DOI:
10.1016/j.fss.2018.01.018

Document Version
Peer reviewed version

Membership-Dependent Stability Conditions for Type-1 and Interval Type-2 T-S Fuzzy Systems

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Abstract

This paper presents an idea to simplify and relax the stability conditions of Takagi-Sugeno (T-S) fuzzy systems based on the membership function extrema[1]. By considering the distribution of membership functions in a unified membership space, a graphical approach is provided to analyze the conservativeness of membership-dependent stability conditions. Membership function extrema are used to construct a simple and tighter convex polyhedron that encloses the membership trajectory and produces less conservative linear matrix inequality (LMI) conditions. The cases of both type-1 and interval type-2 T-S fuzzy systems are considered, and comparison with existing methods is made in the proposed membership vector framework.

Keywords: Stability analysis, T-S fuzzy system, membership function, convex polyhedron

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1. Introduction

Takagi-Sugeno (T-S) fuzzy model [2] is an efficient mathematical tool for the analyze and control of complex nonlinear systems [3, 4]. By this model,

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[1] Here *extrema* refers to the collective name of the largest and smallest value of a function [1].

the original nonlinear systems can be well represented by several local subsystems combined with their membership functions. Specially for the stability problem of nonlinear systems, it can be reduced to the stability analysis of an equivalent T-S fuzzy model. Basic stability conditions in terms of linear matrix inequalities (LMIs) can be derived based on the Lyapunov stability theory [5, 6]. In literature, lots of methods have been applied to reduce the analysis conservativeness [7], for example, the small scale gain theorem in [8], polynomial Lyapunov function and the sum-of-squares approach in [9], and piecewise smooth quadratic (PSQ) Lyapunov function approach in [10]. Most of these obtained results membership-independent. It means that, under given conditions, the T-S fuzzy systems can be always stable regardless of the exact layouts of their membership functions. However, such kind of conditions should be relatively conservative since the information in membership functions has not been considered properly.

An alternative approach on this topic is the membership-dependent stability analysis [11]. In [12, 13, 14], it has been shown that the information of membership functions can be used to reduce the conservativeness in stability analysis of T-S fuzzy systems. Following this approach, different ideas have been proposed to combine the membership information into stability analysis. Among them, one idea is to approximate the complex membership functions by some specific simple functions, for example, staircase function [15], piecewise linear function [16] and polynomial function [17, 18, 19]. In this way, the original complex membership functions can be symbolized by the specific parameters of those simple functions, making them easier to be applied. Another idea is to characterize the membership functions by their upper- (or lower-) bounds information [20], and combine the bounds information into the analysis derivation.

As an extension, the interval type-2 T-S fuzzy model in [21, 22, 23, 24] is proposed to describe a kind of fuzzy systems with uncertain grades of membership. This kind of model can be applied as gesture classifier for disorder recognition [22] and controller for network systems [25]. Generally, the type-2 fuzzy model achieves better system approximation owning to the additional degree of freedom provided by the footprint of uncertainty (FOU) in their type-2 fuzzy sets [24]. The stability analysis of this model is in general done by using boundary information of the lower and upper membership functions [5, 26], using piecewise linear membership functions and polynomial approximations [27]. Clearly the stability analysis of interval type-2 fuzzy system will not be a direct extension of the type-1 case, and both the lower and upper membership functions [5] need to be considered. Then how to find a common approach to consider the membership information for both type-1 and interval type-2 fuzzy system should be an interesting topic, and the result will also provide better solution for some practical applications.

Motivated by the above discussion, this paper provides an alternative approach to combine the membership information of T-S fuzzy systems into stability analysis. By analyzing the membership distribution in a unified membership space, it is easy to construct a convex boundary polyhedron of membership function simply based on the extrema information of each membership function [28]. And corresponding vertices of such a polyhedron can be used to obtain less conservative stability conditions. Compared with existing methods, this method shows advantage in reducing conservativeness and computational burden. In addition, there is no need to concern about the dimensions of membership functions, and the extension to interval type-2 T-S fuzzy systems will be straightforward. Main contribution of this research can be summarized as: 1) Using the membership vector space to compare the conservativeness of different membership-dependent LMI stability conditions. 2) Providing a method to generate less conservative membership dependent LMI stability conditions simply by the extrema values of membership function.

The rest of this paper is organized as follows. In Section 2, some existing membership-dependent stability analysis methods are reviewed, and an alternative conservativeness analysis framework is proposed. In Section 3, the main results about extrema-based stability conditions are introduced. Comparisons of different methods are provided in Section 4. Following the comparison, in Section 5, further discussion is provided to analyze the case of partly overlapping of polyhedrons. In the final section, a conclusion is drawn.

Notations. The notations used throughout this paper are fairly standard. The superscript "T" stands for matrix transposition; $\mathbb{R}^n$ denotes the n-dimensional Euclidean space; $\mathbb{N}$ denotes the set of all natural numbers excluding 0; the notation $P > 0$ (≥ 0) means that P is real symmetric and positive definite (semi-definite). Matrices, if their dimensions are not explicitly stated, are assumed to be compatible for algebraic operations.

2. Problem Formulation and Preliminaries

Here we would like to briefly introduce some existing membership-dependent methods and summarize their results for comparison. To make it easier to follow, the main ideas of these methods will be reviewed based on a simple T-S fuzzy system and a commonly used Lyapunov function, i.e., the quadratic Lyapunov function. Consider the following T-S fuzzy system

$$\dot{x} = \sum_{i=1}^{p} h_i(x) A_i x, \tag{1}$$

where $x \in \mathbb{R}^{1\times n}$ is the system state, $p \in \mathbb{R}$ is the number of fuzzy rules, $h_i(x) : \mathbb{R}^{1\times n} \to \mathbb{R}$ is the membership function in the i-th rule, $A_i \in \mathbb{R}^{n\times n}$ is the system matrix in the i-th rule. Choose the Lyapunov function as $V(t) = x^T P x$, with $P \in \mathbb{R}^{n\times n}$ is a symmetric matrix satisfying $P > 0$. In the existing research, there are two effective branches of membership function dependent methods, namely the membership function approximation methods [15, 16] which consider the membership information by using alternative similar functions, and the membership-bound-dependent method [20] which applies the bound information of membership function into stability analysis. We start with the introduction of membership function approximation methods. In this research, we focus on the analysis of linear matrix inequality (LMI) based conditions, so the staircase and piecewise linear approximation methods will be reviewed. And after this, in the subsequent subsection, the introduction to membership-bound-dependent method will be provided.

2.1. Introduction to the membership function approximation methods

Here the term "membership function approximation" means that, the actual membership functions in the T-S fuzzy systems will be replaced by the alternative functions whose layouts are pretty close to the original ones and are relatively easier to describe as mathematical expressions. In this way, with those alternative approximated functions, it will be possible for us to apply their membership information into the system stability analysis. And the obtained stability condition will be membership-dependent.

2.1.1. Approximation idea in the staircase [15] and piecewise linear [16] methods

For both the staircase and piecewise approximation methods, the whole domain of premise variable x is divided into gridded sub-regions, and in the i-th dimension of x, these sub-regions are separated by sample points satisfying $x_i(t) = x_i^{(\tau)}$ ($\tau = 1, 2, \cdots, d+1$). The regional approximation of $h_i(x)$, $i = 1, 2, \cdots, p$, is described by the values of $h_i(x)$ at the surrounding sample points. To make it simple and highlight the main idea, we consider the special case $h_i(x) = h_i(x_1)$ ($i = 1, 2, \cdots, p$) as an example, which means the membership function is one-dimensional and only depends on $x_1(t)$. For the definition of high-dimensional approximation functions the idea is same, and readers can refer to [16] for more details.

Define $\hat{h}_i(x)$ as the approximated membership function. For the staircase approximation, the approximated function in sub-region $(x_1^{(\tau)},\ x_1^{(\tau+1)}]$ can be chosen as any value of $h_i(x_1)$ ($i = 1, 2, \cdots, p$) in this region. To facilitate the comparison, without loss of generality, the approximated *staircase function* is chosen in the following measure

$$\hat{h}_i^{(\tau)}(x_1) = \mu_\tau^-(x_1) h_i(x_1^{(\tau)}) + \mu_\tau^+(x_1) h_i(x_1^{(\tau+1)}) \tag{2}$$

for all $i = 1, 2, \cdots, p$, where $\mu_\tau^-(x_1) = e(\frac{x_1^{(\tau)} + x_1^{(\tau+1)}}{2} - x_1)$ is a step function changing value at sub-region center $x_1 = \frac{x_1^{(\tau)} + x_1^{(\tau+1)}}{2}$, $\mu_\tau^+(x_1) = 1 - \mu_\tau^-(x_1)$ is the counter-part of $\mu_\tau^-(x_1)$, and $e(t)$ is the step function satisfying $e(t) = 1$ for $t \geq 0$ and $e(t) = 0$ for $t < 0$.

For the *piecewise linear approximation*, the approximated function in region $(x_1^{(\tau)},\ x_1^{(\tau+1)}]$ can be expressed as

$$\hat{h}_i^{(\tau)}(x_1) = \upsilon_\tau^-(x_1) h_i(x_1^{(\tau)}) + \upsilon_\tau^+(x_1) h_i(x_1^{(\tau+1)}) \tag{3}$$

for all $i = 1, 2, \cdots, p$, where $\upsilon_\tau^-(x_1) = \frac{x_1 - x_1^{(\tau+1)}}{x_1^{(\tau)} - x_1^{(\tau+1)}}$ satisfying $\upsilon_\tau^-(x_1^{(\tau)}) = 1$ and $\upsilon_\tau^-(x_1^{(\tau+1)}) = 0$, and $\upsilon_\tau^+(x_1) = 1 - \upsilon_\tau^-(x_1)$ is the counter-part of $\upsilon_\tau^-(x_1)$. For both staircase and piecewise linear approximations, the obtained function $\hat{h}_i(x_1)$ has the same value as $h_i(x_1)$ at the sample points $x_1^{(\tau)}$, $\tau = 1, 2, \cdots, d, d+1$, see Fig. 1.

2.1.2. *Stability condition obtained by the staircase [15] and piecewise linear [16] methods*

For both methods we have $\sum_{i=1}^{p} \hat{h}_i(x) = 1$. Introduce the new matrices $Q_i \triangleq A_i^T P + P A_i$, $(i = 1, 2, \cdots, p)$, then the membership information can be combined in the analysis of Lyapunov function, and the details can be found in Expression (14) of [15]. Overall a sufficient condition for $\dot{V}(t) < 0$ could be

$$\begin{cases} r_i \geq h_i(x_1) - \hat{h}_i(x_1), \qquad \forall\ i = 1, 2, \cdots, p \\ Q_i + M \geq 0, \qquad \forall\ i = 1, 2, \cdots, p \\ \sum_{i=1}^{p} h_i(x_1^{(\tau)}) Q_i < -\sum_{i=1}^{p} r_i (Q_i + M) \end{cases} \tag{4}$$

for all $\tau = 1, 2, \cdots, d, d+1$, where r_i $(i = 1, 2, \cdots, p)$ is a scalar and M is a symmetric matrix of appropriate dimension. In (4), the main difference of staircase and piecewise linear approximation methods is $\hat{h}_i(x_1)$, which will result in different r_i from the first inequality. Consequently, from the third condition of (4), we may find that, the inequality with smaller r_i will be less conservative.

An example from [16] of different approximation methods is provided in Fig.

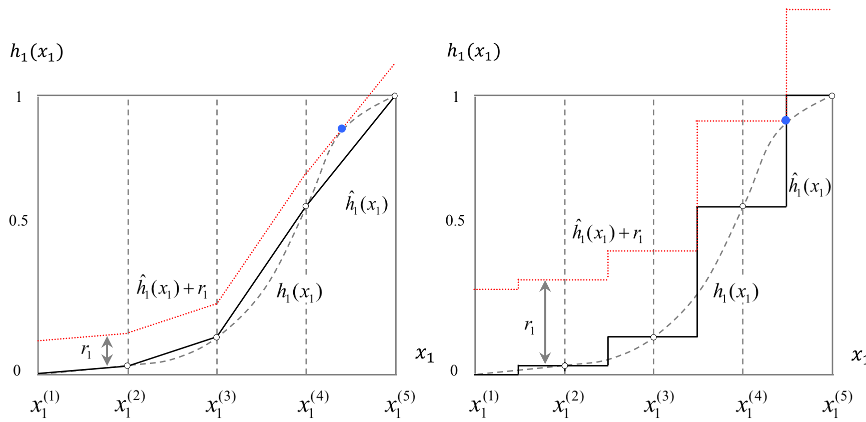


Figure 1: Comparison of the minimum r_1 in staircase (right) and piecewise linear (left) approximation methods. The dashed smooth gray lines are the trajectories of $h_1(x_1)$, the solid lines are the layouts of approximated function $\hat{h}_1(x_1)$, the dotted red lines are the layouts of $\hat{h}_1(x_1)+r_1$, where $r_1 \geq h_1(x_1) - \hat{h}_1(x_1)$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

1, where $h_1(x_1) = e^{-(x_1-10)^{1/25}}$ and x_1 is divided into six sub-regions by points $x_1^{(1)} = -10$, $x_1^{(2)} = -5$, $x_1^{(3)} = 0$, $x_1^{(4)} = 5$ and $x_1^{(5)} = 10$. Clearly we can see that, for smooth membership functions, the r_i value of piecewise linear approximation is generally smaller subject to the same set of sample points, which means less conservative stability conditions. Thus we will choose the piecewise linear approximation method as a representative for the conservativeness comparison in Section 4. In the following part, we will briefly discuss the bound-dependent method.

2.2. Introduction of the membership-bound-dependent relaxation method in [20]

If a bound for $h_i(x)$ is known, then we can name it as β_i which satisfies $h_i(x) \leq \beta_i$ for all x. Directly it holds that

$$h_i(x) \leq \beta_i \sum_{j=1}^{p} h_j(x).$$

Considering a group of arbitrary positive semi-definite matrices $N_i \geq 0$, $(i = 1, 2, \cdots, p)$, a sufficient condition for $\dot{V}(t) < 0$ can be obtained as

$$\begin{cases} \beta_i \geq h_i(x) \\ N_i \geq 0 \\ Q_i - N_i + \sum_{j=1}^{p} \beta_j N_j < 0 \end{cases} \tag{5}$$

for all $i = 1, 2, \cdots, p$.

When it comes to conservativeness analysis, in most existing papers [12, 15, 13], comparisons of different methods are usually analyzed in the numerical approach. Such an approach is simple and direct, and one can compare different methods based on the obtained feasible regions. But the comparison results will be highly dependent on the specific example. And sometimes it is difficult to explain the inner relation of different methods. To avoid those limitations and take one step further, here we would like to introduce another framework for the theoretical comparison which is less dependent on the specific example.

3. Main Results

The idea of considering all the membership functions as the elements of vector [29] makes it possible for us to visually see membership distribution in a unified space. In this section, we will analyze the membership layouts in this unified space and also describe the parameters of an LMI as a point in this space. Firstly we will explain how to use this membership space idea as a framework of conservativeness analysis. Then, an extrema-based method will be proposed to construct a polyhedron convex hull to enclose the membership distribution in this unified space. Extension to the case of interval type-2 T-S fuzzy systems will also be discussed. In the end of this section, corresponding stability conditions will be derived from the obtained polyhedron convex hull.

3.1. Framework of conservativeness analysis

The time derivative of $V(t)$ should be $\dot{V}(t) = \sum_{i=1}^{p} h_i(x)x^T Q_i x$, then a sufficient stability condition can be chosen as

$$\sum_{i=1}^{p} h_i(x)Q_i < 0$$

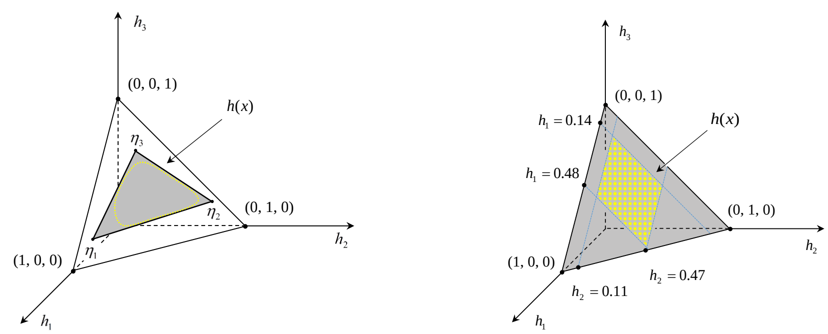
for any x. Generally, the layouts of membership functions $h_1(x)$, $h_2(x)$, $\cdots$, $h_p(x)$ are described in separate figures. In each separate figure the relation of $h_i(x)$ $(i = 1, 2, \cdots, p)$ with premise variable x is analyzed. A different idea is considering all functions $h_1(x), h_2(x), \cdots, h_p(x)$ as the elements of a unified vector [29],

$$h(x) \triangleq (h_1(x), h_2(x), \cdots, h_p(x)).$$

The unified membership vector $h(x)$ means the value distribution of $A(x) \triangleq \sum_{i=1}^{p} h_i(x)A_i$ among matrices A_1, A_2, $\cdots$, A_p. By definition the given membership functions satisfy the following conditions

$$\begin{cases} h_1(x) \geq 0, \quad h_2(x) \geq 0, \quad \cdots, \quad h_p(x) \geq 0 \\ h_1(x) + h_2(x) + \cdots + h_p(x) = 1. \end{cases} \tag{6}$$

Obviously, the trajectory of membership function $h(x)$ can be plotted in a p-dimensional Euclidean space with Cartesian coordinates $(h_1, h_2, \cdots, h_p)$. Choosing $p = 2$ for example, then by condition (6), we know that $h(x)$ is distributed on the line $h_1(x)+h_2(x) = 1$. In addition, with condition (6), the trajectory of $h(x)$ will be constrained in the first quadrant of the coordinate space. For the case of $p = 3$, the trajectory of $h(x)$ is constrained in the regular triangle formed by vertices $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, see Fig. 2a. Generally speaking, the dimension of $h(x)$ should be smaller than p, which results from the condition in (6). The trajectory of $h(x)$ plotted in Fig. 2a is one-dimensional. A simple example of two-dimensional $h(x)$ is $h_1(x) = x_1$, $h_2(x) = x_2$ and $h_3(x) = 1 - h_1(x) - h_2(x)$, with constraints $0.14 \leq x_1 \leq 0.48$ and $0.11 \leq x_2 \leq 0.47$. This example is



(a) One-dimensional $h(x)$. The dotted line is the trajectory of $h(x)$. η_1, η_2, η_3 are the vertices of a convex polyhedron enclosing $h(x)$

(b) Two-dimensional $h(x)$. The dotted area is the layout $h(x)$. $h(x)$ is constrained in the area $0.14 \leq h_1(x) \leq 0.48$ and $0.11 \leq h_2(x) \leq 0.47$ with $h_3(x) = 1 - h_1(x) - h_2(x)$

Figure 2: Distribution of $h(x)$ in three-dimensional space

plotted in Fig. 2b.

Using the membership vector $(h_1, h_2, \cdots, h_p)$, it will be easy to describe the exact value of $A(x)$. For example, the point $(h_1^*, h_2^*, \cdots, h_p^*)$ in the p-dimensional space represents the matrix $\sum_{i=1}^{p} h_i^* A_i$. It means that each point in the p-dimensional space is directly related with a certain system matrix. Specially, considering the case of $p = 2$, the point related with the subsystem matrix A_1 should be $(1, 0)$ and that related with A_2 should be $(0, 1)$.

Remark 1. *The idea of describing the membership functions $h_i(x)$ $(i = 1, 2, \cdots, p)$ as the elements of a joint vector $h(x)$ is not new. If we also describe the parameters of LMIs stability condition as some points in this membership vector space, the convex polyhedron constructed from those points will indicate the conservativeness of the LMIs stability condition. So here we express this membership space as a framework of conservativeness analysis.*

To analyze membership-dependent stability conditions, let us start with the following basic stability criterion for T-S fuzzy systems.

Lemma 1. *[7] If there exists a matrix $P > 0$, such that all the following in-*

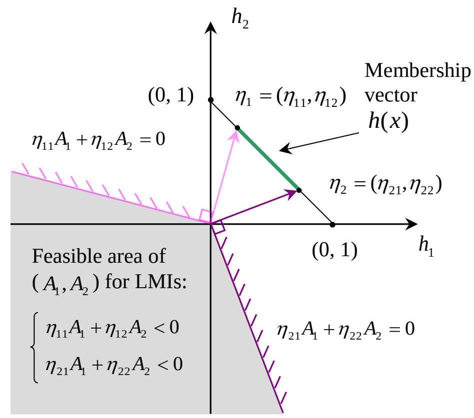


Figure 3: Feasible values of (A_1, A_2) in the case of $p = 2$ and $n = 1$, $(A_1, A_2 \in \mathbb{R}^{1\times 1})$. Points η_1, η_2 are the bounds of $h(x)$

equalities hold

$$Q_i < 0, \qquad \forall\ i = 1, 2, \cdots, p \tag{7}$$

where $Q_i = A^T P_i + P_i A$, then system (1) is asymptotically stable.

We denote the standard basis of a p-dimensional Euclidean space as e_1, e_2, $\cdots$, e_p. The result in Lemma 1 means that, if condition (7) is satisfied, the matrix represented by any point in the polyhedron formed by vertices e_i, $(i = 1, 2, \cdots, p)$ should be stable. Matrix $\sum_{i=1}^{p} h_i(x)A_i$ should be stable since $h(x)$ is constrained in that polyhedron. Intuitively we want to know whether the conservativeness can be reduced by shrinking the area of polyhedron represented by e_1, e_2, $\cdots$, e_p, for example, in 3-dimensional case, shrinking the triangle represented by $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$ to the polygon represented by η_1, η_2, η_3 in Fig. 2a.

To confirm that, we start from the simple case where $p = 2$ and $n = 1$. Both A_1 and A_2 in this case should be numbers. From Fig. 3, it is clear that the shrinking of line segment formed by points $\eta_1 \triangleq (\eta_{11}, \eta_{12})$ and $\eta_2 \triangleq (\eta_{21}, \eta_{22})$ means the enlargement of feasible area of (A_1, A_2). For the general case, similar

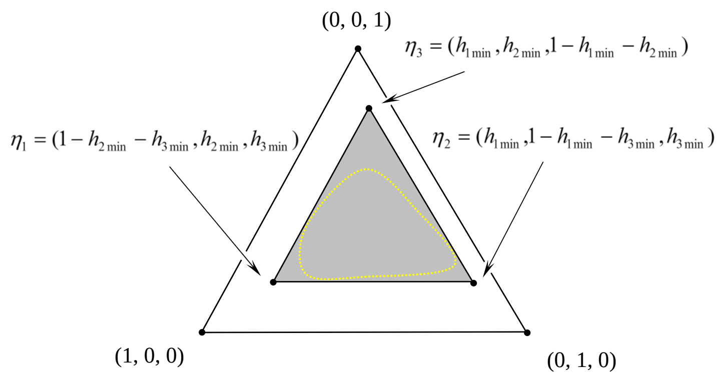


Figure 4: Convex polyhedron obtained by the minimum values of $h_i(x)$, with the dotted line being the trajectory of $h(x)$

result can be ensured by the following lemma.

Lemma 2. *[28] If there exists a matrix $P > 0$, such that the following inequalities hold*

$$\sum_{j=1}^{p} \eta_{ij} \left(A_j^T P + P A_j \right) < 0, \qquad \forall\ i = 1, 2, \cdots, p$$

and all the values of point $h(x) = (h_1(x), h_2(x), \cdots, h_p(x))$ are contained in the convex polyhedron formed by points $\eta_i = (\eta_{i1}, \eta_{i2}, \cdots, \eta_{ip})$, ($0 \leq \eta_{ij} \leq 1$, and $\sum_{j=1}^{p} \eta_{ij} = 1$), $i = 1, 2, \cdots, p$. Then system (1) is asymptotically stable.

Now the idea is quite clear. The conservativeness of membership-dependent stability conditions can be analyzed by their corresponding convex polyhedrons in the coordinate of membership functions. We name the vertices of those polyhedrons as *checking points*, the coordinate components of which can be used to construct the LMIs together with the subsystem matrices. In this membership vector framework, the main task of conservative analysis is to find the equivalent checking points of the obtained LMIs, and compare the convex polyhedrons described by those checking points. If a smaller polyhedron is contained in a bigger polyhedron, then condition related with the smaller one should be less conservative. Besides, there might be the case that two polyhedrons share some

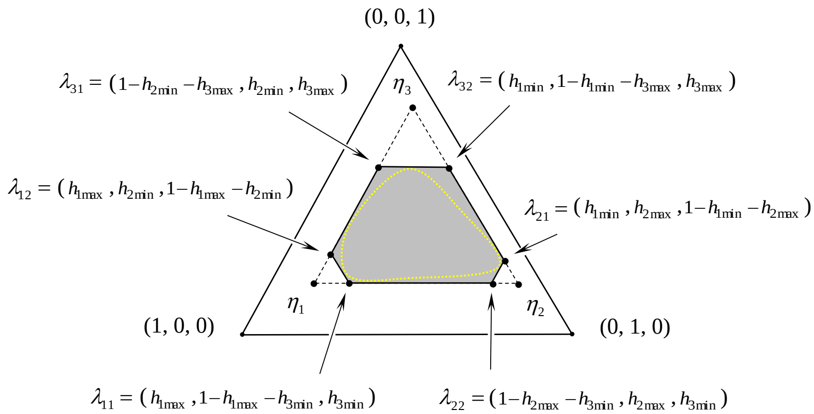


Figure 5: Convex polyhedron obtained by the extrema of $h_i(x)$, with the dotted line being the trajectory of $h(x)$

overlapping area but are not completely contained in each other. In this case, it would be difficult to conclude which one is more conservative. Further analysis for this case will be discussed in Section 5.

In other words, the tighter bounds will lead to more relaxed stability analysis results. And the following sections will discuss how to find tigher bounds. Specially in the next section, we will propose an effective approach to construct the membership-dependent polyhedron simply based on the extrema of membership functions. This approach will be also extended to interval type-2 T-S fuzzy system. Later in Section 4, this alternative approach will be compared with the methods in Sections 2.1 and 2.2.

3.2. Extrema-based convex polyhedron construction method

Define the minimum and maximum values of $h_i(x)$ as

$$h_{i\,\min} \triangleq \min_x \{h_i(x)\}, \qquad h_{i\,\max} \triangleq \max_x \{h_i(x)\}$$

for all $i = 1, 2, \cdots, p$.

In the following part, these extrema values will be used to construct a polyhedron that encloses the complete trajectory of membership function vector $h(x)$. Vertices of such a polyhedron can then be applied into stability analysis as checking points. The calculation algorithm of these vertices can be divided into two parts. In the first part, we need to redefine the subsystems based on the vertices obtained by the minimum value $h_{i\,\min}$, see Fig. 4, where the vertex $\eta_1 = (1 - h_{2\,\min} - h_{3\,\min}, h_{2\,\min}, h_{3\,\min})$ is the intersection point of surfaces $h_2(x) = h_{2\,\min}$, $h_3(x) = h_{3\,\min}$ and $h_1(x) + h_2(x) + h_3(x) = 1$. Vertices η_2 and η_3 are obtained in similar ways. Here *redefine* means that vertices η_1, η_2, η_3 are used as the replacement of vertices $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$.

Then in the second part, we apply the maximum value $h_{i\,\max}$ to further shrink the obtained polyhedron, see Fig. 5, where the vertex $\lambda_{11} = (h_{1\,\max}, 1 - h_{1\,\max} - h_{3\,\min}, h_{3\,\min})$ is the intersection point of surface $h_1(x) = h_{1\,\max}$ and the edge between η_1 and η_2. Vertex λ_{12} is the intersection point of surface $h_1(x) = h_{1\,\max}$ and the edge between η_1 and η_3. Vertices λ_{21}, λ_{22}, λ_{31} and λ_{32} can be obtained in similar ways. But there might be a special case that two of the six new vertices λ_{ij} $(i = 1, 2, 3,\ j = 1, 2)$ become one. This special case will be discussed in the Step 4.2 of Algorithm 1.

The new subsystems related with the new vertices in Fig. 4 should be

$$\bar{A}_q \triangleq \sum_{i=1}^{p} \eta_{qi} A_i = (1 - \sum_{i=1}^{p} h_{i\,\min}) A_q + \sum_{j=1}^{p} h_{j\,\min} A_j \tag{8}$$

for all $q = 1, 2, \cdots, p$. Define the new membership functions as $\bar{h}_i(x)$ $(i = 1, 2, \cdots, p)$ and define the joint vector of these new functions as

$$\bar{h}(x) \triangleq \left(\bar{h}_1(x), \bar{h}_2(x), \cdots, \bar{h}_p(x)\right).$$

From the definition in (8) and equation $\sum_{i=1}^{p} \bar{h}_i(x) \bar{A}_i = \sum_{j=1}^{p} h_j(x) A_j$, we can get the relation of $\bar{h}(x)$ and $h(x)$

$$(\delta I + \Gamma) \bar{h}^T(x) = h^T(x) \tag{9}$$

where $\delta \triangleq 1 - \sum_{i=1}^{p} h_{i\,\min}$ and

$$\Gamma \triangleq \begin{bmatrix} h_{1\,\min} & h_{1\,\min} & h_{1\,\min} & \cdots & h_{1\,\min} \\ h_{2\,\min} & h_{2\,\min} & h_{2\,\min} & \cdots & h_{2\,\min} \\ h_{3\,\min} & h_{3\,\min} & h_{3\,\min} & \cdots & h_{3\,\min} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h_{p\,\min} & h_{p\,\min} & h_{p\,\min} & \cdots & h_{p\,\min} \end{bmatrix}.$$

Note that for matrix Γ, it has the property $\Gamma\Gamma = (1-\delta)\Gamma$, which means

$$(\delta I + \Gamma)(I - \Gamma) = \delta I + \Gamma - \delta\Gamma - (1-\delta)\Gamma = \delta I.$$

In addition, δ is a value satisfying $0 < \delta \leq 1$. Thus by derivation we can get the following equivalent expression of (9),

$$\bar{h}^T(x) = \frac{1}{\delta}(I - \Gamma)h^T(x). \tag{10}$$

From the relation in (10), the maximum value of $\bar{h}_i(x)$ $(i = 1, 2, \cdots, p)$ can be obtained as

$$\begin{aligned} \bar{h}_{i\,\max} &= \frac{1}{\delta} \max_x \left\{ h_i(x) - h_{i\,\min} \sum_{j=1}^{p} h_j(x) \right\} \\ &= \frac{1}{\delta} \max_x \{ h_i(x) - h_{i\,\min} \} \\ &= \frac{1}{\delta} \left(h_{i\,\max} - h_{i\,\min} \right). \end{aligned} \tag{11}$$

By the expression in (10), it can be found that the minimum value of $\bar{h}_i(x)$ should be 0. Before the introduction of vertices calculation algorithm, we need to define two new module functions that will make it easier to refer to the correct index. Assume that a and b are positive integers. Then define the new module functions $[a]_b$ and $(a)_b$ as

$$[a]_b \triangleq a \bmod b, \qquad (a)_b \triangleq [a-1]_b + 1.$$

It should be noted that the ranges of $[a]_b$ and $(a)_b$ are different, which are

$$0 \leq [a]_b \leq b-1 \quad \text{and} \quad 1 \leq (a)_b \leq b.$$

With the above definitions, the vertices calculation method can be summarized as Algorithm 1 in the appendix. The following example will be a simple application of Algorithm 1.

Example 1. *To further explain Algorithm 1, we will go through all the steps based on a simple T-S fuzzy system with 3 rules. Assume that, in the first step, we get*

$$\begin{aligned} h_{1\,\min} &= 0.1, \quad h_{2\,\min} = 0.12, \quad h_{3\,\min} = 0.15 \\ h_{1\,\max} &= 0.5, \quad h_{2\,\max} = 0.4, \quad h_{3\,\max} = 0.5. \end{aligned}$$

Directly we have $\delta = 0.37$ *(defined after Equation (9)). From (11), the maximum values of* $\bar{h}_i(x)$ $(i = 1, 2, 3)$ *can be obtained as*

$$\bar{h}_{1\,\max} = 0.635, \quad \bar{h}_{2\,\max} = 0.444, \quad \bar{h}_{3\,\max} = 0.555.$$

Following Steps 2 and 3, we summarize the relation of different variables in the following table, where variables in each line have the same value. To make it easier to be identified, the initial index 0 *of each variable is marked in bold font.*

Table 1: Relation of the intermediate variables in Algorithm 1

	α_{1i}	α_{2i}	α_{3i}	β_{11i}	β_{12i}	β_{21i}	β_{22i}	β_{31i}	β_{32i}
$\bar{h}_{1\,\max}$	$\alpha_{1\mathbf{0}}$	α_{22}	α_{31}	$\beta_{11\mathbf{0}}$	$\beta_{12\mathbf{0}}$	β_{212}	β_{221}	β_{311}	β_{322}
$\bar{h}_{2\,\max}$	α_{11}	$\alpha_{2\mathbf{0}}$	α_{32}	β_{111}	β_{122}	$\beta_{21\mathbf{0}}$	$\beta_{22\mathbf{0}}$	β_{312}	β_{321}
$\bar{h}_{3\,\max}$	α_{12}	α_{21}	$\alpha_{3\mathbf{0}}$	β_{112}	β_{121}	β_{211}	β_{222}	$\beta_{31\mathbf{0}}$	$\beta_{32\mathbf{0}}$

It is obvious that $\bar{h}_{1\,\max} + \bar{h}_{2\,\max} > 1$, $\bar{h}_{2\,\max} + \bar{h}_{3\,\max} < 1$ *and* $\bar{h}_{3\,\max} + \bar{h}_{1\,\max} >$

1. *According to Step 4, we need to make the following modification*

$$\beta_{111} = 1 - \beta_{11\mathbf{0}} = 0.365, \quad \beta_{112} = 0; \qquad \beta_{121} = 1 - \beta_{12\mathbf{0}} = 0.365, \quad \beta_{122} = 0$$

$$\beta_{212} = 1 - \beta_{21\mathbf{0}} - \beta_{211} = 0.001; \qquad \beta_{221} = 1 - \beta_{22\mathbf{0}} = 0.556, \quad \beta_{222} = 0$$

$$\beta_{311} = 1 - \beta_{31\mathbf{0}} = 0.445, \quad \beta_{312} = 0; \qquad \beta_{322} = 1 - \beta_{32\mathbf{0}} - \beta_{321} = 0.001$$

and the rest of β_{qmk} $(q = 1, 2, 3,\ m = 1, 2,\ k = 0, 1, 2)$ *are unchanged. A flow chart of Steps 1–4 in Algorithm 1 can be described as Figure 6. In Steps 5–7, the subscripts of parameter* β_{qmk} *will be rearranged to get the coordinate parameters on checking points. Denote* $\tilde{\lambda}_{qm} \triangleq (\tilde{\lambda}_{qm1}, \tilde{\lambda}_{qm2}, \tilde{\lambda}_{qm3})$ *for* $q = 1, 2, 3,\ m = 1, 2.$

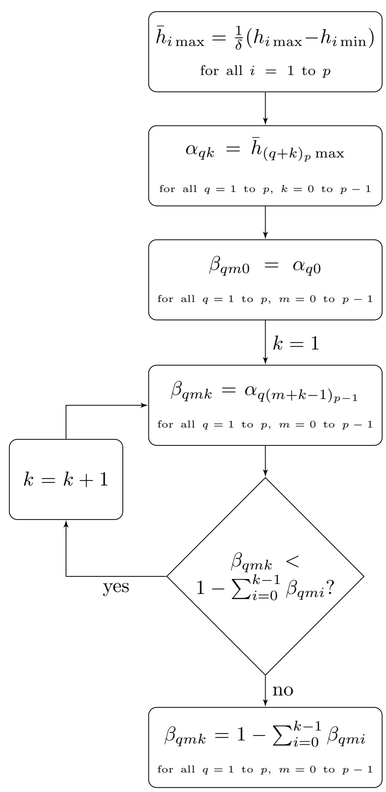


Figure 6: Flow chart of Steps 1–4 in Algorithm 1, explaining the calculation of α_{qk} and β_{qmk}

With Steps 5 and 6, locations of the 6 checking points in the resized coordinate *should be*

$$\tilde{\lambda}_{11} = (\beta_{11\mathbf{0}},\ \beta_{111},\ \beta_{112}); \qquad \tilde{\lambda}_{12} = (\beta_{12\mathbf{0}},\ \beta_{121},\ \beta_{122})$$
$$\tilde{\lambda}_{21} = (\beta_{212},\ \beta_{21\mathbf{0}},\ \beta_{211}); \qquad \tilde{\lambda}_{22} = (\beta_{221},\ \beta_{22\mathbf{0}},\ \beta_{222})$$
$$\tilde{\lambda}_{31} = (\beta_{311},\ \beta_{312},\ \beta_{31\mathbf{0}}); \qquad \tilde{\lambda}_{32} = (\beta_{322},\ \beta_{321},\ \beta_{32\mathbf{0}}).$$

By expression (A.3) in Step 7, the 6 checking points in the original coordinate should be

$$\lambda_{11} = (0.500,\ 0.350,\ 0.150); \qquad \lambda_{12} = (0.500,\ 0.120,\ 0.380)$$
$$\lambda_{21} = (0.101,\ 0.400,\ 0.500); \qquad \lambda_{22} = (0.450,\ 0.400,\ 0.150)$$
$$\lambda_{31} = (0.380,\ 0.120,\ 0.500); \qquad \lambda_{32} = (0.101,\ 0.400,\ 0.500).$$

In the above calculation $\bar{h}_{2\max} + \bar{h}_{3\max} < 1$ *means that two checking points in Fig. 5 come together. Thus we have* $\lambda_{21} = \lambda_{32}$ *in the above results.*

3.3. Extension to the case of interval type-2 T-S fuzzy systems

For interval type-2 T-S fuzzy systems [23, 24, 25, 30, 31], the possible position of $h_i(x)$ is restricted between the upper- and lower-membership functions $h_i^L(x)$ and $h_i^U(x)$, that is

$$0 \leq h_i^L(x) \leq h_i(x) \leq h_i^U(x) \leq 1.$$

Then by integration from $i = 1$ to $i = p$, we know that

$$0 \leq \sum_{i=1}^{p} h_i^L(x) \leq 1 \leq \sum_{i=1}^{p} h_i^U(x) \leq p.$$

It means that point $(h_1^L(x), h_2^L(x), \cdots, h_p^L(x))$ is located below the flat surface

$$h_1(x) + h_2(x) + \cdots + h_p(x) = 1 \tag{12}$$

and point $(h_1^U(x), h_2^U(x), \cdots, h_p^U(x))$ is located above the flat surface in (12). As it is depicted in Fig. 7, for any state $x = x^*$, the possible value of $h(x^*)$ is restricted in the hypercube formed by vertices

$$(h_1^L(x^*), h_2^L(x^*), \cdots, h_p^L(x^*)), \quad (h_1^U(x^*), h_2^U(x^*), \cdots, h_p^U(x^*)).$$

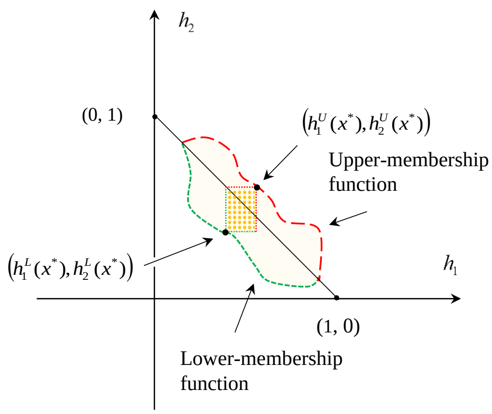


Figure 7: Possible distribution of one-dimensional interval type-2 membership functions, yellow dotted area is the hypercube formed by $(h_1^L(x^*), h_2^L(x^*))$ and $(h_1^U(x^*), h_2^U(x^*))$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

It is obvious that, for any positive scalar γ, the LMI condition constructed by checking point $(\gamma h_1, \gamma h_2, \cdots, \gamma h_p)$ should be same as that by $(h_1, h_2, \cdots, h_p)$. Thus by projecting all the possible values of $h(x^*)$ onto surface $\sum_{i=1}^{p} h_i(x) = 1$, similar algorithm for interval type-2 T-S fuzzy system can be also obtained. For a specific point x^*, the projection of all possible $h(x^*)$ should be enclosed by the projection of hypercube formed by $(h_1^L(x^*), h_2^L(x^*), \cdots, h_p^L(x^*))$ and $(h_1^U(x^*), h_2^U(x^*), \cdots, h_p^U(x^*))$. Denote $\hat{h}_{i,0}(x) \triangleq h_i^L(x)$ and $\hat{h}_{i,1}(x) \triangleq h_i^U(x)$ for $i = 1, 2, \cdots, p$. The vertices of such a hypercube will be

$$(\hat{h}_{1,\tau_1}(x^*), \hat{h}_{2,\tau_2}(x^*), \cdots, \hat{h}_{i,\tau_i}(x^*), \cdots, \hat{h}_{p,\tau_p}(x^*))$$

where $\tau_1, \tau_2, \cdots, \tau_i, \cdots, \tau_p = 0, 1$. Their projection on flat surface $\sum_{i=1}^{p} h_i(x) =$

1 should be

$$(h_{1,j}(x^*), h_{2,j}(x^*), \cdots, h_{i,j}(x^*), \cdots, h_{p,j}(x^*))$$

where $j = 1, 2, 3, \cdots, 2^p$, and for $j = \sum_{i=1}^{p} \tau_i \times 2^{i-1} + 1$ we have

$$h_{i,j}(x) \triangleq \frac{\hat{h}_{i,\tau_i}(x)}{\hat{h}_{1,\tau_1}(x) + \cdots + \hat{h}_{i,\tau_i}(x) + \cdots + \hat{h}_{p,\tau_p}(x)}.$$

Based on the projected membership function, the vertices calculation method can be summarized as Algorithm 2 in the appendix.

3.4. Extrema-based stability conditions

Based on the obtained vertices, now the stability condition can be presented as:

Theorem 1. *If there exists a matrix $P > 0$, such that the following inequality holds*

$$\sum_{k=1}^{p} \lambda_{qmk}(A_k^T P + P A_k) < 0 \tag{13}$$

for all $q = 1, 2, \cdots, p$ and $m = 1, 2, \cdots, p-1$, where λ_{qmk} is obtained from the Algorithm 1, and $(\lambda_{qm1}, \lambda_{qm2}, \cdots, \lambda_{qmp})$ is the obtained checking point. Then system (1) is asymptotically stable.

Remark 2. *By the proposed method, we can directly construct the final stability LMI conditions by the parameters λ_{qmk} which are obtained from the membership functions. As a result, conservativeness can be reduced by considering the membership information and, at the same time, keep the final conditions concise.*

In this case, only the extrema information in membership functions is applied to the stability analysis. Of course, we can find a smaller convex polyhedron [29] (smaller means the small convex polyhedron is contained in the current one) to obtain even less conservative vertices. But this would involve some complex linear-programming-related methods [32]. Those methods are usually not very effective for systems with $p \geq 3$, if so, the computational burden will increase sharply with respect to p.

Remark 3. *If a small convex polyhedron is completely enclosed in a bigger one, then we can reach the conclusion that, the stability condition related with the small polyhedron is less conservative. For the case of partially overlapping, we cannot say which method is better. But improved result can be obtained by considering the overlapping area as a new polyhedron. Further discussion on this topic can be found in Section 5.*

An alternative approach is to divide the original domain of membership function into d ($d \in \mathbb{N}$) sub-regions. For each sub-region $R_\tau(x)$ ($\tau = 1, 2, \cdots, d$), we use the above algorithms to find the local convex polyhedrons. Since the local membership function is part of the original one, all the local polyhedrons should be included in the global polyhedron, which means the reduction of conservativeness. In this way, the computational burden for complex polyhedron will be avoided, and we can simply choose a larger d to achieve less conservative analysis. We now summarize the extended stability analysis method in the following theorem.

Theorem 2. *If there exists a matrix $P > 0$, such that the following inequality holds*

$$\sum_{k=1}^{p} \lambda_{(\tau)qmk}(A_k^T P + P A_k) < 0 \tag{14}$$

for all $\tau = 1, 2, \cdots, d$, $q = 1, 2, \cdots, p$, and $m = 1, 2, \cdots, p-1$, where $\lambda_{(\tau)qmk}$ ($\tau = 1, 2, \cdots, r$) is obtained from Algorithm 1 with the amendment that

$$h_{i\,\min} = \min_{x \in R_\tau(x)} \left\{ h_i(x) \right\}, \qquad h_{i\,\max} = \max_{x \in R_\tau(x)} \left\{ h_i(x) \right\}.$$

Then system (1) is asymptotically stable.

Clearly, Theorem 1 is a special case of Theorem 2 with $d = 1$. Specially in the following section where the premise variable of $h(x)$ in the Example 2 is one-dimensional, we can simply choose the sub-regions $R_\tau(x)$ based on the sample points of piecewise linear approximation method. This will greatly facilitate our comparison.

4. Comparison of Different Membership-Dependent Methods

In this section, we will compare the method introduced in this paper with existing membership-dependent methods: piecewise linear approximation method in [16] and bound-dependent method in [13] and [20]. Both theoretical and numerical analysis will be provided. We will firstly go back to the methods reviewed in Section 2, and find the corresponding checking points of them.

4.1. Get the checking points for methods in Sections 2.1 and 2.2

4.1.1. Piecewise linear approximation method in [16]

To compare the condition (4) in Section 2.1 with that of Theorem 2, we assume that these two methods share the same sub-regions. Thus they have the same number of sub-regions d, and the sample points $x_1^{(\tau)}$ ($\tau = 1, 2, \cdots, d$) of them are same as each other. Without loss of generality, we set $p = 3$. Then a necessary condition of (4) is

$$\begin{cases} r_i \geq h_i(x_1) - \hat{h}_i(x_1), \qquad \forall\ i = 1, 2, \cdots, p \\ \sum_{i=1}^{3} h_i(x_1^{(\tau)}) Q_i < -r_2(Q_2 - Q_1) - r_3(Q_3 - Q_1) \\ \sum_{i=1}^{3} h_i(x_1^{(\tau)}) Q_i < -r_1(Q_1 - Q_2) - r_3(Q_3 - Q_2) \\ \sum_{i=1}^{3} h_i(x_1^{(\tau)}) Q_i < -r_1(Q_1 - Q_3) - r_2(Q_2 - Q_3) \end{cases}$$

for all $\tau = 1, 2, \cdots, d+1$. Equivalently it means that

$$\begin{cases} r_i \geq h_i(x_1) - \hat{h}_i(x_1), \qquad \forall\, i = 1, 2, \cdots, p \\ (h_1(x_1^{(\tau)}) - r_2 - r_3)Q_1 + (h_2(x_1^{(\tau)}) + r_2)Q_2 + (h_3(x_1^{(\tau)}) + r_3)Q_3 < 0 \\ (h_1(x_1^{(\tau)}) + r_1)Q_1 + (h_2(x_1^{(\tau)}) - r_1 - r_3)Q_2 + (h_3(x_1^{(\tau)}) + r_3)Q_3 < 0 \\ (h_1(x_1^{(\tau)}) + r_1)Q_1 + (h_2(x_1^{(\tau)}) + r_2)Q_2 + (h_3(x_1^{(\tau)}) - r_1 - r_2)Q_3 < 0 \end{cases}$$

for all $\tau = 1, 2, \cdots, d+1$. So the equivalent stability checking points are

$$\begin{cases} (h_1(x_1^{(\tau)}) - r_2 - r_3,\ h_2(x_1^{(\tau)}) + r_2,\ h_3(x_1^{(\tau)}) + r_3) \\ (h_1(x_1^{(\tau)}) + r_1,\ h_2(x_1^{(\tau)}) - r_1 - r_3,\ h_3(x_1^{(\tau)}) + r_3) \\ (h_1(x_1^{(\tau)}) + r_1,\ h_2(x_1^{(\tau)}) + r_2,\ h_3(x_1^{(\tau)}) - r_1 - r_2) \end{cases} \tag{15}$$

for all $\tau = 1, 2, \cdots, d+1$, where

$$r_i = \max_{x_1}(h_i(x_1) - \hat{h}_i(x_1)), \qquad \forall\, i = 1, 2, 3.$$

For systems with high dimensional premise variables, the sample points for membership approximation are selected as $x^{(\tau)}$ $(\tau = 1, 2, \cdots, d+1)$. Then equivalent checking points can be similarly obtained as

$$(h_1(x^{(\tau)}) + r_1,\ h_2(x^{(\tau)}) + r_2, \cdots, h_i(x^{(\tau)}) + r_i - \sum_{j=1}^{p} r_j, \cdots,\ h_p(x^{(\tau)}) + r_p) \tag{16}$$

for all $j = 1, 2, \cdots, p$ and $\tau = 1, 2, \cdots, d+1$, where

$$r_i = \max_{x}(h_i(x) - \hat{h}_i(x)), \qquad \forall\, i = 1, 2, \cdots, p.$$

4.1.2. Bound-dependent method in [20]

Now let us come to the stability condition (5) which is obtained by the bound-dependent method in Section 2.2. The condition in (5) can be equivalently expressed as

$$\begin{cases} N_i \geq 0, \quad \forall\ i = 1, 2, \cdots, p \\ Q_i - N_i + \displaystyle\sum_{j=1}^{p} h_{j\max} N_j < 0, \qquad \forall\ i = 1, 2, \cdots, p. \end{cases} \tag{17}$$

As a special case, we set $p = 2$. It is obvious that $1 - h_{i\max} \geq 0$ for $i = 1, 2$. Then by taking the weighted sum of inequalities in (17), one can get the following necessary condition of (17),

$$\begin{cases} N_1 \geq 0, \quad N_2 \geq 0 \\ (1 - h_{1\max})(Q_2 - N_2 + h_{1\max} N_1 + h_{2\max} N_2) \\ \qquad +h_{1\max}(Q_1 - N_1 + h_{1\max} N_1 + h_{2\max} N_2) < 0 \\ (1 - h_{2\max})(Q_1 - N_1 + h_{1\max} N_1 + h_{2\max} N_2) \\ \qquad +h_{2\max}(Q_2 - N_2 + h_{1\max} N_1 + h_{2\max} N_2) < 0 \end{cases}$$

which means

$$\begin{cases} N_1 \geq 0, \quad N_2 \geq 0 \\ (1 - h_{2\max}) Q_1 + h_{2\max} Q_2 < (1 - h_{1\max} - h_{2\max}) N_1 \\ h_{1\max} Q_1 + (1 - h_{1\max}) Q_2 < (1 - h_{1\max} - h_{2\max}) N_2. \end{cases}$$

Clearly we can see that, in this case, condition (5) is a sufficient condition of Theorem 2 with $d = 1$, which means more conservativeness.

Since we are not sure whether $1 - \sum_{i=1}^{p} h_{i\max} + h_{j\max}$, $j = 1, 2, \cdots, p$ are positive or not, the above derivation cannot be extended to the case $p \geq 3$. Hopefully, $(1 - \sum_{i=1}^{p} h_{i\min} + h_{j\min})$ $(j = 1, 2, \cdots, p)$ should be always positive, thus we can make that generalized derivation for the lower-bound-based version ($h_{j\min}$ is used). In this case, the linear matrix inequalities in (17) will be replaced by

$$\begin{cases} N_i \geq 0, \qquad \forall\ i = 1, 2, \cdots, p \\ Q_i + N_i - \displaystyle\sum_{j=1}^{p} h_{j\,\min} N_j < 0, \qquad \forall\ i = 1, 2, \cdots, p \end{cases}$$

Following the same derivation as case $p = 2$ of the upper-bound-based version, we can obtain the following necessary condition

$$\begin{cases} N_i \geq 0, \qquad \forall\ i = 1, 2, \cdots, p \\ \displaystyle\sum_{i=1}^{p} h_{i\,\min} Q_i + (1 - \sum_{i=1}^{p} h_{i\,\min}) Q_j < (\sum_{i=1}^{p} h_{i\,\min} - 1) N_j \end{cases} \tag{18}$$

for all $j = 1, 2, \cdots, p$. Since $\sum_{i=1}^{p} h_{i\,\min} - 1 \leq 0$, a necessary condition of (18) will be

$$\sum_{i=1}^{p} h_{i\,\min} Q_i + (1 - \sum_{i=1}^{p} h_{i\,\min}) Q_j < 0$$

for all $j = 1, 2, \cdots, p$. The corresponding checking points are

$$\begin{cases} (h_{1\,\min}, h_{2\,\min}, \cdots, h_{(p-1)\,\min}, 1 + h_{p\,\min} - \sum_{i=1}^{p} h_{i\,\min}) \\ (h_{1\,\min}, h_{2\,\min}, \cdots, 1 + h_{(p-1)\,\min} - \sum_{i=1}^{p} h_{i\,\min}, h_{p\,\min}) \\ \qquad\qquad \cdots \\ (1 + h_{1\,\min} - \sum_{i=1}^{p} h_{i\,\min}, h_{2\,\min}, \cdots, h_{(p-1)\,\min}, h_{p\,\min}). \end{cases} \tag{19}$$

4.2. Numerical test for the checking points of different methods

To see clearly the relations of checking points of different methods, we will plot and compare them in a unified membership space in the following example.

Example 2. *The comparisons will be based on the following system whose membership function trajectory is a round circle in the 3-dimensional space (the* *yellow round trajectory in Fig. 8):* $\dot{x} = \sum_{i=1}^{3} h_i(x) A_i x$ *where*

$$\begin{aligned} h_1(x) &= \frac{1}{3} + \frac{0.7}{\sqrt{36}} \cos x_1 - \frac{0.7}{\sqrt{12}} \sin x_1, \\ h_2(x) &= \frac{1}{3} + \frac{0.7}{\sqrt{36}} \cos x_1 + \frac{0.7}{\sqrt{12}} \sin x_1, \\ h_3(x) &= \frac{1}{3} - \frac{0.7}{\sqrt{9}} \cos x_1, \end{aligned}$$

and

$$A_1 = \begin{bmatrix} -1 & 2 \\ 0 & -1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -1 & 0.3 \\ b & -1 \end{bmatrix}, \quad A_3 = \begin{bmatrix} -1 & 0 \\ a & -1 \end{bmatrix}.$$

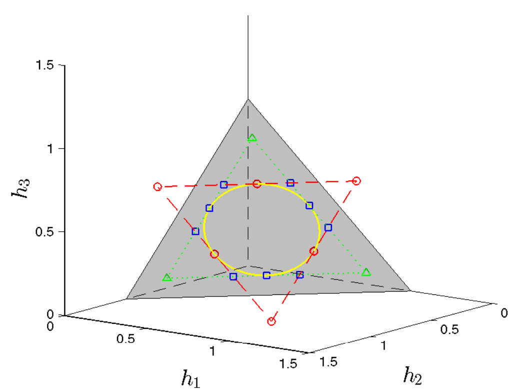


Figure 8: Checking points of different methods (piecewise linear approximation method [16] with $d = 3$: red circle points; bound-dependent method [20]: green triangle points; Theorem 2 with $d = 3$: blue square points). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

The vertices obtained from (15) with $d = 3$ *(where* $x_1^{(1)} = 0$, $x_1^{(2)} = \frac{2\pi}{3}$, $x_1^{(3)} = \frac{4\pi}{3}$*) are plotted as the red circle points in Fig. 8. The vertices obtained from (19) are plotted as the green triangle points in Fig. 8. For the method mentioned in Theorem 2, we choose* $d = 3$ *and sub-regions are divided based on*

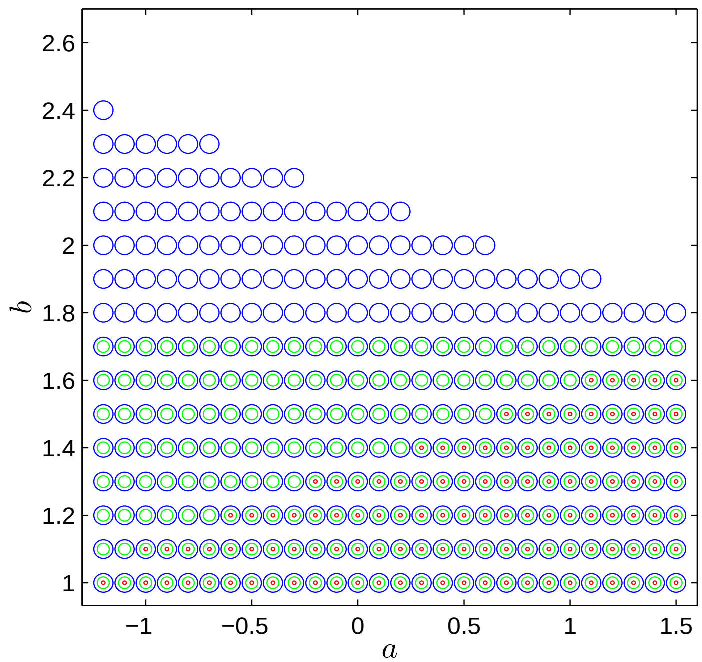


Figure 9: Feasible regions of different methods (piecewise linear approximation method [16] with $d = 3$: red dots; bound-dependent method [20]: small green circles; Theorem 2 with $d = 3$: big blue circles). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

the sample points in piecewise linear approximation method, which are

$$R_1(x) = \{x|x_1^{(1)} \leq x_1 < x_1^{(2)}\}$$
$$R_2(x) = \{x|x_1^{(2)} \leq x_1 < x_1^{(3)}\}$$
$$R_3(x) = \{x|x_1^{(3)} \leq x_1 < x_1^{(1)}\}.$$

The obtained checking points are plotted as blue square points. Clearly, we can find that the convex polytope constructed by the square points is contained in both that of circle points and that of triangle points. It means that the method described in Theorem 2 is superior in the aspect of conservativeness. To further confirm the above conservativeness relation, let us compare the feasible regions *of (a, b) obtained by different methods. The results are described in Fig. 9. It is obvious that the method introduced in Theorem 2 has larger feasible region.*

From the numerical point of view, the method in Theorem 2 would require lower computational burden since there is no additional matrix variable involved in the final linear matrix inequalities. Another merit of Theorem 2 is that there is no need to concern about the dimension of premise variable. In addition, it can be easily extended to interval type-2 T-S fuzzy systems (see Fig. 7). In this sense, one can reach the conclusion that Theorem 2 is a more general method with less conservativeness and computational burden.

5. Further discussion

In the above comparison analysis, we only discussed the case where one polyhedron is fully enclosed in another. In that case, one can reach the conclusion that, stability conditions related with smaller polyhedron should be less conservative. For the case that two polyhedrons share some overlapping parts but are not completely contained in each other, global conservativeness comparison result cannot be obtained. But in such a case, we can get less conservative stability condition by combining the two methods together. The idea is that, polyhedron area without overlapping means that such an area is not necessary for stability checking. Thus, theoretically, we can shrink the enclosing polyhedron to the overlapped area of several ones. Denote $\mathcal{P}_i$ as the equivalent convex polyhedron of the i-th method $(i = 1, 2, \cdots, l)$. This idea will be based on the following set theory.

Lemma 3. *[33] If $h(x)$ is fully contained in all the convex polyhedrons $\mathcal{P}_i$ $(i = 1, 2, \cdots, l)$, then $h(x)$ is fully contained in their overlapping area $\mathcal{P} \triangleq \mathcal{P}_1 \cap \mathcal{P}_2 \cap \cdots \cap \mathcal{P}_l$.*

Denote the d vertices of $\mathcal{P}$ as $\eta_j = (\eta_{j1}, \eta_{j2}, \cdots, \eta_{jp})$ $(j = 1, 2, \cdots, d)$. Then the stability condition can be expressed in the following theorem.

Theorem 3. *System (1) is asymptotically stable if there exists a matrix $P > 0$ such that the following condition is satisfied*

$$\sum_{k=1}^{p} \eta_{jk}(A_k^T P + P A_k) < 0$$

for all $j = 1, 2, \cdots, d$, where $\eta_j = (\eta_{j1}, \eta_{j2}, \cdots, \eta_{jp})$, $j = 1, 2, \cdots, d$, are the vertices of overlapping area $\mathcal{P}$.

Denote the membership-dependent LMI condition related with j-th method ($j = 1, 2, \cdots, l$) as $P < 0$ and $L_j(Q_1, Q_2, \cdots, Q_p) < 0$ where $Q_i = PA_i + A_i^T P$ for $i = 1, 2, \cdots, p$. We also have direct combination of multiple methods.

Theorem 4. *System (1) is asymptotically stable if the following condition is satisfied*

$$\begin{cases} P > 0, \\ L_1 < 0 \ \textit{or}\ L_2 < 0 \ \textit{or}\ \cdots \ \textit{or}\ L_l < 0. \end{cases}$$

where L_i is the abbreviation of $L_i(Q_1, Q_2, \cdots, Q_p)$ which is related to Q_i ($i = 1, 2, \cdots, p$).

For a given fuzzy system with uncertain parameter, the feasible solution of uncertain parameter obtained by Theorem 4 should be the union of feasible solutions obtained by methods 1 to l. Theoretically the method in Theorem 3 will be less conservative than the simple combination result in Theorem 4. To verify such a conclusion, we start with the one dimensional system. For the given subsystems $A_i \in \mathbb{R}^{1\times 1}$ ($i = 1, 2, \cdots, p$), expression

$$h_1(x)A_1 + h_2(x)A_2 + \cdots + h_p(x)A_p = 0 \tag{20}$$

should be a surface perpendicular to vector $(A_1, A_2, \cdots, A_p)$ and passing through the origin point. The condition $P > 0$ and $L_i < 0$ is equivalent to say that $\mathcal{P}_i$ is located on one side of surface (20) (the side satisfying $\sum_{i=1}^{p} h_i(x)A_i < 0$). In the case that, all $\mathcal{P}_i$ $(i = 1, 2, \cdots, l)$ have intersection with surface (20), $\mathcal{P}$ can still be on only one side of surface (20), see Fig. 10.

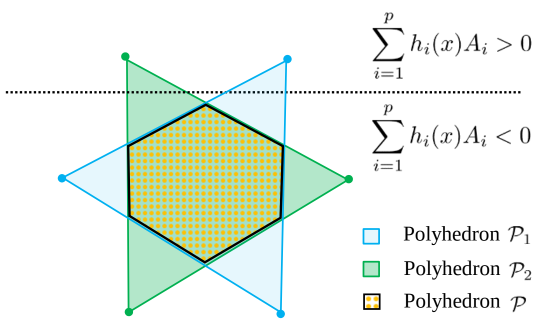


Figure 10: Both polyhedrons $\mathcal{P}_1$ and $\mathcal{P}_2$ have intersection with surface (20), but their overlapping area $\mathcal{P}$ is on only one side of surface (20). Here $A_i \in \mathbb{R}^{1\times 1}$ for all $i = 1, 2, \cdots, p$

It means that when Theorem 4 is not satisfied, the condition in Theorem 3 may still be achieved. This verifies the reduced conservativeness of Theorem 3. The main difficulty in applying Theorem 3 is the algorithm to find the vertices of $\mathcal{P}$ based on the vertices of $\mathcal{P}_i$ $(i = 1, 2, \cdots, l)$. This will be a direction of our future research.

6. Conclusions

By considering the membership functions in a unified space, it is easier to analyze the conservativeness of membership-dependent stability conditions. It has been proven that final LMI conditions can be constructed based on the vertices of a convex polyhedron enclosing the membership trajectory, and conservativeness can be reduced by shrinking the range of such a polyhedron. Following this idea, the extrema values of membership functions have been used to construct a tighter polyhedron to reduce conservativeness. Compared with existing methods, this method shows advantage of less conservativeness, simplicity, and there is no need to concern about the dimension of membership functions. Moreover, it has been extended to the stability analysis of interval type-2 T-S fuzzy systems. Conversely, by deriving the checking points of existing LMIs, this membership vector framework can also been used to compare the conservativeness of membership dependent stability conditions. Finally, further results on this framework has been discussed.

Appendix A. Explanation of Algorithm 1

Algorithm 1: Vertices calculation method for type-1 T-S fuzzy system.

Step 1. Calculate the minimum and maximum value of $h_i(x)$

$$h_{i\min} = \min_x \{h_i(x)\}, \qquad h_{i\max} = \max_x \{h_i(x)\}$$

for all $i = 1, 2, \cdots, p$.

Step 2. The minimum value of $\bar{h}_q(x)$ should be 0. Referring to (11), we can calculate the maximum value of $\bar{h}_q(x)$ as

$$\alpha_{qk} = \bar{h}_{(q+k)_p \max} = \frac{1}{\delta}(h_{(q+k)_p \max} - h_{(q+k)_p \min}) \tag{A.1}$$

for all $q = 1, 2, \cdots, p$ and $k = 0, 1, \cdots, p-1$.

Step 3. Create new variable β_{qmk} as

$$\beta_{qm0} = \alpha_{q0} \quad \text{and} \quad \beta_{qmk} = \alpha_{q(m+k-1)_{p-1}} \tag{A.2}$$

for all $q = 1, 2, \cdots, p$, $m = 1, 2, \cdots, p-1$, and $k = 1, 2, \cdots, p-1$.

Step 4.

For all $q = 1, 2, \cdots, p$ and $m = 1, 2, \cdots, p-1$,

4.1 set $k = 1$.

4.2 check whether the condition $\beta_{qmk} \geq 1 - \sum_{i=0}^{k-1} \beta_{qmi}$ is satisfied,

if $\beta_{qmk} \leq 1 - \sum_{i=0}^{k-1} \beta_{qmi}$, then set $k = k + 1$ and repeat *Step 4.2*;

if $\beta_{qmk} > 1 - \sum_{i=0}^{k-1} \beta_{qmi}$, then set $\beta_{qmk} = 1 - \sum_{i=0}^{k-1} \beta_{qmi}$.

4.3 for $i = k+1, k+2, \cdots, p$, set $\beta_{qmi} = 0$.

Step 5. Create new variable γ_{qmk} as

$$\gamma_{qm0} = \beta_{qm0}, \quad \text{and} \quad \gamma_{qmk} = \beta_{qm(k-m+1)_{p-1}}$$

for all $q = 1, 2, \cdots, p$, $m = 1, 2, \cdots, p-1$ and $k = 1, 2, \cdots, p-1$.

Step 6. Create new variable $\tilde{\lambda}_{qmk}$ as

$$\tilde{\lambda}_{qmk} = \gamma_{qm[k-q]_p}$$

for all $q = 1, 2, \cdots, p$, $m = 1, 2, \cdots, p-1$ and $k = 1, 2, \cdots, p$.

Step 7. Create new variable λ_{qmk} as

$$\lambda_{qmk} = \delta \tilde{\lambda}_{qmk} + h_{k\,\min} \tag{A.3}$$

for all $q = 1, 2, \cdots, p$, $m = 1, 2, \cdots, p-1$ and $k = 1, 2, \cdots, p$. And the final $p \times (p-1)$ vertices (checking points) should be

$$\lambda_{qm} = (\lambda_{qm1}, \lambda_{qm2}, \cdots, \lambda_{qmp})$$

for all $q = 1, 2, \cdots, p$ and $m = 1, 2, \cdots, p-1$.

Remark 4. *When the maximum values are used, all the original resized vertices (obtained by minimum values)*

$$(h_{1\,\min}, h_{2\,\min}, \cdots, 1 + h_{i\,\min} - \sum_{j=1}^{p} h_{j\,\min}, \cdots, h_{p\,\min})$$

will disappear. Note that each original vertex is connected with $p-1$ edges, then $p-1$ new vertices will be created for each maximum value $h_{i\,\max}$. Step 4 is just the method used to calculate the location of those $p \times (p-1)$ new vertices.

The "if" condition is used to check whether the new vertex is an intersection point of some maximum level surfaces $h_i(x) = h_{i\,\max}$. In such a case, we have $\sum_{i=0}^{k-1} \beta_{qmi} \leq 1$ $(k \geq 1)$, and the new vertex will not be a point on the original polyhedron edges.

Appendix B. Explanation of Algorithm 2

Algorithm 2: Vertices calculation method for interval type-2 T-S fuzzy system .

Step 1. Calculate the minimum and maximum values of possible $h_i(x)$,

$$h_{i\,\min} = \min_{j=1}^{\kappa} \left\{ \min_x \{h_{i,j}(x)\} \right\}, \qquad h_{i\,\max} = \max_{j=1}^{\kappa} \left\{ \max_x \{h_{i,j}(x)\} \right\}$$

for all $i = 1, 2, \cdots, p$, where $\kappa = 2^p$.

The remaining **Steps 2 − 7** are the same as those of Algorithm 1. Similarly, $p \times (p-1)$ vertices $(\lambda_{qm1}, \lambda_{qm2}, \cdots, \lambda_{qmp})$ $(q = 1, 2, \cdots, p$ and $m = 1, 2, \cdots, p-1)$ will be obtained.

Acknowledgements

Thanks to my colleagues and the anonymous reviewers for their efforts in improving this paper. Funding: This work was supported in part by the National Natural Science Foundation of China [grant numbers 61525303], the Top-Notch Young Talents Program of China (L. Wu), the Heilongjiang Outstanding Youth Science Fund [grant numbers JC201406], the Fok Ying Tung Education Foundation [grant numbers 141059], China Scholarship Council (201406120001), and King's College London (K-CSC).

References

[1] Wikipedia, Maxima and minima — wikipedia, the free encyclopedia, [Online; accessed 21-April-2017] (2017).

URL https://en.wikipedia.org/w/index.php?title=Maxima_and_minima&oldid=770619671

[2] T. Takagi, M. Sugeno, Fuzzy identification of systems and its applications to modelling and control, IEEE Transactions on Systems, Man and Cybernetics SMC-15 (1) (1985) 116–132. doi:10.1109/TSMC.1985.6313399.

[3] Q. Zhou, H. Li, C. Wu, L. Wang, C. K. Ahn, Adaptive fuzzy control of nonlinear systems with unmodeled dynamics and input saturation using small-gain approach, IEEE Transactions on Systems, Man, and Cybernetics: Systems 47 (8) (2017) 1979–1989. doi:10.1109/TSMC.2016.2586108.

[4] Q. Zhou, C. Wu, P. Shi, Observer-based adaptive fuzzy tracking control of nonlinear systems with time delay and input saturation, Fuzzy Sets and Systems 38 (Supplement C) (2017) 49 – 68. doi:10.1016/j.fss.2016.11.002.

[5] H. Lam, L. Seneviratne, Stability analysis of interval type-2 fuzzy-model-based control systems, IEEE Transactions on Systems, Man, and Cybernetics, Part B: Cybernetics 38 (3) (2008) 617–628. doi:10.1109/TSMCB.2008.915530.

[6] X. Yang, L. Wu, H. Lam, X. Su, Stability and stabilization of discrete-time T-S fuzzy systems with stochastic perturbation and time-varying delay, IEEE Transactions on Fuzzy Systems 22 (1) (2014) 124–138. doi:10.1109/TFUZZ.2013.2249518.

[7] K. Tanaka, H. O. Wang, Fuzzy Control Systems Design and Analysis, New York: Wiley, 2001.

[8] X. Su, P. Shi, L. Wu, Y.-D. Song, A novel control design on discrete-time Takagi-Sugeno fuzzy systems with time-varying delays, IEEE Transactions on Fuzzy Systems 21 (4) (2013) 655–671. doi:10.1109/TFUZZ.2012.2226941.

[9] K. Tanaka, H. Yoshida, H. Ohtake, H. Wang, A sum-of-squares approach to modeling and control of nonlinear dynamical systems with polynomial fuzzy systems, IEEE Transactions on Fuzzy Systems 17 (4) (2009) 911–922. doi:10.1109/TFUZZ.2008.924341.

[10] W.-J. Wang, C.-H. Sun, Relaxed stability and stabilization conditions for a ts fuzzy discrete system, Fuzzy Sets and Systems 156 (2) (2005) 208 – 225. doi:10.1016/j.fss.2005.05.002.

[11] H. Lam, F. Leung, Stability analysis of fuzzy control systems subject to uncertain grades of membership, IEEE Transactions on Systems, Man, and Cybernetics, Part B: Cybernetics 35 (6) (2005) 1322–1325. doi:10.1109/TSMCB.2005.850181.

[12] C. Arino, A. Sala, Extensions to "stability analysis of fuzzy control systems subject to uncertain grades of membership, IEEE Transactions on Systems, Man, and Cybernetics, Part B: Cybernetics 38 (2) (2008) 558–563. doi:10.1109/TSMCB.2007.913596.

[13] M. Narimani, H. Lam, Relaxed LMI-based stability conditions for Takagi-Sugeno control systems using regional-membership-function-shape-dependent analysis approach, IEEE Transactions on Fuzzy Systems 17 (5) (2009) 1221–1228. doi:10.1109/FUZZY.2009.5277410.

[14] A. Sala, C. Arino, Relaxed stability and performance LMI conditions for Takagi-Sugeno fuzzy systems with polynomial constraints on membership function shapes, IEEE Transactions on Fuzzy Systems 16 (5) (2008) 1328–1336. doi:10.1109/TFUZZ.2008.926585.

[15] H. Lam, M. Narimani, Quadratic-stability analysis of fuzzy-model-based control systems using staircase membership functions, IEEE Transactions on Fuzzy Systems 18 (1) (2010) 125–137. doi:10.1109/TFUZZ.2009.2037744.

[16] H. Lam, Polynomial fuzzy-model-based control systems: Stability analysis via piecewise-linear membership functions, IEEE Transactions on Fuzzy Systems 19 (3) (2011) 588–593. doi:10.1109/TFUZZ.2011.2118215.

[17] H. Lam, C. Liu, L. Wu, X. Zhao, Polynomial fuzzy-model-based control systems: stability analysis via approximated membership functions considering sector nonlinearity of control input, IEEE Transactions on Fuzzy Systems 23 (6) (2015) 2202–2214. doi:10.1109/TFUZZ.2015.2407907.

[18] M. Narimani, H. Lam, R. Dilmaghani, C. Wolfe, LMI-based stability analysis of fuzzy-model-based control systems using approximated polynomial membership functions, IEEE Transactions on Systems, Man, and Cybernetics, Part B: Cybernetics 41 (3) (2011) 713–724. doi:10.1109/TSMCB.2010.2086443.

[19] M. Narimani, H. Lam, SOS-based stability analysis of polynomial fuzzy-model-based control systems via polynomial membership functions, IEEE Transactions on Fuzzy Systems 18 (5) (2010) 862–871. doi:10.1109/TFUZZ.2010.2050890.

[20] A. Sala, C. Arino, Relaxed stability and performance conditions for Takagi-Sugeno fuzzy systems with knowledge on membership function overlap, IEEE Transactions on Systems, Man, and Cybernetics, Part B: Cybernetics 37 (3) (2007) 727–732. doi:10.1109/TSMCB.2006.887949.

[21] B. Q. Hu, C. Y. Wang, On type-2 fuzzy relations and interval-valued type-2 fuzzy sets, Fuzzy Sets and Systems 236 (2014) 1 – 32, theme: Algebraic Aspects of Fuzzy Sets. doi:10.1016/j.fss.2013.07.011.

[22] P. Rakshit, S. Saha, A. Konar, S. Saha, A type-2 fuzzy classifier for gesture induced pathological disorder recognition, Fuzzy Sets and Systems 305 (2016) 95 – 130, theme: Classification, Recognition and Clustering. doi:10.1016/j.fss.2016.05.001.

[23] S. Coupland, R. Joh, Geometric type-1 and type-2 fuzzy logic systems, IEEE Transactions on Fuzzy Systems 15 (1) (2007) 3–15. doi:10.1109/TFUZZ.2006.889764.

[24] J. Mendal, R. John, F. Liu, Interval type-2 fuzzy logic systems made simple, IEEE Transactions on Fuzzy Systems 14 (6) (2006) 808–821. doi:10.1109/TFUZZ.2006.879986.

[25] Q. Lu, P. Shi, H. Lam, Y. Zhao, Interval type-2 fuzzy model predictive control of nonlinear networked control systems, IEEE Transactions on Fuzzy Systems 23 (6) (2015) 2327–2328. doi:10.1109/TFUZZ.2015.2417975.

[26] H. K. Lam, H. Li, C. Deters, E. L. Secco, H. A. Wurdemann, K. Althoefer, Control design for interval type-2 fuzzy systems under imperfect premise matching, IEEE Transactions on Industrial Electronics 61 (2) (2014) 956–968. doi:10.1109/TIE.2013.2253064.

[27] B. Xiao, H. K. Lam, H. Li, Stabilization of interval type-2 polynomial-fuzzy-model-based control systems, IEEE Transactions on Fuzzy Systems 25 (1) (2017) 205–217. doi:10.1109/TFUZZ.2016.2554153.

[28] X. Yang, H. Lam, L. Wu, Novel membership-function-dependent stability condition for T-S fuzzy systems, in: Proceedings of 2016 IEEE World Congress on Computational Intelligence, Vancouver, Canada, 2016. doi:10.1109/FUZZ-IEEE.2016.7737966.

[29] A. Sala, C. Arino, Local stability of open- and closed-loop fuzzy systems, in: Proceedings of the 2006 IEEE International Symposium on Intelligent Control, Munich, Germany, 2006. doi:10.1109/CACSD-CCA-ISIC.2006.4777013.

[30] H. Li, L. Wu, H. Lam, Y. Gao, Analysis and Synthesis for Interval Type-2 Fuzzy-Model-Based Systems, Springer, Singapore, 2016. doi:10.1007/978-981-10-0593-0.

[31] T. Wang, S. Tong, J. Yi, H. Li, Adaptive inverse control of cable-driven parallel system based on type-2 fuzzy logic systems, IEEE Transactions on Fuzzy Systems 23 (5) (2015) 1803–1816. doi:10.1109/TFUZZ.2014.2379284.

[32] D. Luenberger, Linear and Nonlinear Programming, 2nd Edition, Springer, 2003.

[33] R. Stoll, Set Theory and Logic, Dover books on advanced mathematics, Dover Publications, 1979.