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Double-Scaling Limit of the $O(N)$ -Symmetric Anharmonic Oscillator

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Abstract. In an earlier paper it was argued that the conventional double-scaling limit of an $O(N)$ -symmetric quartic quantum field theory is inconsistent because the critical coupling constant is negative and thus the integral representing the partition function of the critical theory does not exist. In this earlier paper it was shown that for an $O(N)$ -symmetric quantum field theory in zero-dimensional spacetime one can avoid this difficulty if one replaces the original quartic theory by its \mathcal{PT} -symmetric analog. In the current paper an $O(N)$ -symmetric quartic quantum field theory in one-dimensional spacetime [that is, $O(N)$ -symmetric quantum mechanics] is studied using the Schrödinger equation. It is shown that the global \mathcal{PT} -symmetric formulation of this differential equation provides a consistent way to perform the double-scaling limit of the $O(N)$ -symmetric anharmonic oscillator. The physical nature of the critical behavior is explained by studying the \mathcal{PT} -symmetric quantum theory and the corresponding and equivalent Hermitian quantum theory.

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1. Introduction

Typically, the double-scaling limit of a quantum field theory is a correlated limit characterized by a *universal* function of one parameter γ , which is a combination of the original parameters in the Hamiltonian or Lagrangian [1]. This universal function, which is *entire* (analytic for all γ), reveals the essential features of the theory while being insensitive to specific details. In quantum field theory the correlated limit of an $O(N)$ -symmetric vector model represents a discretized branched polymer [2].

This paper examines the double-scaling limit of an $O(N)$ -symmetric quartic quantum field theory in one-dimensional spacetime. As in our previous paper on the double-scaling limit in zero-dimensional spacetime [3], we argue here that the conventional double-scaling limit in one-dimensional spacetime is inconsistent because

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the critical value of the coupling constant is negative. We then show that the double-scaling limit *is* consistent for the associated \mathcal{PT} -symmetric quantum field theory. However, we do not follow the procedures used in Ref. [3] because this would require the introduction of a functional-integral representation for the partition function, which is an infinite-dimensional integral. Analyzing Stokes wedges in this context is complicated and difficult. Fortunately, the one-dimensional quantum field theory is equivalent to an $O(N)$ -symmetric quantum-mechanical anharmonic oscillator, and thus the theory is described by a Schrödinger equation [4]. It is easier and more physically transparent to study the properties of a wave function than the boundary conditions and convergence of an infinite-dimensional integral.

The conventional coordinate-space anharmonic-oscillator Hamiltonian is

$$H = - \sum_{j=1}^{N+1} \frac{\partial^2}{\partial x_j^2} + \frac{\mu^2}{2} \sum_{j=1}^{N+1} x_j^2 + \frac{\lambda}{4} \left(\sum_{j=1}^{N+1} x_j^2 \right)^2. \quad (1)$$

To obtain the double-scaling limit of this theory, we begin by constructing the uncorrelated large- N expansion in powers of $1/N$. This expansion is a divergent asymptotic series in powers of $1/N$ and has the general form [5]

$$\sum_{k=0}^{\infty} a_k N^{-k}, \quad (2)$$

where the coefficients a_k have a nontrivial dependence on the parameters μ and λ in the Hamiltonian. Only the first few terms in the $1/N$ expansion are calculable with reasonable effort.

To study the Hamiltonian (1) for $N \gg 1$, we rewrite the time-independent Schrödinger equation $H\psi = E\psi$ in polar coordinates by substituting $\sum_{j=1}^{N+1} x_j^2 = Nr^2$ (see Ref. [6]) and we let $\lambda = g/N$ and $\mathcal{E} \equiv E/N$. We then seek a spherically-symmetric solution $\psi(r)$, which obeys the differential equation

$$- \frac{1}{N^2} \psi''(r) - \frac{1}{Nr} \psi'(r) + \frac{\mu^2}{2} r^2 \psi(r) + \frac{g}{4} r^4 \psi(r) = \mathcal{E} \psi(r). \quad (3)$$

We can convert (3) to the form of a radial Schrödinger equation by making the change of variable $\Phi(r) = r^{N/2} \psi(r)$. The resulting Schrödinger equation has the form

$$- \frac{1}{N^2} \Phi''(r) + V(r) \Phi(r) = \mathcal{E} \Phi(r), \quad (4)$$

where for large N

$$V(r) = \frac{1}{4r^2} + \frac{\mu^2}{2} r^2 + \frac{g}{4} r^4. \quad (5)$$

(Originally, the coefficient of r^{-2} in the potential is $\frac{1}{4} - \frac{1}{2N}$, so for large N we replace this coefficient by $\frac{1}{4}$.) Note that $1/N$ plays the role of \hbar in the Schrödinger equation (4) and thus the large- N expansion (2) is just the standard semiclassical (WKB) expansion.

To find the double-scaling limit of the WKB expansion we must take the limits $N \rightarrow \infty$ and $g \rightarrow g_{\text{crit}}$ in a *correlated* fashion. In this limit the WKB series (2) undergoes a transmutation in which all terms become of comparable size; the series is no longer

dominated by early terms and N^{-k} in the k th term is balanced by a large coefficient a_k [1]. In this correlated limit the perturbation series still diverges. The question is whether we can apply a summation procedure such as Borel summation to the series. In Ref. [3] we found that in the double-scaling limit the series was *not* Borel summable because the terms did not alternate in sign, and the lack of Borel summability was the indication that the conventional quartic $O(N)$ -symmetric theory did not in fact have a double-scaling limit.

We emphasize that the solution to the Schrödinger equation (4) is required to be normalizable and thus it must vanish as $r \rightarrow \infty$. However, the correlated limit is defined by the pair of equations [7]

$$V'(r) = -\frac{2}{r^3} + \mu^2 r + gr^3 = 0 \quad \text{and} \quad V''(r) = -\frac{6}{r^4} + \mu^2 + 3gr^2 = 0. \quad (6)$$

Solving these equations simultaneously, we find that the critical value of g is *negative*:

$$g_{\text{crit}} = -(2/3)^{3/2} \mu^3 \approx -0.544331 \mu^3. \quad (7)$$

This result reveals the problem with taking the double-scaling limit of the quantum theory whose potential $V(r)$ is given in (5). When g is positive, $V(r)$ confines bound states and the eigenfunction is normalizable (Fig. 1, left panel). However, as g moves downward toward its critical value (7), the potential turns over as $r \rightarrow \infty$. When g becomes negative, the quartic term in $V(r)$ allows particles to tunnel out to $r = \infty$. The states of the theory become quasi-stable (Fig. 1, right panel) and the solutions to the Schrödinger equation (4) are not normalizable. As g continues to decrease towards its critical value, the potential barrier decreases in size (Fig. 2, left panel) and at the critical value the potential barrier *disappears entirely* (Fig. 2, right panel). Thus, there are no longer any quasi-stable states; quantum particles flow unimpeded and without tunneling out to infinity. This gives a physical picture of the transition that occurs at the double-scaling limit and demonstrates the physical problem with taking the double-scaling limit of the conventional Hermitian $O(N)$ -symmetric anharmonic oscillator.

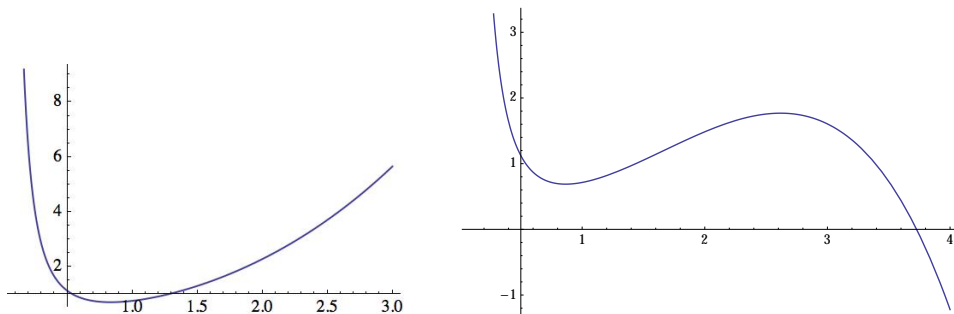


Figure 1. Plot of the potential $V(r)$ in (5) for $\mu = 1$ and for two values of g . Left panel: $g = g_{\text{crit}} + 0.6$, where $g_{\text{crit}} = -0.544331$; the potential rises as $r \rightarrow \infty$ and thus it confines bound states. Right panel: $g = g_{\text{crit}} + 0.4$; the potential falls as $r \rightarrow \infty$, so particles tunnel out to ∞ and states become quasi-stable.

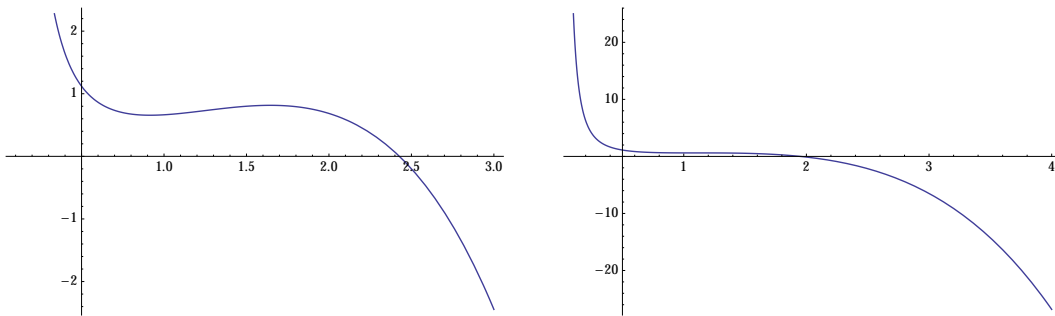


Figure 2. The potential $V(r)$ in (5) for $\mu = 1$ and for two values of g . Left panel: $g = g_{\text{crit}} + 0.2$, where $g_{\text{crit}} = -0.544331$; for this value of g the potential just barely has quasi-stable states. Right panel: $g = g_{\text{crit}}$; particles move unimpeded off to $r = \infty$ without tunneling. Note that at the critical value of g the maximum and minimum of the potential coalesce into an inflection point.

We show in this paper that while the conventional $O(N)$ -symmetric anharmonic oscillator does not possess a double-scaling limit, the corresponding \mathcal{PT} -symmetric $O(N)$ -symmetric anharmonic oscillator *does* have a double-scaling limit. Unlike the conventional anharmonic oscillator, the \mathcal{PT} -symmetric anharmonic oscillator has a *negative* quartic term in the potential. However, it still possesses normalizable eigenfunctions because the Schrödinger equation for the \mathcal{PT} -symmetric theory is posed in the complex plane and is normalized in appropriate complex Stokes wedges. We show how to construct the \mathcal{PT} -symmetric anharmonic oscillator in Sec. 2. Then by constructing the equivalent Hermitian quantum theory in Sec. 3, we explain the physical transformation that occurs at the critical value of the coupling constant in the double-scaling limit. Concluding remarks are given in Sec. 4.

2. \mathcal{PT} -symmetric formulation of the $O(N)$ -symmetric anharmonic oscillator

Following the procedure for the case of the $O(N)$ -symmetric zero-dimensional quantum field theory in Ref. [3], we confront the problem of negative g_{crit} [see (7)] by considering the family of $O(N)$ -symmetric \mathcal{PT} -symmetric Hamiltonians

$$H = - \sum_{j=1}^{N+1} \frac{\partial^2}{\partial x_j^2} + \frac{1}{2} \sum_{j=1}^{N+1} x_j^2 + \frac{\lambda}{2 + \epsilon} \left(\sum_{j=1}^{N+1} x_j^2 \right) i^\epsilon \left(\sum_{j=1}^{N+1} x_j^2 \right)^{\epsilon/2}, \quad (8)$$

where $\epsilon \geq 0$ is a real parameter. At $\epsilon = 0$ this Hamiltonian describes an $(N + 1)$ -component harmonic-oscillator, but as ϵ approaches 2 the Hamiltonian becomes quartic and a negative quartic term arises naturally. As in Sec. 1 (and as in Ref. [3]), we introduce polar coordinates $\sum_{j=1}^{N+1} x_j^2 = Nr^2$ and let $g = \lambda N^{\epsilon/2}$. The spherically symmetric solution $\Phi(r) = r^{N/2} \psi(r)$ satisfies the differential-equation eigenvalue problem

$$- \frac{1}{N^2} \Phi''(r) + V_\epsilon(r) \Phi(r) = \mathcal{E} \Phi(r), \quad (9)$$

where \mathcal{E} is the eigenvalue and

$$V_\epsilon(r) \equiv \left(\frac{1}{4} - \frac{1}{2N} \right) \frac{1}{r^2} + \frac{1}{2} r^2 + \frac{g}{2+\epsilon} r^2 (ir)^\epsilon. \quad (10)$$

Let us analyze the differential equation (9) Rlocally near the origin by using Frobenius theory. In the vicinity of $r = 0$ this differential equation is approximated by the equidimensional equation

$$-\frac{1}{N^2} \Phi''(r) + \left(\frac{1}{4} - \frac{1}{2N} \right) \frac{1}{r^2} \Phi(r) = 0. \quad (11)$$

As $r \rightarrow 0$, $\Phi(r)$ is approximated by r^α , where α is the Frobenius index. The possible values of α are $N/2$ and $1 - N/2$. Thus, there is one solution that is finite at the origin and another that is divergent. We then make the assumption that N is even so that $r^{N/2}$ is single-valued at the origin. This allows us to extend the range of r from the positive- r axis to the whole real- x axis. The case of even N is special because the parity reflection operator in $(N + 1)$ -dimensional space, which has the effect of changing the sign of all spatial components of the coordinate vector, is distinct from a rotation. [When N is odd, parity reflection in $(N + 1)$ -dimensional space is just a rotation and thus there is no distinct parity operator.] The existence of a distinct \mathcal{P} operator is crucial in a \mathcal{PT} -symmetric theory. (It should be noted that a similar result regarding the dimension of space, namely, that N is odd, was found in Ref. [3] using a different argument.)

We then let ϵ approach 2 in order to obtain the quartic (anharmonic) theory. In this limit the boundary conditions on the differential equation (9) extended from r space to x space, which are imposed in Stokes wedges that rotate downward off the real- x axis and into the complex- x plane, are treated as explained in Ref. [8]. This procedure defines a \mathcal{PT} -symmetric quantum theory that has a positive real spectrum. In the next section we demonstrate rigorously the positivity of the spectrum at $\epsilon = 2$ by constructing an exact equivalent Hermitian quantum theory.

3. Construction of an equivalent Hermitian theory

We will now prove that when the positive variable r is replaced by the complex variable x , the \mathcal{PT} -symmetric Schrödinger equation (9) has a positive real spectrum when $\epsilon = 2$. In doing so we will understand in physical terms how to interpret the critical behavior in the double-scaling limit.

We begin with the large- N \mathcal{PT} -symmetric Schrödinger equation

$$-\frac{1}{N^2} \psi''(x) + \left(\frac{1}{4x^2} + \frac{1}{2} x^2 - \frac{g}{4} x^4 \right) \psi(x) = \mathcal{E} \psi(x) \quad (12)$$

and we substitute $\psi(x) = x^\beta \phi(x)$. Hence, $\psi''(x) = \beta(\beta - 1)x^{\beta-2}\phi(x) + 2\beta x^{\beta-1}\phi'(x) + x^\beta \phi''(x)$. Then, if we let $\beta(\beta - 1) = N^2/4$, we obtain

$$\left(-\frac{1}{N^2} \frac{d^2}{dx^2} - \frac{2\beta}{N^2 x} \frac{d}{dx} + \frac{1}{2} x^2 - \frac{g}{4} x^4 \right) \phi(x) = \mathcal{E} \phi(x). \quad (13)$$

Next, generalizing the work of Ref. [9] for transforming a quartic theory with a negative coupling constant to one with a positive coupling constant, we substitute $x = -2i\sqrt{1+it}$. Note that as t ranges along the real- t axis from $t = -\infty$ to $t = \infty$, x traces a path in the complex- x plane that originates and terminates in both Stokes wedges. This transformation gives

$$\frac{d}{dx} = \frac{ix}{2} \frac{d}{dt} \quad \text{and} \quad \frac{d^2}{dx^2} = \frac{i}{2} \frac{d}{dt} + (1+it) \frac{d^2}{dt^2},$$

and we obtain the following differential equation on the real- t axis:

$$-\frac{1+it}{N^2} \phi''(t) - i \frac{1+2\beta}{2N^2} \phi'(t) - [2(1+it) + 4g(1+2it-t^2)] \phi(t) = \mathcal{E} \phi(t).$$

We then take a Fourier transform of this equation according to $\tilde{\phi}(s) \equiv \int_{-\infty}^{\infty} dt e^{its} \phi(t)$. The resulting equation for $\tilde{\phi}$ is

$$\left(\frac{3-2\beta}{2N^2} s + \frac{1}{N^2} s^2 + \frac{1}{N^2} s^2 \frac{d}{ds} - 2 - 2 \frac{d}{ds} - 4g - 8g \frac{d}{ds} - 4g \frac{d^2}{ds^2} \right) \tilde{\phi}(s) = \mathcal{E} \tilde{\phi}(s).$$

Finally, we eliminate the one-derivative term and convert to an equation of Schrödinger type. To do so, we substitute $\tilde{\phi}(s) = A(s)\chi(s)$, which gives

$$\tilde{\phi}'(s) = A'(s)\chi(s) + A(s)\chi'(s) \quad \text{and} \quad \tilde{\phi}''(s) = A''(s)\chi(s) + 2A'(s)\chi'(s) + A(s)\chi''(s).$$

Demanding that the $\chi'(s)$ terms drop out gives the following condition on $A(s)$:

$$A'(s) = \left(\frac{s^2}{8N^2g} - \frac{1}{4g} - 1 \right) A(s). \quad (14)$$

The equation for $\chi(s)$ then reduces to

$$-4g\chi''(s) + \left(\frac{1-2\beta}{2N^2} s + \frac{1}{16gN^4} s^4 - \frac{1}{4gN^2} s^2 + \frac{1}{4g} \right) \chi(s) = \mathcal{E}\chi(s). \quad (15)$$

Making the replacement $s \rightarrow sN$, we get

$$-\frac{4g}{N^2} \chi''(s) + \left[\frac{1-2\beta}{2N} s + \frac{1}{4g} \left(\frac{1}{2} s^2 - 1 \right)^2 \right] \chi(s) = \mathcal{E}\chi(s) \quad (16)$$

and for large N , $\beta = N/2$. The derivative operator on the left side of this equation is the equivalent (isospectral) Hermitian Hamiltonian, which has a *positive* quartic term, and we identify the effective potential in the large- N limit as

$$V(s) = -\frac{s}{2} + \frac{1}{4g} \left(\frac{1}{2} s^2 - 1 \right)^2. \quad (17)$$

The condition for criticality [7] is $V'(s) = V''(s) = 0$. This condition reproduces the result in (7) with the appropriate change in sign; namely, that $g_{\text{crit}} = (2/3)^{3/2} \approx 0.544331$. (See Figs. 3 and 4.)

Let us now examine the effective potential near the critical point and determine the universal function that arises in the double-scaling limit. To do so, we study the

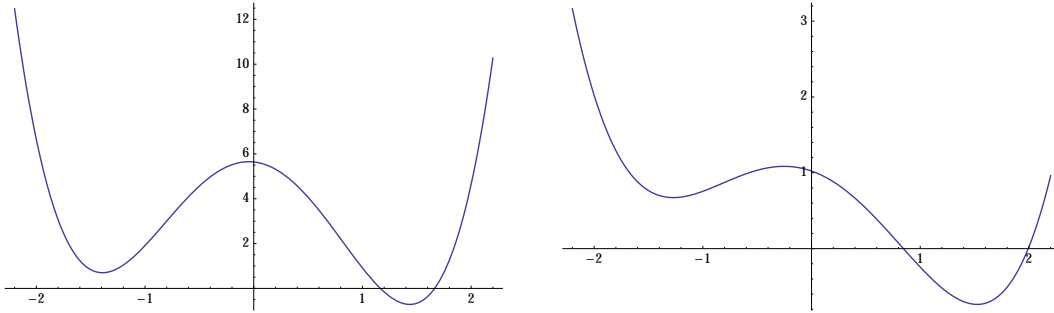


Figure 3. Left panel: $g = g_{\text{crit}} - 0.5$ where $g_{\text{crit}} = 0.544331$; this is a double well. Right panel: $g = g_{\text{crit}} - 0.3$ where $g_{\text{crit}} = 0.544331$; the barrier between the wells is disappearing.

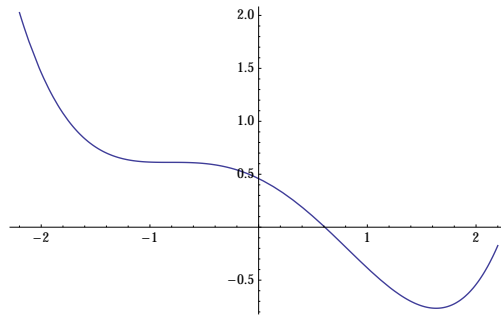


Figure 4. $g = g_{\text{crit}}$ where $g_{\text{crit}} = 0.544331$; the barrier between the wells is gone and there is just a shoulder. The double well has become a single well.

behavior of the potential $V(s)$ in (18) near the critical point $g = g_{\text{crit}}$. We let $s = s_{\text{crit}} + \Delta$ and $g = g_{\text{crit}} + G$ with G small and negative. For nonzero G the potential has two local minima at $s = s_1$ and $s = s_3$ and a local maximum at $s = s_2$. These extremal points satisfy $s_1 < s_2 < s_3$. At the critical point $g = g_{\text{crit}}$ the two extrema at $s = s_1$ and $s = s_2$ coalesce.

It is convenient to write $G = -\delta^2$, where $\delta \ll 1$. Away from criticality the positions of the extrema are determined by the condition $\frac{\partial}{\partial s} V(s_{\text{crit}} + \Delta, g_{\text{crit}} - \delta^2) = 0$. Hence,

$$2\delta^2 + \Delta^2 \left(-\sqrt{6} + \Delta \right) = 0. \quad (18)$$

We seek a perturbative expansion for Δ in the form $\Delta = \sum_{n=1}^{\infty} a_n \delta^n$, where the coefficients a_n are real. Substituting this series for Δ into (18) allows us to calculate the coefficients $a_1 = \pm(2/3)^{1/4}$ and $a_2 = 1/6$. This gives

$$\Delta = \Delta_{\pm} = \pm \left(\frac{2}{3} \right)^{1/4} \delta + \frac{1}{6} \delta^2 + \dots$$

At the extrema $s_1 = s_{\text{crit}} + \Delta_-$ and $s_2 = s_{\text{crit}} + \Delta_+$ the potential takes the values

$$V(s_{\text{crit}} + \Delta_{\pm}) = \sqrt{\frac{3}{8}} + \frac{3}{8} \delta^2 \pm \frac{1}{2} \left(\frac{3}{2} \right)^{1/4} \delta^3 + \dots$$

The mean value of the potential between s_1 and s_2 is approximately $\sqrt{3/8} - 3G/8$. We take $\mathcal{E} = \sqrt{3/8} - 3G/8$ and, since we are studying the $N \rightarrow \infty$ limit, this value of \mathcal{E} represents a high-lying level. The turning points at this energy are roots of

$$V(s, g_{\text{crit}} + G) - \mathcal{E} = 0. \quad (19)$$

These roots have the form $s = s_{\text{crit}} + b_1\delta + b_2\delta^2 + b_3\delta^3 + \dots$. Thus, (19) has the form

$$\begin{aligned} & V(s_{\text{crit}} + b_1\delta + b_2\delta^2 + b_3\delta^3, g_{\text{crit}} - \delta^2) - \mathcal{E} \\ &= \frac{3}{8} \left(\sqrt{6}b_1 - b_1^3 \right) \delta^3 + \frac{3}{64} \left(6^{3/2} + \sqrt{6}b_1^4 + 8\sqrt{6}b_2 - 24b_1^2b_2 \right) \delta^4 + \dots \end{aligned}$$

From this equation we determine that the three possible values for b_1 are

$$b_1 = 6^{1/4}, 0, \text{ and } -6^{1/4}. \quad (20)$$

Since the formula for b_2 in terms of b_1 is

$$b_2 = -\frac{\sqrt{6}}{8} \frac{(6 + b_1^4)}{\sqrt{6} - 3b_1^2},$$

the values of b_2 corresponding to those for b_1 from (20) are

$$b_2 = 3/4, \quad -3/4, \text{ and } 3/4. \quad (21)$$

The resulting values for the turning points in s are $s_- < s_0 < s_+$, where

$$s_0 = s_{\text{crit}} + \frac{3}{4}G + \dots \quad \text{and} \quad s_{\pm} = s_{\text{crit}} - \frac{3}{4}G \pm 6^{1/4}(-G)^{1/2} + \dots$$

Near the critical point $V(s, g_{\text{crit}} + G) - \mathcal{E}$ can be approximated by $-\frac{3}{8}s(s^2 + 6^{1/2}G)$ to leading order in $(-G)^{1/2}$, where we have made a shift in the origin of s to s_{crit} . Hence, in the neighborhood of the critical point the Schrödinger equation reads

$$\frac{4g}{N^2} \frac{d^2\chi}{ds^2} + \frac{3}{8}s(s^2 + 6^{1/2}G)\chi = 0. \quad (22)$$

On making the change of variable $t = 6^{1/4}(-G)^{-1/2}s$, we obtain the Schrödinger equation valid near the critical point $g = g_{\text{crit}}$:

$$-\chi''(s) + \gamma s(1 - s^2)\chi(s) = 0, \quad (23)$$

where

$$\gamma = \frac{27}{64}6^{3/4}N^2(-G)^{5/2}. \quad (24)$$

Thus, the double-scaling limit is obtained by letting $G \rightarrow 0$ and $N \rightarrow \infty$ in such a way that $N(-G)^{5/4}$ is held fixed. Because the universal function $\chi(s)$ in this double-scaling limit satisfies the differential equation (24), which has no singular points in the finite complex- s plane, $\chi(s)$ is an entire function of s .

4. Discussion and conclusions

In this paper we have identified and clarified a problem with the double-scaling limit in a prototypical quantum field theory in one-dimensional spacetime; namely, that the critical coupling constant g_{crit} in the double-scaling limit is *negative*. We have argued that for a conventional Hermitian one-dimensional quartic theory the double-scaling limit does not exist because near g_{crit} the potential is upside-down and thus the wave function is not normalizable; particles are not confined and can tunnel out to $r = \infty$. However, we have shown that if we approach the critical theory in a \mathcal{PT} -symmetric fashion, the resulting double-scaling limit gives a physically acceptable quantum theory and this theory is equivalent (isospectral) to a conventional Hermitian quantum theory with a confining potential.

We are not interested in the complex quantum theory that one obtains by analytically continuing the coupling constant g in the potential $V = x^2 + gx^4$ from positive g to negative g in the complex- g plane. (The eigenvalues of the resulting Hamiltonian are complex. Worse yet, they are not unique because they are path dependent; one obtains different eigenvalues depending on whether one rotates around $g = 0$ in a clockwise or an anticlockwise direction [4].) Rather, our approach is to keep the coupling constant g fixed and to perform the \mathcal{PT} -symmetric limit of $x^2 + gx^2(ix)^\epsilon$ as ϵ goes from 0 to 2. The integration path lies on the real axis when $\epsilon = 0$ and the path rotates downward into the complex plane as ϵ increases. When $\epsilon = 2$, the contour comes inward from $x = \infty$ in the 60° Stokes wedge $-\pi < \arg x < -2\pi/3$ and goes back out to $x = \infty$ in the 60° Stokes wedge $-\pi/3 < \arg x < 0$. As a consequence, the eigenfunctions are normalizable. In this paper we have used $O(N)$ symmetry to perform the N -dimensional version of this \mathcal{PT} -symmetric analytic continuation, thus to obtain the quartic theory that is studied in this paper.

It is interesting that in zero-dimensional spacetime we found in Ref. [3] that the double-scaling limit is characterized by the asymptotic behavior $G \sim N^{-1/3}$, while in one-dimensional spacetime we have shown in this paper that $G \sim N^{-4/5}$. We do not know yet what happens in higher dimensional spacetime. Stokes wedges represent *global* constraints and understanding the structure of these wedges becomes very difficult when the dimension of spacetime increases [5]. Many studies of the double-scaling limit in field theory rely on applying formal saddle-point methods to functional integrals and these techniques involve only *local* analysis.

\mathcal{PT} -symmetric field theories, at least in low dimension, provide an arena in which the double-scaling limit can be performed consistently. We have used a different methodology in the one-dimensional spacetime case from the zero-dimensional spacetime case. To extend our work to D -dimensional spacetime with $D \geq 2$, we will need to develop even more powerful methods. The usual heuristic treatment of functional integrals is inadequate to discuss the global questions addressed in this paper. In future work we shall pursue two different research directions: (i) \mathcal{PT} -symmetric versions of matrix models [10], and (ii) Schwinger-Dyson equations for \mathcal{PT} -symmetric field theories

for $D \geq 2$ [11].

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