SIEGEL MODULAR FORMS AND THE GONALITY OF CURVES

N. I. SHEPHERD-BARRON

1 Introduction

Denote by M_g and A_g the coarse moduli spaces of genus g curves and principally polarized abelian g-folds, respectively, and let a superscript S denote their Satake compactifications. The main result of [CSB] is that the intersection $M_{g+m}^S \cap A_g^S$, taken inside A_{g+m}^S, is far from transverse; it contains the mth order infinitesimal neighbourhood of M_g^S in A_g^S. So $\cup_m (M_{g+m}^S \cap A_g^S)$ is the formal completion of A_g^S along M_g^S and there are no stable Siegel modular forms that vanish along every moduli space M_g. The proof depends upon the construction by Fay of certain, very special, degenerating families of curves for which he could establish a formula for (a part of) the derivative of the period matrix as a certain explicit tensor of rank one [F, p. 53]. For an arbitrary degeneration the derivative is a tensor of higher rank, usually maximal, and it is more difficult to make use of this; cf. the assertion on p. 1 of the erratum to [G-SM]. Interpreting Fay's formula in terms of the projective geometry of the canonical model of the singular fibre then gives the result.

Here we prove similar results for the loci $V_{g,n,tot}$ in M_g of n-gonal curves of genus g with a point of total ramification, for any fixed $n \geq 3$, as follows.

Theorem 1.1 *(= 3.7) There is no stable Siegel modular form that vanishes on every locus $V_{g,n,tot}$. In particular, there is no stable Siegel modular form that vanishes on every trigonal locus.*

This sharpens [CSB], but depends upon it. For hyperelliptic curves, however, Codogni has shown [C] that the story becomes very different. He has found many millions of stable modular forms that vanish on the hyperelliptic locus in every genus, for example, the difference $\Theta_P - \Theta_Q$ of two theta series where P, Q are positive, even and unimodular quadratic forms of rank 32 with no roots.

Recall that a curve C is n-gonal if there is a map $C \to \mathbb{P}^1$ of degree n. If $g = 2n-2$ is even, then a general curve of genus g is n-gonal in finitely many ways; if $g > 2n-2$ then the n-gonal curves form a proper subvariety (the Hurwitz scheme) $V_{g,n}$ of M_g. The n-gonal curves for which the given map to $\mathbb{P}^1$ has a point of total ramification form the subvariety $V_{g,n,tot}$ mentioned above. Its closure in M_g^S will be denoted by $V_{g,n,tot}^S$.

Compared to the arguments in [CSB], the proof here depends upon combining Fay's construction with those by Schiffer to get certain variations of a curve where what is essentially the derivative of the period matrix can be calculated explicitly. Controlling the construction of these Fay–Schiffer variations (see below) is crucial in controlling the derivative.

2 Variations

Suppose that C is a curve (= compact Riemann surface) of genus g, that a, b, c are distinct points of C and that z_a, z_b, z_c are local co-ordinates on C at a, b, c respectively. There are various well known kinds of variation that can be constructed from these data, and we recall some of them now.

The first is a *Fay variation* of C centred at $(a, z_a; b, z_b)$. This is a particular proper morphism $\mathcal{C} \to \Delta$ from a smooth complex surface to a disc such that the fibre over 0 is the nodal curve $C/(a \sim b)$ and for every $t \neq 0$ the fibre $\mathcal{C}_t = C_t$ is of genus $g+1$. It is constructed as follows [F, p. 50].

Fix $\delta > 0$ with $\delta << 1$. Let D_{δ^2} be a disc of radius δ^2 and complex co-ordinate t. In $C \times D_{\delta^2}$ consider two closed subsets, one defined by the inequality $|z_a| \leq |t|/\delta$ and the other by the inequality $|z_b| \leq |t|/\delta$. Delete these closed subsets from $C \times D_{\delta^2}$ to get the complex manifold $\mathcal{C}^0$. There are open subsets U_a and U_b of $\mathcal{C}^0$ defined by the further inequalities $|z_a| < \delta$ and $|z_b| < \delta$, respectively.

Let S be the open part of the complex surface with co-ordinates X, Y defined by the inequalities $|X|, |Y| < \delta$. There is a morphism $S \to D_{\delta^2}$ given by $t = XY$. Now map U_a and U_b to S by the formulae

$$X = z_a,\ Y = t/z_a,$$

$$X = t/z_b,\ Y = z_b$$

and then glue $\mathcal{C}^0$ to S via these maps; by definition, the result is $\mathcal{C}$, and $\mathcal{C}$ is provided with a proper morphism to $\Delta = D_{\delta^2}$.

Another kind is a *Schiffer variation* of C centred at (c, z_c). This is a particular proper morphism $\mathcal{C} \to \Delta$ where now all fibres are smooth of genus g. It is also constructed via a glueing procedure.

Start with $C \times D_{\delta^2/4}$ and delete the closed subset defined by the inequality $|z_c| \leq \sqrt{|t|}$ to obtain the complex manifold $\mathcal{C}^0$. In $\mathcal{C}^0$ there is the open subset V_c defined by

$$\sqrt{|t|} < |z_c| < \delta - \sqrt{|t|}.$$

The principle of the argument says that, as z_c goes once around the circle R of radius $\delta - \sqrt{|t|}$ and centre 0, so $w = z_c + t/z_c$ has exactly one zero inside R, so that the image of R in the w-plane is a simple closed curve $\Gamma(t)$ around 0, and varies smoothly with t for $0 \leq |t| < \delta^2/4$.

Say that $D(t)$ is the open neighbourhood of 0 with boundary $\Gamma(t)$. Then $\cup_{0 \leq |t| < \delta^2/4} D(t)$ is an open submanifold V of $\mathbb{C} \times D_{\delta^2/4}$. Map V_c to V via $w = z_c + t/z_c$; this is unramified, since the branch locus is given by $z_c^2 + t = 0$, and glueing $\mathcal{C}^0$ to V via the map $V_c \to V$ that has just been constructed gives the Schiffer variation of which we speak.

If now $(a_1, ..., a_n)$ are distinct points of C and $z_j = z_{a_j}$ is a local co-ordinate at each, then we can simultaneously construct a Fay variation centred at

$(a_{n-1}, z_{n-1}; a_n, z_n)$ and a Schiffer variation centred at $(a_1, z_1; ...; a_{n-2}, z_{n-2})$. This is a proper map $f : \mathcal{C}^+ \to \Delta^{n-1}$, where now Δ^{n-1} is an $(n-1)$-dimensional complex polydisc with co-ordinates $t_1, t_2, ..., t_{n-1}$, the map f is smooth over the locus $t_{n-1} \neq 0$ and the fibres over $t_{n-1} = 0$ are copies of the nodal curve $C/(a_{n-1} \sim a_n)$. We call it the Fay–Schiffer variation of C centred at $(a_1, z_1; ...; a_n, z_n)$.

Theorem 2.1 *With respect to a suitable fixed homology basis and a correspondingly normalized basis $\omega = (\omega_1, ..., \omega_g)$ of the abelian differentials on C, the period matrix $T(t)$ of C_t can be written in 2×2 block form as*

$$T(t) = \begin{bmatrix} \tau + \sum_1^{n-1} t_j \sigma_j & AJ(t) + t_{n-1} s \\ {}^t(AJ(t) + t_{n-1} s) & \frac{1}{2\pi i}(\log t_{n-1} + c_1 + c_2 t_{n-1}) \end{bmatrix} + O(t^2)$$

where for $1 \leq j \leq n-2$ the matrix σ_j is of rank 1 and is given by

$$(\sigma_j)_{pq} = 2\pi i \left(\frac{\omega_p}{dz_j}(a_j) . \frac{\omega_q}{dz_j}(a_j) \right),$$

the matrix σ_{n-1} is of rank 1 and is given by

$$(\sigma_{n-1})_{pq} = 2\pi i \left(\frac{\omega_p}{dz_{n-1}}(a_{n-1}) - \frac{\omega_p}{dz_n}(a_n) \right) \left(\frac{\omega_q}{dz_{n-1}}(a_{n-1}) - \frac{\omega_q}{dz_n}(a_n) \right),$$

${}^tM$ is the transpose of the matrix M, $AJ(t) = AJ_0(a_n - a_{n-1}) + \sum_{j=1}^{n-2} t_j AJ_j$, AJ_0 is the Abel–Jacobi map $AJ_0(y - x) = \int_x^y \omega$ on C, each AJ_j is a holomorphic function of the parameters a_i, z_i for $i = 1, ..., n-2$, s, c_1, c_2 are holomorphic functions of the parameters a_j, z_j in the construction and c_1 also depends on $t_1, ..., t_{n-2}$.

PROOF: Consider the Schiffer variation of C centred at $(a_1, z_1; ...; a_{n-2}, z_{n-2})$. This gives a genus g family $\Gamma \to \Delta^{n-2}$ where Δ^{n-2} is an $(n-2)$-dimensional polydisc with co-ordinates $t_1, ..., t_{n-2}$ and the period matrix of Γ_t is

$$\left[\tau + \sum_1^{n-2} t_j \sigma_j \right] + O(t^2).$$

(This is due to Patt [P].) By construction, this Schiffer variation is trivial outside neighbourhoods of the points $a_1, ..., a_{n-2}$, and so the points a_{n-1}, a_n and the local co-ordinates z_{n-1}, z_n come along for the ride. So now we make a Fay variation of $\Gamma \to \Delta^{n-2}$ centred at $(a_{n-1}, z_{n-1}; a_n, z_n)$ to get $\mathcal{C} \to \Delta^{n-1}$. The period matrix $T(t)$ of the curve C_t of genus $g+1$ is then

$$\begin{bmatrix} \tau + \sum_1^{n-2} t_j \sigma_j + t_{n-1} \sigma_{n-1} & AJ_{\Gamma_t}(a_n(t) - a_{n-1}(t)) + t_{n-1} s \\ {}^t\left(AJ_{\Gamma_t}(a_n(t) - a_{n-1}(t)) + t_{n-1} s\right) & \frac{1}{2\pi i}\left(\log t_{n-1} + c_1 t_{n-1} + c_2\right) \end{bmatrix} + O(t^2)$$

where AJ_{Γ_t} is the Abel–Jacobi map for the curve Γ_t and each of the terms $AJ_{\Gamma_t}(a_n(t) - a_{n-1}(t))$, s, c_1 and c_2 is a holomorphic function of $t_1, ..., t_{n-2}$ and the parameters $a_1, ..., a_{n-2}$ and $z_1, ..., z_{n-2}$. However, for $t_1 = \cdots = t_{n-2} = 0$ the family $\mathcal{C} \to \Delta^{n-1}$ is just the usual Fay variation of C centred at $(a_{n-1}, z_{n-1}; a_n, z_n)$, and so the Abel–Jacobi term $AJ_{\Gamma_t}(a_n(t) - a_{n-1}(t))$ is independent of the a_j and the z_j; $AJ_{\Gamma_0}(a_n(0) - a_{n-1}(0)) = AJ_C(a_n - a_{n-1})$ and so

$$AJ_{\Gamma_t}(a_n(t) - a_{n-1}(t)) = AJ_C(a_n - a_{n-1}) + \sum_1^{n-2} t_j AJ_j + O(t^2).$$

□

Now suppose that $h : C \to B$ is a morphism of Riemann surfaces of degree n, that $e \in B$ is a point over which h is unramified and that $h^{-1}(e) = \{a_1, ..., a_n\}$. For any local co-ordinate z_e on B at e, define the local co-ordinate z_j on C at a_j to be the pull-back of z_e restricted to a neighbourhood of a_j.

Take the corresponding Fay–Schiffer variation $\mathcal{C}^+ \to \Delta^{n-1}$ of C centred at $(a_1, z_1; ...; a_n, z_n)$, and let $\mathcal{C} \to \Delta$ be the one-parameter family obtained by restricting $\mathcal{C}^+ \to \Delta^{n-1}$ to the diagonal disc Δ in Δ^{n-1} defined by $t_1 = \cdots = t_{n-1} = t$. Let $\mathcal{B} \to \Delta$ be the Schiffer variation of B centred at (e, z_e).

Proposition 2.2 *There is a degree n morphism $H : \mathcal{C} \to \mathcal{B}$ relative to Δ that at $t = 0$ is the morphism $C/(a_{n-1} \sim a_n) \to B$ induced by h.*

PROOF: The Schiffer variation $\mathcal{B} \to \Delta$ is constructed by deleting a disc and then glueing in a new disc with co-ordinate $w = z_e + t/z_e$; the variation $\mathcal{C} \to \Delta$ is constructed by the same formula except where the points a_{n-1}, a_n are identified over $t = 0$. Here we have a complex surface S with co-ordinates X, Y with $XY = t$, and the glueing was given by $X = z_{n-1}, Y = t/z_{n-1}$ and $X = t/z_n, Y = z_n$. So to construct $H : \mathcal{C} \to \mathcal{B}$ it is enough to give the map from S to the w-disc. This is achieved by writing $w = X + Y$. □

Note that for all t, including $t = 0$, the morphism $H_t : C_t \to B_t$ coincides with h outside a union of small open sets. In particular, the ramification data of H_t coincides with those of h away from this union.

Proposition 2.3 *$V_{g,n,tot} \times M_1$ lies in the closure of $V_{g+1,n,tot}$.*

PROOF: Suppose that the curve C is a point in $V_{g,n,tot}$, that $f : C \to \mathbb{P}^1$ is of degree n and that f is totally ramified at $P \in C$. Say $f(P) = e$, so that $f^{-1}(e) = n[P]$. Suppose also that the curve E is a point in M_1. Fix $Q \in E$, and regard E as an elliptic curve with origin Q. Then choose a primitive n-torsion point R on E, so that $n[Q] \sim n[R]$ and there is a rational function $h : E \to \mathbb{P}^1$ such that $h^{-1}(0) = n[Q]$ and $h^{-1}(\infty) = n[R]$. We assume, as we may, that $e \neq 0, \infty$.

We shall construct a variation similar (but not identical) to that described on pp. 37–41 of [F], omitting the topological details. Choose local co-ordinates z_e and z_0 on $\mathbb{P}^1$ at e and 0, respectively. Then there is a local co-ordinate w_P on C at P with $z_e = w_P^n$ and a local co-ordinate w_Q on E at Q with $z_0 = w_Q^n$. Use these to construct variations $\mathcal{C} \to \Delta$ and $\mathcal{B} \to \Delta$, where $\mathcal{B}$ is obtained by glueing $\mathbb{P}^1 \times \Delta$ and $\mathbb{P}^1 \times \Delta$ to the surface $S_n = (X_n Y_n = t^n)$ by

$$X_n = z_e,\ Y_n = t^n/z_e,$$

$$X_n = t^n/z_0,\ Y_n = z_0$$

and $\mathcal{C}$ is obtained by glueing $C \times \Delta$ and $E \times \Delta$ to the surface $S_1 = (X_1 Y_1 = t)$ by

$$X_1 = w_P,\ Y_1 = t/w_P,$$

$$X_1 = t/w_Q,\ Y_1 = w_Q.$$

Via the morphism $S_1 \to S_n$ given by $X_n = X_1^n,\ Y_n = Y_1^n$ there is a morphism $\pi : \mathcal{C} \to \mathcal{B}$ obtained by glueing the morphisms $f \times 1_\Delta : C \times \Delta \to \mathbb{P}^1 \times \Delta$ and $h \times 1_\Delta : E \times \Delta \to \mathbb{P}^1 \times \Delta$. Moreover, since $h \times 1_\Delta$ is totally ramified along $\{R\} \times \Delta$ and the variation $\mathcal{C} \to \Delta$ is trivial outside neighbourhoods of P and of Q, the morphism $\mathcal{C}_t \to \mathcal{B}_t$ is totally ramified somewhere. Since $\mathcal{B}_t \cong \mathbb{P}^1$, the result is proved. □

3 Modular forms vanishing on $V_{g,n,tot}$

We fix an integer n with $3 \leq n \leq g-1$. We are especially interested in those values of n for which a general curve of genus g possesses at most finitely many g^1_n's, so that $n \leq g/2+1$. Then if C is a non-hyperelliptic curve possessing a pencil Π that is a complete g^1_n, the linear span $\langle D \rangle$ of each element D of Π is a copy of $\mathbb{P}^{n-2}$, and as D varies over Π these copies sweep out a rational scroll $\Sigma(\Pi)$ of dimension $n-1$ in $\mathbb{P}^{g-1}$. For example, if $n=3$ then $\Sigma(\Pi)$ is a surface (and is the intersection of the quadrics that contain C).

Suppose that $G = G_{g+1}$ is a Siegel modular form on A_{g+1} such that the restriction $G|_{M_{g+1}}$ of G to M_{g+1} has multiplicity at least m along $V_{g+1,n,tot}$. That is, G and all its partial derivatives F of order $\leq m-1$ with respect to the entries T_{pq} of a period matrix T in $\mathfrak{H}_{g+1}$ in directions tangent to M_{g+1} vanish along $V_{g+1,n,tot}$. We can define the Siegel Φ-operator on the derivatives by

$$\Phi(F)(\tau) = \lim_{t \to i\infty} F\begin{pmatrix} \tau & 0 \\ 0 & t \end{pmatrix}.$$

Lemma 3.1 *$\Phi(F)$ is a derivative of $\Phi(G)$ of order $\leq m-1$ in directions tangent to M_g and vanishes along $V_{g,n,tot}$.*

PROOF: By construction, $\Phi(F)$ can be computed by restricting to $A_g \times A_1$, then restricting to $A_g \times \{j\}$ for some $j \in A_1$, and finally letting $j \to \infty$. Since the intersection of M_{g+1} and $A_g \times A_1$ certainly contains $M_g \times M_1$, the first part of the lemma is proved. The second part then follows from Proposition 2.3. □

Theorem 3.2 *Under these assumptions, the restriction* $\Phi(G) = G|_{M_g}$ *has multiplicity at least* $m+1$ *along* $V_{g,n,tot}$.

PROOF: We need to show that $\Phi(F)$ is singular along $V_{g,n,tot}$. Now F has a Fourier expansion

$$F(T) = \sum_{X \in S_{g+1}} a(X) \exp \pi i \operatorname{tr}(XT),$$

where T is a point in Siegel space $\mathfrak{H}_{g+1}$ and S_n is the lattice of positive semi-definite $n \times n$ symmetric matrices over $\mathbb{Z}$ whose diagonal is even.

Take a curve C in $V_{g,n,tot}$, and choose any reduced divisor $D = \sum_1^n a_j$ in the specified g_n^1 on C. Let $h : C \to B = \mathbb{P}^1$ be the morphism defined by this g_n^1 and say that $D = h^{-1}(e)$ and that h is totally ramified at P. We have, according to Proposition 2.2, a 1-parameter Fay–Schiffer variation $\mathcal{C} \to \Delta$ of C centred at $(a_1, z_1; ...; a_n, z_n)$ with a degree n morphism to the Fay–Schiffer variation $\mathcal{B} \to \Delta$ of B centred at (e, z_e). Since $B = \mathbb{P}^1$, the variation $\mathcal{B} \to \Delta$ is trivial, so that for $t \neq 0$ the curve C_t lies in $V_{g+1,n}$. Moreover, because the variation is constructed to be trivial outside a neighbourhood of D, the curve C_t lies in $V_{g+1,n,tot}$.

Now the argument follows [CSB] closely.

Take $T = T(t)$ to be the period matrix of C_t as above. Note that since $t_1 = \cdots = t_{n-1} = t$, we can re-arrange c_1 and c_2 so that both of them are independent of t, and are holomorphic functions only of the parameters (e, z_e). Then

$$F_{g+1}(T) = \sum_{X \in S_{g+1}} a(X) \exp \pi i \sum_{p,q=1}^{g+1} x_{pq} T_{pq}$$

where $X = (x_{pq})$. Our aim is to examine the coefficient of t in the expansion of this expression in powers of t, so calculate modulo t^2. Since $\exp 2\pi i T_{g+1,g+1} \equiv t.\exp c_1.\exp(c_2 t)$ modulo t^2, it follows that

$$(F_{g+1})(T) \equiv \sum_{x_{g+1,g+1}=0} + \sum_{x_{g+1,g+1}=2}$$

modulo t^2, since all terms with $x_{g+1,g+1} \geq 4$ vanish modulo t^2. Here $\sum_{x_{g+1,g+1}=r}$ denotes the sum over $X \in S_{g+1}$ with $x_{g+1,g+1} = r$, for $r = 0$ or 2. Therefore, modulo t^2,

$$\sum_{x_{g+1,g+1}=0} \equiv \sum_{X \in S_g} a(X) \exp \pi i \sum_{p,q=1}^{g} x_{pq}(\tau_{pq} + t\sigma_{pq})$$

and

$$\sum_{x_{g+1,g+1}=2} \equiv t.\exp c_1. \textstyle\sum_{X \in S_{g+1}, x_{g+1,g+1}=2} a(X)$$
$$.\exp\left(2\pi i \textstyle\sum_{p=1}^{g} x_{p,g+1} \int_{a_{n-1}}^{a_n} \omega_p\right).\exp\left(\pi i \textstyle\sum_{p,q=1}^{g} x_{pq}\tau_{pq}\right).$$

So the coefficient of t is $A + B \exp c_1$, where

$$A = \sum_{x_{g+1,g+1}=0} a(X) \left(\pi i \sum_{p,q=1}^{g} x_{pq}\sigma_{pq} \right) \left(\exp \pi i \sum_{p,q=1}^{g} x_{pq}\tau_{pq} \right),$$

$$B = \sum_{x_{g+1,g+1}=2} a(X) \left(\exp 2\pi i \sum_{p=1}^{g} x_{p,g+1} \int_{a_{n-1}}^{a_n} \omega_p \right) \left(\exp \pi i \sum_{p,q=1}^{g} x_{pq}\tau_{pq} \right).$$

By assumption, $A + B \exp c_1$ vanishes identically.

Now rescale the local co-ordinate z_e; that is, given any non-zero scalar λ, replace z_e by $\lambda^{-1} z_e$. Such a rescaling will produce a different family $\mathcal{C} \to \Delta$ with C_t in $V_{g+1,n,tot}$ for all $t \neq 0$, but the quantity $A + (\exp c_1)B$ will still vanish for the rescaled family. Moreover, B is invariant under this rescaling, as is revealed by a cursory inspection. Also c_1 is a holomorphic function of λ because the entries of a period matrix are holomorphic functions of the parameters.

Lemma 3.3 $A = B = 0$.

PROOF: From the description above of σ_{pq}, this rescaling multiplies σ_{pq} by λ^2, so that A can be written as

$$A = C\lambda^2$$

with C independent of λ. So we have an identity

$$C\lambda^2 = -B \exp(c_1(\lambda))$$

of holomorphic functions on the 1-dimensional algebraic torus $\mathbb{G}_m = \mathbf{Spec}\, \mathbb{C}[\lambda^{\pm}]$, where B, C are constant functions on $\mathbb{G}_m$. The result follows at once. □

Now A can also be written as

$$A = \left.\frac{\partial}{\partial t}\right|_{t=0} \left(\sum_{X \in S_g} a(X) \exp \pi i \sum_{pq,=1}^{g} x_{pq}(\tau_{pq} + t\sigma_{pq}) \right)$$

$$= \left.\frac{\partial}{\partial t}\right|_{t=0} F_g(\tau + t\sigma).$$

That is, σ lies in the Zariski tangent space H at the point τ to the divisor in $\mathfrak{H}_g$ defined by the function F_g. It is important to note that, from this description, H depends upon C but is independent of any of the other parameters (points, local co-ordinates) used to construct the variation. Thus H contains every σ that arises from different choices of these other parameters.

Assume that C has no non-trivial automorphisms. Then there are the standard classical natural identifications of tangent spaces to moduli given by

$$T_{[C]}M_g = H^0({\Omega^1_C}^{\otimes 2})^\vee,$$

$$T_{[C]}A_g = \mathrm{Sym}^2\, H^0(\Omega^1_C)^\vee.$$

The inclusion $T_{[C]}M_g \hookrightarrow T_{[C]}A_g$ is dual to the natural multiplication (which is surjective, by Max Noether's theorem) $\mathrm{Sym}^2 H^0(\Omega^1_C) \to H^0(\Omega^{1\ \otimes 2}_C)$. So the vector space of quadrics in $\mathbb{P}^{g-1}$ can be regarded as the space of linear forms on $T_{[C]}A_g$, and then $T_{[C]}M_g$ is the subspace of $T_{[C]}A_g$ defined by the vanishing of those quadrics in $\mathbb{P}^{g-1}$ that contain C.

We know that the tangent space H to the divisor $(F_g = 0)$ at the point τ in $\mathfrak{H}_g$ contains every matrix σ that arises as above. Projectivize, and use the classical descriptions above of the tangent spaces to moduli. Then (the projectivization of) H is a hyperplane in $\mathbb{P}(\mathrm{Sym}^2 H^0(C, K_C))^\vee$ that contains every point $\sigma(n-1,n) = \sigma = (\sigma_{pq})$ of the form

$$\sigma_{pq} = (\omega_p(a_n) - \omega_p(a_{n-1}))\,(\omega_q(a_n) - \omega_q(a_{n-1})) + \sum_{j=1}^{n-2} \omega_p(a_j)\omega_q(a_j),$$

where we have omitted a factor of $2\pi i$ and the factors of dz_e that should appear as denominators. We can also regard H as a quadric in the $\mathbb{P}^{g-1}$ in which C is canonically embedded, and then what we have to prove is that H contains C.

We shall in fact prove a stronger statement, namely that H contains the scroll $\Sigma(\Pi)$ (which certainly contains C) that is mentioned in the first paragraph of this section.

In $\mathbb{P}^{g-1}$, any element $D = \sum_{j=1}^n a_j$ of the given pencil Π spans a copy $L = L_D$ of $\mathbb{P}^{n-2}$; the points $a_1, \dots, a_n$ are, therefore, in general position in L. Regard L as the projectivization of an $(n-1)$-dimensional vector space W and the points a_j as projectivizations of vectors $w_j = (\omega_1(a_j), \dots, \omega_g(a_j))$ in W. Consider the second Veronese embedding $Ver_2(L_D)$ in a copy $\mathbb{P}^N_D$ of $\mathbb{P}^N$, where $N+1 = n(n-1)/2$ and $\mathbb{P}^N_D$ is a linear subspace of the projectivized tangent space $\mathbb{P}(T_{[C]}A_g)$. Then H contains the point (in the projectivization of $\mathrm{Sym}^2 W$)

$$\sigma(n-1,n) = (w_{n-1} - w_n)^2 + \sum_1^{n-2} w_j^2;$$

the same argument shows that H also contains every other point $\sigma(k,l)$, for $k < l$, that is obtained from $\sigma(n-1,n)$ by permutation of the vectors $w_1, ..., w_n$. The $\sigma(k,l)$ form a set of $N+1$ points in $\mathbb{P}^N_D$.

Lemma 3.4 *These $N+1$ points span $\mathbb{P}^N_D$.*

PROOF: We can assume that $\sum w_j = 0$. So W is the irreducible $(n-1)$-dimensional representation of the symmetric group $\mathfrak{S}_n$ as a Coxeter group of type A_{n-1}. Let $\mathbb{1}$ denote the trivial 1-dimensional representation, so that $W \oplus \mathbb{1}$ is the standard permutation representation V with standard basis $(v_1, ..., v_n)$ and $\mathbb{1}$ is the line generated by $e_1 = \sum v_i$. Let $\pi_1 : V \to W$ be the projection, so that $\pi_1(v_i) = w_i$. Let $V_2 \subset \mathrm{Sym}^2 V$ be the module of cross-terms.

Lemma 3.5 $\mathrm{Sym}^2 V = e_1.V \oplus V_2$.

PROOF: It is enough to show that $e_1.V \cap V_2 = 0$, since both sides have the same dimension. So take $Q \in e_1.V \cap V_2$; evaluating the polynomial Q at each of the points $(1, 0, ..., 0), ..., (0, ..., 0, 1)$ in turn shows that $Q = 0$, because $Q \in V_2$, and Lemma 3.5 is proved. □

So the projection $\pi_1 : V \to W$ induces an inclusion $\pi_2 : V_2 \to \mathrm{Sym}^2 W$; since both terms have the same dimension, π_2 is an isomorphism. Hence the cross-terms $w_k w_l$ form a basis of $\mathrm{Sym}^2 W$.

Now $\sigma(k, l) = -2\pi_2(e_2) - 2w_k w_l$. Suppose that there is a linear relation

$$\sum_{k,l} \lambda_{kl}\sigma(k, l) = 0;$$

then $\sum \lambda_{kl}\pi_2(e_2) = -\sum \lambda_{kl} w_k w_l$. Since $\pi_2(e_2) = \sum w_k w_l$ and the $w_k w_l$ are linearly independent, we get

$$\lambda_{ij} = -\sum_{k,l} \lambda_{kl}$$

for all i, j. Then $\lambda_{ij} = 0$, and the lemma is proved. □

It follows that H contains $\mathbb{P}^N_D$, and therefore contains $Ver_2(L_D)$ for every reduced divisor D in Π, the g^1_n under consideration. So indeed H, when regarded as a quadric in $\mathbb{P}^{g-1}$, contains the rational scroll $\Sigma(\Pi)$. □

Corollary 3.6 *Assume that $n \geq 3$ and that $m \geq 1$. Then the intersection $V^S_{g+m,n,tot} \cap M_g$ contains the mth order infinitesimal neighbourhood of $V_{g,n,tot}$ in M_g.*

PROOF: Suppose that Φ is some modular form on A_{g+1} such that $(\Phi)_0 \cap M_{g+1}$ is singular, with multiplicity m, along $V_{g+1,n,tot}$. That is, Φ and all its derivatives of order at most $m-1$, taken in directions along M_{g+1}, vanish along $V_{g+1,n,tot}$.

Suppose that F is such a derivative. Then it follows from what we have shown that the restriction $F|_{M_g}$ is singular along $V_{g,n,tot}$. That is, the restriction $\Phi|_{A_g}$ of Φ to A_g and all derivatives of $\Phi|_{A_g}$ of order at most m, taken in directions along M_g, vanish along $V_{g,n,tot}$. This follows from Lemma 3.1. □

Theorem 3.7 *Fix $n \geq 3$. Then there is no stable Siegel modular form that vanishes on the totally ramified n-gonal locus $V_{g,n,tot}$ for every g.*

PROOF: Suppose that F is such a modular form. Then, by Corollary 3.6, F vanishes on M_g for every g. But the main result of [C-SB] is that then $F = 0$. □

The main result of [G-SM] is that the the Schottky form $F = \Theta_{E_8^2} - \Theta_{D_{16}^+}$ (the difference of two theta series associated to the positive even unimodular lattices E_8^2 and D_{16}^+ of rank 16) that, by results of Schottky [S] and Igusa [I1], [I2], defines M_4 inside A_4, does not vanish along M_5. They prove further that it cuts out the exactly trigonal locus $V_{5,3}$ in M_5, and does so with multiplicity 1.

Corollary 3.8 *In genus* 6 *the Schottky form* F *does not vanish along the totally ramified trigonal locus.*

PROOF: Suppose that F_6 vanishes along $V_{6,3,tot}$. Then, by Theorem 3.2, the restriction $F_5|_{M_5}$ of F_5 to M_5 is singular along $V_{5,3,tot}$. Then the trigonal locus $V_{5,3}$ is singular along the subvariety $V_{5,3,tot}$. But the trigonal locus is smooth outside the hyperelliptic locus, and we are done. □

For $g = 6$ there is another subvariety of M_g that is distinguished by the fact that the canonical model is not an intersection of quadrics, namely the locus of plane quintics. Our techniques, however, cannot let us decide whether F vanishes along this locus; more generally, they cannot handle g^r_d's with $r \geq 2$.

4 The even genus case

Suppose that $g = 2(n-1)$ is even. Then a general curve of genus g has a finite, but non-zero, number of g^1_n's, while the locus $V_{g+1,n}$ is an irreducible divisor in M_{g+1} (and a general curve in $V_{g+1,n}$ has a unique g^1_n).

Fix a general curve C of genus $g = 2(n-1)$, and let $\Pi_1, ..., \Pi_r$ be the g^1_n's on it. (The number r is a known function of g, but all we need is that $r \geq 4$ when $g \geq 6$.) As above, the members of each Π_i sweep out a scroll $\Sigma_i = \Sigma(\Pi_i)$ in $\mathbb{P}^{g-1}$ that contains C.

Lemma 4.1 *If* $g \geq 6$*, then there is no quadric in* $\mathbb{P}^{g-1}$ *that contains every* Σ_i*.*

PROOF: Choose any $a \in C$. For every i there is a unique $D_i \in \Pi_i$ passing through a. Say $D_i = a + \sum_{j=2}^n b_{ij}$ and $L_i = \langle D_i \rangle$. Suppose that there is a hyperplane H in $\mathbb{P}^{g-1}$ that contains each L_i; then

$$H.C \geq a + \sum_{ij} b_{ij},$$

so that $2g - 2 \geq 1 + r(n-1)$. Since $r \geq 4$ this is impossible, and there is no such hyperplane. Since the L_i are linear, this means that $\cup L_i$ has embedding dimension $g-1$ at a.

Now suppose that Q is a quadric that contains every Σ_i. Then Q contains $\cup L_i$, and so has embedding dimension $g-1$ at every point of C. However, the singular locus of a quadric is linear, and we are done. □

This is false for $g = 4$; there are two g^1_3's, but the scrolls Σ_1 and Σ_2 coincide, and are the unique quadric containing C.

Theorem 4.2 *The* n*-gonal divisor* $V_{g+1,n}$ *in* M_{g+1} *has contact with* A_g *along* M_g*.*

PROOF: We need to show that for any modular form $F = F_{g+1}$ on A_{g+1} that vanishes along $V_{g+1,n}$, the restriction F_g of F to A_g is singular along M_g. But this follows from the proof of Theorem 3.2 (the entire proof, except for the final paragraph): if F_g is smooth on A_g at the point $[C]$ of M_g, then the tangent hyperplane H to A_g corresponds, if it is non-zero, to a quadric in $\mathbb{P}^{g-1}$ that contains every scroll $\Sigma(\Pi_i)$. But we have just seen that there is no such quadric. $\square$

I am very grateful to Giulio Codogni for many conversations on these subjects and to the organizers of the Edinburgh meeting on modular forms in November 2012 for the stimulating environment that led to the writing of this paper.

References

[C] G. Codogni, *Non-perturbative Schottky problem and stable equations for the hyperelliptic locus*, arXiv:1306.1183

[CSB] G. Codogni and N.I. Shepherd-Barron, *The non-existence of stable Schottky forms*, arXiv:1112.6137; Compositio Math., to appear.

[F] J. Fay, *Theta functions on Riemann surfaces*, LNM **352**, Springer, 1973.

[G-SM] S. Grushevsky and R. Salvati Manni, *The superstring cosmological constant and the Schottky form in genus* 5, Am. J. Math. **133** (2011), 1007-1037 (erratum, **134** (2012), 1139-1142), and arXiv:0809.1391v5.

[I1] J.-I. Igusa, *Schottky's invariant and quadratic forms*, in E. B. Christoffel Int. Symp., Aachen (1981), 352-362.

[I2] ———, *On the irreducibility of Schottky's divisor*, J. Fac. Sci. Univ. Tokyo Section IA Math., **28** (1981), 531-545.

[S] F. Schottky, *Zur Theorie der Abelschen Functionen von vier Variabeln*, J. Reine Angew. Math. **102** (1888), 304-352.

D.P.M.M.S., University of Cambridge, Cambridge CB3 0WB U.K.
E-mail address: nisb@dpmms.cam.ac.uk