Y}_{\text{rig}}$, we let

$$U_t(Q) = c_{p|t} U_p(Q),$$

which is independent of the order.

**Corollary 3.31.** Let $\overline{x}$ be a vertex point of the cube $[0, 1]^{\mathbb{B}}$, $t \subset O_L$ be an ideal dividing $(p)$. Then, for any $Q \in \deg^{-1}(\overline{x}) - V_{\overline{x}}^{sp,t}$, and any $Q' \in U_t(Q)$, we have $\deg(Q') > \deg(Q)$.

**Proof.** This is immediate from Proposition 3.29 since $\deg_\beta$ is unchanged under $U_p$ for all $\beta \in \mathbb{B}_q$ with $q \neq p$. \qed

**Corollary 3.32.** Let $Q$ be a rigid point in $\mathcal{Y}_{\text{rig}}$ such that $\deg(Q)$ is not an integer. Then, for any $Q' \in U_p(Q)$, we have $\deg(Q') > \deg(Q)$.

**Proof.** Indeed, since $\deg(Q)$ is not an integer, there exists at least one $p \in \mathbb{B}$ such that $\deg_p(Q)$ is not an integer. Hence, by Proposition 3.29, we have $\deg_p(Q') > \deg_p(Q)$, for all $Q' \in U_p(Q)$. \qed
3.33. Integral model of $Z_K$. We will need the integral model of $Z_K$ over $\mathcal{O}_K$ defined in [24]. Let $H \subset \mathcal{A}_{\text{univ}}[p]$ be the universal isotropic $(\mathcal{O}_L/p)$-cyclic subgroup over $Y$. We have a canonical decomposition $H = \prod_{p \in S} H[p]$, where each $H[p] = \text{Spec}(\mathcal{O}_H[p])$ is a scheme of $(\mathcal{O}_L/p)$-vector spaces of dimension 1 over $Y$. By Raynaud’s classification of such group schemes [28, 1.4.1], there exist invertible sheaves $\mathcal{L}_\beta$ over $Y$, for every $\beta \in \mathbb{B}_p$, together with $\mathcal{O}_Y$-linear morphisms $\Delta_\beta : \mathcal{L}_\beta^\otimes p \to \mathcal{L}_{\sigma \beta}$ and $\Gamma_\beta : \mathcal{L}_{\sigma \beta} \to \mathcal{L}_\beta^\otimes p$ such that $\Delta_\beta \circ \Gamma_\beta$ and $\Gamma_\beta \circ \Delta_\beta$ are both multiplication by $p$, and the $\mathcal{O}_Y$-algebra $\mathcal{O}_H[p]$ is isomorphic to

$$\text{Sym}_{\mathcal{O}_Y}(\oplus_{\beta \in \mathbb{B}} \mathcal{L}_\beta)/(1 - \Delta_\beta)\mathcal{L}_\beta^\otimes (p-1)).$$

In fact, $\mathcal{L}_\beta$ is the direct summand of the augmentation ideal $\mathcal{J}_H[p] \subset \mathcal{O}_H[p]$, where $(\mathcal{O}_L/p)^\times$ acts via the Teichmüller character $\chi_\beta : (\mathcal{O}_L/p)^\times \to W(\kappa_K)^\times$. Now we consider the closed subscheme $H'[p]$ of $H[p]$ defined by the equation

$$(\oplus_{\beta \in \mathbb{B}} \Delta_\beta - 1)(\oplus_{\beta \in \mathbb{B}} \mathcal{L}_\beta^\otimes (p-1)).$$

Explicitly, let $U = \text{Spec}(R)$ be an affine open subset of $Y$ such that $H[p]|_U = \text{Spec}(R)[T_\beta : \beta \in \mathbb{B}_p]/(T_\beta^p - y_\sigma \beta T_\sigma \beta : \beta \in \mathbb{B}_p)$, with $y_\beta \in R$, for $\beta \in \mathbb{B}_p$. Then, we have

$$H'[p]|_U = \text{Spec}(R)[T_\beta : \beta \in \mathbb{B}_p]/(T_\beta^p - y_\sigma \beta T_\sigma \beta, \prod_{\beta \in \mathbb{B}_p} T_\beta^{p-1} - \prod_{\beta \in \mathbb{B}_p} y_\beta).$$

From this local description, we see that $H'[p]$ is a finite flat scheme over $Y$ of rank $p^f - 1$. We set $H' := \prod_{p \in S} H'[p]$. Prop 3.34. We now begin the proof of Theorem 3.27.

**Proposition 3.34 ([24], 5.1.5, 2.3.3).** The $Y$-scheme $H' \to Y$ represents the functor which associates to each $Y$-scheme $S$, the set of $(\mathcal{O}_L/p)$-generators of $H \times_Y S$ in the sense of Drinfeld-Katz-Mazur [19, 1.10]. Consequently, the scheme $H'$ is an integral model of $Z$ over $\mathcal{O}_K$, which is finite flat of degree $\prod_{p \in S} (p^f - 1)$ over $Y$.

**Remark 3.35.** In the proof of Proposition 3.27, we only need the existence of an integral model of $Z_K$ which is finite flat over $Y$.

In the sequel, we will define $Z := H'$. We let $\mathfrak{Z}$ denote the formal completion of $Z$ along its special fibre, and $\mathfrak{Z}_{\text{rig}}$ its associated rigid generic fibre. Therefore, $\mathfrak{Z}_{\text{rig}}$ is the quasi-compact admissible open subdomain of $\mathfrak{Z}$ where the universal HBAV has good reduction.

Now we begin the proof of Theorem 3.27.

**Proof of Theorem 3.27.** Let $\alpha_*(\omega_\mathfrak{Z})$ be the push-forward of the sheaf $\omega_\mathfrak{Z}$ on $\mathfrak{Z}$ via the finite map $\alpha : \mathfrak{Z} \to \mathfrak{Z}_{\text{rig}}$. For any admissible open subset $U \subset \mathfrak{Z}_{\text{rig}}$, we can identify $H^0(\alpha^{-1}(U), \omega_\mathfrak{Z})$ with $H^0(U, \alpha_*(\omega_\mathfrak{Z}))$. This allows us to work over $\mathfrak{Z}_{\text{rig}}$. By assumption, $f$ is a section of $\alpha_*(\omega_\mathfrak{Z})$ defined over $\bigcup_{T \in C \mathbb{W}_1^{-1} (\mathfrak{Z}_{\text{rig}}^{[\sigma \leq 1]})}$.

To prove Theorem 3.27, it is enough to show that $f$ extends to a global section of $\alpha_*(\omega_\mathfrak{Z})$ over $\mathfrak{Z}_{\text{rig}}$, since, then, we can conclude by rigid GAGA that $f$ is classical. Note that

$$\mathfrak{Z}_{\text{rig}}(g-1, g) \subset \mathfrak{Z}_{\text{rig}}^{[\sigma \leq 1]} \subset \mathfrak{Z}_{\text{rig}}^{[\sigma \leq 1]}$$
and $\mathfrak{C}_{\text{rig}}[0,1] \subset w^{-1}(\mathfrak{C}_{\text{rig}})'$ by (2.2.1). Therefore, to complete the proof, it’s sufficient to prove by descending induction that if $f$ is a section of $\alpha_+ \mathfrak{L}$ over $\mathfrak{C}_{\text{rig}}(r,g)$, then $f$ can be further extended to $\mathfrak{C}_{\text{rig}}(r-1,g)$ for an integer $1 \leq r \leq g-1$.

**Step 1. Extension to vertex points of degree $r$.** Let $\underline{x} = (x_{\beta})_{\beta \in \mathbb{B}} \in \{0,1\}^\mathbb{B}$ be a vertex point of the cube $[0,1]^\mathbb{B}$ of degree $r$. We will show that $f$ extends to $\deg^{-1}(\underline{x})$ (2.5.1). We distinguish several cases.

**Case 1.** There exist two distinct ideals $p, q \in \mathcal{S}$, such that $\underline{x}$ is neither $p$-ordinary, nor $q$-ordinary. (See Definition 2.7(1)). First, we have $\deg^{-1}(\underline{x}) = \text{sp}^{-1}(W_{\underline{x}}) \text{ by (2.5.1)}$. By Corollary 3.31, we have $\deg_p(Q') > \deg_p(Q)$ for all $Q' \in \text{sp}^{-1}(W_{\underline{x}}) - V_x^{sp,pq}$, and $Q' \in U_{pq}(Q)$. Therefore, we can extend $f$ to $\text{sp}^{-1}(W_{\underline{x}}) - V_x^{sp,pq}$ via the functional equation

$$f = \frac{1}{\lambda_p} U_{pq}(f),$$

where $\lambda_t = \prod_{p|t} \lambda_p$ for any ideal $t$ dividing $(p)$, and $\lambda_p := \lambda_{(p)}$. Note that $V_x^{sp,pq} \subset \text{sp}^{-1}(W_x^{sp,pq})$, and the closed subscheme $W_x^{sp,pq}$ is negligible by Corollary 2.9(2). On the other hand, thanks to Pappas’s integral model $Z$ of $Z_K$, the sheaf $\alpha_+ \mathfrak{L}_{\text{rig}} = \alpha_+ \mathfrak{L} \otimes_{\mathcal{O}_{\mathfrak{C}_{\text{rig}}}} \alpha_+ \mathcal{O}_{3_{\text{rig}}}$ on $\mathfrak{C}_{\text{rig}}$ admits an integral formal model $\alpha_+ \mathfrak{L} \otimes_{\mathcal{O}_\mathfrak{Q}} \alpha_+(\mathcal{O}_3)$, which is a locally free $\mathcal{O}_\mathfrak{Q}$-module of rank $p^g - 1$. Let $\mathfrak{M}_{\underline{x}}$ be the open formal subscheme of $\mathfrak{C}_{\text{rig}}$ corresponding to $W_{\underline{x}}$. Then $\text{sp}^{-1}(W_{\underline{x}})$ is identified with the associated rigid generic fibre $\mathfrak{M}_{\underline{x}}^{\text{rig}}$, hence it’s admissible. Applying the rigid Koecher principle (19, Lemma A.3) to the admissible formal scheme $\mathfrak{M}_{\underline{x}}$ and the sheaf $\alpha_+ \mathfrak{L} \otimes_{\mathcal{O}_\mathfrak{Q}} \alpha_+(\mathcal{O}_3)$ restricted to $\mathfrak{M}_{\underline{x}}$, we see that $f$ extends automatically to $\text{sp}^{-1}(W_{\underline{x}})$.

**Case 2.** There exists a $p \in \mathcal{S}$, such that $\underline{x}$ is neither $p$-ordinary, nor of type 1 at $p$. Just like Case 1, we have $\deg^{-1}(\underline{x}) = \text{sp}^{-1}(W_{\underline{x}})$, and $f$ can be extended analytically to $\text{sp}^{-1}(W_{\underline{x}} - W_x^{sp,p})$ by Corollary 3.31. Now, Corollary 2.9(1) says that $W_x^{sp,p}$ is already negligible, so we can conclude as above using the rigid Koecher principle.

**Case 3.** There exists a unique $p \in \mathcal{S}$ such that $\underline{x}$ is of type 1 at $p$, and $q$-ordinary for all $q \neq p$. In this case, we have either $\sum_{\beta \in \mathbb{B}_q} x_{\beta} \in \{0, f_q\}$ for all $q \neq p$. We set

$$T := \{q \in \mathcal{S} : \{p\} : \sum_{\beta \in \mathbb{B}_q} x_{\beta} = 0\}.$$

Then, we have the following inclusions by the formula (2.7.1)

$$\text{sp}^{-1}(W_{\underline{x}}^{\mid \mathfrak{q} \mid \leq 1}) \subset w^{-1}(\mathfrak{C}_{\text{rig}})'_T,$$

$$\text{sp}^{-1}(w^{-1}_p(W_{\underline{x}}^{\mid \mathfrak{q} \mid \leq 1})) \subset w^{-1}_T(\mathfrak{C}_{\text{rig}})'_T.$$

Hence, by assumption, the form $f$ is defined over

$$\text{sp}^{-1}(W_{\underline{x}}^{\mid \mathfrak{q} \mid \leq 1}) \cup \text{sp}^{-1}(w^{-1}_p(W_{\underline{x}}^{\mid \mathfrak{q} \mid \leq 1})) = \text{sp}^{-1}(W_{\underline{x}}^{\mid \mathfrak{q} \mid \leq 1} \cup w^{-1}_p(W_{\underline{x}}^{\mid \mathfrak{q} \mid \leq 1})).$$
Since Proposition 2.11 says that the complement of the open subscheme \( W_{\mathbf{T}}^{[\mathbf{r}]_p \leq 1} \cup W_{w_{p}(\mathbf{x})}^{\mathbf{r} \leq 1} \) in \( W_{\mathbf{x}} \) is negligible, we can conclude, using the rigid Koecher principle, that \( f \) extends to \( \tilde{\operatorname{deg}}^{-1}(\mathbf{x}) = \text{sp}^{-1}(W_{\mathbf{x}}) \).

**Case 4.** The vertex \( \mathbf{x} \) is \( p \)-ordinary for all \( p \in \mathcal{S} \). Similarly to Case 3., for all \( p \in \mathcal{S} \), we have \( \sum_{\beta \in \mathcal{B}_p} x_{\beta} \in \{0, f_p\} \). We set

\[
T_{\mathbf{x}} := \{ p \in \mathcal{S} : \sum_{\beta \in \mathcal{B}_p} x_{\beta} = 0 \}.
\]

Note that \( T_{\mathbf{x}} \neq \mathcal{S} \) because \( r = \sum_{p \not\in T_{\mathbf{x}}} f_p > 0 \). Let \( L \in \{0, 1\}^\mathcal{B} \) be the vector with all components equal to 1, and \( \mathcal{V}_{T_{\mathbf{x}}} \) be the multiset of invertals given by \( I_p = (f_p - 1, f_p) \) for \( p \not\in T_{\mathbf{x}} \) and \( I_p = [0, 1) \) for \( p \in T_{\mathbf{x}} \). Then, we have

\[
\tilde{\operatorname{deg}}^{-1}(\mathbf{x}) = w_{T_{\mathbf{x}}}^{-1}(1) \subset \mathfrak{g}_{\text{rig}}^{\mathcal{V}_{T_{\mathbf{x}}}} = w_{T_{\mathbf{x}}}^{-1}(\mathfrak{g}_{\text{rig}}^{\mathcal{V}}) \subset w_{T_{\mathbf{x}}}^{-1}((\mathfrak{g}_{\text{rig}}^{\mathcal{V}})_{|\mathbf{r} \leq 1}).
\]

This shows that \( \mathfrak{g}_{\text{rig}}^{\mathcal{V}_{T_{\mathbf{x}}}} \) is a strict neighborhood of \( \tilde{\operatorname{deg}}^{-1}(\mathbf{x}) \), over which \( f \) is defined by assumption.

**Step 2.** In step 1, we have analytically extended the form \( f \) to

\[
\tilde{\mathfrak{g}}_{\text{rig}}(r, g)^+ := \tilde{\mathfrak{g}}_{\text{rig}}(r, g) \cup \left( \bigcup_{\substack{\mathbf{x} \in \{0, 1\}^\mathcal{B} \\ \tilde{\operatorname{deg}}(\mathbf{x}) = r}} \tilde{\operatorname{deg}}^{-1}(\mathbf{x}) \right),
\]

where \( \tilde{\operatorname{deg}}^{-1}(\mathbf{x}) = \tilde{\mathfrak{g}}_{\text{rig}}^{\mathcal{V}_{T_{\mathbf{x}}}} \) if \( \mathbf{x} \) is a vertex point of Case 4 above, and \( \tilde{\operatorname{deg}}^{-1}(\mathbf{x}) = \operatorname{deg}^{-1}(\mathbf{x}) = \text{sp}^{-1}(W_{\mathbf{x}}) \) otherwise. Let \( \epsilon \) be any rational number with \( 0 < \epsilon < 1 \), and we prove that \( f \) can be further extended to \( \tilde{\mathfrak{g}}_{\text{rig}}^{\mathfrak{g}_{\text{rig}}^{\mathcal{V}}(r - \epsilon, r + \epsilon)} \). This will complete the proof, since we see that \( f \) extends to \( \tilde{\mathfrak{g}}_{\text{rig}}^{\mathfrak{g}_{\text{rig}}^{\mathcal{V}}(r - 1, g)} \) by letting \( \epsilon \to 1^- \).

For each vertex point \( \mathbf{x} \) of degree \( r \), let \( V_{\mathbf{x}}^{sp} = \tilde{V}_{\mathbf{x}}^{sp}(p) \) be the special locus of \( \tilde{\operatorname{deg}}^{-1}(\mathbf{x}) \) for \( (p) \) defined in 3.30. We choose another admissible open subset \( V_{\mathbf{x}}^{sp} \) of \( \tilde{\operatorname{deg}}^{-1}(\mathbf{x}) \) such that

(1) we have natural inclusions of strict neighborhoods: \( V_{\mathbf{x}}^{sp} \subset \tilde{V}_{\mathbf{x}}^{sp} \subset \tilde{\operatorname{deg}}^{-1}(\mathbf{x}) \);

(2) the complement \( \tilde{\mathfrak{g}}_{\text{rig}} - \tilde{V}_{\mathbf{x}}^{sp} \) is quasi-compact.

For instance, if \( \mathbf{x} \) is not in Case 4 above, we take \( \tilde{V}_{\mathbf{x}}^{sp} = \text{sp}^{-1}(W_{\mathbf{x}}^{sp,S}) \), where \( W_{\mathbf{x}}^{sp,S} \) is the closed subscheme of \( W_{\mathbf{x}} \) given in Definition 2.7 if \( \mathbf{x} \) is in Case 4, we take \( \tilde{V}_{\mathbf{x}}^{sp} \) to be \( \tilde{\mathfrak{g}}_{\text{rig}}^{\mathcal{V}_{T_{\mathbf{x}}}} \), where \( \mathcal{V}_{T_{\mathbf{x}}} \) is the multiset of invertals with \( I_p = (f_p - 1/2, f_p) \) for \( p \not\in T_{\mathbf{x}} \) and \( I_p = [0, 1/2) \) for \( p \in T_{\mathbf{x}} \).

We consider the admissible open covering \( \tilde{\mathfrak{g}}_{\text{rig}}^{[r - \epsilon, r + \epsilon]} = \mathcal{U}_1 \cup \mathcal{U}_2 \) with

\[
\mathcal{U}_1 = \tilde{\mathfrak{g}}_{\text{rig}}^{[r - \epsilon, r + \epsilon]} - \left( \bigcup_{\tilde{\operatorname{deg}}(\mathbf{x}) = r} \tilde{V}_{\mathbf{x}}^{sp} \right),
\]

\[
\mathcal{U}_2 = \left( \bigcup_{\tilde{\operatorname{deg}}(\mathbf{x}) = r} \tilde{V}_{\mathbf{x}}^{sp} \right).
\]
and
\[ \mathcal{U}_2 = \tilde{\mathfrak{H}}_{\text{rig}}(r, g) + \cap \tilde{\mathfrak{H}}_{\text{rig}}[r - \varepsilon, g]. \]
Note that \( \mathcal{U}_1 \) is quasi-compact by property (2) of the \( \tilde{V}_{\text{sp}} \)'s. By Corollaries 3.31 and 3.32 we have \( \deg(Q') > \deg(Q) \) for all rigid points \( Q \in \mathcal{U}_1 \) and all \( Q' \in U_p(Q) \). By the maximal modulus principle in rigid geometry, there exists an \( \eta > 0 \) such that \( \deg(Q') - \deg(Q) \geq \eta \) for all \( Q \in \mathcal{U}_1 \) and \( Q' \in U_p(Q) \). Since \( \tilde{\mathfrak{H}}_{\text{rig}}[r - \varepsilon, g] = \mathcal{U}_1 \cap \mathcal{U}_2 \) is stable under the Hecke correspondence \( U_p \), we have \( U_p^M(\mathcal{U}_1) \subset \mathcal{U}_2 \) if \( M > 2\varepsilon/\eta \). Since \( f \) has been defined over \( \mathcal{U}_2 \) by Step 1, we can extend \( f \) to \( \mathcal{U}_1 \) by setting \( f = \frac{1}{\lambda_p} U_p^M(f) \). This completes the proof. \( \square \)

4. Residual Modularity

Let \( F \) be a totally real field (on which we will put various conditions). In this section, we prove that certain mod-\( p \) representations of \( G_F = \text{Gal}(\overline{\mathbb{Q}}/F) \) are modular.

4.1. Modularity of icosahedral mod 5 representations of \( G_F \). We begin with a Lemma.

**Lemma 4.2.** Let \( F \) be a totally real field. Suppose that \( [F(\zeta_5) : F] = 4 \), and \( [F_p(\sqrt{5}) : F_p] = 2 \) for every place \( p \) of \( F \) above 5; in particular, every \( p \) of \( F \) above 5 does not split completely in the quadratic extension \( F(\sqrt{5}) \) of \( F \). Suppose that \( \overline{\rho} : G_F \to GL_2(\mathbb{F}_5) \) is a continuous representation of \( G_F = \text{Gal}(\overline{\mathbb{Q}}/F) \) satisfying

- totally odd,
- \( \overline{\rho} \) has projective image \( A_5 \),
- for every prime \( p \) of \( F \) above 5, the projective image of the decomposition subgroup \( D_p \) of \( G_F \) has order 2, and the quadratic extension of \( F_p \) fixed by the kernel of \( D_p \) is not \( F_p(\sqrt{5}) \).

Then, there is a finite soluble totally real extension \( F \supset M \subset \overline{\mathbb{Q}} \) and an elliptic curve \( E \) over \( M \) satisfying the following conditions:

- \( M \supset F(\sqrt{5}) \) and \( \sqrt{5} \) splits completely in \( M \),
- \( \overline{\rho}_{E,5} : G_M = \text{Gal}(\overline{\mathbb{Q}}/M) \to \text{Aut}(E[5]) \) is equivalent to a twist of \( \overline{\rho}|_{G_M} \) by some character,
- \( \overline{\rho}_{E,3} : \text{Gal}(\overline{\mathbb{Q}}/M(\zeta_3)) \to \text{Aut}(E[3]) \) is absolutely irreducible,
- \( E \) has good ordinary reduction at the primes \( o \) of \( M \) above 3, and potentially ordinary reduction at the primes of \( M \) above 5.

**Proof.** Firstly, as in [33], find a biquadratic extension \( F_1 \) of \( F \) containing \( F(\sqrt{5}) \) such that, when restricted to \( F_1 \), \( \text{proj} \overline{\rho}_{G_{F_1}} : G_{F_1} = \text{Gal}(\overline{\mathbb{Q}}/F_1) \to PSL_2(\mathbb{F}_5) \simeq A_5 \) lifts to a representation \( \overline{\rho}_1 : G_{F_1} \to GL_2(\mathbb{F}_5) \) with determinant the mod 5 cyclotomic character \( \tilde{\epsilon}_5 \). Choose, by class-field theory, a finite soluble totally real extension \( F_2 \) of \( F_1 \) such that every prime place of \( F_1 \) dividing \( \sqrt{5} \) splits completely in \( F_2 \), and such that \( \overline{\rho}_1 \) is trivial when restricted to the decomposition group of every place in \( F_2 \) above 3. The construction follows from class field theory; see Lemma 2.2 in [10] of “Chevalley’s lemma”, for example. Let \( M \) be the Galois closure of \( F_2 \) over \( F \). Note that \( M \) is soluble over \( F \).
As in section 1 of [37], let \( Y_{\mathbf{p}} / M \) (resp., \( X_{\mathbf{p}} / M \)) denote the twist of the (resp., compactified) modular curve \( Y_5 \) (resp., \( X_5 \)) with full level 5 structure. As proved in Lemma 1.1 [37], the “twist” cohomology class is in fact trivial, and therefore \( X_{\mathbf{p}} \cong X_5 \) and \( Y_{\mathbf{p}} \) is isomorphic over \( M \) to a Zariski open subset of the projective 1-line \( \mathbb{P}^1 \). In particular, \( Y_{\mathbf{p}} \) has infinitely many rational points.

Let \( Y_{\mathbf{p}, 0}(3) \) denote the degree 4 cover over \( Y_{\mathbf{p}} \) which parameterises the isomorphism classes \((E, \phi_5, C)\) of elliptic curves \( E \) equipped with an isomorphism \( \phi_5 : E[5] \cong \mathbf{p} \) taking the Weil pairing on \( E[5] \) to \( \epsilon : \wedge^2 \mathbf{p} \to \mu_5 \), and a finite flat subgroup scheme \( C \subset E[3] \) of order 3.

Let \( Y_{\mathbf{p}, \text{split}}(3) \) denote the étale cover over \( Y_{\mathbf{p}} \) which parameterises the isomorphism classes \((E, \phi_5, C, D)\) where \((E, \phi_5)\) is as in \( Y_{\mathbf{p}} \), and where \((C, D)\) is an unordered pair, fixed by \( G_M \), of finite flat subgroup schemes of \( E[3] \) of order 3 which intersect trivially. Then, it follows from Lemma 12 in [31] that \( Y_{\mathbf{p}, \text{split}}(3) \) and \( Y_{\mathbf{p}, 0}(3) \) have only finitely many rational points.

For every prime \( \mathfrak{p} \) of \( M \) above 3, the elliptic curve \( y^2 = x^3 + x^2 - x \) defines an element of \( Y_{\mathbf{p}}(M_5) \) with good ordinary reduction, and we let \( \mathcal{U}_{\mathfrak{p}} \subset Y_{\mathbf{p}}(M_5) \) denote a (non-empty) open neighbourhood (for the 3-adic topology) of the point, consisting of elliptic curves with good ordinary reduction at \( \mathfrak{p} \).

For every prime \( \mathfrak{p} \) of \( M \) above 5, the restriction of \( \mathbf{p} \mid G_M \) to the decomposition subgroup \( D_{M, \mathfrak{p}} \) of \( G_M \) at \( \mathfrak{p} \) is of the form

\[
\mathbf{p} \mid D_{M, \mathfrak{p}} \sim \chi \delta \oplus \chi,
\]

up to twist by quadratic characters, where \( \delta : D_{M, \mathfrak{p}} \to \mathbb{F}_5^\times \) is a character of order 2, and \( \chi : D_{M, \mathfrak{p}} \to \mathbb{F}_5^\times \) is a tamely ramified character of order 4. This follows from the construction of \( \mathbf{p} \). Let \( I_{M, \mathfrak{p}} \) denote the inertia subgroup at \( \mathfrak{p} \). Since the determinant \( \delta \chi^2 \) is the mod 5 cyclotomic character, if \( \delta \mid I_{M, \mathfrak{p}} \) is of order 1, i.e., \( \delta \) is unramified (resp. \( \delta \mid I_{M, \mathfrak{p}} \) is of order \( \geq 2 \)), we may assume that \( \chi \mid I_{M, \mathfrak{p}} \) is of order 4 (resp. \( \chi \mid I_{M, \mathfrak{p}} \) is unramified). To find a rational point of \( Y_{\mathbf{p}}(M_5) \), it suffices to construct a twist by a character of an elliptic curve over \( M_5 \) with CM (so that the action of \( D_{M, \mathfrak{p}} \) on its 5-torsions is split) with good ordinary reduction. For this we use Taylor’s example \( y^2 = x^3 + x \) in the proof of Lemma 2.3 in [10], and define \( \mathcal{U}_{\mathfrak{p}} \) to be a sufficiently small non-empty open neighbourhood (for the 5-adic topology) of the point.

By Hilbert irreducibility theorem (Theorem 1.3 in [10]; see also Theorem 3.5.7 in [36]), we may then find a rational point in \( Y_{\mathbf{p}}(M) \) which lies in \( \mathcal{U}_{\mathfrak{p}} \) for every \( \mathfrak{p} \) above 15 and does not lie in the images of \( Y_{\mathbf{p}, 0}(3)(M) \to Y_{\mathbf{p}}(M) \) and \( Y_{\mathbf{p}, \text{split}}(3)(M) \to Y_{\mathbf{p}}(M) \). The elliptic curve over \( M \) corresponding to the rational point is what we are looking for.

\[ \square \]

**Theorem 4.3.** Let \( F \) be a totally real field. Suppose that \( [F(\zeta_5) : F] = 4 \) and \( [F_p(\sqrt{5}) : F_p] = 2 \) for every place \( p \) of \( F \) above 5. Let \( \mathbf{p} : G_F \to GL_2(\mathbb{F}_5) \) be a continuous representation of the absolute Galois group \( G_F = \text{Gal}(\overline{\mathbb{Q}}/F) \) of \( F \) which satisfies the following conditions.

- totally odd
- \( \mathbf{p} \) has projective image \( A_5 \)
The projective image of the decomposition subgroup $D_p$ of $G_F$ for every $p\nmid 5$ has order 2 and the quadratic extension of $F_p$ fixed by the kernel of $D_p \rightarrow G_F \rightarrow A_5$ is not $F_p(\sqrt{5})$.

Then $\overline{\rho}$ is modular.

Proof. This can be proved exactly as in [34]. Choose an elliptic curve $E$ over a finite soluble totally real extension $M$ of $F$ as in the preceding lemma. Replace $M$ by its finite totally real soluble extension, if necessary, to assume that the mod 3 representation $\overline{\rho}_{E,3}$ is unramified at every prime of $M$ above 3. As argued in the proof of Theorem 3.5.5 in [21], one can do this with the absolute irreducibility of $\overline{\rho}_{E,3}|_{\text{Gal}(\overline{\mathbb{Q}}/M)}$ intact.

By the Langlands-Tunnell theorem, there exists a weight 1 cuspidal Hilbert eigenform $f_1$ which gives rise to $\overline{\rho}_{E,3}$. By 3-adic Hida theory, we may find a cuspidal Hilbert eigenform $f_2$ of weight 2 and of level prime to 3, ordinary at every prime of $M$ above 3, which gives rise to $\overline{\rho}_{E,3}$. As $E$ is ordinary at 3, $f_2$ renders $\rho_{E,3} : G_M \rightarrow GL(T_3E)$ “strongly residually modular” in the sense of Kisin [21], and it follow from Theorem 3.5.5 in [21] that $T_3E$ is modular. By Falting’s isogeny theorem, $E$ is therefore modular. As $\overline{\rho}_{E,5}$ is modular, $\overline{\rho}|_{G_M}$ is modular. The soluble descent to $F$ is exactly as in [40]. □

4.4. Modularity of totally odd icosahedral mod 2 representations.

Theorem 4.5. Let $F$ be a totally real field. Suppose that $[F(\zeta_5) : F] = 4$. Let $\overline{\rho} : G_F \rightarrow SL_2(\mathbb{F}_4)$ be a continuous representation. Suppose that $\overline{\rho}$ is unramified at every place of $F$ above 5. Then $\overline{\rho}$ is modular.

Proof. By theorem 3.4 in [37], there exits a principally polarised abelian surface $A$ over $F$ with real multiplication $\mathbb{Z}[(1 + \sqrt{5})/2]$ (compatible with the polarisation) such that the action of $G_F$ on $A[2] \simeq \mathbb{F}_4^2$ is equivalent to $\overline{\rho}$; and the action of $G_F$ on $A[\sqrt{5}] \simeq \mathbb{F}_5^2$ is given via a homomorphism

$$\overline{\rho}_{A,\sqrt{5}} : G_F \rightarrow GL_2(\mathbb{F}_5)$$

which is surjective and whose image contains $SL_2(\mathbb{F}_5)$.

It suffices to prove that $A$ as in the previous lemma is modular. Firstly, the Weil paring on $A(\overline{F})(\sqrt{5})$ shows that $\det \overline{\rho}_{A,\sqrt{5}}$ is the cyclotomic character. By the assumption that $[F(\zeta_5) : F] = 4$, the determinant is indeed surjective, and therefore $\overline{\rho}_{A,\sqrt{5}}$ is (absolutely) irreducible. Secondly, since $\overline{\rho}$ is unramified at every place $p$ of $F$ above 5, the image of the inertia subgroup $I_p$ on the 2-adic Tate module is conjugated, up to twist by characters, to lie in the upper-trianglar matrices in $GL_2(\mathbb{Z}_2)$ with 1 on the diagonal; $A$ has, potentially, good ordinary or multiplicative reduction at every $p$ above 5, and $\overline{\rho}_{A,\sqrt{5}}$ is potentially ordinary at every place $p$ of $F$ above 5. The restriction to $\text{Gal}(\overline{F}/F(\sqrt{5}))$ of $\overline{\rho}_{A,\sqrt{5}}$ therefore is irreducible; in fact it is absolutely irreducible (see Proposition 7 in [31] for example).

By Kisin’s modular lifting theorems in [21] and Khare-Wintenberger’s modular lifting theorems [16], it suffices to prove that $\overline{\rho}_{A,\sqrt{5}}$ has a potentially 5-Barsotti-Tate, potentially 5-ordinary modular lifting. As in the previous section, we may find an elliptic curve $E$ over a finite soluble totally real extension $F_1$ of $F$ satisfying the following conditions:

- $\overline{\rho}_{E,5} : G_{F_1} = \text{Gal}(\overline{F}/F_1) \rightarrow \text{Aut}(E[5])$ is equivalent to $\overline{\rho}_{A,\sqrt{5}}|_{G_{F_1}}$,
the image by $\overline{\rho}_{E,5}$ of the decomposition group at every prime of $F_1$ above 5 is trivial.

- $\overline{\rho}_{E,3}: \text{Gal}(\overline{F}/F_1(\zeta_3)) \to \text{Aut}(E[3])$ is absolutely irreducible.

- the image by $\overline{\rho}_{E,5}$ of the decomposition group at every prime of $F_1$ above 3 is trivial.

- $E$ has good ordinary reduction at every prime of $F_1$ above 3 and has potentially good ordinary reduction at every prime of $F_1$ above 5.

Then modularity of $E/F_1$ follows as in the proof of the theorem in the previous section. 

\[ \square \]

5. Hida theory and $\Lambda$-adic companion forms

5.1. Hilbert modular forms. Let $L$ be a totally real field. In the following, we will define Hilbert modular forms as true automorphic forms for the group $\text{Res}_{L/Q}\text{GL}_{2,F}$. See Remark 3.4 for the relationship between these forms and the geometric Hilbert modular forms defined in §3.4.

As before, $\mathcal{O}_L$ denotes the integers of $L$, and $\mathfrak{d}_L$ the different of $L$. Let $\mathbb{A}_L = \mathbb{A}_L^\infty \times L_\infty$ denote the adeles of $L$. By $\infty$, we shall also mean the product of the infinite primes of $L$.

For an ideal $M$ of $\mathcal{O}_L$, let $L_M$ denote the strict ray class field of conductor $M$.

Let $U_1(M)$ denote the open compact subgroup of matrices $\left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \text{GL}_2(\mathcal{O}_L \otimes \hat{\mathbb{Z}})$ such that $c \equiv 0 \mod M$ and $d \equiv 1 \mod M$. Let $C_{L,M}$ denote the strict ray class group $\mathbb{A}_L^\times/L^\times(\mathbb{A}_L^\infty \times U_1(M))L_\infty^\times \mod M_\infty$.

Let $p$ be a rational prime and fix an algebraic closure $\overline{\mathbb{Q}}_p$ and an isomorphism $\overline{\mathbb{Q}}_p \to \mathbb{C}$.

If $k \in \mathbb{Z}$, let $S_{k,1}(U_1(M); \mathbb{C})$ denote the $\mathbb{C}$-vector space, in the sense of Hida [14], of parallel weight $k = \sum_{\tau \in \text{Hom}_\mathbb{Q}(F,\mathbb{R})} k_\tau$ cusp forms of level $U_1(M)$. Let $S_k(U_1(M))$ (resp., $S_k(U_1(M); R)$ for a ring $R \subset \overline{\mathbb{Q}}_p$) denote the subspace of forms $f$ in $S_{k,1}(U_1(N); \mathbb{C})$ whose Fourier coefficients $c(n, f) \in \mathbb{Z}$ (resp. $\in R$) for all integral ideals $n$ of $\mathcal{O}_L$. These spaces come equipped with an action of $C_{L,M}$ via the Diamond operators $q \mapsto \langle q \rangle$ for a prime $q \nmid M$, $T_q$ for a prime $q \nmid M$, and $U_q$ for a prime $q | M$.

Let $h_k(M)$ denote the sub $\mathbb{Z}$-algebra of $\text{End}(S_k(U_1(M)))$ generated over $\mathbb{Z}$ by all these operators. For a prime $q \nmid M$, define $S_q$ by $(\mathcal{N}_L/q)^{k-1}(q)$.

Fix an ideal $N$ of $\mathcal{O}_L$ coprime to $p$. For the ring $\mathcal{O}$ of integers of a finite extension $K$ of $\mathbb{Q}_p$, Hida (See section 3 in [14]) defines the idempotent $e$ and we set

$$h_0^\mathcal{O}(N) = \lim_{\leftarrow r} e(h_2(Np^r) \otimes \mathcal{O}).$$

We have a natural map (induced by the Diamond operator)

$$\langle \rangle : C_{L,Np^\infty} \overset{\text{def}}{=} \lim_{\leftarrow r} C_{L,Np^r} = \mathbb{A}_L^\times/L^\times(\mathbb{A}_L^\infty \times U_1(Np^\infty))L_\infty^\times \longrightarrow h_0^\mathcal{O}(N)^\times.$$
where by \( h_0^\infty \cap U_1(Np^\infty) \), we mean the set of elements in \( \mathbb{A}_L^{\infty,\times} \cap U_1(N) \) which are 1 at every prime \( p \) of \( L \) above \( p \).

We let \( \text{Tor}_{L,Np^\infty} \) (resp. \( \text{Fr}_{L,Np^\infty} \)) denote the torsion subgroup (resp. the maximal \( \mathbb{Z}_p \)-free subgroup of rank \( 1 + \delta \) with \( \delta = 0 \) if the Leopoldt conjecture holds) of \( C_{L,Np^\infty} \), and let \( \Lambda_\mathcal{O} \) denote the completed group algebra over \( \mathcal{O} \) of \( \text{Fr}_{L,Np^\infty} \). Note that \( h_0^\infty(N) \) is a \( \Lambda_\mathcal{O} \)-module via \( (\_\_\_\_) \).

In [14], Hida proves that \( h_0^\infty(N) \) is a torsion free module of finite type over \( \Lambda_\mathcal{O} \).

We will let

\[
\text{Art} : \mathbb{A}_L^{\infty,\times}/L^\times L_\infty^{+\times} \simeq \text{Gal}(\overline{L}/L)^{ab}
\]
denote the (global) Artin map, normalised compatibly with the local Artin maps which are normalised to take uniformisers to arithmetic Frobenius elements. By abuse of notation, we shall let \( \text{Art} \) also denote the induced isomorphism \( C_{L,Np^\infty} \simeq \text{Gal}(L(N(\mu_p^\infty))/L) \).

Let \( \epsilon \) denote the cyclotomic character

\[
\epsilon : \text{Gal}(L(N(\mu_p^\infty))/L) \to \mathbb{Z}_p^\times \to \overline{\mathbb{Q}}_p^\times.
\]

We will let \( \epsilon_{\text{cyclo}} \) denote the character

\[
G_L = \text{Gal}(\overline{L}/L) \to \text{Gal}(L(N(\mu_p^\infty))/L) \simeq C_{L,Np^\infty} \hookrightarrow \mathcal{O}[C_{L,Np^\infty}]^\times = \Lambda_\mathcal{O}[\text{Tor}_{L,Np^\infty}]^\times.
\]

Note that \( q \mapsto S_q \) extends to \( S : C_{L,Np^\infty} \to h_0^\infty(N)^\times \) and then to \( S : \Lambda_\mathcal{O}[\text{Tor}_{L,Np^\infty}] \to h_0^\infty(N) \).

If \( \mathfrak{m} \) is a maximal ideal of \( h_0^\infty(N) \) with residue field \( k(\mathfrak{m}) \), there is (Taylor [39], Carayol [5], Wiles [42], Rogawski-Tunnell [29]) a continuous representation \( \overline{\mathfrak{f}}_\mathfrak{m} : G_L \to GL_2(k(\mathfrak{m})) \) such that, for all prime ideals \( \mathfrak{q} \) not dividing \( Np \), the representation is unramified at \( \mathfrak{q} \) and \( \text{tr}\overline{\mathfrak{f}}_\mathfrak{m}(\text{Frob}_\mathfrak{q}) = T_q \).

For a finite extension \( \mathcal{O}_L \) of the field of fractions of \( \Lambda_\mathcal{O} \) and the integral closure \( \mathcal{O}_L \) of \( \Lambda_\mathcal{O} \) in \( L \) and for a \( \Lambda_\mathcal{O} \)-algebra homomorphism \( F_\text{Hida} : h_0^\infty(N) \to \mathcal{O}_L \), which we often call a \( \Lambda \)-adic eigenform, if the unique maximal ideal \( \mathfrak{m} \subset h_0^\infty(N) \) above \( \ker F_\text{Hida} \) is non-Eisenstein, i.e., \( \overline{\mathfrak{f}}_\mathfrak{m} \) is absolutely irreducible, it follows from results of Nyssen [23] and Rouquier [30] that there is a continuous representation

\[
\rho_{F_\text{Hida}} : G_L \to GL_2(h_0^\infty(N)_\mathfrak{m}) \xrightarrow{F_\text{Hida}} GL_2(\mathcal{O}_L),
\]

which is unramified for all primes \( q \nmid Np \) and satisfies \( \text{tr}\rho_{F_\text{Hida}}(\text{Frob}_\mathfrak{q}) = T_q \) and \( \det \rho_{F_\text{Hida}} = S \circ \epsilon_{\text{cyclo}} \). Moreover, it follow from [32] that, for every \( p \mid p \),

\[
\rho_{F_\text{Hida}}|_{D_p} \sim \begin{pmatrix} \chi_{p,2} & * \\ 0 & \chi_{p,1} \end{pmatrix}
\]

such that \( \chi_{p,1}\chi_{p,2} = ((F_\text{Hida} \circ S) \circ \epsilon_{\text{cyclo}})|_{D_p} \) where \( \chi_{p,1} \) is, in particular, the unramified character of the decomposition group \( D_p \) at \( p \) sending \( \text{Frob}_p \) to \( F(U_p) \).
5.2. $\Lambda$-adic companion forms. Let $F$ be a totally real field with ring of integers $\mathcal{O}_F$. Let $S_F$ be the set of prime ideals of $\mathcal{O}_F$ above $p$. Assume that $p$ is unramified in $F$. In the following, we construct a finite totally real soluble extension $L$ of $F$ and $2^{[S_L]}$ overconvergent Hilbert modular forms on $Res_{L/\mathbb{Q}}GL_{2,L}$ whose various twists by characters of finite order give rise to the Galois representation in question.

**Theorem 5.3.** Let $p > 2$ be a rational prime. Let $K$ be a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}_K$ and maximal ideal $\mathfrak{m}_K$. Suppose that

$$\rho : G_F = \text{Gal}(\overline{\mathbb{Q}}/F) \rightarrow GL_2(\mathcal{O}_K)$$

is a continuous representation satisfying

- $\rho$ is ramified at only finitely many primes;
- $\bar{\rho} = (\rho \mod \mathfrak{m}_K)$ has a modular lifting which is potentially ordinary and potentially Barsotti-Tate at every prime of $F$ above $p$, and $\bar{\rho}$ is absolutely irreducible when restricted to $\text{Gal}(\overline{\mathbb{F}}/F(\zeta_p))$;
- for every prime $p|p$, $\rho|_{D_p}$ is the direct sum of 1-dimensional characters $\alpha_p$ and $\beta_p : D_p \rightarrow \mathcal{O}_K^\times$ such that $\alpha_p(I_p)$ and $\beta_p(I_p)$ are finite, and $\alpha_p$ and $\beta_p$ are distinct mod $\mathfrak{m}_K$.

Then, we may find

- a finite totally real soluble extension $L$ of $F$ in which every prime in $\mathcal{O}_F$ above $p$ split completely;
- an ideal $N$ of $\mathcal{O}_L$ coprime to $p$ such that $\rho|_{\text{Gal}(\overline{\mathbb{Q}}/L)}$ is unramified outside the primes of $L$ dividing $N$;
- for every subset $T$ of $S = S_L$,
  - a character $\chi_T : \text{Gal}(\overline{\mathbb{Q}}/L) \rightarrow \mathcal{O}_F^\times$ of conductor a finite power of $p$ such that $\chi_T|_{D_p} = \chi|_{D_p}$ for $p \in T$ and $\chi_T|_{D_p} = \chi|_{D_p}$ for $p \in T^c$,
  - a $\Lambda$-adic form $F_{\text{Hida},T} : h^0_{\mathcal{O}_K}(N) \rightarrow \Lambda_{\mathcal{O}_K}$ of level $N$,
  - a homomorphism $f_T : h^0_{\mathcal{O}_K}(N) \rightarrow \mathcal{O}_K$ satisfying
    - $f_T(T_l) = \text{tr} \rho(Frob_l)/\chi_T(Frob_l)$ for all $l$ not dividing $N(p)$;
    - $f_T(S_l) = \text{det} \rho(Frob_l)/\chi_T(Frob_l)^2$ for all $l$ not dividing $N(p)$;
    - $f_T(U_l) = 0$ for $l$ dividing $N$;
    - $f_T(U_p) = (\alpha_p/\chi_T)(Frob_p)$ for every $p \in T$ and $f_T(U_p) = (\beta_p/\chi_T)(Frob_p)$ for every $p \in T^c$

**Proof.** By class-field theory, we can and will choose a finite totally real field extension $L$ of $F$ of even degree in which every prime in $\mathcal{O}_F$ above $p$ splits completely in $L$, and a finite set $S$ containing both the set of infinite places of $L$ and the set $S_L$ of primes of $L$ above $p$ as in the section 4 of [33], satisfying the following conditions:

- $\rho$ is unramified outside $S$ and, for every finite prime $q \in S - S_L$, the image by $\rho$ of the inertia subgroup at $q$ is unipotent;
- there exists a cuspidal automorphic representation $\pi$ of $GL_2(k_L)$ such that $\pi_q$, for every finite prime $q \in S - S_L$, is a special representation of $GL_2(L_q)$ of conductor $q$ and such that its associated Galois representation $\rho_q$ of $G_L = \text{Gal}(\overline{\mathbb{Q}}/L)$ defines
a modular lifting of $\overline{\rho}$ which is potentially ordinary and potentially Barsotti-Tate at every prime of $L$ above $p$.

Let $N_S$ denote $\prod_{q \in S, q \not{\equiv} 0 \pmod{p}} q$. Define the characters $\chi_T : G_L \to \mathcal{O}_K^\times$ of finite order by $\chi_T|_{I_p} = \alpha_p|_{I_p}$ for $p \in T$ and $\chi_T|_{I_p} = \beta_p|_{I_p}$ and $p \in T^c$. This follows immediately from class-field theory; more precisely a theorem of Chevalley [6] (See also Lemma 2.1 in [10]). In fact we may choose $\chi_T$ inductively on the size of $T^c$ so that, for every subset $T$ of $S$, $\chi_T|_{D_p} = \chi_S|_{D_p}$ for $p \in T$ and $\chi_T|_{D_p} = \chi_0|_{D_p}$ for $p \in T^c$; furthermore $\chi_T$, when restricted to $D_l$ for $l$ not dividing $N_p$, is made independent of $T$.

Let $\rho_T$ denote

$$\rho \otimes_{G_L} \chi_T^{-1} : G_L \to GL_2(\mathcal{O}_K)$$

and $\overline{\rho}_T$ denotes its reduction

$$G_L \to GL_2(\mathcal{O}_K/\mathfrak{m}_K)$$

mod $\mathfrak{m}_K$. For every subset $T \subseteq S$, $\rho_T$ is ordinary at every prime $p$ of $L$ above $p$, and $\rho_T$ is ‘strongly residually modular’ in the sense of Kisin (3.5.4 in [21]). Indeed, first observe that there exists a nearly ordinary automorphic representation $\pi_T$ of $GL_2(\mathbb{A}_L)$ whose associated $p$-adic Galois representation, as constructed by Hida and Wiles, is $\rho_x \otimes \chi_T$ and whose associated mod $p$ Galois representation is $\overline{\rho}_T$. If $\pi_T$ is principal series (resp., special) at a prime $p$ of $L$ above $p$, then the Fontaine-Laffaille theory (resp., Mokrane’s extension [22] to the ‘semi-stable case’ of the Fontaine-Laffaille theory) proves that $\pi_T$ is indeed ordinary at $p$. It follows from Hida theory (by which we specialise the Hida $p$-ordinary family of $\pi_T$ at level $N_S p$ and weight 2) together with the main theorem of Jarvis [15] about lowering levels of mod $p$ Hilbert modular forms (in the case when the specialisation is special at $p|p$), and the main theorem of Gee [11] that there is a homomorphism

$$\overline{f}_T : h_{2,\mathcal{O}_K}(N_S) \to \mathcal{O}_K/\mathfrak{m}_K,$$

which is ‘$p$-old’, such that

- $\overline{f}_T(l_1) = \text{tr}(\text{Frob}_l)/\chi_T(\text{Frob}_q)$ for $l \nmid N_S p$,
- $\overline{f}_T(l_1) = \det(\text{Frob}_l)/\chi_T(\text{Frob}_q)^2$ for $l \nmid N_S p$,
- and $\overline{f}_T(U_p) = (\alpha_p/\chi_0)(\text{Frob}_p)$ for $p \in T$ and $\overline{f}_T(U_p) = (\beta_p/\chi_0)(\text{Frob}_p)$ for $p \in T^c$.

We will let $\mathfrak{m}_{S,T}$ denote the kernel of this homomorphism. Then

$$\overline{\rho}_{m_{S,T}} : \text{Gal}(\overline{\mathbb{Q}}/L) \to GL_2(h^0_{\mathcal{O}_K}(N_S)\mathfrak{m}_{S,T}) \to GL_2(h^0_{\mathcal{O}_K}(N_S)/\mathfrak{m}_{S,T}) \hookrightarrow GL_2(\mathcal{O}_K/\mathfrak{m}_K)$$

is isomorphic to $\overline{\rho}_T$. It follows from results in [33] that there is a finite extension $\mathcal{L}$ of the field of fractions of $\Lambda_{\mathcal{O}_K}$, a $\Lambda$-adic “Hida family” $F_{\text{Hida},T} : h^0_{\mathcal{O}_K}(N_S) \to \mathcal{L}$, and a height one prime $\mathfrak{o}_T$ of $\mathcal{O}_\mathcal{L}$, defined by $\rho_T$ lifting $\overline{\rho}_T$, such that the reduction mod $\mathfrak{o}_T$ of the Galois representation

$$\text{Gal}(\overline{\mathbb{Q}}/L) \to GL_2(h^0_{\mathcal{O}_K}(N_S)\mathfrak{m}_{S,T}) \xrightarrow{F_{\text{Hida},T}} GL_2(\mathcal{O}_\mathcal{L}) \to GL_2(\mathcal{O}_\mathcal{L}/\mathfrak{o}_T)$$

is isomorphic to $\rho_T$. Let $f_T$ be the composite $h^0_{\mathcal{O}_K}(N_S) \to \mathcal{L} \xrightarrow{\mathfrak{o}_T} \mathcal{O}_K$. By increasing the level at $l|N_S$ if necessary to ensure that the image of $U_l \in h^0_{\mathcal{O}_K}(N)$ is 0.

\[\square\]
6. Modularity of Artin representations

We shall firstly prove that the $f_T$'s constructed above define overconvergent Hilbert modular forms.

Let $K$ be a finite extension of $\mathbb{Q}_p$ with ring $\mathcal{O}$ of integers. Let $S^{p, \text{adic}}_k(U_1(N); K)$ denote the completion of $\lim_{n \to \infty} S_k(U_1(Np^n); K)$ with respect to the norm given by $\|f\| = \sup |c(n, f)|$. By Theorem 11.11 in [11], it may be identified with the subspace of cuspidal “convergent” (as opposed to overconvergent) forms in $S^1_k(\Gamma_1(N); K)$ whose $q$-expansions are invariant under the action of $\mathcal{O}_{L, 1, N}^{\times, +}/(\mathcal{O}_{L, 1, N})^2$ and therefore

$$S_k(U_1(Np^n); K) \subset S^1_k(\Gamma_1(N); K)\mathcal{O}_{L, 1, N}^{\times, +}/(\mathcal{O}_{L, 1, N})^2 \subset S^{p, \text{adic}}_k(U_1(N); K).$$

The operators $T_q$ for $q \nmid Np$, $U_q$ for $q \nmid N$, $U_p$ for $p|p$, and the Diamond operator naturally act on these spaces (See (3.5) in [14] for example). In particular, the $K$-vector subspace of $S_k(U_1(Np^n); K)$ of forms with a ray class character $\psi$ mod $Np^n$ is thought of as a subspace of classical, and therefore overconvergent, cusp forms of level $\Gamma_1(Np^n)$ with the character the restriction to $(\mathcal{O}_L/Np^n)^\times$ of $\psi$.

Hida [14] shows that his idempotent $e$ is also defined on $S^{p, \text{adic}}_k(U_1(N); K)$ and that, given a $\Lambda$-adic eingenform $F_{\text{Hida}} : h^0_\mathfrak{o}(N) \to \mathcal{O}_L$ as above, for any height one prime $\mathfrak{p}$ of $\mathcal{O}_L$ such that $\mathcal{O}_L/\mathfrak{p} \hookrightarrow \mathcal{O}$, $(F_{\text{Hida}} \mod \mathfrak{p})$ defines an eigenform in $eS^{p, \text{adic}}_k(U_1(N); K)$ with its eigenvalue for a Hecke operator its image by $F_{\text{Hida}} \mod \mathfrak{p}$.

Conversely, for any eigenform $f$ in $S_k(U_1(Np^n); \mathcal{O})$, $k \geq 2$, with character $\psi : C_{N(p^n)} \to \mathbb{Z}_p[\psi]^\times \subset \mathcal{O}^\times$, there exist a finite extension $L$ of the field of fractions of $\Lambda_\mathcal{O}$ with the integral closure $\mathcal{O}_L$ of $\Lambda_\mathcal{O}$ in $L$; a $\Lambda_\mathcal{O}$-algebra homomorphism

$$F_{\text{Hida}} : h^0_\mathfrak{o}(N) \to \mathcal{O}_L;$$

and a height one prime $\mathfrak{p} \subset \mathcal{O}_L$ whose restriction to $\Lambda_\mathcal{O}$ is $\ker ((\epsilon|_{F_{\text{Hida}}N_{\mathfrak{p}^\infty}})^{k-2}(\chi|_{F_{\text{Hida}}N_{\mathfrak{p}^\infty}}))$ for a character $\chi : C_{N(p^n)} \to \mathcal{T}_p^{\times}$ which factors through $\psi : C_{N(p^n)} \to \mathcal{O}^\times$, such that, there exist embeddings $\mathcal{O}_L/\mathfrak{p} \hookrightarrow \mathcal{T}_p$ and $\mathcal{O} \to \mathcal{T}_p$, and $(F_{\text{Hida}} \mod \mathfrak{p})$ defines $f$ up to a constant in $\mathcal{T}_p$.

By the theory of canonical subgroups, the operator

$$U_{(p)} = \prod_{p \mid (N)} U_p : S_k^1(\Gamma_1(N); K) \to S_k^1(\Gamma_1(N); K)$$

is completely continuous. By Serre’s theory [35], there is an idempotent $e^\dagger$ commuting with $U_{(p)}$ and we may write $S_k^1(\Gamma_1(N); K) = e^\dagger S_k^1(\Gamma_1(N); K) + (1 - e^\dagger) S_k^1(\Gamma_1(N); K)$ where $e^\dagger S_k^1(\Gamma_1(N); K)$ is finite dimensional and all the (generalised) eigenvalues of $U_{(p)}$ are units, while $U_{(p)}$ is topologically nilpotent on $(1 - e^\dagger) S_k^1(\Gamma_1(N); K)$. We see that $e^\dagger = e|_{S_k^1(\Gamma_1(N); K)}$.

**Lemma 6.1.** $eS^{p, \text{adic}}_k(U_1(N); K) \subset eS^1_k(\Gamma_1(N); K)\mathcal{O}_{L, 1, N}^{\times, +}/(\mathcal{O}_{L, 1, N})^2$
Proof. See [33] □

It follows from the $R = T$ theorem in [33] and the lemma that, given a $p$-adic Galois representation $\rho : G_F \to GL_2(\mathbb{O}_K)$ as in the main theorem, there exists a finite totally real soluble extension $L$ of $F$ in which every prime of $\mathcal{O}_L$ above $p$ splits completely (in particular $p$ is unramified in $L$), a finite set $S$ of places of $L$ containing the places above $p$ and the infinite places, and $2^{|S_L|}$ overconvergent Hilbert eigenforms $\{f_T\}_{T \subseteq S}$ of weight one and of level $N$ (divisible by the conductor of $\rho|_{G_L}$) such that, for every subset $T \subseteq S$, the Galois representation associated to $f_T$ is $\rho_T = \rho|_{G_L} \otimes_{G_L} \chi_T^{-1}$, and $f_T$ and $f_{T'}$ are “in companion”. They are specialisations to weight one of $\Lambda$-adic companion forms $F_{\text{Hida},T}$, and satisfy

- $c(\mathcal{O}_K, f_T) = c(\mathcal{O}_K, f_{T'}) = 1$, and $c(n, f_T) = c(n, f_{T'}) = 0$ if $n$ is not coprime to $N$,
- $c(l, f_T) = \text{tr}(\text{Frob}_l) / \chi_T(\text{Frob}_l)$ and $c(l, f_{T'}) = \text{tr}(\text{Frob}_l) / \chi_T(\text{Frob}_l)$ for a prime $l$ not dividing $Np$,
- $c(n, f_T)(\chi_T \circ \text{Art})(n) = c(n, f_{T'})(\chi_T \circ \text{Art})(n)$ for an ideal $n$ coprime to $Np$,
- for every $p | p$, the character of $f_T$ at $p$ is $\chi_T / \chi_T$ and the character of $f_{T'}$ at $p$ is $\chi_T / \chi_T$,
- the characters $f_T$ and $f_{T'}$ away from $(p)$ are equal (and, since it is independent of $T$, we shall denote it by $\chi_N$),
- for every $p | p$, $c(p, f_T) = (\alpha_p / \chi_T)(\text{Frob}_p)$, $c(p, f_{T'}) = (\beta_p / \chi_T)(\text{Frob}_p)$. We also have $c(p, f_T)c(p, f_{T'}) = \chi_N(p)$.

**Theorem 6.2.** Let $F$ be a totally real field in which 5 is unramified. Let $\rho : G_F = \text{Gal}(\overline{\mathbb{Q}}/F) \to GL_2(\mathbb{C})$ be a totally odd and continuous representation satisfying the following conditions:

- $\rho$ has the projective image $A_5$.
- For every place $\mathfrak{p}$ of $F$ above 5, the projective image of the decomposition group $D_\mathfrak{p}$ at $\mathfrak{p}$ has order 2. Furthermore, the quadratic extension of $F_\mathfrak{p}$ fixed by the kernel of $D_\mathfrak{p} \hookrightarrow G_F \overset{\text{proj}}{\to} A_5$ is not $F_\mathfrak{p}(\sqrt{5})$.

Then, there exists a holomorphic Hilbert cuspidal eigenform $f$ of weight 1 such that $\rho$ arises from $f$ in the sense of Rogawski-Tunnell, and the Artin L-function $L(\rho, s)$ is entire.

**Proof.** Fix an isomorphism $\overline{\mathbb{Q}}_5 \simeq \mathbb{C}$. Then, by Theorems 4.3 and 5.3 there is a finite soluble totally real field extension $L$ of $F$, and $2^{|S_L|}$ overconvergent cusp eigenforms $\{f_T\}_{T \subseteq S}$ whose associated Galois representations are twists of $\rho|_{\text{Gal}(\overline{\mathbb{Q}}/L)}$ by characters of finite orders. By Theorem 3.13 they are indeed classical cusp eigenforms. By automorphic descent, they give rise to $f$. □

**References**


[34] S. Sasaki, On Artin representations and nearly ordinary Hecke algebras over totally real fields II, pre-print.