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Livingstone Boomla, Alice Jane

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Selmer groups, zeta elements and refined Stark conjectures

Alice Livingstone Boomla

Supervised by David Burns

A thesis presented for the degree of

Doctor of Philosophy in Pure Mathematics

January 2018

Abstract

In this thesis we study explicit connections between the values at $s = 0$ of the higher derivatives of Dirichlet L-functions and the higher Fitting ideals of Selmer groups of the multiplicative group over finite abelian extensions of number fields.

We also prove new structural results for such Selmer groups, showing that their higher Fitting ideals admit natural direct sum decompositions.

The first of our main results allows us to show that certain canonical invariants that are associated to (generalised) Rubin-Stark elements by Vallières in [28] can be completely, though in general only conjecturally, described in terms of the higher Fitting ideals of the Selmer groups of $\mathbb{G}_m$.

Following on from this observation, we then formulate a refined conjecture, which extends the existing theory of abelian Stark conjectures in two key ways.

Firstly, our conjecture deals for the first time in a consistent way with L-functions that do not necessarily have 'minimal' order of vanishing at $s = 0$ and secondly it includes an important 'boundary case' that has been excluded from all previous formulations of conjectures in this area.

We also present evidence, both theoretic and numerical, for the conjectures that we formulate. In particular, we prove that our conjectures would follow from the validity of the relevant special case of equivariant Tamagawa number conjecture and are therefore, for example, unconditionally true in the classical setting of abelian extensions of $\mathbb{Q}$.

Acknowledgements

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Chapter 1

Introduction

1.1 Historical motivation

Through the study of particular values of analytic objects called zeta functions, or L-functions we can gain much important information about number fields. The most famous example of this is the class number formula

$$\lim_{z \to 1}(z-1)\zeta_K(z) = \frac{2^{r_1} \cdot (2\pi)^{r_2} \cdot h_K \cdot R_K}{w_K \cdot \sqrt{|D_K|}} \qquad (1.1)$$

first proved for quadratic fields in 1839 by Dirichlet and later for any number field K by Dedekind. Before we explain how this is used, we will explain a little of the historical background.

Infinite series have been studied for centuries, their sums often containing surprising and elegant results.

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{6} \qquad (1.2)$$

This is the value at $z = 2$ of the Riemann zeta function

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$$

and was first calculated by Leonhard Euler in the early eighteenth century. He introduced the zeta function, as a function of the real variable z, more than a century before Riemann extended the definition to a complex variable and showed a relationship between its zeroes and the distribution of prime numbers. Euler also showed that the zeta function had an infinite product expansion

$$\zeta(z) = \prod_{p \text{ prime}} \frac{1}{1 - p^{-z}}.$$

Following on from Euler's work, the zeta function has been generalised and replicated and zeta functions and L-functions now play a central role in modern number theory. Their study has proved fruitful in numerous ways and, as we shall see, their special values often encode useful arithmetic and algebraic information not easily obtainable through other methods.

The first generalisation came in 1837 by Dirichlet, who introduced the use of the letter L to describe such series and used them to prove that there are an infinite number of primes in any (primitive) arithmetic progression. He defined the L-function

$$L(z, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^z} = \prod_{p \text{ prime}} \frac{1}{1 - \chi(p)p^{-z}}$$

and showed that the Euler product converged for $\text{Re}(z) > 1$, where χ is a Dirichlet

character modulo m, a multiplicative map $\chi : \mathbb{Z} \to \mathbb{C}$, which is periodic on some integer m and such that $\chi(n) \neq 0$ if and only if n and m are coprime. If χ is trivial then this specialises to the Riemann zeta function.

Example 1.1.1. Let χ_4 be the non-principal character modulo 4, so that $\chi(n) = 1$ if $n = 1 \mod 4$, -1 when $n = 3 \mod 4$ and 0 otherwise. Then

$$L(1, \chi_4) = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \cdots = \frac{\pi}{4}$$

Another generalisation came from the Dedekind zeta function, attached to a general number field K.

$$\zeta_K(z) = \prod_{\mathfrak{P} \text{ a prime of } K} \frac{1}{1 - \mathbf{N}\mathfrak{P}^{-z}}$$

(see Chapter 2 for necessary notations and definitions). If $K = \mathbb{Q}$ this again specialises to the Riemann zeta function. Further for any abelian number field K, we have

$$\zeta_K(z) = \prod_{\chi} L(z, \chi), \tag{1.3}$$

where χ runs over the characters associated to the Galois group of $K/\mathbb{Q}$.

We can now return to the class number formula in (1.1). The class number h_K of a number field K is of great interest to number theorists. It can be thought of as measuring, in some way, the failure of unique factorisation in the ring of integers $\mathcal{O}_K$. If $\mathcal{O}_K$ is a unique factorisation domain, for example $\mathcal{O}_K = \mathbb{Z}$ when $K = \mathbb{Q}$, then $h_K = 1$.

Example 1.1.2. Let $K = \mathbb{Q}(i)$. The character χ_4 in Example 1.1.1 is the unique non trivial character of $\mathbb{Q}(i)/\mathbb{Q}$. Therefore (1.3) becomes $\zeta_{\mathbb{Q}(i)}(z) = \zeta(z)L(z, \chi_4)$. It is well

known that the Riemann zeta function has a simple pole at $z = 1$. Since $L(1, \chi_4) = \frac{\pi}{4}$ we have

$$\lim_{z \to 1}(z - 1)\zeta_{\mathbb{Q}(i)}(z) = \frac{\pi}{4}.$$

This is consistent with the class number formula. Since $\mathbb{Q}(i)$ is an imaginary quadratic field, $r_1 = 0$ and $r_2 = 1$. The regulator is trivial, the roots of unity are $\{\pm 1, \pm i\}$, $w_{\mathbb{Q}(i)} = 4$, $D_{\mathbb{Q}(i)} = -4$ and its ring of integers $\mathbb{Z}[i]$ is a unique factorisation domain so $h_{\mathbb{Q}(i)} = 1$.

We can relate the values of zeta functions and L-functions at $z = 1$ and $z = 0$ using the functional equation. This gives some surprising results. For example the simple pole at $z = 1$ of $\zeta(z)$ implies that

$$\zeta(0) = 1 + 1 + 1 + 1 + \cdots = -\frac{1}{2} \tag{1.4}$$

Via the functional equation we can rewrite the class number formula much more simply as the Taylor expansion at $z = 0$ of $\zeta_K(z)$,

$$\zeta_K(z) = -\frac{h_K R_K}{w_K} z^{r_1 + r_2 - 1} + \text{(higher order terms)}. \tag{1.5}$$

Example 1.1.3. Let $K = \mathbb{Q}$, so $\zeta_K(0) = \zeta(0) = -\frac{1}{2}$. Then $r_1 + r_2 - 1 = 0$ so (1.5) expresses the fact that the regulator is trivial, there are two roots of unity and the class number is one.

The factorisation of $\zeta_K(z)$ given in (1.3) implies that $\frac{h_K R_K}{w_K}$ can be factorised as the product of the leading terms at $z = 0$ of the L-functions corresponding to each character

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χ. This therefore motivates an area of number theory studying the leading terms of L-functions at special values, such as $z = 0$.

1.2 Stark Conjectures

In the 1970s and 80s Stark wrote a seminal series of four papers [25] in which he formulated conjectures concerning special values of Artin L-functions, a generalisation of the L-functions defined above to arbitrary extensions of number fields. In the final paper he formulated a conjecture for abelian L-functions with order of vanishing one at $z = 0$. Stark's work was reinterpreted and extended by Tate in [26] and a large number of related conjectures were subsequently formulated and studied by many different authors. A central component of these conjectures are so called 'Stark elements', which 'evaluate' the leading term of the L-functions.

Of particular interest to us is the so-called 'Rubin-Stark conjecture', formulated by Rubin in [24, Conj. B']. This is an 'abelian rank r' Stark's conjecture in that one considers S-truncated L-functions for characters that factor through some abelian extension of number fields K/k, where S is a finite set of places of k containing r places which split completely in K so that the truncated L-functions each vanish to order at least r at $z = 0$. The conjecture then asserts the existence of a canonical 'Rubin-Stark element' that acts as an 'evaluator' for the values at $z = 0$ of the r-th derivatives of the truncated L-functions.

In later work Emmons and Popescu [14] considered the more general situation in which one assumes that the truncated L-functions vanish to order at least r at $z = 0$ but

not that S contains a prescribed subset of splitting places. In this setting they defined a natural generalisation of the Rubin-Stark element and conjectured that it satisfies precise integrality conditions.

A little earlier Burns [1] had shown that a special case of the very general 'equivariant Tamagawa number conjecture' (from [4]) implied the validity of the Rubin-Stark conjecture, as well as that of a host of related conjectures due to Gross, to Tate and to Popescu among others. The approach used in [1] was then adapted by Vallières in [28] to show that the same case of the equivariant Tamagawa number conjecture also implied the validity of a refined version of the conjecture of Emmons and Popescu.

In some more recent work Burns, Kurihara and Sano have significantly improved upon the proof in [1] and provided new methods for studying abelian Stark type conjectures (see [6]).

1.3   This thesis

In this thesis we use the methods of Burns, Kurihara and Sano involving higher Fitting ideals and Selmer groups to formulate abelian Stark conjectures in a more general setting. The main contents are as follows.

In Chapter 2 we recall the conjectures referred to above in greater detail, as well as giving necessary notations and definitions. In Chapter 3 we provide the necessary preliminary results, including the formulation of the relevant case of the equivariant Tamagawa number conjecture, known as the leading term conjecture.

In Chapter 4 we will apply the new methods of Burns, Kurihara and Sano to study Stark

elements of the type defined by Popescu, Emmons and Vallières and, in so doing, we will show that the proof of the main result of Vallières in [28] can be both significantly simplified and improved (for details see §4.3). This allows us to formulate a stronger conjecture.

In particular, while the conjecture of Emmons and Popescu asserts that for any finite abelian extension of number fields, K/k, images under certain homomorphisms of their Stark elements form an ideal of $\mathbb{Z}[\mathrm{Gal}(K/k)]$, in this paper we are led to conjecture that this ideal can be described precisely in terms of the r-th Fitting ideal over $\mathbb{Z}[\mathrm{Gal}(K/k)]$ of the canonical Selmer groups that one can associate to the multiplicative group $\mathbb{G}_m$ over K (see §3.2.1 for details on Selmer groups).

To do this we fix such a K/k with Galois group G, let S be a non-empty finite set of places of k containing $S_\infty(k)$ and $S_{\mathrm{ram}}(K/k)$ and let T be a finite set of places of k disjoint from S. We assume $\mathcal{O}_{K,S,T}^\times$ is $\mathbb{Z}$-torsion free (see §2.1.1 for definitions).

We let r be the minimum order of vanishing of the S-truncated T-modified L-functions and use their r-th derivatives to define, for certain subsets I of S of cardinality r, a canonical element

$$\eta_{K/k,S,T}^I$$

in the exterior power module $\mathbb{C} \otimes \bigwedge_G^r \mathcal{O}_{K,S,T}^\times$ (the precise definition is given in §4.2). These elements agree with the elements defined by Vallières in [28], but are defined in greater generality.

Our main result from Chapter 4 is stated below and described in full in Theorem 4.2.6. It states that our conjectural description of the ideal generated by the elements $\eta_{K/k,S,T}^I$ follows from the validity of the leading term conjecture (LTC(K/k)) (and from which

we are able to deduce that it is unconditionally true in some important cases). A crucial part of this approach involves showing that the elements $\eta^I_{K/k,S,T}$ can be computed as the 'canonical projection' of the zeta element $z_{K/k,S,T}$ arising from the formulation of $\mathrm{LTC}(K/k)$.

Theorem 1.3.1. Let K/k be an abelian extension of number fields. Assume the equivariant Tamagawa number conjecture holds for K/k. Assume the conditions of Theorem 4.2.4 hold. Then we have

$$\mathrm{Fit}^r_G(\mathrm{Sel}^T_S(K)^{\mathrm{tr}}) = \bigoplus_{I \in \wp_r(S_{\min})} \left\{ \Phi(\eta^I_{K/k,S,T}) : \Phi \in \bigwedge_G^r \mathrm{Hom}(\mathcal{O}^\times_{K,S,T}, \mathbb{Z}[G]) \right\}.$$

The theorem above contains a key limitation, that like other conjectures in the area, it excludes an important boundary case ($S = S_{\min}$).

In Chapter 5 we develop a theory that extends the work in Chapter 4 and all other existing conjectures in the area in two key ways. Firstly it deals with L-functions that do not necessarily have minimal order of vanishing and secondly it treats on an equal footing the natural boundary case previously excluded. For K/k, S and T as above and particular subsets I of S of any cardinality a we will define 'Stark elements of arbitrary rank' $\eta^I_{K/k,S,T} \in \mathbb{C} \otimes \bigwedge_G^a \mathcal{O}^\times_{K,S,T}$.

Our main algebraic result in Chapter 5, Theorem 5.2.1 proves that the higher Fitting ideals, in the sense discussed by Northcott in [22] admit a natural direct sum decomposition. This decomposition naturally leads us to make a higher order abelian Stark conjecture for arbitrary order of vanishing a. This conjecture specialises to give refined versions of previous conjectures.

In a generalisation of the work in the previous chapter, we show that our conjecture follows from the validity of $\mathrm{LTC}(K/k)$ and prove the following result.

Theorem 1.3.2. Assume $\mathrm{LTC}(K/k)$ and the hypotheses of Theorem 5.2.1. Fix a non-negative integer a with $a < |S|$ and let $\eta^I = \eta^I_{K/k,S,T}$ be the Stark element of rank a and set $\eta^a_k := \eta^a_{k,S,T}$ all as defined in §5.1.

Then for each place v in S one has

$$(1 - e_v + e_{\mathbf{1}})e_{(a),S} \cdot \mathrm{Fit}^a_G(\mathrm{Sel}^T_S(K))^{\#}$$

$$= c^a_{S,v} e_{\mathbf{1}} \cdot \left\{ \Phi_G(\eta^a_k) : \Phi \in \bigwedge_G^a \mathrm{Hom}_G(\mathcal{O}^\times_{K,S,T}, \mathbb{Z}[G]) \right\}$$

$$\oplus \bigoplus_{I \in \wp_a(S,v)} \left\{ \Phi(\eta^I) : \Phi \in \bigwedge_G^a \mathrm{Hom}_G(\mathcal{O}^\times_{K,S,T}, \mathbb{Z}[G]) \right\},$$

where $c^a_{S,v} \in \mathbb{Z}$ is as defined in §5.2 and $x \mapsto x^{\#}$ is the $\mathbb{Z}$-linear involution on $\mathbb{Z}[G]$ that inverts elements of G.

In particular, for every I in $\wp^*_a(S)$ and every Φ in $\bigwedge_G^a \mathrm{Hom}_G(\mathcal{O}^\times_{K,S,T}, \mathbb{Z}[G])$ one has both

$$\mathfrak{n}^a_{S,T}(K/k) \cdot \Phi(\eta^I) \subseteq \mathbb{Z}[G] \quad \text{and} \quad \mathfrak{m}^a_{S,T}(K/k) \cdot \Phi(\eta^I) \subseteq \mathrm{Fit}^a_G(\mathrm{Sel}^T_S(K)),$$

where $\mathfrak{n}^a_{S,T}(K/k)$ and $\mathfrak{m}^a_{S,T}(K/k)$ are the ideals of $\mathbb{Z}[G]$ explicitly defined after Theorem 5.2.1.

In Chapter 6, we conclude by exploring two explicit examples. The first example shows how the results of Chapter 5 can be applied when $a > r$ in cases of non-minimal order of vanishing to obtain predictions finer than previously existing conjectures. The second

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example is a boundary case excluded from Chapter 4, and again shows how the results of Chapter 5 may be applied to make integrality predictions beyond that allowed by the relevant case of the Rubin-Stark conjecture.

1.4   Future directions

While the results of this paper hold for abelian extensions of number fields, there is also by now a rich theory of non-abelian Stark conjectures and more time would have allowed for an attempt to generalise our results into this setting, as outlined below.

In a recent preprint [8] Burns and Sano extend their previous work with Kurihara in [6] concerning zeta elements to the non-abelian setting. In particular, in this article they associate natural notions of 'non-abelian zeta elements', of 'Selmer groups' and of 'Weil-ètale cohomology complexes' to the multiplicative group $\mathbb{G}_m$ over finite (non-abelian) Galois extensions L/K of global fields.

At the same time, they also introduce natural non-commutative generalisations of several well-known constructions in commutative algebra including the notions of exterior powers, determinant modules and higher Fitting invariants.

By combining these algebraic and arithmetic concepts they are able to define, among other things, a natural notion of 'non-abelian Rubin-Stark elements' and to formulate a non-abelian generalisation of the Rubin-Stark conjecture that incorporates both extensions and refinements of the 'universal refined non-abelian Stark conjectures' that are discussed by Burns in [2].

They are also able to derive from this general conjectural formalism explicit formulae

in terms of the leading terms of Artin L-series at zero for the higher non-commutative Fitting invariants of the Selmer groups of $\mathbb{G}_m$ for L/K and, by using non-abelian zeta elements, to show these formulae would follow from the validity of the equivariant Tamagawa number conjecture for the pair $(h^0(\mathrm{Spec}(L)), \mathbb{Z}[\mathrm{Gal}(L/K)])$.

It seems to us highly likely that the general approach of Burns and Sano could be developed in order to formulate a natural extension to non-abelian Galois extensions of the results, and conjectures, that we discuss in this thesis.

In this way one could expect to both extend and refine the general theory of non-abelian Stark conjectures developed by Burns and Sano in [8] by dealing both with Artin L-functions that do not necessarily have minimal order of vanishing at zero and also with the (non-abelian version of the) 'boundary case' that has been excluded from all previous treatments.

In this thesis we often use the fact that the equivariant Tamagawa number conjecture for the pair $(h^0(\mathrm{Spec}(L)), \mathbb{Z}[\mathrm{Gal}(L/K)])$ has been verified for several important families of abelian extensions L/K. At this stage, however, there is still much less evidence for the equivariant Tamagawa number conjecture for $(h^0(\mathrm{Spec}(L)), \mathbb{Z}[\mathrm{Gal}(L/K)])$ for non-abelian Galois extensions L/K.

Nevertheless, in this context, the fundamental work of Ritter and Weiss in [23] (and the later results of Kakde in [18]) establishing the validity, under natural hypotheses, of the main conjecture of non-commutative Iwasawa theory for totally-real fields already plays a key role.

In particular, by combining the main result of [23] with the explicit descent computations in non-commutative Iwaswa theory performed in [3] and recent results in [10]

of Dasgupta, Kakde and Ventullo concerning the Gross-Stark conjecture, Burns and Sano have obtained strong evidence in [8] for the 'minus part' of the equivariant Tamagawa number conjecture for $(h^0(\mathrm{Spec}(L)), \mathbb{Z}[\mathrm{Gal}(L/K)])$ for certain natural families of non-abelian Galois CM extensions L/K of totally real fields.

It would be reasonable to expect that such results could in turn be used to provide evidence for any natural non-abelian generalisations that one is able to formulate of the conjectures that are discussed in this thesis.

Chapter 2

Background

2.1 Notations and definitions

For an abelian group G we will refer to a $\mathbb{Z}[G]$-module as G-module and write $\otimes_G$, Hom_G, $\wedge_G$ and $\det_G$ for tensor products, Hom, exterior powers and determinant modules respectively. For any G-module M and subgroup $H \subset G$ we will write M^H for the submodule of M comprising of elements fixed by H. We write $\mathcal{I}_H$ for the augmentation ideal of H. The norm element of H is given by

$$N_H = \sum_{h \in H} h \in \mathbb{Z}[G].$$

If E is $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$, we write EM for the $E[G]$-module $E \otimes_{\mathbb{Z}} M$. We let $M^\vee :=$ $\operatorname{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ denote the Pontryagin dual of M and we let $M^* := \operatorname{Hom}_G(M, \mathbb{Z}[G])$. We write $x \mapsto x^\#$ for the $\mathbb{Z}$-linear involution on $\mathbb{Z}[G]$ that inverts elements of G. If A is a set let $|A|$ denote the cardinality of A and $\wp_r(A)$ denote the set of subsets of

A of cardinality r. Let $\wp_r(m)$ denote the set of r-tuples $(n_1, ..., n_r)$ of integers between 1 and m that satisfy $n_1 < ... < n_r$.

2.1.1 Galois modules

We let K/k be a finite abelian extension of number fields with Galois group G. We fix a finite set S of places of k such that S contains all infinite places $S_\infty(k)$ and $S_{\mathrm{ram}}(K/k)$, the places that ramify in K/k. For any place v of k and place w of K lying above v, write G_v for the decomposition subgroup of w in G, which is defined by

$$G_v := \{g \in G : g \cdot w = w\}.$$

Since G is abelian this is independent of our choice of w lying above v.

If v is a finite unramified place in K/k there is a unique automorphism $\mathrm{Fr}_v \in G$ called the Frobenius automorphism such that

$$x^{\mathrm{Fr}_v} \equiv x^{\mathbf{N}v} \pmod{w} \quad \text{for all } x \in \mathcal{O}_K$$

where $\mathbf{N}v$ is the size of the residue field. Again since G is abelian Fr_v is independent of our choice of w lying above v.

For any place v the symbol $|\cdot|_v$ denotes the normalised absolute value at v and it is

defined by

$$|x|_v = \begin{cases} \mathbf{N}v^{-\mathrm{ord}_v(x)} & \text{if } v \text{ is a finite place} \\ |\tau(x)| & \text{if } v \text{ is a real place corresponding to the real embedding } \tau \\ |\tau(x)|^2 & \text{if } v \text{ is a complex place corresponding to the complex embedding } \tau \end{cases}$$

(2.1)

where ord_v denotes the normalised additive valuation at v.

We write S_K for the set of places of K lying above those in S and we let $Y_{K,S}$ be the free abelian group on the set S_K. We let $X_{K,S}$ be the kernel of the homomorphism from $Y_{K,S}$ to $\mathbb{Z}$ that sends each place in S_K to one. Then we have a short exact sequence of G-modules

$$0 \to X_{K,S} \to Y_{K,S} \to \mathbb{Z} \to 0.$$

If w_0 is an arbitrary place in S_K it is straightforward to show that the elements $w - w_0$, where w runs over all places of S_K different from w_0, form a $\mathbb{Z}$-basis of $X_{K,S}$.

The ring of S_K-integers is defined by

$$\mathcal{O}_{K,S} := \{x \in K : \mathrm{ord}_w(x) \geq 0 \text{ for all finite places } w \text{ of } K \text{ not contained in } S_K\}$$

and we write $\mathcal{O}_{K,S}^\times$ for the group of S_K-units.

We fix a non-empty finite auxiliary set of places T with $S \cap T = \emptyset$ and such that $\mathcal{O}_{K,S,T}^\times$ is $\mathbb{Z}$-torsion-free. The (S_K, T_K)-units are defined by

$$\mathcal{O}_{K,S,T}^\times := \{x \in \mathcal{O}_{K,S} : x \equiv 1 \bmod w \text{ for all } w \in T_K\}.$$

We let $\mathrm{Cl}_S(K)$ denote the S_K-class group and $\mathrm{Cl}_S^T(K)$ denote the (S_K, T_K)-class group, which is defined by

$$\mathrm{Cl}_S^T(K) := \frac{\{\text{fractional ideals of } \mathcal{O}_{K,S} \text{ prime to } T_K\}}{\{x \cdot \mathcal{O}_{K,S} : x \equiv 1 \bmod w \text{ for all } w \in T_K\}}$$

All of these groups are stable under the action of G, so form G-modules.

2.1.2 Characters

As G is abelian, all irreducible representations of G are one dimensional. Hence the set of characters

$$\widehat{G} := \mathrm{Hom}(G, \mathbb{C}^\times)$$

forms a group under multiplication. We let $\mathbf{1}$ represent the trivial character of G. For each $\chi \in \widehat{G}$ define

$$e_\chi := \frac{1}{|G|} \sum_{g \in G} \chi(g^{-1}) g.$$

The idempotents e_χ satisfy the following properties:

Lemma 2.1.1. Let χ and $\psi \in \widehat{G}$. Then

(i) $e_\chi^2 = e_\chi$

(ii) If $\chi \neq \psi$, then $e_\chi e_\psi = 0$

(iii) If $g \in G$, then $g \cdot e_\chi = \chi(g) e_\chi$.

This gives us an orthogonal decomposition of the group ring $\mathbb{C}[G]$

$$\mathbb{C}[G] = \bigoplus_{\chi \in \widehat{G}} \mathbb{C} \cdot e_\chi.$$

For any G-module M and character $\chi \in \widehat{G}$, we write M_χ for the χ component of $\mathbb{C}M$. This is the submodule of $\mathbb{C}M$ defined by $M_\chi := e_\chi \cdot \mathbb{C}M$. Then viewing M as a $\mathbb{C}[G]$-module

$$M = \bigoplus_{\chi \in \widehat{G}} M_\chi.$$

For any subgroup H of G we define the idempotent

$$e_H := \frac{1}{|H|} \sum_{g \in H} g \in \mathbb{Q}[G]$$

and for any place $v \in S$ we let $e_v := e_{G_v}$.

2.1.3 L-functions

Definition 2.1.2. For any character $\chi \in \widehat{G}$ we define the S-truncated T-modified L-function

$$L_{K/k,S,T}(\chi, z) = \prod_{v \in T} \left(1 - \chi(\mathrm{Fr}_v)\mathbf{N}v^{1-z}\right) \prod_{v \notin S} \left(1 - \chi\left(\mathrm{Fr}_v\right)\mathbf{N}v^{-z}\right)^{-1} \in \mathbb{C}[G].$$

We let

$$r_S(\chi) := \mathrm{ord}_{z=0} L_{K/k,S,T}(\chi, z)$$

and note it this does not depend on the set T. We denote the leading coefficient of the Taylor expansion of $L_{K/k,S,T}(\chi, z)$ at $z = 0$ by

$$L^*_{K/k,S,T}(\chi, 0) := \lim_{z \to 0} z^{-r_S(\chi)} L_{K/k,S,T}(\chi, z).$$

The order of vanishing of S-truncated T-modified L-functions is well understood and given by the following lemma. (See, for example [27, Chap. I, Prop. 3.4] for proof.)

Lemma 2.1.3. Let K/k be an abelian extension of number fields and S a finite set of primes of k containing all infinite primes. Then

$$r_S(\chi) = \dim_{\mathbb{C}}(\mathbb{C}\mathcal{O}_{K,S} \cdot e_\chi) = \begin{cases} |S| - 1, & \text{if } \chi = \mathbf{1} \\ |\{v \in S : G_v \subseteq \ker(\chi)\}| & \text{if } \chi \neq \mathbf{1}. \end{cases}$$

For any non-negative integer a we define

$$\widehat{G}_{a,S} := \left\{ \chi \in \widehat{G} : r_S(\chi) = a \right\}$$

as the set of characters whose S-truncated T-modified L-functions vanish at $s = 0$ to order a. We write $\widehat{G}_{(a),S}$ for the union of $\widehat{G}_{a',S}$ for $a' \geq a$ and define idempotents

$$e_{a,S} := \sum_{\chi \in \widehat{G}_{a,S}} e_\chi \quad \text{and} \quad e_{(a),S} := \sum_{\chi \in \widehat{G}_{(a),S}} e_\chi.$$

It follows from Lemma 2.1.3 that these are rational idempotents.

We gather the L-functions for different characters together to obtain the S-truncated

T-modified equivariant L-function given by

$$\theta_{K/k,S,T}(z) := \sum_{\chi \in \widehat{G}} L_{K/k,S,T}(\chi^{-1}, z) \cdot e_\chi$$

and define its leading term to be

$$\theta^*_{K/k,S,T}(0) := \sum_{\chi \in \widehat{G}} L^*_{K/k,S,T}(\chi^{-1}, 0) \cdot e_\chi.$$

Then $\theta^*_{K/k,S,T}(0)$ belongs to $\mathbb{R}[G]^\times$.

For every non-negative integer a we also define

$$\theta^a_{K/k,S,T}(z) := z^{-a} \sum_{\chi \in \widehat{G}_{(a),S}} L_{K/k,S,T}(\chi^{-1}, z) \cdot e_\chi. \qquad (2.2)$$

We note that Lemma 2.1.3 implies both that this $\mathbb{C}[G]$-valued function is holomorphic at $z = 0$ and that

$$\theta^a_{K/k,S,T}(0) = e_{a,S} \cdot \theta^a_{K/k,S,T}(0) = e_{a,S} \cdot \theta^*_{K/k,S,T}(0). \qquad (2.3)$$

2.1.4   Dirichlet regulator map

The Dirichlet logarithm induces an isomorphism of $\mathbb{C}[G]$ modules

$$\lambda_{K,S} : \mathbb{C}\mathcal{O}^\times_{K,S,T} \longrightarrow \mathbb{C}X_{K,S}$$

such that at each element $u \in \mathcal{O}_{K,S,T}^{\times}$ we have

$$\lambda_{K,S}(u) = -\sum_{w \in S_K} \log |u|_w \cdot w,$$

where $|\cdot|_w$ is as defined in (2.1).

For any non-negative integer a the map $\lambda_{K,S}$ also induces an isomorphism on the a-th exterior power which we will also denote by $\lambda_{K,S}$

$$\lambda_{K,S} : \bigwedge_{\mathbb{C}[G]}^{a} \mathbb{C}\mathcal{O}_{K,S,T}^{\times} \xrightarrow{\sim} \bigwedge_{\mathbb{C}[G]}^{a} \mathbb{C}X_{K,S}.$$

When K and S are clear from the context we will just write λ.

2.2 Previous conjectures

Throughout this section as in §2.1 we let K/k be a finite abelian extension of number fields with Galois group G. Let S be a non-empty finite set of places of k containing $S_\infty(k)$ and $S_{\mathrm{ram}}(K/k)$ and let T be a finite set of places of k disjoint from S. Assume $\mathcal{O}_{K,S,T}^{\times}$ is $\mathbb{Z}$-torsion free. For every place v in S fix a place w of K lying above v.

2.2.1 The Rubin-Stark conjecture

In this subsection we recall the formulation of the classical Rubin-Stark conjecture ([24, Conj. B']). This conjecture extended previous work by Stark in [25] by studying the first non-vanishing coefficient of L-functions with higher (non-negative) orders of vanishing. (Stark had studied the case where the L-functions had simple zeroes at $s = 0$.) Rubin

guaranteed the order of vanishing by assuming that the set of places S contained a subset V of r places that split in K/k. Then he defined a canonical element

$$\epsilon^V_{K/k,S,T}$$

of $\bigwedge^r_G \mathcal{O}^\times_{K,S,T} \otimes \mathbb{C}$, now known as the 'Rubin-Stark element,' which acts as an evaluator to the r-th derivatives of the S-truncated T-modified L-functions. A priori this element has coefficients in $\mathbb{C}$, but Rubin predicted that it satisfied certain integrality properties. Fix a non-negative integer r and assume that S contains a subset V consisting of r places of k that split completely in K. Further assume that $|S| \geq r+1$. Then Lemma 2.1.3 implies that $r_S(\chi) \geq r$ for every character χ and therefore that $z^{-r} L_{K/k,S,T,}(\chi, z)$ is holomorphic at $z = 0$ and that the 'r-th order Stickelberger element' $\theta^{(r)}_{K/k,S,T} :=$ $\lim_{z \to 0} z^{-r} \sum_{\chi \in \widehat{G}} L_{K/k,S,T}(\chi^{-1}, z) \cdot e_\chi \in \mathbb{C}[G]$ is equal to the element $\theta^r_{K/k,S,T}(0)$ defined in (2.2).

Fix an ordering of $S = \{v_0, v_1, \ldots, v_n\}$ so that $V = \{v_1, \ldots, v_r\}$.

Definition 2.2.1. We define the '(r-th order) Rubin-Stark element'

$$\epsilon^V_{K/k,S,T} \in \bigwedge^r_{\mathbb{C}[G]} \mathbb{C}\mathcal{O}^\times_{K,S,T} = \mathbb{C}\bigwedge^r_G \mathcal{O}^\times_{K,S,T}$$

by

$$\lambda(\epsilon^V_{K/k,S,T}) = \theta^r_{K/k,S,T}(0) \bigwedge_{1 \leq i \leq r} (w_i - w_0).$$

Remark 2.2.2. The Rubin-Stark element, as defined above, does depend on our choice of w lying above v in K. However, this does not affect the validity of the conjecture.

Furthermore $\epsilon^V_{K/k,S,T}$ does not depend on the choice of $v_0 \in S \backslash V$ in the ordering of S.

Although the Rubin-Stark element does not necessarily have coefficients in $\mathbb{Z}$, Rubin defined a lattice to predict its integrality properties.

Definition 2.2.3. The Rubin Lattice is defined by

$$\Lambda_{K/k,S,T} := \left\{ u \in \mathbb{Q} \bigwedge_G^r \mathcal{O}^\times_{K,S,T} : \Phi(u) \in \mathbb{Z}[G] \text{ for all } \Phi \in \bigwedge_G^r \mathrm{Hom}_G(\mathcal{O}^\times_{K,S,T}, \mathbb{Z}[G]) \right\},$$

where Φ is regarded as an element of $\mathrm{Hom}_G(\bigwedge_G^r \mathcal{O}^\times_{K,S,T}, \mathbb{Z}[G])$ as described in §3.1.1.

Conjecture 2.2.4 (The Rubin-Stark conjecture for $(K/k, S, T, V)$)**.** One has

$$\epsilon^V_{K/k,S,T} \in \Lambda_{K/k,S,T}.$$

Remark 2.2.5. It is straightforward to show that the formulation above is equivalent to [24, Conj. B'] and that the element described there is equal to the unique element described above. The validity of the conjecture does not depend upon the choice of places lying above $v \in S$ or the ordering of the elements in V.

Remark 2.2.6. Stark's original first order abelian conjecture did not include a set T and only required a set S with one distinguished place which split completely in K/k. However the Stark units that it predict are not unique, indeed they are only unique up to multiplication by a root of unity. By introducing an auxiliary set T such that $\mathcal{O}^\times_{K,S,T}$ is trivial, Rubin guaranteed uniqueness.

Remark 2.2.7. The Rubin-Stark conjecture for $(K/k, S, T, V)$ is known to be true

(i) If S contains more than r places that split completely in K/k. This is proved by Rubin in [24, Prop. 3.1].

(ii) If $r = 0$ so $V = \emptyset$. If k is totally real then the conjecture claims that $\theta_{K/k,S,T}(0) \in \mathbb{Z}[G]$, which is a celebrated result of Deligne and Ribet [11]. If k is not totally real then S has at least one complex place that splits completely, and so we are done by (i).

(iii) $[K : k] \leq 2$. This is proved by Rubin in [24, Cor. 3.2 and Thm. 3.5].

(iv) K is an abelian extension over $\mathbb{Q}$. This is proved by Burns in [1, Thm. A].

2.2.2 Refined abelian Stark conjectures

In both the approach of Stark in [25] and Rubin in [24] the order of vanishing is forced by the inclusion of split places in the set S. This led others to ask if such approaches might be refined to include situations where the L-functions vanish at $z = 0$ but without the split places in S. In [16] Erickson extends Stark's abelian rank one conjecture to the case where S does not contain a distinguished split place. In [14] Emmons and Popescu formulate an analogue of the Rubin-Stark conjecture for higher orders of vanishing r, but again without the split places.

The Emmons-Popescu conjecture

In order to guarantee the vanishing at $z = 0$ to order r of all of the S-truncated T-modified L-functions, Emmons and Popescu use the idea of an r-cover, which provides the minimum conditions necessary for this vanishing to occur.

Definition 2.2.8. Let r be a non-negative integer. Then S is an r-cover for K/k if the following two conditions are satisfied:

1. For all $\chi \in \widehat{G}$, there exist (at least) r distinct places $v \in S$ such that $G_v \subseteq \ker(\chi)$

2. $|S| \geq r + 1$.

In place of the set V of split places, they define a set $S_{\min}$ consisting of places that contribute to the order of vanishing of the L-functions associated to non-trivial characters with order of vanishing exactly r.

Firstly for any $\chi \in \widehat{G}\backslash\{\mathbf{1}\}$ let

$$S_\chi := \{v \in S : G_v \subseteq \ker(\chi)\},$$

then we make the following definition.

Definition 2.2.9. Let S be an r-cover for K/k and suppose further that $|S| > r + 1$. Then

$$S_{\min} = \bigcup_{\chi \in \widehat{G}_{r,S}\backslash\{\mathbf{1}\}} S_\chi.$$

We now suppose that S is an r-cover for K/k and that $S \neq S_{\min}$. We fix an ordering for $S = \{v_0, v_1, \ldots, v_n\}$ such that $S_{\min} = \{v_1, \ldots, v_m\}$ for some integer m with $r \leq m \leq n$.

Remark 2.2.10. If the conditions of the Rubin-Stark conjecture are satisfied, so that S contains a set V consisting of r places that split completely in K/k and $|S| \geq r + 1$, then S is an r-cover for K/k. Further if $r = \min\left\{r_S(\chi) : \chi \in \widehat{G}\right\}$ and S contains precisely r places that split completely then $S_{\min} = V$.

Conjecture 2.2.11 (The Emmons-Popescu conjecture for $(K/k, S, T, r)$). Let S be an r-cover for K/k and suppose $S \neq S_{\min}$ and $|S| > r + 1$. We define the Emmons-Popescu evaluator $\eta_{K/k,S,T,r}$ to be the unique element of $e_{r,S} \cdot \mathbb{C} \bigwedge_G^r \mathcal{O}_{K,S,T}^\times$ such that

$$\lambda(\eta_{K/k,S,T,r}) = \theta_{K/k,S,T}^r(0) \cdot \sum_{I \in \wp_r(S_{\min})} w_I,$$

where $w_I = \bigwedge_{w \in I}(w - w_0)$ and the ordering on the exterior product is that imposed by the ordering on S. Then

$$\eta_{K/k,S,T,r} \in \Lambda_{K/k,S,T}.$$

Remark 2.2.12. The above formulation is equivalent to the formulation in [13] (this follows from [28, §6.2]). In [14] it includes the case where $|S| = r + 1$, but the definitions are slightly more involved since we have $\mathbf{1} \in \widehat{G}_{r,S}$ although the trivial character does not contribute to the composition of $S_{\min}$. However, as shown in the Lemma below in this case S must contain at least r places that split completely in K/k, so that conditions of the Rubin-Stark conjecture are satisfied and the definition of the Emmons-Popescu evaluator in [14] coincides with that of the Rubin-Stark element. In all cases, if S contains r places that split completely in K/k then $\eta_{K/k,S,T,r}$ is a Rubin-Stark element for K/k.

Lemma 2.2.13. If S is an r-cover for K/k and $|S| = r + 1$ then the subset S_{sp} of S comprising of places that split completely in K has cardinality at least r.

Proof. See [14, Lemma 2.2]. $\qquad\square$

Vallières' Conjecture

Using the subsets I of order r of $S_{\min}$ Emmons and Popescu were able to show the relationship between their evaluator and Rubin-Stark elements of particular subfields (see for example [13, Thm. 3.2.5]). Vallières went further that this and used the subsets I to decompose the Emmons-Popescu evaluators into a sum of orthogonal elements. Following Vallières we will continue with the set up of Conjecture 2.2.11 above. For each $I \in \wp_r(S_{\min})$ we define

$$\widehat{G}_{r,S,I} := \left\{ \chi \in \widehat{G}_{r,S} : G_v \subseteq \ker(\chi), \text{ for all } v \in I \right\}.$$

We note that since $|S| > r + 1$, $\mathbf{1} \notin \widehat{G}_{r,S}$. We obtain a partition of the set

$$\widehat{G}_{r,S} = \bigcup_{I \in \wp_r(S_{\min})} \widehat{G}_{r,S,I}.$$

We may think of the sets I as defining equivalence classes for the set of characters in $\widehat{G}_{r,S}$. We define idempotents corresponding to these sets. For each $I \in \wp_r(S_{\min})$ let

$$e_I = e_{r,S,I} := \sum_{\chi \in \widehat{G}_{r,S,I}} e_\chi.$$

It follows from Lemma 2.1.3 that e_I is a rational idempotent.

Following Vallières (see [28]) we define the following elements:

Definition 2.2.14. For each $I \in \wp_r(S_{\min})$ let

$$\eta^I_{K/k,S,T} \in e_I \cdot \mathbb{C} \bigwedge_G^r \mathcal{O}^\times_{K,S,T}$$

be the unique element such that

$$\lambda(\eta^I_{K/k,S,T}) = e_I \cdot \theta^r_{K/k,S,T}(0) \cdot w_I,$$

where $w_I = \bigwedge_{w \in I}(w - w_0)$ and the ordering on the exterior product is that imposed by the ordering on S.

Remark 2.2.15. If we sum over all $I \in \wp_r(S_{\min})$ we recover the Emmons-Popescu evaluator.

$$\eta_{K/k,S,T} = \sum_{I \in \wp_r(S_{\min})} \eta^I_{K/k,S,T} \in e_{r,S} \cdot \mathbb{C} \bigwedge_G^r \mathcal{O}^\times_{K,S,T}.$$

Conjecture 2.2.16 (Vallières for $(K/k, S, T, I)$). Let S be an r-cover for K/k and suppose $S \neq S_{\min}$ and $|S| > r + 1$. Let $I \in \wp_r(S_{\min})$. Then

$$\eta^I_{K/k,S,T} \in \Lambda_{K/k,S,T}.$$

Remark 2.2.17. Suppose S contains precisely r places that split completely in K/k so that the conditions of the Rubin-Stark conjecture are satisfied. Then $S_{\min}$ consists of precisely these r places and therefore there is only one $I \in \wp_r(S_{\min})$. Then $\eta^I_{K/k,S,T} = \eta_{K/k,S,T,r}$ is a Rubin-Stark element for K/k and Conjecture 2.2.16 reduces to the Rubin-Stark conjecture.

Remark 2.2.18. The condition $S \neq S_{\min}$ is not trivial. In the classical case, where S contains r places that split completely, there is always a distinguished place v_0 that does not contribute to the vanishing of any L-function that vanishes to order precisely r. In our more general case, without this condition the existence of such a place is not guaranteed. This place is used to show that the denominators of the rational idempotents e_I do not lead us to lose integrality properties. There are examples of number field extensions K/k and sets S and T that satisfy all the conditions above except $S \neq S_{\min}$ but where $\eta_{K/k,S,T,r} \notin \Lambda_{K/k,S,T}$. These examples, originally from Dummit and Hayes in [12], are computed and discussed in [15, §4.2]].

2.2.3 Rubin-Stark elements, the leading term conjecture and Fitting ideals

In [1] Burns showed that the Rubin-Stark conjecture and several other conjectures concerning special values of L-functions at $z = 0$ follow from a special case of the more general 'equivariant Tamagawa number conjecture' called the leading term conjecture (LTC(K/k)). His methods were later used by Vallières in [28] to show that the validity of LTC(K/k) would also imply the validity of Conjectures 2.2.11 and 2.2.16.

While the Rubin-Stark conjecture predicts that $\epsilon_{K/k,S,T}^V \in \Lambda_{K/k,S,T}$, it doesn't provide a description of the ideal of $\mathbb{Z}[G]$ formed by the elements $\Phi(\epsilon_{K/k,S,T}^V) \in \mathbb{Z}[G]$ as Φ runs over $\bigwedge_G^r \mathrm{Hom}_G(\mathcal{O}_{K,S,T}^\times, \mathbb{Z}[G])$.

In more recent work, Burns, Kurihara and Sano provide a new proof that the Rubin-Stark conjecture follows from LTC(K/k) (see [6]). The key idea is to express the Rubin-Stark element as the 'canonical projection' of the zeta element $z_{K/k,S,T}$ arising in

a particular form of the leading term conjecture. This explicit description allows them to calculate the ideal of $\mathbb{Z}[G]$ described above in terms of the r-th Fitting ideal of $\mathrm{Sel}_S^T(K)^{\mathrm{tr}}$, the 'transpose' of a canonical 'S-relative T-trivialised integral dual Selmer group' for the multiplicative group over K. We denote this r-th Fitting ideal by $\mathrm{Fit}_G^r(\mathrm{Sel}_S^T(K)^{\mathrm{tr}})$. (See §3.2 for full details.)

Theorem 2.2.19. Let $K/k, S, T, V$ and r be as in § 2.2.1 and assume that $\mathrm{LTC}(K/k)$ is valid. Then

$$\mathrm{Fit}_G^r(\mathrm{Sel}_S^T(K)^{\mathrm{tr}}) = \left\{ \Phi(\epsilon_{K/k,S,T}^V) : \Phi \in \bigwedge_G^r \mathrm{Hom}_G(\mathcal{O}_{K,S,T}^\times, \mathbb{Z}[G]) \right\}.$$

Chapter 3

Preliminaries

3.1 Algebraic prerequisites

3.1.1 Exterior powers

When working with exterior powers, we will need to deal with changes of sign arising when we use ordered bases. We will use the following notations (for further details see [28, §6.1]).

Let $I = (i_1, ..., i_{r_1}) \in \wp_{r_1}(m)$ and $J = (j_1, ..., j_{r_2}) \in \wp_{r_2}(m)$. Define

$$(-1)^{I+J} := (-1)^{i_1 + ... + i_{r_1} + j_1 + ... + j_{r_2}}.$$

If in addition we have that $J \subseteq I$ then let $\tau_{I,J}$ denote the permutation $I \mapsto J \cdot (I \setminus J)$,

where $\cdot$ means concatenation. If t is a positive integer let

$$[t] = (1, 2, ..., t).$$

Then

$$\mathrm{sgn}(\tau_{[m],I}) = (-1)^{I+[r_1]}. \tag{3.1}$$

We will also need some constructions for exterior powers of homomorphisms. Suppose M is a $\mathbb{Z}[G]$-module and $f \in M^*$. Then for every positive integer t we can define a $\mathbb{Z}[G]$-module homomorphism

$$\bigwedge_G^t M \longrightarrow \bigwedge_G^{t-1} M$$

by

$$m_1 \wedge ... \wedge m_t \mapsto \sum_{i=1}^{t} (-1)^{i-1} f(m_i) m_1 \wedge ... \wedge m_{i-1} \wedge m_{i+1} \wedge ... \wedge m_t.$$

We will still denote this morphism by f.

Furthermore this construction allows us to regard elements of

$\bigwedge_G^s M^*$ as elements of $\mathrm{Hom}_G\left(\bigwedge_G^t M, \bigwedge_G^{t-s} M\right)$ for non negative integers t and s with $t \geq s$. We do this by defining a homomorphism

$$\bigwedge_G^s M^* \longrightarrow \mathrm{Hom}_G\left(\bigwedge_G^t M, \bigwedge_G^{t-s} M\right)$$

by

$$f_1 \wedge ... \wedge f_s \mapsto (m \mapsto f_s \circ ... \circ f_1(m)).$$

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When $s = t$ this is the map

$$(f_1 \wedge ... \wedge f_t)(m_1 \wedge ... \wedge m_t) = \det(f_i(m_j)).$$

Finally we will use the following results from [6] which we state without proof.

Proposition 3.1.1. Let Q be a commutative $\mathbb{Z}[G]$-algebra. Let $m_1, ..., m_t \in M$ and $f_1, ..., f_s \in \mathrm{Hom}_G(M, Q)$. Then we have

$$(f_1 \wedge ... \wedge f_s)(m_1 \wedge ... \wedge m_t) = \sum_{\sigma \in \mathfrak{S}_{t,s}} \mathrm{sgn}(\sigma) m_{\sigma(s+1)} \wedge ... \wedge m_{\sigma(t)} \otimes \det(f_i(m_{\sigma(j)}))_{1 \leq i,j \leq s},$$

where

$$\mathfrak{S}_{t,s} := \{\sigma \in \mathfrak{S}_t : \sigma(1) < ... < \sigma(s) \text{ and } \sigma(s+1) < ... < \sigma(t)\}.$$

Proof. See [6, Prop. 4.1]. $\qquad \square$

Lemma 3.1.2. Let D be a field, and A an n-dimensional D-vector space. If we have a D-linear map

$$\Psi : A \longrightarrow D^{\oplus m},$$

where $\Psi = \bigoplus_{i=1}^m \psi_i$ with $\psi_1, ..., \psi_m \in \mathrm{Hom}_D(A, D)$ $(m \leq n)$, then we have

$$\mathrm{im}\left(\bigwedge_{1 \leq i \leq m}^n \psi_i : \bigwedge_D^n A \longrightarrow \bigwedge_D^{n-m} A \right) = \begin{cases} \bigwedge_D^{n-m} \ker(\Psi) & \text{if } \Psi \text{ is surjective,} \\ 0 & \text{if } \Psi \text{ is not surjective.} \end{cases}$$

Proof. See [6, Lem. 4.2]. $\qquad \square$

Lemma 3.1.3. Let P be a finitely generated projective $\mathbb{Z}[G]$-module and $j : \mathcal{O}_{K,S,T}^\times \hookrightarrow P$ be an injection whose cokernel is $\mathbb{Z}$-torsion-free. If we regard $\mathcal{O}_{K,S,T}^\times$ as a submodule of P via j, then we have

$$\Lambda_{K/k,S,T} = \left(\mathbb{Q} \bigwedge_G^r \mathcal{O}_{K,S,T}^\times \right) \cap \bigwedge_G^r P.$$

Proof. See [6, Lem. 4.7(ii)]. □

3.1.2   Restriction and corestriction maps

Here we recall the definitions of the restriction and corestriction maps. We follow the notation in [28].

Let $k \subseteq L \subseteq K$ be a tower of number fields, where K/k is abelian. As before let G be the Galois group of K/k and then let H be the Galois group of K/L and $\Gamma = G/H$ be the Galois group of L/k.

The restriction map

$$\mathrm{res}_{K/L} : \mathbb{C}[G] \longrightarrow \mathbb{C}[\Gamma]$$

is the $\mathbb{C}[G]$-algebra morphism defined by

$$\sigma \mapsto \sigma|_L$$

for $\sigma \in G$.

The corestriction map

$$\mathrm{cor}_{K/L} : \mathbb{C}[\Gamma] \longrightarrow \mathbb{C}[G]$$

is the $\mathbb{C}[G]$-module morphism defined by

$$\gamma \mapsto \sum_{\substack{\sigma \in G \\ \sigma|_L = \gamma}} \sigma = \tilde{\gamma} \cdot N_H,$$

where $\tilde{\gamma}$ is any extension of γ and $N_H = \sum_{h \in H} h$.

Remark 3.1.4. The restriction and corestriction map satisfy the following properties

(i) For $\lambda_1, \lambda_2 \in \mathbb{C}[\Gamma]$ we have

$$\mathrm{cor}_{K/L}(\lambda_1 \cdot \lambda_2) = \frac{1}{|H|} \mathrm{cor}_{K/L}(\lambda_1) \cdot \mathrm{cor}_{K/L}(\lambda_2).$$

(ii) For $\sigma \in \mathbb{C}[G]$ we have

$$\mathrm{cor}_{K/L} \circ \mathrm{res}_{K/L}(\sigma) = N_H \cdot \sigma.$$

(iii) For $\gamma \in \mathbb{C}[\Gamma]$ we have

$$\mathrm{res}_{K/L} \circ \mathrm{cor}_{K/L}(\gamma) = |H| \cdot \gamma.$$

We will also need the following result.

Lemma 3.1.5. We have the following isomorphism of abelian groups

$$\mathrm{Hom}_G(\mathcal{O}_{L,S,T}^\times, \mathbb{Z}[G]) \xrightarrow{\sim} \mathrm{Hom}_\Gamma(\mathcal{O}_{L,S,T}^\times, \mathbb{Z}[\Gamma]),$$

given by

$$\phi \mapsto \frac{1}{|H|} \cdot \mathrm{res}_{K/L} \circ \phi,$$

for $\phi \in \mathrm{Hom}_G(\mathcal{O}_{L,S,T}^{\times}, \mathbb{Z}[G])$. The inverse map is given by

$$\phi \mapsto \mathrm{cor}_{K/L} \circ \phi$$

for $\phi \in \mathrm{Hom}_{\Gamma}(\mathcal{O}_{L,S,T}^{\times}, \mathbb{Z}[\Gamma])$.

Proof. This follows from Remark 3.1.4(ii) and (iii). $\qquad\qquad\qquad\qquad\qquad$ $\square$

3.2 The leading term conjecture

3.2.1 Selmer groups

K/k be a finite abelian extension of number fields with Galois group G. Let S be a non-empty finite set of places of k containing $S_{\infty}(k)$ and $S_{\mathrm{ram}}(K/k)$ and let T be a finite set of places of k disjoint from S.

Burns has constructed a canonical complex which allows $\mathrm{LTC}(K/k)$ to be expressed in terms of a zeta element and the determinant module of this complex. Burns first uses this complex in [3], but it is developed further in [6] where it is expressed in terms of an 'integral dual Selmer group for $\mathbb{G}_m$' (and the notation $\mathcal{S}_{S,T}(\mathbb{G}_{m/K})$ is used). Here we give the necessary definitions and properties of the complex. For full details see [6, §2].

Definition 3.2.1. The 'S-relative T-trivialised integral dual Selmer group for $\mathbb{G}_m$' is defined by

$$\mathrm{Sel}_S^T(K) := \mathrm{coker}\left(\prod_{w \notin S_K \cup T_K} \mathbb{Z} \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(K_T^{\times}, \mathbb{Z}) \right),$$

where w runs over all places of K except those explicitly excluded, $K_T^{\times}$ is the subgroup

of $K^\times$ defined by

$$K_T^\times := \left\{ a \in K^\times : \mathrm{ord}_w(a-1) > 0 \text{ for all } w \in T_K \right\},$$

and the homomorphism in the right hand side is defined by

$$(x_w)_w \mapsto \left(a \mapsto \sum_{w \notin S_K \cup T_K} \mathrm{ord}_w(a) x_w \right).$$

$\mathrm{Sel}_S^T(K)$ can be better understood, conjecturally at least, as a cohomology group of a canonical complex of G-modules using 'Weil-étale cohomology'. It is a natural analogue for $\mathbb{G}_m$ of the integral Selmer groups of abelian varieties that are defined by Mazur and Tate in [20].

Proposition 3.2.2. There exists a perfect complex $R\Gamma_{c,T}((\mathcal{O}_{K,S})_\mathcal{W}, \mathbb{Z})$ such that

(i) $R\Gamma_{c,T}((\mathcal{O}_{K,S})_\mathcal{W}, \mathbb{Z})$ is acyclic outside degrees one, two and three.

(ii) There are canonical isomorphisms

$$H^i(R\Gamma_{c,T}((\mathcal{O}_{K,S})_\mathcal{W}, \mathbb{Z})) \simeq \begin{cases} Y_{K,S}/\Delta_S(\mathbb{Z}) & \text{if } i = 1 \\ \mathrm{Sel}_S^T(K) & \text{if } i = 2 \\ (K_{T,\mathrm{tors}}^\times)^\vee & \text{if } i = 3 \end{cases}$$

where Δ_S is the natural diagonal map and $(K_{T,\mathrm{tors}}^\times)^\vee$ is the Pontryagin dual of the torsion subgroup of $K_T^\times$.

(iii) There is an exact sequence

$$0 \longrightarrow \mathrm{Cl}_{S,T}(K)^{\vee} \longrightarrow \mathrm{Sel}_S^T(K) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathcal{O}_{K,S,T}^{\times}, \mathbb{Z}) \longrightarrow 0.$$

Proof. See [6, Prop. 2.2 and Prop. 2.4] □

We define a 'dual' of the complex $R\Gamma_{c,T}((\mathcal{O}_{K,S})_{\mathcal{W}}, \mathbb{Z})$ Let

$$R\Gamma_T((\mathcal{O}_{K,S})_{\mathcal{W}}, \mathbb{G}_m) := R\mathrm{Hom}_{\mathbb{Z}}(R\Gamma_{c,T}((\mathcal{O}_{K,S})_{\mathcal{W}}, \mathbb{Z}), \mathbb{Z})[-2].$$

Definition 3.2.3. We define the 'transpose' of $\mathrm{Sel}_S^T(K)$ by

$$\mathrm{Sel}_S^T(K)^{\mathrm{tr}} := H_T^1(R\Gamma_T((\mathcal{O}_{K,S})_{\mathcal{W}}, \mathbb{G}_m)) = H^{-1}(R\mathrm{Hom}_{\mathbb{Z}}(R\Gamma_{c,T}((\mathcal{O}_{K,S})_{\mathcal{W}}, \mathbb{Z}), \mathbb{Z})).$$

Proposition 3.2.4. The complex $R\Gamma_T((\mathcal{O}_{K,S})_{\mathcal{W}}, \mathbb{G}_m)$ is acyclic outside degrees zero and one. There exist canonical isomorphisms

$$H^i(R\Gamma_T((\mathcal{O}_{K,S})_{\mathcal{W}}, \mathbb{G}_m)) \simeq \begin{cases} \mathcal{O}_{K,S,T}^{\times} & \text{if } i = 0 \\ \mathrm{Sel}_S^T(K)^{\mathrm{tr}} & \text{if } i = 1 \end{cases}$$

and there is a canonical exact sequence

$$0 \longrightarrow \mathrm{Cl}_{S,T}(K) \longrightarrow \mathrm{Sel}_S^T(K)^{\mathrm{tr}} \longrightarrow X_{K,S} \longrightarrow 0. \qquad (3.2)$$

Proof. See [6, Rem. 2.7] □

3.2.2 The 'zeta element'

If $D_{K,S,T}^\bullet : D^0 \longrightarrow D^1$ is a representative of the complex $R\Gamma_T((\mathcal{O}_{K,S})_\mathcal{W}, \mathbb{G}_m)$, we define the regulator isomorphism

$$\rho_{K,S} : \mathbb{C}\mathrm{det}_G \left(D_{K,S,T}^\bullet\right) \xrightarrow{\sim} \mathbb{C}[G]$$

as follows:

$$
\begin{aligned}
\mathbb{C}\mathrm{det}_G \left(D_{K,S,T}^\bullet\right) \quad &\xrightarrow{\sim} \quad \mathrm{det}_{\mathbb{C}[G]} \left(\mathbb{C}D_{K,S,T}^0\right) \otimes_{\mathbb{C}[G]} \mathbb{C}\mathrm{det}_{\mathbb{C}[G]}^{-1} \left(\mathbb{C}D_{K,S,T}^1\right) \\
&\xrightarrow{\sim} \quad \mathrm{det}_{\mathbb{C}[G]} \left(\mathbb{C}\mathcal{O}_{K,S,T}^\times\right) \otimes_{\mathbb{C}[G]} \mathrm{det}_{\mathbb{C}[G]} \left(\mathbb{C}\,\mathrm{im}(d^0)\right) \\
&\qquad \otimes_{\mathbb{C}[G]} \mathrm{det}_{\mathbb{C}[G]}^{-1} \left(\mathbb{C}\,\mathrm{im}(d^0)\right) \otimes_{\mathbb{C}[G]} \mathrm{det}_{\mathbb{C}[G]}^{-1} \left(\mathbb{C}X_{K,S}\right) \\
&\xrightarrow{\sim} \quad \mathrm{det}_{\mathbb{C}[G]} \left(\mathbb{C}\mathcal{O}_{K,S,T}^\times\right) \otimes_{\mathbb{C}[G]} \mathrm{det}_{\mathbb{C}[G]}^{-1} \left(\mathbb{C}X_{K,S}\right) \\
&\xrightarrow{\sim} \quad \mathrm{det}_{\mathbb{C}[G]} \left(\mathbb{C}X_{K,S}\right) \otimes_{\mathbb{C}[G]} \mathrm{det}_{\mathbb{C}[G]}^{-1} \left(\mathbb{C}X_{K,S}\right) \\
&\xrightarrow{\sim} \quad \mathbb{C}[G].
\end{aligned}
$$

The first isomorphism is the definition of $\mathrm{det}_G \left(D_{K,S,T}^\bullet\right)$. The second follows from the properties of determinants and Proposition 3.2.4. The third isomorphism follows from the evaluation map and the fourth by applying the Dirichlet logarithm λ. The final isomorphism is again the evaluation map.

Definition 3.2.5. The *zeta element* $z_{K/k,S,T} \in \mathbb{C}\mathrm{det}_G \left(R\Gamma_T((\mathcal{O}_{K,S})_\mathcal{W}, \mathbb{G}_m)\right)$ is the unique element such that

$$\rho_{K,S}(z_{K/k,S,T}) = \theta_{K/k,S,T}^*(0).$$

Conjecture 3.2.6 (Leading Term Conjecture - LTC(K/k)). In $\mathbb{C}\mathrm{det}_G(R\Gamma_T((\mathcal{O}_{K,S})_{\mathcal{W}}, \mathbb{G}_m))$ one has

$$\mathbb{Z}[G]z_{K/k,S,T} = \mathrm{det}_G(R\Gamma_T((\mathcal{O}_{K,S})_{\mathcal{W}}, \mathbb{G}_m)).$$

Lemma 3.2.7. Assume LTC(K/k). Then the leading term conjecture is also true for K^H/k for every subgroup H of G.

Proof. This follows from [6, Rem 3.2 and Prop. 3.4]. $\qquad\qquad\square$

3.3 A convenient resolution of $R\Gamma_T((\mathcal{O}_{K,S})_{\mathcal{W}}, \mathbb{G}_m)$

Let K/k, S and T be as in §3.2 and further assume that $\mathcal{O}_{K,S,T}^{\times}$ is $\mathbb{Z}$-torsion free. Fix an order $S = \{v_0, v_1, ..., v_n\}$ and for each v_i fix a place w_i of K lying above it.

Before we show that the leading term conjecture implies the refined abelian Stark conjectures due to Vallières, Emmons and Popescu, we fix a convenient resolution $D_{K,S,T}^{\bullet}$ of the complex $R\Gamma_T((\mathcal{O}_{K,S})_{\mathcal{W}}, \mathbb{G}_m)$.

In [1] Burns constructs a Yoneda two extension of $\mathcal{O}_{K,S,T}^{\times}$ by $X_{K,S}$ using a free $\mathbb{Z}[G]$-module F and a homomorphism π which maps F onto $X_{K,S}$. Vallières follows this method in [28]. This method first works under the assumption that $\mathrm{Cl}_S^T(K)$ is trivial, and then shows how this may be removed.

In [6] Burns et al. adapt this method to create a representative for $D_{K,S,T}^{\bullet}$ by constructing a Yoneda two-extension of $\mathcal{O}_{K,S,T}^{\times}$ by $H^1(D_{K,S,T}^{\bullet})$ and making the necessary adaptions to F and π. In the case that $\mathrm{Cl}_S^T(K)$ is trivial, this method recovers the method used by Burns in [1] and Vallières in [28]. We follow this second method.

3.3.1 Presentations of Selmer groups

Here we recall the construction of the free $\mathbb{Z}[G]$-module F by Burns et al. ([6, §5.4]) that is used to give presentations of $\mathrm{Sel}_S^T(K)^{\mathrm{tr}}$.

Let $d \in \mathbb{Z}$ be sufficiently large. Let F be a free $\mathbb{Z}[G]$-module with basis $\{b_i\}_{1 \leq i \leq d}$. We will construct a surjective $\mathbb{Z}[G]$-homomorphism

$$\pi : F \longrightarrow H^1(D_{K,S,T}^\bullet)$$

in two parts. Firstly write $F = F_1 \oplus F_2$, where F_1 is the free $\mathbb{Z}[G]-$module generated by $\{b_i\}_{1 \leq i \leq n}$ and F_2 is the free $\mathbb{Z}[G]-$module generated by $\{b_i\}_{n < i \leq d}$.

Since F_1 is free we may choose a homomorphism $\pi_1 : F_1 \longrightarrow H^1(C_{K,S,T}^\bullet)$ such that the composition

$$F_1 \xrightarrow{\pi_1} H^1(D_{K,S,T}^\bullet) \longrightarrow X_{K,S}$$

sends b_i to $w_i - w_0$.

Now let A be the kernel of the composition

$$H^1(D_{K,S,T}^\bullet) \longrightarrow X_{K,S} \longrightarrow Y_{K,S \backslash \{v_0\}},$$

where the last map sends places above v_0 to 0. We chose d to be sufficiently large so that we can now choose a surjection

$$\pi_2 : F_2 \longrightarrow A.$$

This then gives us a surjective map π defined by:

$$\pi := \pi_1 \oplus \pi_2 : F = F_1 \oplus F_2 \longrightarrow H^1(D_{K,S,T}^\bullet).$$

3.3.2   Constructing the resolution

We use F and the map π to construct a representative for the Yoneda extension class of $D_{K,S,T}^\bullet$ in $\mathrm{Ext}_G^2\left(H^1(D_{K,S,T}^\bullet), \mathcal{O}_{K,S,T}^\times\right)$. Since $D_{K,S,T}^\bullet$ is perfect we can represent it by an exact sequence

$$0 \longrightarrow \mathcal{O}_{K,S,T}^\times \longrightarrow P \xrightarrow{\ \psi\ } F \xrightarrow{\ \pi\ } H^1(D_{K,S,T}^\bullet) \longrightarrow 0 \tag{3.3}$$

where P is a cohomologically trivial $\mathbb{Z}[G]$-module. Since $\mathcal{O}_{K,S,T}^\times$ is torsion-free, P is torsion-free and hence projective.

Lemma 3.3.1. For every prime p there is a natural isomorphism of $\mathbb{Z}_{(p)}[G]$-modules

$$P_{(p)} \simeq F_{(p)}.$$

Proof. We have that $\mathbb{C}\mathcal{O}_{K,S,T}^\times \simeq \mathbb{C}H^1(D_{K,S,T}^\bullet)$. Therefore for every prime p

$$\mathbb{Q}_{(p)}\mathcal{O}_{K,S,T(p)}^\times \simeq \mathbb{Q}_{(p)}H^1(D_{K,S,T}^\bullet)_{(p)}.$$

Since $\mathbb{Q}_{(p)}[G]$ is semisimple we get

$$\mathbb{Q}_{(p)}P_{(p)} \simeq \mathbb{Q}_{(p)}F_{(p)}.$$

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Finally our required result follows from Swan's Theorem since P and F are both projective and torsion-free. (See [9] for details of Swan's Theorem.) □

Remark 3.3.2. Since

$$\det{}_G(D^\bullet_{K,S,T}) \simeq \det{}_G(P) \otimes_G \det{}_G^{-1}(F),$$

if $\text{LTC}(K/k)$ is valid then we have that the $\mathbb{Z}[G]$-module P is isomorphic to F.

Chapter 4

Zeta elements and Selmer groups

In this chapter we present and prove our key results, published in [19], which bring together and extend the ideas of Vallières, which we saw in Conjecture 2.2.16, and the new methods involving zeta elements, Selmer groups and Fitting ideals developed by Burns, Kurihara and Sano, which we saw in Theorem 2.2.19.

4.1 Definitions and preliminaries

In this chapter, as in §2.1, we let K/k be a finite abelian extension of number fields with Galois group G. Let S be a non-empty finite set of places of k containing $S_\infty(k)$ and $S_{\mathrm{ram}}(K/k)$ and let T be a finite set of places of k disjoint from S. Assume $\mathcal{O}_{K,S,T}^\times$ is $\mathbb{Z}$-torsion free. For every place v in S fix a place w of K lying above v.

We let
$$r = r_S(K/k) := \min\left\{ r_S(\chi) : \chi \in \widehat{G} \right\}$$

so that r is the minimal order of vanishing at $s = 0$ of the S-truncated T-modified L-functions for any character χ of G.

Remark 4.1.1. Since $r_S(\mathbf{1}) \geq r$ we have that $|S| \geq r + 1$. We therefore need to extend the definitions from §2.2.2 to include the case where $|S| = r + 1$. In this case $\mathbf{1} \in \widehat{G}_{r,S}$ and therefore our definitions of S_χ and $S_{\min}$ have to be extended to include the trivial character.

Definition 4.1.2. For any $\chi \in \widehat{G}_{r,S}$ we define the subsets S_χ of S as follows.

If $\chi \neq \mathbf{1}$ then we let

$$S_\chi := \{v \in S : G_v \subseteq \ker(\chi)\}$$

as before.

If $\mathbf{1} \in \widehat{G}_{r,S}$ it follows that $|S| = r + 1$. If in addition all places in S split completely in K, then we pick $v_0 \in S$ arbitrarily. Otherwise we pick some v_0 that does not split completely in K. This v_0 is unique by Lemma 2.2.13. In both cases then we define

$$S_{\mathbf{1}} = S \setminus v_0 = \{v_1, ..., v_r\}.$$

Then we let

$$S_{\min} = \bigcup_{\chi \in \widehat{G}_{r,S}} S_\chi \subset S. \tag{4.1}$$

Remark 4.1.3. For clarity we describe $S_{\min}$ in each of the three possible cases.

(i) If $|S| = r + 1$ and all places in S split completely in K we have $S_{\mathbf{1}} = S \setminus v_0 = \{v_1, ..., v_r\}$ for arbitrary $v_0 \in S$. Then since in this case $\widehat{G}_{r,S} = \{\mathbf{1}\}$ we have $S_{\min} = S_{\mathbf{1}} = \{v_1, ..., v_r\}$.

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(ii) If $|S| = r + 1$ and some (unique) $v_0 \in S$ does not split completely in K and we have $S_{\mathbf{1}} = S \setminus v_0 = \{v_1, ..., v_r\}$. In this case $\mathbf{1} \in \widehat{G}_{r,S}$ and for any $\chi \in \widehat{G}_{r,S}$ we have $S_\chi = \{v_1, ..., v_r\}$ and so again we have $S_{\min} = \{v_1, ..., v_r\}$.

(iii) If $|S| > r + 1$ then $\mathbf{1} \notin \widehat{G}_{r,S}$, $|S_{\min}| \geq r$ and this agrees with Definition 2.2.9.

Then for each $I \in \wp_r(S_{\min})$ define

$$\widehat{G}_{r,S,I} := \left\{ \chi \in \widehat{G}_{r,S} : S_\chi = I \right\}.$$

So for $|S| \neq r + 1$ we have

$$\widehat{G}_{r,S,I} = \left\{ \chi \in \widehat{G}_{r,S} : G_v \subseteq \ker(\chi), \text{ for all } v \in I \right\}.$$

In all cases we have

$$G_{r,S} = \bigcup_{I \in \wp_r(S_{\min})} \widehat{G}_{r,S,I}. \tag{4.2}$$

Remark 4.1.4. We can think of the sets I as defining equivalence classes for the set of characters in $\widehat{G}_{r,S}$. The union in (4.2) is disjoint. This follows from the definition of $S_{\min}$. If $|S| \neq r + 1$ and $\chi \in \widehat{G}_{r,S}$ then there is a unique $I \in \wp_r(S_{\min})$ such that $G_v \subseteq \ker(\chi)$ for all $v \in I$. If $|S| = r + 1$ then $S_{\min} = S_{\mathbf{1}} = \{v_1, ..., v_r\}$ as described in Remark 4.1.3. Therefore there is only one I in $\wp_r(S_{\min})$.

Assume that $S \neq S_{\min}$. We fix an ordering of $S = \{v_0, v_1, ..., v_n\}$ such that $S_{\min} = \{v_1, ..., v_m\}$. This means that for each $I \in \wp_r(S_{\min})$ and non-trivial character $\chi \in \widehat{G}_{r,S,I}$ we have $G_{v_0} \nsubseteq \ker(\chi)$. Fix places $w_0, w_1, ..., w_n$ lying above $\{v_0, v_1, ..., v_n\}$ in K.

We will use the correspondence $v_i \leftrightarrow i$ to identify $\wp_r(S)$ with the set $\wp_r(\{0, 1, \ldots, n\})$.

Let $I \in \wp_r(S_{\min})$.

Lemma 4.1.5. For all $\chi \in \widehat{G}_{r,S,I} \backslash \{\mathbf{1}\}$ we have

$$e_\chi \cdot \mathbb{C} \operatorname{im}(\psi) = \bigoplus_{i \notin I} e_\chi \cdot \mathbb{C}[G] \cdot b_i$$

where ψ is the $\mathbb{Z}[G]$-homomorphism in (3.3)

Proof. We follow the proof from [28, Lem. 6.5].

Let $x \in e_\chi \cdot \mathbb{C} \operatorname{im}(\psi) = e_\chi \cdot \mathbb{C} \ker(\pi)$. Then $x = e_\chi \cdot \sum_{i=1}^{d} z_i b_i$ for some $z_i \in \mathbb{C}$.

Then

$$
\begin{aligned}
0 = \pi(x) \\
= \sum_{i \in I} e_\chi \cdot z_i(w_i - w_0) \\
= \sum_{i \in I} e_\chi \cdot z_i w_i
\end{aligned}
$$

where the second and third equalities follow from the definition of π and since $e_\chi \cdot w = 0$ if $G_v \nsubseteq \ker(\chi)$.

Then since $\{w_i \cdot e_\chi : i \in I\}$ is a $\mathbb{C}$-basis for $e_\chi \cdot \mathbb{C} X_{K,S} = e_\chi \cdot \mathbb{C} Y_{K,S}$, we get $z_i = 0$ for all $i \in I$. Therefore

$$x \in \bigoplus_{i \notin I} e_\chi \cdot \mathbb{C}[G] \cdot b_i,$$

and

$$e_\chi \mathbb{C} \operatorname{im}(\psi) \subseteq \cdot \bigoplus_{i \notin I} e_\chi \cdot \mathbb{C}[G] \cdot b_i.$$

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Finally we obtain equality by dimension count. $\square$

4.2 Statement of main results

Definition 4.2.1. For each $I \in \wp_r(S_{\min})$ let

$$\eta^I_{K/k,S,T} \in e_I \cdot \mathbb{C} \bigwedge_G^r \mathcal{O}^\times_{K,S,T}$$

be the unique element such that

$$\lambda(\eta^I_{K/k,S,T}) = e_I \cdot \theta^r_{K/k,S,T}(0) \cdot w_I,$$

where $w_I = (w_{i_1} - w_0) \wedge (w_{i_2} - w_0) \wedge \cdots \wedge (w_{i_r} - w_0)$.

Remark 4.2.2. This definition extends the definition in Definition 2.2.14 to include the case where $|S| = r + 1$. In this case, as described in Remark 4.1.4, $S_{\min}$ consists of r places that split completely in K/k and so $\eta^I_{K/k,S,T}$ is a Rubin-Stark element for $(K/k, S, T, I)$.

Remark 4.2.3. If $r = 0$ then Definition 4.1.2 and (4.1) give $S_{\min} = \emptyset$ and $\wp_0(S_{\min}) = \{\emptyset\}$. So if $I \in \wp_0(S_{\min})$ then $I = \emptyset$ and $e_I = e_{0,S}$. It then follows from (2.3) that $\eta^I_{K/k,S,T}$ is a Rubin-Stark element for $(K/k, S, T, I)$.

Our main algebraic result in this chapter tells us more about the structure of the r-th Fitting ideal of $\mathrm{Sel}^T_S(K)^{\mathrm{tr}}$.

Theorem 4.2.4. Let K/k be a finite abelian extension of number fields. Let S be a non-empty finite set of places of k containing $S_\infty(k)$ and $S_{\mathrm{ram}}(K/k)$ and let T be a finite set of places of k disjoint from S. Assume $\mathcal{O}_{K,S,T}^\times$ is $\mathbb{Z}$-torsion free. Assume $S \neq S_{\min}$ and let $r = r_S(K/k)$. Then for each $I \in \wp_r(S_{\min})$

$$e_{D_I} \cdot \mathrm{Fit}_G^r(\mathrm{Sel}_S^T(K)^{\mathrm{tr}}) \subseteq \mathrm{Fit}_G^r(\mathrm{Sel}_S^T(K)^{\mathrm{tr}}).$$

Here D_I denotes the group generated by the decomposition groups of the places in I .

This result leads us to make the following conjecture which improves upon Conjecture 2.2.16.

Conjecture 4.2.5. Assume the conditions of Theorem 4.2.4 hold. Let $\eta_{K/k,S,T}^I \in e_I \cdot \mathbb{C} \bigwedge_G^r \mathcal{O}_{K,S,T}^\times$ be defined as above. Then we have

$$\mathrm{Fit}_G^r(\mathrm{Sel}_S^T(K)^{\mathrm{tr}}) = \bigoplus_{I \in \wp_r(S_{\min})} \left\{ \Phi(\eta_{K/k,S,T}^I) : \Phi \in \bigwedge_G^r \mathrm{Hom}(\mathcal{O}_{K,S,T}^\times, \mathbb{Z}[G]) \right\}.$$

The main evidence that we can give for this conjecture is the following result.

Theorem 4.2.6. The validity of $\mathrm{LTC}(K/k)$ implies the validity of Conjecture 4.2.5.

Remark 4.2.7. If S contains r places that split completely in K/k then by Remark 4.2.2, Theorem 4.2.6 reduces the result of Burn, Kurihara and Sano, stated in Theorem 2.2.19. As described in Remark 4.2.3 this is automatically the case if $r = 0$.

Remark 4.2.8. We recall from Remark 2.2.18 that the condition $S \neq S_{\min}$ is not trivial and that there exist examples that show that relaxing the condition would mean that Conjecture 4.2.5 would not hold.

We recall that Burns and Greither have proved in [5] that LTC(K/k) holds away from the prime 2 when K is an abelian extension of $\mathbb{Q}$ (and k is any intermediate field of $K/\mathbb{Q}$) and that Flach proved the same result at the prime 2 in [17].

Given this, the following result is an immediate consequence of Theorem 4.2.6.

Corollary 4.2.9. If K is an abelian extension of $\mathbb{Q}$, then Conjecture 4.2.5 is valid for any intermediate field k of $K/\mathbb{Q}$.

4.3 The leading term conjecture implies the refined abelian Stark conjectures

In this section we prove the following result:

Theorem 4.3.1 (Vallières). Assume the conditions of Theorem 4.2.4 hold. Let $I \in \wp_r(S_{\min})$. Let $\eta^I_{K/k,S,T} \in e_I \cdot \mathbb{C} \bigwedge^r_G \mathcal{O}^\times_{K,S,T}$ be defined as above. Then LTC(K/k) implies that

$$\eta^I_{K/k,S,T} \in \Lambda_{K/k,S,T}.$$

Remark 4.3.2. Theorem 4.3.1 is originally proved by Vallières in [28, Thm. 6.12] but with slightly different conditions. We include the case that $|S| = r + 1$ as we have extended the definition of $\eta^I_{K/k,S,T}$ to include this. Also as we define r to be the minimal order of vanishing of the S-truncated L-functions, we do not need to assume that S is an 'r-cover'. (This condition essentially asserts that all orders of vanishing are at least r.)

Here we use methods from [6] to improve and simplify the proof. Crucially this proof

allows us to express $\eta^I_{K/k,S,T}$ in terms of the zeta element $z_{K/k,S,T}$. This new proof is necessary as it provides us with the construction we need to to prove Theorem 4.2.6.

Proof. Assume LTC(K/k) holds.

If $|S| = r+1$ then as discussed in Remark 4.2.2, S contains at least r places that split completely in K and $\eta^I_{K/k,S,T}$ is a Rubin-Stark element for $(K/k, S, T, I)$. Then the statement $\eta^I_{K/k,S,T} \in \Lambda_{K/k,S,T}$ is just the Rubin-Stark conjecture, and the theorem has been proved by Burns in [1].

So we may assume $|S| > r+1$ and therefore that $\mathbf{1} \notin \widehat{G}_{r,S}$. We adapt the method from the proof of [6, Theorem 5.11].

It follows from Remark 3.3.2 that P is free of rank d. It also follows from LTC(K/k) that we can define $z_b \in \bigwedge^d_G P$ to be the element that corresponds to the zeta element $z_{K/k,S,T} \in \det_G(D^\bullet_{K,S,T})$ via the isomorphism

$$\kappa : \bigwedge^d_G P \xrightarrow{\sim} \bigwedge^d_G P \otimes \bigwedge^d_G F^* \simeq \det_G(D^\bullet_{K,S,T}),$$

where the first isomorphism is given by

$$a \mapsto a \otimes \bigwedge_{1 \le i \le d} b^*_i,$$

where $b^*_i \in F^*$ is the dual basis of $b_i \in F$, and the second isomorphism is given by

$$\det_G(D^\bullet_{K,S,T}) \simeq \det_G(P) \otimes_G \det^{-1}_G(F).$$

For each $1 \le i \le d$, we define

$$\psi_i := b_i^* \circ \psi \in P^*.$$

Then the theorem follows from the next proposition. $\qquad\square$

Proposition 4.3.3. Regard $\mathcal{O}_{K,S,T}^\times \subset P$. Then we have

(i) $e_I \cdot \left(\bigwedge_{i \notin I} \psi_i \right) (z_b) \in \Lambda_{K/k,S,T} \left(\subset \bigwedge_G^r P \right)$

(ii) $\eta_{K/k,S,T}^I = e_I \cdot (-1)^{I + [r] + r(d-r)} \left(\bigwedge_{i \notin I} \psi_i \right) (z_b)$.

Proof. Let $\chi \in \widehat{G}_{r,S,I}$. We define a map by

$$\Psi := \bigoplus_{i \notin I} \psi_i : e_\chi \cdot \mathbb{C}P \to e_\chi \cdot \mathbb{C}[G]^{\oplus(d-r)}.$$

We will show that Ψ is surjective and then apply Lemma 3.1.2. We have that

$$e_\chi \cdot \mathbb{C}\operatorname{im}(\psi) = \bigoplus_{i \notin I} e_\chi \cdot \mathbb{C}[G] \cdot b_i,$$

from Lemma 4.1.5 which implies surjectivity. Therefore by Lemma 3.1.2 we have $e_\chi \cdot$ $\left(\bigwedge_{i \notin I} \psi_i \right) (z_b) \in e_\chi \cdot \mathbb{C} \bigwedge_G^r \mathcal{O}_{K,S,T}^\times$. Summing over all $\chi \in \widehat{G}_{r,S,I}$ we get

$$e_I \cdot \left(\bigwedge_{i \notin I} \psi_i \right) (z_b) \in e_I \cdot \mathbb{C} \bigwedge_G^r \mathcal{O}_{K,S,T}^\times.$$

Since e_I is a rational character it follows from Lemma 3.1.3 and Lemma 4.3.4 below that

$$e_I \cdot \left(\bigwedge_{i \notin I} \psi_i \right) (z_b) \in \left(\mathbb{Q} \bigwedge_G^r \mathcal{O}_{K,S,T}^\times \right) \cap \bigwedge_G^r P = \Lambda_{K/k,S,T}.$$

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By Lemma 4.3.5 below we have that

$$\lambda \left(e_I \cdot (-1)^{I+[r]+r(d-r)} \left(\bigwedge_{i \notin I} \psi_i \right) (z_b) \right) = e_I \cdot \theta^*_{K/k,S,T} \cdot w_I.$$

Since λ is an isomorphism it follows that

$$\eta^I_{K/k,S,T} = e_I \cdot (-1)^{I+[r]+r(d-r)} \left(\bigwedge_{i \notin I} \psi_i \right) (z_b).$$

$\square$

Lemma 4.3.4. Assume LTC(K/k). Then

$$e_I \cdot \left(\bigwedge_{i \notin I} \psi_i \right) (z_b) = \left(\bigwedge_{i \notin I} \psi_i \right) (z_b).$$

Proof. This proof uses methods developed in [1], [6] and [28, Lem. 6.8].

By Proposition 3.1.1 it suffices to prove that for every $\sigma \in \mathfrak{S}_{d,r}$ we have that

$$e_I \cdot \det(\psi_i(b_{\sigma(j)}))_{i \notin I, r < j \leq d} = \det(\psi_i(b_{\sigma(j)}))_{i \notin I, r < j \leq d}.$$

We prove this prime by prime after localisation.

Fix a prime p. By Lemma 3.3.1 the $\mathbb{Z}_{(p)}[G]$-modules $P_{(p)}$ and $F_{(p)}$ are free of the same rank d. Thus we may assume that $P_{(p)} = F_{(p)}$. The basis $\{b_1, ..., b_d\}$ of F thus gives a basis for both $P_{(p)}$ and $F_{(p)}$, for which we will use the same notation.

Fix $\sigma \in \mathfrak{S}_{d,r}$. Let $\psi_{I,\sigma,p}$ denote the composite map

$$\psi_{I,\sigma,p} : \bigoplus_{r<j\leq d} \mathbb{Z}_{(p)}[G] \cdot b_{\sigma(j)} \xrightarrow{\psi_{res}} F_{(p)} \xrightarrow{q} \bigoplus_{i\notin I} \mathbb{Z}_{(p)}[G] \cdot b_i$$

where q is the natural projection.

The matrix $(\psi_i(b_{\sigma(j)}))_{i\notin I, r<j\leq d}$ corresponds to the morphism $\psi_{I,\sigma,p}$. We will show that if $\chi \notin \widehat{G}_{r,S,I}$ then

$$\psi^{\chi}_{I,\sigma,p} := e_\chi \cdot \left(\mathbb{C} \otimes_{\mathbb{Z}_{(p)}} \psi_{I,\sigma,p}\right)$$

is singular (and so has determinant equal to zero). This will suffice since $e_I + \sum_{\chi\notin\widehat{G}_{r,S,I}} e_\chi = 1$. We will examine two separate cases.

First fix $\chi \notin \widehat{G}_{r,S}$. By argument in Lemma 4.1.5 it follows that $\dim_{\mathbb{C}}(\mathbb{C}\,\mathrm{im}(\psi) \cdot e_\chi) = d - r_S(\chi)$ and so $\dim_{\mathbb{C}}(\mathbb{C}\ker(\psi) \cdot e_\chi) = r_S(\chi) > r$. Suppose $\psi^{\chi}_{I,\sigma,p}$ is not singular. Then

$$(\mathbb{C}\ker(\psi) \cdot e_\chi) \cap \left(\bigoplus_{r<j\leq d} \mathbb{C}[G] \cdot b_{\sigma(j)} \cdot e_\chi\right) = 0.$$

It follows that

$$\dim_{\mathbb{C}}\left(\mathbb{C}\ker(\psi) \cdot e_\chi + \bigoplus_{r<j\leq d} \mathbb{C}[G] \cdot b_{\sigma(j)} \cdot e_\chi\right) = \dim_{\mathbb{C}}(\mathbb{C}\ker(\psi) \cdot e_\chi) + d - r$$

$$> r + d - r$$

$$= d,$$

which is a contradiction.

Now let $\chi \in \widehat{G}_{r,S}\setminus\widehat{G}_{r,S,I}$, then $\dim_{\mathbb{C}}(\mathrm{im}(\psi) \cdot e_\chi) = d - r$ by Lemma 4.1.5, (which we

may apply since $\chi \in \widehat{G}_{r,S,J}$ for some $J \neq I$ in $\wp_r(S_{\min})$. Again suppose $\psi^\chi_{I,\sigma,p}$ is not singular. Then

$$\mathbb{C} \ker(\psi) \cdot e_\chi \cap \left(\bigoplus_{r<j\leq d} \mathbb{C}[G] \cdot b_{\sigma(j)} \right) \cdot e_\chi = 0,$$

so by counting dimensions

$$\mathbb{C} \ker(\psi) \cdot e_\chi + \left(\bigoplus_{r<j\leq d} \mathbb{C}[G] \cdot b_{\sigma(j)} \cdot e_\chi \right) = \mathbb{C} F \cdot e_\chi.$$

We have $\chi \notin \widehat{G}_{r,S,I}$. Therefore $\exists i_0 \in I$ such that $G_{v_{i_0}} \not\subseteq \ker(\chi)$. We also have that $\pi(b_{i_0}) = w_{i_0} - w_0$ in $\mathbb{C}\mathrm{Sel}^T_S(K)^{\mathrm{tr}} = \mathbb{C} X_{K,S}$. Therefore $\pi(e_\chi \cdot b_{i_0}) = e_\chi \cdot (w_{i_0} - w_0) = 0$ as $G_{v_0}, G_{v_{i_0}} \not\subseteq \ker(\chi)$. This gives $e_\chi \cdot b_{i_0} \in \mathrm{im}(\psi^\chi)$.

Therefore we can find x in $\mathbb{C} F \cdot e_\chi$ such that $\psi^\chi(x) = e_\chi \cdot b_{i_0}$ and write $x = y' + y$ for some $y' \in \ker(\psi^\chi)$ and $y \in \left(\bigoplus_{r<j\leq d} \mathbb{C}[G] \cdot b_{\sigma(j)} \right) \cdot e_\chi$.

Then $\psi^\chi(x) = \psi^\chi(y)$ and since $q(e_\chi \cdot b_{i_0}) = 0$, we get

$$\psi^\chi_{I,\sigma,p}(y) = 0.$$

But $y \neq 0$ since $e_\chi \cdot b_{i_0} \neq 0$, therefore $\psi^\chi_{I,\sigma,p}$ is singular. $\qquad\square$

Lemma 4.3.5. We have

$$e_I \cdot \lambda \left((-1)^{I+[r]+r(d-r)} \left(\bigwedge_{i\notin I} \psi_i \right)(z_b) \right) = e_I \cdot \theta^*_{K/k,S,T} \cdot w_I.$$

Proof. Let $\chi \in \widehat{G}_{r,S,I}$.

We have

$$0 \longrightarrow \mathcal{O}_{K,S,T}^\times \longrightarrow P \xrightarrow{\psi} F \xrightarrow{\pi} H^1(D_{K,S,T}^\bullet) \longrightarrow 0.$$

This breaks up into two short exact sequences, which after tensoring with $\mathbb{C}$ and taking χ-components give

$$0 \longrightarrow (\mathcal{O}_{K,S,T}^\times)_\chi \longrightarrow P_\chi \xrightarrow{\psi} \mathrm{im}(\psi)_\chi \longrightarrow 0 \tag{4.3}$$

and

$$0 \longrightarrow \mathrm{im}(\psi)_\chi \longrightarrow F_\chi \xrightarrow{\pi} (X_{K,S})_\chi \longrightarrow 0. \tag{4.4}$$

We now fix ordered $\mathbb{C}$-bases for the modules above. (Note that we identify $e_\chi \cdot \mathbb{C}[G]$ with $\mathbb{C}$ by letting e_χ correspond to 1.) Let $J := \{1, ..., d\} \backslash I = \{j_1, j_2, ..., j_{d-r}\}$ with $j_1 < j_2 < ... < j_{d-r}$.

By the definition of π we have that $\pi(e_\chi \cdot b_i) = e_\chi \cdot (w_i - w_0)$ for $1 \leq i \leq n$. We also have that $e_\chi \cdot w_i = 0$ for $i \notin I$. Therefore we write

$$x_{i,\chi} := e_\chi \cdot (w_i - w_0),$$

and we fix an ordered $\mathbb{C}$-basis for $(\mathbb{C}X_{K,S})_\chi$ as $\{x_{i_1,\chi}, x_{i_2,\chi}, ..., x_{i_r,\chi}\}$.

We recall that $\mathbb{C}[G]$ is semisimple. Choose a section

$$\iota_2 : \mathbb{C}X_{K,S} \longrightarrow \mathbb{C}F$$

such that

$$\iota_2(x_{i,\chi}) = b_{i,\chi}$$

for $i \in I$, where $b_{i,\chi} := e_\chi \cdot b_i$.

An ordered $\mathbb{C}$-basis for F_χ is $\{b_{1,\chi}, b_{2,\chi}, ..., b_{d,\chi}\}$. Then $\mathrm{im}(\psi)_\chi$ has $\mathbb{C}$-rank $d - r$ and we fix $\{b_{j_1,\chi}, b_{j_2,\chi}, ..., b_{j_{d-r},\chi}\}$ as an ordered basis. We also have that P_χ has rank d and $(\mathcal{O}_{K,S,T}^\times)_\chi$ has rank r. Fix a basis $\{u_{i_1,\chi}, u_{i_2,\chi}, ..., u_{i_r,\chi}\}$ of $(\mathcal{O}_{K,S,T}^\times)_\chi$. We can then choose a section

$$\iota_1 : \mathbb{C}\,\mathrm{im}(\psi) \longrightarrow \mathbb{C}P$$

with $\{u_{i_1,\chi}, ..., u_{i_r,\chi}, \iota_1(b_{j_1,\chi}), ..., \iota_1(b_{j_{d-r},\chi})\}$ forming a basis for P_χ.

We can now use these two sections to define an isomorphism f of $\mathbb{C}$-modules from $\mathbb{C}P$ to $\mathbb{C}F$.

$$
\begin{array}{ccccc}
\mathbb{C}P & \xrightarrow{\;\sim\;} & \mathbb{C}\mathcal{O}_{K,S,T}^\times & \oplus & \iota_1(\mathbb{C}\,\mathrm{im}(\psi)) \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \iota_2 \circ \lambda} & & \downarrow{\scriptstyle \psi} \\
\mathbb{C}F & \xrightarrow{\;\sim\;} & \iota_2(\mathbb{C}X_{K,S}) & \oplus & \mathbb{C}\,\mathrm{im}(\psi)
\end{array}
$$

Then f also defines an isomorphism from P_χ to F_χ and we can define

$$f_i := b_i^* \circ f \in \mathrm{Hom}_{\mathbb{C}}(P_\chi, \mathbb{C}).$$

We will show that diagram (4.5) below commutes. This will allow us to obtain the equality given in the proposition by mapping the element z_b across both the top and bottom arrows.

$$\begin{array}{ccc}
\overset{d}{\underset{\mathbb{C}}{\bigwedge}} P_\chi & \xrightarrow{\;(-1)^{r(d-r)}\cdot\rho_{K,S}\circ\kappa\;} & \mathbb{C} \\
\Big\| & & \Big\| \\
\overset{d}{\underset{\mathbb{C}}{\bigwedge}} P_\chi & \xrightarrow{\;\wedge_{i=1}^{i=d} f_i\;} & \mathbb{C}
\end{array} \qquad (4.5)$$

We need only show that it commutes for a basis of $\bigwedge_{\mathbb{C}}^d P_\chi$.

Since $\{u_{i_1,\chi}, ..., u_{i_r,\chi}, \iota_1(b_{j_1,\chi}), ..., \iota_1(b_{j_{d-r},\chi})\}$ is a $\mathbb{C}$-basis for P_χ, we have that the element

$u_{i_1,\chi} \wedge ... \wedge u_{i_r,\chi} \wedge \iota_1(b_{j_1,\chi}) \wedge ... \wedge \iota_1(b_{j_{d-r},\chi})$ is a $\mathbb{C}$-basis for $\bigwedge_{\mathbb{C}}^d P_\chi$.

Let us first apply the top line. Unless otherwise stated all individual wedge products are taken in order of increasing index.

We start with the basis element

$$\bigwedge_{i\in I} u_{i,\chi} \wedge \bigwedge_{j\in J} \iota_1(b_{j,\chi}) \in \overset{d}{\underset{\mathbb{C}}{\bigwedge}} P_\chi.$$

Recall that the map κ was defined via two isomorphisms. We first apply the isomorphism given by $a \mapsto a \otimes \bigwedge_{1\leq i\leq d} b_i^*$. This gives

$$\bigwedge_{i\in I} u_{i,\chi} \wedge \bigwedge_{j\in J} \iota_1(b_{j,\chi}) \otimes \bigwedge_{1\leq k\leq d} b_{k,\chi}^* \in \overset{d}{\underset{\mathbb{C}}{\bigwedge}} P_\chi \otimes_{\mathbb{C}[G]} \overset{d}{\underset{\mathbb{C}}{\bigwedge}} (F^*)_\chi.$$

The second part of κ is given by the isomorphism

$$\det_G(P) \otimes_G \det_G^{-1}(F) \xrightarrow{\sim} \det_G(D_{K,S,T}^\bullet).$$

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However the definition of the regulator isomorphism $\rho_{K,S}$ in §3.2.2 shows that its first part is given by the inverse of this map. Therefore our next step is to apply the second part of $\rho_{K,S}$, which is given by the isomorphisms coming from the split short exact sequences (4.3) and (4.4). We note that by (3.1) $\bigwedge_{i \in I} u_{i,\chi} \wedge \bigwedge_{j \in J} \iota_1(b_{j,\chi}) \otimes \bigwedge_{1 \leq k \leq d} b_{k,\chi}^*$ is equal to

$$(-1)^{J+[d-r]} \bigwedge_{i \in I} u_{i,\chi} \wedge \bigwedge_{j \in J} \iota_1(b_{j,\chi}) \otimes \bigwedge_{j \in J} b_{j,\chi}^* \wedge \bigwedge_{i \in I} b_{i,\chi}^*$$

and is therefore mapped to the element

$$(-1)^{J+[d-r]} \bigwedge_{i \in I} u_{i,\chi} \otimes \bigwedge_{j \in J} b_{j,\chi} \otimes \bigwedge_{j \in J} b_{j,\chi}^* \otimes \bigwedge_{i \in I} x_{i,\chi}^*$$

belonging to

$$\bigwedge^r_{\mathbb{C}} (\mathcal{O}_{K,S,T}^\times)_\chi \otimes_{\mathbb{C}} \bigwedge^{d-r}_{\mathbb{C}} \mathrm{im}(\psi)_\chi \otimes_{\mathbb{C}} \bigwedge^{d-r}_{\mathbb{C}} (\mathrm{im}(\psi)^*)_\chi \otimes_{\mathbb{C}} \bigwedge^r_{\mathbb{C}} (X_{K,S}^*)_\chi.$$

We then apply the evaluation map to obtain

$$(-1)^{J+[d-r]} \bigwedge_{i \in I} u_{i,\chi} \otimes \bigwedge_{i \in I} x_{i,\chi}^* \in \bigwedge^r_{\mathbb{C}} (\mathcal{O}_{K,S,T}^\times)_\chi \otimes_{\mathbb{C}} \bigwedge^r_{\mathbb{C}} (X_{K,S}^*)_\chi.$$

Applying the Dirichlet logarithm gives

$$(-1)^{J+[d-r]} \bigwedge_{i \in I} \lambda(u_{i,\chi}) \otimes \bigwedge_{i \in I} x_{i,\chi}^* \in \bigwedge^r_{\mathbb{C}} (X_{K,S})_\chi \otimes_{\mathbb{C}} \bigwedge^r_{\mathbb{C}} (X_{K,S}^*)_\chi.$$

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This is equal to

$$(-1)^{J+[d-r]}\det(C_\chi)\bigwedge_{i\in I}(x_{i,\chi}) \otimes \bigwedge_{i\in I} x_{i,\chi}^* \in \bigwedge_{\mathbb{C}}^r (X_{K,S})_\chi \otimes_{\mathbb{C}} \bigwedge_{\mathbb{C}}^r (X_{K,S}^*)_\chi$$

where $C_\chi = (c_{ik,\chi})_{i,k\in I}$ is the matrix defined by $\lambda(u_{i,\chi}) = \sum_{k\in I} c_{ik,\chi} x_{k,\chi}$.

Finally applying the evaluation map and multiplying by $(-1)^{r(d-r)}$ gives

$$(-1)^{I+[r]}\det(C_\chi) \in \mathbb{C},$$

where we note that $(-1)^{[d-r]+[r]} = (-1)^{[d]+r(d-r)}$ and $(-1)^{[d]} = (-1)^{I+J}$.

Let us now turn our attention to the bottom row of diagram (4.5). We have $\wedge_{i=1}^{i=d} f_i = \wedge_{i=1}^{i=d} b_{i,\chi}^* \circ \wedge_{i=1}^{i=d} f$. In order to apply $\wedge_{i=1}^{i=d} f$ we calculate $\det(f)$ with respect to our chosen ordered bases.

It is easier to do this by first considering the matrix of f with respect to a different ordering of the basis of F_χ. We consider the matrix of f with respect to the ordered bases $\{u_{i_1,\chi}, ..., u_{i_r,\chi}, \iota_1(b_{j_1,\chi}), ..., \iota_1(b_{j_{d-r},\chi})\}$ of P_χ and $\{b_{i_1,\chi}, ..., b_{i_r,\chi}, b_{j_1,\chi}, ..., b_{j_{d-r},\chi}\}$ of F_χ.

We have for $i \in I$ that

$$f(u_{i,\chi}) = \iota_2\left(\sum_{k\in I} c_{ik,\chi} x_{k,\chi}\right)$$
$$= \sum_{k\in I} c_{ik,\chi} b_{k,\chi}$$

where the first equality follows since $u_{i,\chi} \in (\mathcal{O}_{K,S,T}^\times)_\chi$ and the second by the definition

of ι_2. For $j \in J$ we have

$$f(\iota_1(b_{j,\chi})) = b_{j,\chi}.$$

Thus the matrix of f with respect to this ordering is:

$$
\begin{array}{c}
\\
u_{i_1,\chi} \\
\vdots \\
u_{i_r r,\chi} \\
\iota_1(b_{j_1,\chi}) \\
\vdots \\
\iota_1(b_{j_{d-r},\chi})
\end{array}
\begin{array}{cccccc}
b_{i_1,\chi} & \cdots & b_{i_r,\chi} & b_{j_1,\chi} & \cdots & b_{j_{d-r},\chi} \\
\left(\begin{array}{ccc|ccc}
 & & & & & \\
 & C_\chi & & & 0 & \\
 & & & & & \\
\hline
 & & & & & \\
 & 0 & & & \mathrm{Id} & \\
 & & & & & \\
\end{array}\right)
\end{array}
$$

The determinant of this matrix is equal to $\det(C_\chi)$. In order to obtain the matrix of f with respect to our original ordered basis $\{b_{1,\chi}, ..., b_{d,\chi}\}$ of F_χ we have to permute the columns and so we obtain that

$$\det(f) = (-1)^{I+[r]}\det(C_\chi).$$

We can now apply the bottom arrow to our basis element. Again we start with the element

$$\bigwedge_{i \in I} u_{i,\chi} \wedge \bigwedge_{j \in J} \iota_1(b_{j,\chi}) \in \bigwedge_{\mathbb{C}}^{d} P_\chi.$$

We then apply $\wedge_{i=1}^{i=d} f$ to get

$$\bigwedge_{i \in I} f(u_{i,\chi}) \wedge \bigwedge_{j \in J} f(\iota_1(b_{j,\chi})) = (\det(f)) \bigwedge_{i=1}^{i=d} b_{i,\chi} \in \bigwedge_{\mathbb{C}}^{d} F_\chi.$$

Applying $\wedge_{i=1}^{i=d} b_{i,\chi}^*$ gives

$$\det(f) = (-1)^{I+[r]} \det(C_\chi) \in \mathbb{C},$$

thus proving the commutativity of the diagram.

We can use this result to map the element $e_\chi \cdot z_b \in \bigwedge_{\mathbb{C}}^d P_\chi$ in both directions around diagram (4.5).

By the definition of z_b under the top line we have

$$e_\chi \cdot z_b \mapsto e_\chi \cdot z_{K/k,S,T} \mapsto (-1)^{r(d-r)} e_\chi \cdot \theta_{K/k,S,T}^*(0).$$

Under the bottom line we have

$$e_\chi \cdot z_b \mapsto \left(\wedge_{i=1}^{i=d} f_i \right) (e_\chi \cdot z_b).$$

Therefore

$$\left(\bigwedge_{i=1}^{i=d} f_i\right)(e_\chi \cdot z_b) = (-1)^{r(d-r)} e_\chi \cdot \theta^*_{K/k,S,T}$$

$$\Rightarrow \quad (-1)^{I+[r]+r(d-r)}\left(\bigwedge_{i\in I} f_i\right)\left(\left(\bigwedge_{i\notin I} f_i\right)(e_\chi \cdot z_b)\right) = e_\chi \cdot \theta^*_{K/k,S,T}$$

$$\Rightarrow \quad (-1)^{I+[r]+r(d-r)}(\iota_2 \circ \lambda)\left(\left(\bigwedge_{i\notin I} f_i\right)(e_\chi \cdot z_b)\right) = e_\chi \cdot \theta^*_{K/k,S,T}$$

$$\Rightarrow \quad (-1)^{I+[r]+r(d-r)}\lambda\left(\left(\bigwedge_{i\notin I} f_i\right)(e_\chi \cdot z_b)\right) = e_\chi \cdot \theta^*_{K/k,S,T}\bigwedge_{i\in I}(w_i - w_0)$$

$$\Rightarrow \quad (-1)^{I+[r]+r(d-r)}\lambda\left(e_I \cdot \left(\bigwedge_{i\notin I} f_i\right)(z_b)\right) = e_I \cdot \theta^*_{K/k,S,T} \cdot w_I$$

$$\Rightarrow \quad (-1)^{I+[r]+r(d-r)}\lambda\left(e_I \cdot \left(\bigwedge_{i\notin I} f_i\right)(z_b)\right) = e_I \cdot \theta^a_{K/k,S,T} \cdot w_I$$

where third line follows since $\left(\bigwedge_{i\notin I} f_i\right)(z_b) \in \bigwedge^r \ker(\psi)$ and the last line by (2.2).

Finally let us consider $e_\chi \cdot (\bigwedge_{i\notin I} f_i)(z_b)$ for $\chi \in \widehat{G}_{r,S,I}$. Recall that f is $\iota_2 \circ \lambda$ on $\mathbb{C}\mathcal{O}^\times_{K,S,T} = \mathbb{C}\ker(\psi)$ and ψ on $\iota_2(\mathbb{C}\operatorname{im}(\psi))$. We claim that $(b_i^* \circ \iota_2 \circ \lambda)^\chi = 0$ for $i \notin I$. Considering just the χ component we have that λ maps to $(X_{K,S})_\chi$ which has basis $\{x_{i,\chi} : i \in I\}$. These basis elements are mapped to $\{b_{i,\chi} : i \in I\}$ by ι_2. However these elements are not seen by b_i^* for $i \notin I$, which proves our claim. Therefore

$$e_I \cdot \left(\bigwedge_{i\notin I} f_i\right)(z_b) = e_I \cdot \left(\bigwedge_{i\notin I} \psi_i\right)(z_b)$$

thus proving the proposition. $\qquad\square$

4.4 Evaluators and Fitting ideals

In this section we investigate the ideal generated by $\Phi(\eta^I_{K/k,S,T})$ as Φ runs over $\bigwedge^r_G \mathrm{Hom}_G(\mathcal{O}^\times_{K,S,T}, \mathbb{Z}[G])$. We will apply and adapt the techniques developed by Burns et al. in [6] in the following way. We begin by defining, for every $I \in \wp_r(S_{\min})$, a subextension of K/k in which the hypotheses of the Rubin-Stark Conjecture hold. Following Emmons, Popescu and Vallières (see [14] and [28]) we show the relationship between the Rubin-Stark elements of the subextensions, and the elements $\eta^I_{K/k,S,T}$. This allows us to apply the results in [6].

Applying these methods introduces denominators that remove some of the integrality properties. However, through Theorem 4.2.4 we are able to regain some integrality results.

4.4.1 Subextensions of K/k

Let a be a non-negative integer such that $a < |S_{\min}|$. For each $I \in \wp_a(S_{\min})$ we define a subextension of K/k in which there are at least a places that split completely. Let

- $D_I = \langle G_v : v \in I \rangle$

- $L_I = K^{D_I}$

- $\Gamma_I = G/D_I$

- $e_{D_I} = \frac{1}{|D_I|} \sum_{d \in D_I} d$

In this chapter we will use these constructions with $a = r$, however in the next chapter we will use them for any a as defined above.

Remark 4.4.1. The definition of e_{D_I} implies that it can be decomposed as

$$e_{D_I} = \sum_{\substack{\chi \in \widehat{G} \\ \chi(D_I)=1}} e_\chi = \sum_{\substack{\chi \in \widehat{G} \\ \chi(G_v)=1, \forall v \in I}} e_\chi.$$

By comparing this with the definition of the idempotent e_I, it is easy to see that

$$e_I \cdot e_{D_I} = e_I.$$

For all $i \in \{0, ..., n\}$ fix w_i^I as the unique place of L_I lying between v_i and w_i.

Let

$$\epsilon_{L_I/k,S,T}^I \in \mathbb{C} \bigwedge_{\Gamma_I}^r \mathcal{O}_{L_I,S,T}^\times$$

be the unique element such that

$$\lambda_{L_I,S}\left(\epsilon_{L_I/k,S,T}^I\right) = \theta_{L_I/k,S,T}^r \bigwedge_{i \in I}(w_i^I - w_0^I). \tag{4.6}$$

Since the set of places $\{v_i : i \in I\}$ split completely in L_I/k, $\epsilon_{L_I/k,S,T}^I$ is a Rubin-Stark element for L_I/k.

Theorem 2.2.19 applied to the Rubin-Stark element $\epsilon_{L_I/k,S,T}^I$ therefore says that $\mathrm{LTC}(L_I/k)$ implies

$$\mathrm{Fit}_{\Gamma_I}^r\left(\mathrm{Sel}_S^T(L_I)^{\mathrm{tr}}\right) = \left\{\Psi(\epsilon_{L_I/k,S,T}^I) : \Psi \in \bigwedge_{\Gamma_I}^r \mathrm{Hom}_{\Gamma_I}(\mathcal{O}_{L_I,S,T}^\times, \mathbb{Z}[\Gamma_I])\right\} \tag{4.7}$$

In order to apply this result to our element $\eta_{K/k,S,T}^I$ we use the following result linking

the Rubin-Stark element of each subextension to the corresponding element $\eta^I_{K/k,S,T}$ of the original extension.

Lemma 4.4.2. If we identify $\mathbb{C} \cdot \bigwedge^r_{\Gamma_I} \mathcal{O}^\times_{L_I,S,T}$ with a subspace of $\mathbb{C} \cdot \bigwedge^r_G \mathcal{O}^\times_{K,S,T}$ in the obvious way, then:

$$\eta^I_{K/k,S,T} = \frac{1}{|D_I|^r} \epsilon^I_{L_I/k,S,T}.$$

Proof. Identifying $\mathbb{C}[\Gamma_I]$ with $e_{D_I}\mathbb{C}[G]$ in the natural way, one has $\theta^r_{L_I/k,S,T}(0) = e_{D_I}\theta^r_{K/k,S,T}(0)$ and the analogue of the idempotent $e_{r,S}$ for L_I/k is equal to $e_{D_I}e_Ie_{r,S}$ and so

$$\theta^r_{L_I/k,S,T}(0) = e_{D_I}e_Ie_{r,S} \cdot \theta^r_{K/k,S,T}(0) = e_{D_I}e_I \cdot \theta^r_{K/k,S,T}(0) = e_I \cdot \theta^r_{K/k,S,T}(0) \qquad (4.8)$$

(with the latter equality since $e_{D_I}e_I = e_I$ by Remark 4.4.1).

Recall $N_{D_I} := \sum_{h \in D_I} h (= |D_I|e_{D_I})$. Substituting (4.8) into (4.6) gives

$$\lambda_{L_I,S}\left(\epsilon^I_{L_I/k,S,T}\right) = e_I \cdot \theta^r_{K/k,S,T}(0) \cdot \bigwedge_{i \in I} \left(w^I_i - w^I_0\right).$$

Then applying the corestriction map and using the functorality of the Dirichlet regulator map under change of field (as expressed by the commutativity of the diagram in [26,

Chap. I, §6.5]) to explicitly compare the maps $\lambda_{L_I,S}$ and $\lambda_{K,S}$ gives

$$\lambda_{K,S}\left(\epsilon^I_{L_I/k,S,T}\right) = e_I \cdot \theta^r_{K/k,S,T}(0) \cdot \bigwedge_{i\in I} N_{D_I}(w_i - w_0)$$

$$= (N_{D_I})^r \cdot e_I \cdot \theta^r_{K/k,S,T}(0) \cdot \bigwedge_{i\in I}(w_i - w_0)$$

$$= |D_I|^r \cdot e_I \cdot \theta^r_{K/k,S,T}(0) \cdot \bigwedge_{i\in I}(w_i - w_0)$$

$$= \lambda_{K,S}\left(|D_I|^r \cdot \eta^I_{K/k,S,T}\right).$$

Here the second equality is clear, the third is valid because $(N_{D_I})^r = |D_I|^r e_{D_I}$ and $e_{D_I}e_I = e_I$ and the last follows directly from Definition 4.2.1.

Then, since $\lambda_{K,S}$ is bijective, the displayed equality implies $\epsilon^I_{L_I/k,S,T} = |D_I|^r \cdot \eta^I_{K/k,S,T}$, as required. $\qquad\square$

Remark 4.4.3. The Rubin-Stark conjecture asserts that $\epsilon^I_{L_I/k,S,T} \in \Lambda_{L_I/k,S,T}$. However this relationship between elements allows Vallières to make the stronger conjecture that $\epsilon^I_{L_I/k,S,T} \in |D_I| \cdot \Lambda_{L_I/k,S,T}$ and to prove that this conjecture follows from LTC(K/k).

This relationship allows us to apply results already proven for the Rubin-Stark element $\epsilon^I_{L_I/k,S,T}$, and via the restriction and corestriction maps apply them to the element $\eta^I_{K/k,S,T}$.

4.4.2 Proof of Theorem 4.2.4

In this section we prove Theorem 4.2.4.

We prove this prime by prime after localisation, i.e. we prove that for every prime p

we have

$$e_{D_I} \cdot \mathrm{Fit}^r_{\mathbb{Z}_{(p)}[G]}(\mathrm{Sel}^T_S(K)^{\mathrm{tr}}_{(p)}) \subseteq \mathrm{Fit}^r_{\mathbb{Z}_{(p)}[G]}(\mathrm{Sel}^T_S(K)^{\mathrm{tr}}_{(p)}).$$

We have the following presentation for $\mathrm{Sel}^T_S(K)^{\mathrm{tr}}$

$$P \xrightarrow{\psi} F \xrightarrow{\pi} \mathrm{Sel}^T_S(K)^{\mathrm{tr}} \longrightarrow 0.$$

After localisation, we assume, as we may by Lemma 3.3.1, that $F_{(p)} = P_{(p)}$. Thus we obtain the presentation

$$F_{(p)} \xrightarrow{\psi} F_{(p)} \xrightarrow{\pi} \mathrm{Sel}^T_S(K)^{\mathrm{tr}}_{(p)} \longrightarrow 0.$$

The $\mathbb{Z}[G]$-basis $\{b_1, ..., b_d\}$ for F also gives a $\mathbb{Z}_{(p)}[G]$-basis for $F_{(p)}$, for which we use the same notation. Let $A_{(p)}$ be the matrix corresponding to this presentation.

For any $\sigma \in \mathfrak{S}_{d,r}$, we have that $\det(\psi_i(b_{\sigma(j)}))_{i \notin I, r < j \leq d}$ is a $(d-r) \times (d-r)$ minor of $A_{(p)}$. Furthermore the proof of Lemma 4.3.4 gives us that

$$e_I \cdot \det\left(\psi_i\left(b_{\sigma(j)}\right)\right)_{i \notin I, r < j \leq d} = \det\left(\psi_i\left(b_{\sigma(j)}\right)\right)_{i \notin I, r < j \leq d}.$$

Since we also have that $e_I \cdot e_{D_I} = e_I$ (see Remark 4.4.1) it's not hard to see that

$$e_{D_I} \cdot \det\left(\psi_i\left(b_{\sigma(j)}\right)\right)_{i \notin I, r < j \leq d} = \det\left(\psi_i\left(b_{\sigma(j)}\right)\right)_{i \notin I, r < j \leq d}.$$

The other $(d-r) \times (d-r)$ minors of $A_{(p)}$ are of the form $\det\left(\psi_i\left(b_{\sigma(j)}\right)\right)_{i \in J, r < j \leq d}$ for some $\sigma \in \mathfrak{S}_{d,r}$ and some $J \in \wp_{d-r}(S_{\min})$ such that $I \cap J \neq \emptyset$. In this case we have that

$$e_{D_I} \cdot \det\left(\psi_i\left(b_{\sigma(j)}\right)\right)_{i \in J, r < j \leq d} = 0.$$

To see this we fix $i \in I \cap J$. Then $\mathrm{im}(\psi_i) \subseteq \mathcal{I}_{G_{v_i}}$ (see [6, Rem. 5.19] for proof). Let $x \in \mathcal{I}_{G_{v_i}}$. Then $x = \sum_{\delta \in G_{v_i}} (\delta - 1) \cdot x_\delta$ for some $x_\delta \in \mathbb{Z}_{(p)}$. Then

$$
\begin{aligned}
e_{D_I} \cdot x &= \sum_{\delta \in G_{v_i}} e_{D_I} \cdot (\delta - 1) \cdot x_\delta \\
&= \sum_{\delta \in G_{v_i}} \sum_{\substack{\chi \in \widehat{G} \\ \chi(G_v)=1, \forall v \in I}} e_\chi \cdot (\delta - 1) \cdot x_\delta \\
&= \sum_{\delta \in G_{v_i}} \sum_{\substack{\chi \in \widehat{G} \\ \chi(G_v)=1, \forall v \in I}} e_\chi \cdot (\chi(\delta) - 1) \cdot x_\delta \\
&= 0,
\end{aligned}
$$

where the final line follows since $\chi(\delta) = 1$.

We then obtain our final result since

$$e_{D_I} \cdot \mathrm{Fit}^r_{\mathbb{Z}_{(p)}[G]}(\mathrm{Sel}^T_S(K)^{\mathrm{tr}}_{(p)})$$

$$= \sum_{\substack{B \in \{(d-r)\times \\ (d-r)\text{ minors of } A_{(p)}\}}} e_{D_I} \cdot \det(B) \cdot \mathbb{Z}_{(p)}$$

$$= \sum_{\sigma \in \mathfrak{S}_{d,r}} e_{D_I} \cdot \det\left(\psi_i\left(b_{\sigma(j)}\right)\right)_{i \notin I, r < j \leq d} \cdot \mathbb{Z}_{(p)}[G]$$

$$+ \sum_{\substack{J \in \wp_{d-r}(S_{\min}) \\ I \cap J \neq \emptyset \\ \sigma \in \mathfrak{S}_{d,r}}} e_{D_I} \cdot \det\left(\psi_i\left(b_{\sigma(j)}\right)\right)_{i \in J, r < j \leq d} \cdot \mathbb{Z}_{(p)}[G]$$

$$= \sum_{\sigma \in \mathfrak{S}_{d,r}} \det\left(\psi_i\left(b_{\sigma(j)}\right)\right)_{i \notin I, r < j \leq d} \cdot \mathbb{Z}_{(p)}[G] + 0$$

$$\subseteq \mathrm{Fit}^r_{\mathbb{Z}_{(p)}[G]}(\mathrm{Sel}^T_S(K)^{\mathrm{tr}}_{(p)}).$$

4.4.3 The proof of Theorem 4.2.6

As a first step we prove the following result.

Lemma 4.4.4.

$$\mathrm{Fit}^r_G(\mathrm{Sel}^T_S(K)^{\mathrm{tr}}) = e_{r,S} \cdot \mathrm{Fit}^r_G(\mathrm{Sel}^T_S(K)^{\mathrm{tr}}).$$

Proof. We prove that for all $x \in \mathrm{Fit}^r_G(\mathrm{Sel}^T_S(K)^{\mathrm{tr}})$ and all χ such that $r_S(\chi) > r$ we have that $xe_\chi = 0$. If this holds then

$$x = x \cdot 1 = xe_{r,S} + \sum_{\substack{\chi \in \widehat{G} \\ r_S(\chi) > r}} xe_\chi = xe_{r,S},$$

so that $\mathrm{Fit}^r_G(\mathrm{Sel}^T_S(K)^{\mathrm{tr}}) = e_{r,S} \cdot \mathrm{Fit}^r_G(\mathrm{Sel}^T_S(K)^{\mathrm{tr}})$ as required.

Let χ be such that $r_S(\chi) > r$ and consider the surjective ring homomorphism

$$\mathbb{Z}[G] \longrightarrow R_\chi := e_\chi \cdot \mathbb{Z}[\chi][G]$$

that sends x to $e_\chi \cdot x$.

Under this map the standard properties of Fitting ideals give us that

$$\mathrm{Fit}^r_G(\mathrm{Sel}^T_S(K)^{\mathrm{tr}}) \longrightarrow \mathrm{Fit}^r_{R_\chi}(R_\chi \otimes_G \mathrm{Sel}^T_S(K)^{\mathrm{tr}}).$$

We show that $\mathrm{Fit}^r_{R_\chi}(R_\chi \otimes_G \mathrm{Sel}^T_S(K)^{\mathrm{tr}}) = 0$.

By a result of Northcott (see [22, Ex. 3, p.61]), and the fact that a Fitting ideal is determined by its localisations, it is enough to show that $R_\chi \otimes_G \mathrm{Sel}^T_S(K)^{\mathrm{tr}}$ surjects onto a projective R_χ module of rank greater than r. We show this by computing the dimensions over $\mathbb{C}$.

Note that Lemma 2.1.3 gives

$$\dim_{\mathbb{C}}(\mathbb{C}\mathrm{Sel}^T_S(K)^{\mathrm{tr}} \cdot e_\chi) = \dim_{\mathbb{C}}(\mathbb{C}X_{K,S} \cdot e_\chi) = r_S(\chi) > r.$$

It follows that $R_\chi \otimes_G \mathrm{Sel}^T_S(K)^{\mathrm{tr}})$ spans a $\mathbb{C} \simeq \mathbb{C}[G] \cdot e_\chi$-space of dimension greater than r, proving the required result. $\square$

Remark 4.4.5. For each $I \in \wp_r(S_{\min})$ we have that

$$e_{D_I} \cdot \mathrm{Fit}^r_G(\mathrm{Sel}^T_S(K)^{\mathrm{tr}}) = e_I \cdot \mathrm{Fit}^r_G(\mathrm{Sel}^T_S(K)^{\mathrm{tr}}).$$

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To see this note that $e_{D_I} = e_I + (e_{D_I} - e_I)$ and $e_{D_I} - e_I$ is the sum of idempotents e_χ such that $r_S(\chi) > r$ and $\chi(D_I) = 1$ and follow the argument at the end of the proof of Lemma 4.4.4 above.

Theorem 4.4.6. With the same set up as before LTC(K/k) implies that:

$$e_{D_I} \cdot \mathrm{Fit}_G^r(\mathrm{Sel}_S^T(K)^{\mathrm{tr}}) = \left\{ \Phi(\eta_{K/k,S,T}^I) : \Phi \in \bigwedge_G^r (\mathcal{O}_{K,S,T}^\times)^* \right\}$$

$$\subseteq \mathrm{Fit}_G^r(\mathrm{Sel}_S^T(K)^{\mathrm{tr}}).$$

Proof. Assume $LTC(K/k)$.

We need only prove that

$$e_{D_I} \cdot \mathrm{Fit}_G^r(\mathrm{Sel}_S^T(K)^{\mathrm{tr}}) = \left\{ \Phi(\eta_{K/k,S,T}^I) : \Phi \in \bigwedge_G^r (\mathcal{O}_{K,S,T}^\times)^* \right\}$$

since the containment follows from Theorem 4.2.4, proved above.

Suppose $|S| = r + 1$. Then S must contain at least r places that split completely. In this case we note that $e_I = e_{r,S}$ and therefore Lemma 4.4.4 and Remark 4.4.5 imply that

$$e_{D_I} \cdot \mathrm{Fit}_G^r(\mathrm{Sel}_S^T(K)^{\mathrm{tr}}) = e_I \cdot \mathrm{Fit}_G^r(\mathrm{Sel}_S^T(K)^{\mathrm{tr}}) = \mathrm{Fit}_G^r(\mathrm{Sel}_S^T(K)^{\mathrm{tr}}).$$

Therefore the stronger result $\left\{ \Phi(\eta_{K/k,S,T}^I) : \Phi \in \bigwedge_G^r (\mathcal{O}_{K,S,T}^\times)^* \right\} = \mathrm{Fit}_G^r(\mathrm{Sel}_S^T(K)^{\mathrm{tr}})$ has been proved by Burns et al in [6] so we are done.

Now we assume that $|S| > r + 1$. Lemma 3.2.7 implies LTC(L_I/k). Thus we can apply the result from Burns et al. in (4.7) to each element $\epsilon_{L_I/k,S,T}^I$. We use Lemma 4.4.2 and the properties of restriction and corestriction maps to apply the result to the elements

$\eta^I_{K/k,S,T}$. This will allow us to show that

$$\left\{ \Phi(\eta^I_{K/k,S,T}) : \Phi \in \bigwedge^r_G (\mathcal{O}^\times_{K,S,T})^* \right\} = \frac{1}{|D_I|} \mathrm{cor}_{K/L_I} \left(\mathrm{Fit}^r_{\Gamma_I}(\mathrm{Sel}^T_S(L_I)^{\mathrm{tr}}) \right).$$

First we show that

$$\frac{1}{|D_I|} \mathrm{cor}_{K/L_I} \left(\mathrm{Fit}^r_{\Gamma_I}(\mathrm{Sel}^T_S(L_I)^{\mathrm{tr}}) \right) \subseteq \left\{ \Phi(\eta^I_{K/k,S,T}) : \Phi \in \bigwedge^r_G (\mathcal{O}^\times_{K,S,T})^* \right\}.$$

By (4.7) we may write any element of $\mathrm{Fit}^r_{\Gamma_I}(\mathrm{Sel}^T_S(L_I)^{\mathrm{tr}})$ as $\Psi(\epsilon^I_{L_I/k,S,T})$ for some $\Psi \in \bigwedge^r_{\Gamma_I} \mathrm{Hom}_{\Gamma_I}(\mathcal{O}^\times_{L_I,S,T}, \mathbb{Z}[\Gamma_I])$. Therefore it is enough to show the inclusion above holds for the element

$$\psi_1 \wedge \dots \wedge \psi_r(\epsilon^I_{L_I/k,S,T}) \in \mathrm{Fit}^r_{\Gamma_I}(\mathrm{Sel}^T_S(L_I)^{\mathrm{tr}})$$

for any $\psi_1, \dots, \psi_r \in \mathrm{Hom}_{\Gamma_I}(\mathcal{O}^\times_{L_I,S,T}, \mathbb{Z}[\Gamma_I])$.

We apply the corestriction map to get

$$\mathrm{cor}_{K/L_I}\left(\psi_1 \wedge ... \wedge \psi_r\left(\epsilon^I_{L_I/k,S,T}\right)\right)$$

$$= \frac{1}{|D_I|^{r-1}}\left(\mathrm{cor}_{K/L_I} \circ \psi_1\right) \wedge ... \wedge \left(\mathrm{cor}_{K/L_I} \circ \psi_r\right)\left(\epsilon^I_{L_I/k,S,T}\right)$$

$$= \frac{1}{|D_I|^{r-1}}\left(\mathrm{cor}_{K/L_I} \circ \psi_1\right) \wedge ... \wedge \left(\mathrm{cor}_{K/L_I} \circ \psi_r\right)\left(|D_I|^r \eta^I_{K/k,S,T}\right)$$

$$= |D_I|\left(\mathrm{cor}_{K/L_I} \circ \psi_1\right) \wedge ... \wedge \left(\mathrm{cor}_{K/L_I} \circ \psi_r\right)\left(\eta^I_{K/k,S,T}\right)$$

$$= |D_I|\left(\widetilde{\mathrm{cor}_{K/L_I} \circ \psi_1}\right) \wedge ... \wedge \left(\widetilde{\mathrm{cor}_{K/L_I} \circ \psi_r}\right)\left(\eta^I_{K/k,S,T}\right)$$

$$\in |D_I| \cdot \left\{\Phi(\eta^I_{K/k,S,T}) : \Phi \in \bigwedge_G^r (\mathcal{O}^\times_{K,S,T})^*\right\}$$

where the first equality follows from Remark 3.1.4(i), the second equality from the relation $\eta^I_{K/k,S,T} = \frac{1}{|D_I|^r}\epsilon^I_{L_I/k,S,T}$ proved in Lemma 4.4.2, the map $\mathrm{cor}_{K/L_I}\circ\psi_i \in \mathrm{Hom}_G(\mathcal{O}^\times_{L_I,S,T}, \mathbb{Z}[G])$ for $i = 1, ..., r$ by Lemma 3.1.5, and $\widetilde{\mathrm{cor}_{K/L_I} \circ \psi_i}$ is any lift to $(\mathcal{O}^\times_{K,S,T})^*$.

Therefore

$$\frac{1}{|D_I|}\mathrm{cor}_{K/L_I}\left(\mathrm{Fit}^r_{\Gamma_I}(\mathrm{Sel}^T_S(L_I)^{\mathrm{tr}})\right) \subseteq \left\{\Phi(\eta^I_{K/k,S,T}) : \Phi \in \bigwedge_G^r (\mathcal{O}^\times_{K,S,T})^*\right\}$$

as required.

We now show the inclusion in the other direction. Again it is enough to show that

$$\phi_1 \wedge ... \wedge \phi_r\left(\eta^I_{K/k,S,T}\right) \in \frac{1}{|D_I|}\mathrm{cor}_{K/L_I}\left(\mathrm{Fit}^r_{\Gamma_I}(\mathrm{Sel}^T_S(L_I)^{\mathrm{tr}})\right)$$

for arbitrary $\phi_1, ..., \phi_r \in (\mathcal{O}^\times_{K,S,T})^*$.

Fix $\phi_1, ..., \phi_r \in (\mathcal{O}^\times_{K,S,T})^*$ and for each i denote the restriction of ϕ_i to $\mathcal{O}^\times_{L_I,S,T}$ by the

same symbol ϕ_i. Then Lemma 3.1.5 gives that

$$\frac{1}{|D_I|}\mathrm{res}_{K/L_I} \circ \phi_i \in \mathrm{Hom}_{\Gamma_I}(\mathcal{O}_{L_I,S,T}^{\times}, \mathbb{Z}[\Gamma_I])$$

for $i = 1, ..., r$. Therefore by (4.7)

$$\left(\frac{1}{|D_I|}\mathrm{res}_{K/L_I} \circ \phi_1\right) \wedge ... \wedge \left(\frac{1}{|D_I|}\mathrm{res}_{K/L_I} \circ \phi_r\right)\left(\epsilon_{L_I/k,S,T}^I\right) \in \mathrm{Fit}_{\Gamma_I}^r(\mathrm{Sel}_S^T(L_I)^{\mathrm{tr}}).$$

Applying the corestriction map then gives an element of $\mathrm{cor}_{K/L_I}\left(\mathrm{Fit}_{\Gamma_I}^r(\mathrm{Sel}_S^T(L_I)^{\mathrm{tr}})\right)$.

Therefore

$$\mathrm{cor}_{K/L_I}\left(\left(\frac{1}{|D_I|}\mathrm{res}_{K/L_I} \circ \phi_1\right) \wedge ... \wedge \left(\frac{1}{|D_I|}\mathrm{res}_{K/L_I} \circ \phi_r\right)\left(\epsilon_{L_I/k,S,T}^I\right)\right)$$
$$= \frac{1}{|D_I|^{r-1}}\left(\left(\frac{1}{|D_I|}\mathrm{cor}_{K/L_I} \circ \mathrm{res}_{K/L_I} \circ \phi_1\right) \wedge ... \wedge \left(\frac{1}{|D_I|}\mathrm{cor}_{K/L_I} \circ \mathrm{res}_{K/L_I} \circ \phi_r\right)\left(\epsilon_{L_I/k,S,T}^I\right)\right)$$
$$= \frac{1}{|D_I|^{r-1}}\left(\left(\frac{N_{D_I}}{|D_I|} \cdot \phi_1\right) \wedge ... \wedge \left(\frac{N_{D_I}}{|D_I|} \cdot \phi_r\right)\left(\epsilon_{L_I/k,S,T}^I\right)\right)$$
$$= \frac{N_{D_I}}{|D_I|^r} \cdot \phi_1 \wedge ... \wedge \phi_r\left(\epsilon_{L_I/k,S,T}^I\right)$$
$$= \frac{1}{|D_I|^{r-1}} \cdot \phi_1 \wedge ... \wedge \phi_r\left(\epsilon_{L_I/k,S,T}^I\right)$$
$$= |D_I| \cdot \phi_1 \wedge ... \wedge \phi_r\left(\eta_{K/k,S,T}^I\right) \in \mathrm{cor}_{K/L_I}\left(\mathrm{Fit}_{\Gamma_I}^r(\mathrm{Sel}_S^T(L_I)^{\mathrm{tr}})\right),$$

where the first equality follows from Remark 3.1.4(i), the second equality from Remark 3.1.4(ii) and the third since $\left(\frac{N_{D_I}}{|D_I|^r}\right)^2 = \frac{N_{D_I}}{|D_I|^r}$. The fourth equality follows since $\frac{N_{D_I}}{|D_I|^r} = e_{D_I}$ and $e_{D_I} \cdot \epsilon_{L_I/k,S,T}^I = \epsilon_{L_I/k,S,T}^I$. The final equality follows from Lemma 4.4.2 and this gives the required inclusion.

The final requirement that

$$\frac{1}{|D_I|}\mathrm{cor}_{K/L_I}\left(\mathrm{Fit}^r_{\Gamma_I}(\mathrm{Sel}^T_S(L_I)^{\mathrm{tr}})\right) = e_{D_I} \cdot \mathrm{Fit}^r_G(\mathrm{Sel}^T_S(K)^{\mathrm{tr}})$$

follows immediately from Remark 3.1.4(ii) if we note that

$$\mathrm{res}_{K/L_I}(\mathrm{Fit}^r_G(\mathrm{Sel}^T_S(K)^{\mathrm{tr}})) = \mathrm{Fit}^r_{\Gamma_I}(\mathrm{Sel}^T_S(L_I)^{\mathrm{tr}}).$$

□

To deduce Theorem 4.2.6 we now need only make the following computation

$$\begin{aligned}
\mathrm{Fit}^r_G(\mathrm{Sel}^T_S(K)^{\mathrm{tr}}) &= e_{r,S} \cdot \mathrm{Fit}^r_G(\mathrm{Sel}^T_S(K)^{\mathrm{tr}}) \\
&= \left(\sum_{I \in \wp_r(S_{\min})} e_I\right) \cdot \mathrm{Fit}^r_G(\mathrm{Sel}^T_S(K)^{\mathrm{tr}}) \\
&= \bigoplus_{I \in \wp_r(S_{\min})} \left(e_I \cdot \mathrm{Fit}^r_G(\mathrm{Sel}^T_S(K)^{\mathrm{tr}})\right) \\
&= \bigoplus_{I \in \wp_r(S_{\min})} \left(e_I \cdot e_{D_I} \cdot \mathrm{Fit}^r_G(\mathrm{Sel}^T_S(K)^{\mathrm{tr}})\right) \\
&= \bigoplus_{I \in \wp_r(S_{\min})} \left(e_I \cdot \left\{\Phi(\eta^I_{K/k,S,T}) : \Phi \in \bigwedge_G^r(\mathcal{O}^\times_{K,S,T})^*\right\}\right) \\
&= \bigoplus_{I \in \wp_r(S_{\min})} \left\{\Phi(\eta^I_{K/k,S,T}) : \Phi \in \bigwedge_G^r(\mathcal{O}^\times_{K,S,T})^*\right\}.
\end{aligned}$$

Here the first equality is Lemma 4.4.4 and the second is clear from the definition of

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the idempotents $e_{r,S}$ and e_I. The third equality follows from the mutual orthogonality of the idempotents e_I, from the equality in Remark 4.4.5 and from the inclusion in Theorem 4.2.4. The fourth equality is by Remark 4.4.1, the fifth by Theorem 4.4.6 and the final equality since $\eta^I_{K/k,S,T} = e_I \cdot \eta^I_{K/k,S,T}$.

This completes the proof of Theorem 4.2.6.

Chapter 5

Non-minimal vanishing and higher Fitting ideals

This chapter extends the work of the previous chapter in two key ways. Firstly it studies L-functions with any order of vanishing a, not just those with minimal order of vanishing at zero, and secondly it includes the boundary case $S = S_{\min}$.

We show that the higher Fitting ideals of the Selmer group defined in Chapter 3 have a natural direct sum decomposition. We use this result to formulate a higher-order abelian Stark conjecture, linking the evaluator, defined using the leading term of L-functions of order of vanishing a, to the a-th Fitting ideal of the Selmer group of the multiplicative group of an abelian extension of number fields.

Much of the work in this chapter is joint work with my supervisor David Burns contained in [7].

5.1 Definitions

In this chapter as in §2.1 we let K/k be a finite abelian extension of number fields with Galois group G. Let S be a non-empty finite set of places of k containing $S_\infty(k)$ and $S_{\mathrm{ram}}(K/k)$ and let T be a finite set of places of k disjoint from S. Assume $\mathcal{O}_{K,S,T}^\times$ is $\mathbb{Z}$-torsion free. For every place v in S fix a place w_v of K lying above v.

Let a be a non-negative integer. Recall from Definition 4.1.2 that for every $\chi \in \widehat{G} \setminus \{\mathbf{1}\}$ we have the set $S_\chi = \{v \in S : G_v \subseteq \ker(\chi)\}$.

Definition 5.1.1. If G is not the trivial group then we write $S_{\min}^a$ for the union of S_χ as χ ranges over $\widehat{G}_{a,S} \setminus \{\mathbf{1}\}$. If G is trivial, we define $S_{\min}^a$ to be the empty set except if $a = |S| - 1$ when we define $S_{\min}^a$ to be $S \setminus \{v^*\}$ for some fixed place v^* in S (the choice of which will not matter in the sequel).

Remark 5.1.2. If $a = r$ the definition of $S_{\min}^r$ above agrees with the definition in (4.1) in all except one case. If $r \neq |S| - 1$ then the definition of $S_{\min}^r$ above is identical to the definition in (4.1) since $\mathbf{1} \notin \widehat{G}_{r,S}$. If $|S| = r + 1$, then Lemma 2.2.13 implies the subset S_{sp} of S has cardinality at least r. Further, if in this case $|S_{\mathrm{sp}}| = r$, then $S_{\mathrm{sp}} = S_{\min}^r$ and therefore agrees with the definition in (4.1). If $|S_{\mathrm{sp}}| > r$ and G is trivial, then (subject to the same choice of $v^* \in S$) the two definitions also agree. However if $|S_{\mathrm{sp}}| = r + 1$ and G is non-trivial then the $S_{\min}^r$ defined above is the empty set, whereas in (4.1) $S_{\min}$ consists of r split places.

We abbreviate $\wp_a(S_{\min}^a)$ to $\wp_a^*(S)$ and for each $v \in S$ we set

$$\wp_a(S, v) := \{I \in \wp_a^*(S) : v \notin I\}.$$

Finally for each I in $\wp_a^*(S)$ we define a set

$$\widehat{G}'_{a,S,I} := \left\{ \chi \in \widehat{G}_{a,S} \setminus \{\mathbf{1}\} : S_\chi = I \right\}$$

and an idempotent of $\mathbb{Q}[G]$ by

$$e'_I := e_{\mathbf{1}} + \sum_{\chi \in G'_{a,S,I}} e_\chi. \tag{5.1}$$

Remark 5.1.3. If $a = r$ the idempotent e'_I differs from from the idempotent e_I defined previously as it always includes $e_{\mathbf{1}}$ as a summand, whereas the latter only includes it if $I = S_{\mathbf{1}}$.

Definition 5.1.4. For each a with $a < |S|$ and each I in $\wp_a^*(S)$ we then define $\eta_{K/k,S,T}^I$ to be the unique element of $e'_I e_{a,S} \left(\mathbb{C} \cdot \bigwedge_G^a \mathcal{O}_{K,S,T}^\times \right)$ that satisfies

$$\lambda_{K,S}^a(\eta_{K/k,S,T}^I) = e'_I \cdot \theta_{K/k,S,T}^a(0) \cdot \bigwedge_{v \in I}(w_v - w).$$

Here, when defining the exterior product we endow I with the ordering induced by that on S and w is any choice of place of K that lies above a place in $S \setminus I$ (such a choice is possible since $|I| = a$ is assumed to be strictly less than $|S|$).

Remark 5.1.5. We have that $\eta_{K/k,S,T}^I$ is independent of the choice of w since if w' is any other such place, then $w - w'$ is annihilated by $e'_I e_{a,S}$. To justify the latter claim note that $w - w'$ belongs to the kernel of the natural surjective homomorphism $\pi_{S,I} : X_{K,S} \to Y_{K,I}$. In particular, if χ is any character in $\widehat{G}'_{a,S,I}$, then $\dim_{\mathbb{C}}(e_\chi(\mathbb{C} \cdot X_{K,S})) = \dim_{\mathbb{C}}(e_\chi(\mathbb{C} \cdot Y_{K,I})) = a$ so $e_\chi(\mathbb{C} \cdot \ker(\pi_{S,I})) = 0$ and hence $e_\chi(w - w') = 0$.

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Since the sum of e_χ over all such χ is equal to $e_I'(1-e_\mathbf{1})$, and hence to $e_I'e_{a,S}$ if $a < |S|-1$, it suffices to show that $e_\mathbf{1}$ annihilates $w - w'$ if $a = |S| - 1$ and this is true since, in this case, w and w' must both lie above the unique place in $S \setminus I$.

Remark 5.1.6. Suppose $a = r$, $S_{\min}^r \neq S$ and $I \in \wp_a^*(S)$. Then the definition of $\eta_{K/k,S,T}^I$ above agrees with Definition 4.2.1 (possibly subject to a choice of v^*). To see this in the case that $|S| \neq r+1$ we note that $e_\mathbf{1} \cdot \theta_{K/k,S,T}^r(0) = 0$. Suppose $|S| = r+1$. If G is non-trivial and $|S_{\mathrm{sp}}| = r+1$ then $S_{\min}^r$ is empty. Otherwise $S_{\min}^r = S_{\min}$ consists of r split places (if G is trivial and $|S_{\mathrm{sp}}| = r+1$ this is subject to ordering of $S_{\min}$ agreeing with choice of v^*, see remark 5.1.2) and $e_I = e_I'$.

Next we define for each a an element $\eta_{k,S,T}^a$ of $\mathbb{C} \cdot \bigwedge_\mathbb{Z}^a \mathcal{O}_{k,S}^\times$ by setting

$$
\eta_{k,S,T}^a := \begin{cases} \eta_{k/k,S,T}^I & \text{if } a = |S| - 1 \\ 0 & \text{if } a \neq |S| - 1, \end{cases}
$$

where I is the unique element of $S_{\min}^a$ for the extension k/k (and the definition in the case $a \neq |S| - 1$ is motivated by the fact that $e_{a,S} = 0$ in the latter extension). Finally for each Φ in $\bigwedge_G^a \mathrm{Hom}_G(\mathcal{O}_{K,S,T}^\times, \mathbb{Z}[G])$ write Φ_G for its image under the natural (surjective) homomorphism

$$
\bigwedge_G^a \mathrm{Hom}_G(\mathcal{O}_{K,S,T}^\times, \mathbb{Z}[G]) \to \bigwedge_G^a \mathrm{Hom}_G(\mathcal{O}_{k,S,T}^\times, \mathbb{Z} \cdot |G|e_G) \to \bigwedge_\mathbb{Z}^a \mathrm{Hom}_\mathbb{Z}(\mathcal{O}_{k,S,T}^\times, \mathbb{Z})
$$

where the first map is induced by restriction from $\mathcal{O}_{K,S,T}^\times$ to $\mathcal{O}_{k,S,T}^\times$ and the second by the isomorphism $\mathbb{Z} \cdot |G|e_G \cong \mathbb{Z}$ sending $|G|e_G$ to 1.

5.2 Statement of main results

We can now state our main algebraic result.

Theorem 5.2.1. For each non-negative integer a and each place v in S there is a direct sum decomposition

$$(1 - e_v + e_{\mathbf{1}})e_{(a),S} \cdot \mathrm{Fit}_G^a(\mathrm{Sel}_S^T(K)) = c_{S,v}^a e_{\mathbf{1}} \cdot \mathrm{Fit}_G^a(\mathrm{Sel}_S^T(K)) \oplus \bigoplus_{I \in \wp_a(S,v)} e_I' \cdot \mathrm{Fit}_G^a(\mathrm{Sel}_S^T(K))$$

$$(5.2)$$

where $c_{S,v}^a$ is equal to $|\{v\} \backslash S_{\min}^a| - \min\{|S_{\min}^a|, |S \backslash S_{\min}^a|\} - \delta_{a0}$ if $a < |S|$ and is otherwise equal to 0.

For each non-negative integer a we briefly write $\mathcal{F}_a$ in place of $\mathrm{Fit}_G^a(\mathrm{Sel}_S^T(K))$. For each place v in S we then define

$$\mathfrak{n}_v^a = \mathfrak{n}_v^a(K/k)$$

and

$$\mathfrak{m}_v^a = \mathfrak{m}_v^a(K/k)$$

to be the ideals of $\mathbb{Z}[G]$ comprising elements x with

$$x \cdot (1 - e_v + e_{\mathbf{1}})e_{(a),S} \cdot \mathcal{F}_a \subseteq \mathbb{Z}[G]$$

and

$$x \cdot (1 - e_v + e_{\mathbf{1}})e_{(a),S} \cdot \mathcal{F}_a \subseteq \mathcal{F}_a$$

respectively.

Then Theorem 5.2.1 implies that for I in $\wp_a^*(S)$ one has $\mathfrak{n}_v^a \cdot e_I' \cdot \mathcal{F}_a \subseteq \mathbb{Z}[G]$ and $\mathfrak{m}_v^a \cdot e_I' \cdot \mathcal{F}_a \subseteq \mathcal{F}_a$ for all $v \in S \setminus I$ and hence also that $\mathfrak{n}_I^a \cdot e_I' \cdot \mathcal{F}_a \subseteq \mathbb{Z}[G]$ and $\mathfrak{m}_I^a \cdot e_I' \cdot \mathcal{F}_a \subseteq \mathcal{F}_a$ with $\mathfrak{n}_I^a := \sum_{v \in S \setminus I} \mathfrak{n}_v^a$ and $\mathfrak{m}_I^a := \sum_{v \in S \setminus I} \mathfrak{m}_v^a$.

Thus, if we define ideals of $\mathbb{Z}[G]$ by setting

$$\mathfrak{n}_{S,T}^a(K/k) := \bigcap_{I \in \wp_a^*(S)} \mathfrak{n}_I^a$$

and

$$\mathfrak{m}_{S,T}^a(K/k) := \bigcap_{I \in \wp_a^*(S)} \mathfrak{m}_I^a,$$

then Theorem 5.2.1 directly implies the following result.

Corollary 5.2.2. Assume the notation and hypotheses of Theorem 5.2.1. Then for all I in $\wp_a^*(S)$ one has both

$$\mathfrak{n}_{S,T}^a(K/k) \cdot e_I' \cdot \mathrm{Fit}_G^a(\mathrm{Sel}_S^T(K)) \subseteq \mathbb{Z}[G]$$

and

$$\mathfrak{m}_{S,T}^a(K/k) \cdot e_I' \cdot \mathrm{Fit}_G^a(\mathrm{Sel}_S^T(K)) \subseteq \mathrm{Fit}_G^a(\mathrm{Sel}_S^T(K)).$$

Remark 5.2.3. In any general setting an explicit computation of $\mathfrak{n}_{S,T}^a(K/k)$ and $\mathfrak{m}_{S,T}^a(K/k)$ would be rather involved since it requires some knowledge of the higher Fitting ideals of $\mathrm{Sel}_S^T(K)$. It is, however, straightforward to construct elements of $\mathfrak{n}_{S,T}^a(K/k)$ and $\mathfrak{m}_{S,T}^a(K/k)$ in a purely combinatorial way since both $\mathfrak{n}_v^a(K/k)$ and $\mathfrak{m}_v^a(K/k)$ contains the lowest common multiple of the denominators of the coefficients of the element $(1 - e_v + e_1)e_{(a),S}$ of $\mathbb{Q}[G]$. This element is often close to a generator

of $\mathfrak{n}_v^a(K/k)$ and $\mathfrak{m}_v^a(K/k)$ (see, for instance, Examples 5.2.6 and Proposition 6.2.1) and in all cases allows one to make the results of Theorem 5.2.1 and Corollary 5.2.2 much more explicit. In particular, such computations, taken together with Remark 5.3.2(iii) and (iv) below, show that the inclusions of Corollary 5.2.2 are of interest since in many cases there exists an I in $\wp_a^*(S)$ for which neither $\mathfrak{n}_{S,T}^a(K/k) \cdot e_I'$ or $\mathfrak{m}_{S,T}^a(K/k) \cdot e_I'$ is contained in $\mathbb{Z}[G]$. This observation will later play a key role in our discussion of refined Stark conjectures.

Theorem 5.2.1 leads us to make the following conjecture.

Conjecture 5.2.4. Assume the hypotheses of Theorem 5.2.1. Fix a non-negative integer a with $a < |S|$ and set $\eta_k^a := \eta_{k,S,T}^a$ and for each I in $\wp_a^*(S)$ also $\eta^I := \eta_{K/k,S,T}^I$. Then for each place v in S one has

$$(1 - e_v + e_\mathbf{1})e_{(a),S} \cdot \mathrm{Fit}_G^a(\mathrm{Sel}_S^T(K))^\#$$

$$= c_{S,v}^a e_\mathbf{1} \cdot \left\{ \Phi_G(\eta_k^a) : \Phi \in \bigwedge_G^a \mathrm{Hom}_G(\mathcal{O}_{K,S,T}^\times, \mathbb{Z}[G]) \right\}$$

$$\oplus \bigoplus_{I \in \wp_a(S,v)} \left\{ \Phi(\eta^I) : \Phi \in \bigwedge_G^a \mathrm{Hom}_G(\mathcal{O}_{K,S,T}^\times, \mathbb{Z}[G]) \right\}. \quad (5.3)$$

In particular, for every I in $\wp_a^*(S)$ and every Φ in $\bigwedge_G^a \mathrm{Hom}_G(\mathcal{O}_{K,S,T}^\times, \mathbb{Z}[G])$ one has both

$$\mathfrak{n}_{S,T}^a(K/k) \cdot \Phi(\eta^I) \subseteq \mathbb{Z}[G] \quad \text{and} \quad \mathfrak{m}_{S,T}^a(K/k) \cdot \Phi(\eta^I) \subseteq \mathrm{Fit}_G^a(\mathrm{Sel}_S^T(K)). \quad (5.4)$$

Remark 5.2.5. Consider the case that $a = r$ and let us compare this to conjectures in

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previous chapters. If $S \neq S_{\min}^r$, then for each $v \in S \setminus S_{\min}^r$ one has

$$(1 - e_v + e_1) \cdot \mathrm{Fit}_G^r(\mathrm{Sel}_S^T(K)) = \mathrm{Fit}_G^r(\mathrm{Sel}_S^T(K))$$

and

$$\mathfrak{n}_{S,T}^r(K/k) = \mathfrak{m}_{S,T}^r(K/k) = \mathbb{Z}[G]$$

(see Remark 5.3.2(iii)). In this case the equality (5.3) with $a = r$ is implied by Conjecture 4.2.5. (The only difference is due to the difference in the definition of the set $S_{\min}$ as discussed in Remark 5.1.2.) The first containment in (5.4) specialises to give the conjecture formulated under the hypotheses $S \neq S_{\min}^r$ and $|S| > r + 1$ by Vallières and stated in Conjecture 2.2.16 (and hence also the conjecture formulated by Emmons and Popescu, given in Conjecture 2.2.11). The second containment follows from Conjecture 4.2.5. In particular, if $S_{\min}^r$ comprises r places that split completely in K/k, in which case $\eta_{K/k,S,T}^I$ for $I = S_{\min}^r$ is a 'Rubin-Stark element' for K/k, then the first inclusion in (5.4) with $a = r$ recovers the 'Rubin-Stark Conjecture' formulated by Rubin in [24]. In general, however, the first inclusion in (5.4) is strictly finer (even in the case $a = r$) than the Rubin-Stark Conjecture for all suitable subfields of K/k (cf. Proposition 6.2.1).

The following example shows how Conjecture 5.2.4 allows us to make concrete predictions in the boundary case excluded in the previous chapter, i.e. when $a = r$ and $S = S_{\min}^r$.

Example 5.2.6. Let $k = \mathbb{Q}(\alpha)$ with $\alpha^3 - 19\alpha + 21 = 0$ and write K for its strict Hilbert class field. Then k is a totally real non-Galois extension of $\mathbb{Q}$, K/k is unramified outside S_∞ and $\mathrm{Gal}(K/k)$ is isomorphic to $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. In addition, if one sets $S := S_\infty$,

then $r = r_S(K/k)$ is equal to 1 and $S = S_{\min}^1$ and $|G_v| = 2$ for each v in S. In this case Remark 5.3.2(iv) applies with $|S| = 3$ and $r = 1$ and implies $\mathfrak{n}_{S,T}^1(K/k)$ contains 2. On the other hand, Erickson has used numerical computations of Dummit and Hayes [12] to show that for some choices of T and of I in $\wp_1^*(S)$ the element $\eta_{K/k,S,T}^I$ does not belong to Rubin's lattice $\Lambda_{K/k,S,T,1}$ (see the discussion of [16, §7], and [15, §4.2] for the details), and hence that there exists Φ in $\mathrm{Hom}_G(\mathcal{O}_{K,S,T}^\times, \mathbb{Z}[G])$ with $\Phi(\eta_{K/k,S,T}^I) \notin \mathbb{Z}[G]$. In particular, since the first inclusion of (5.4) implies that $2 \cdot \Phi(\eta_{K/k,S,T}^I) \in \mathbb{Z}[G]$ for all such Φ, it is in a natural sense best possible.

Conjecture 5.2.4 also allows us to make concrete predictions in the case that $a > r$, another situation not discussed in the previous chapter. Proposition 6.1.1 shows how in this situation we can obtain predictions finer that those implied by the Rubin-Stark conjecture for the relevant subfield.

Theorem 5.2.7. If LTC(K/k) holds, then so does Conjecture 5.2.4.

As already discussed in §4.2, earlier work of Burns, of Burns and Greither and of Flach can be used to show the validity of LTC(K/k) in the case that K is an abelian extension of $\mathbb{Q}$. The last result therefore has the following immediate consequence.

Corollary 5.2.8. Conjecture 5.2.4 is valid if K is an abelian extension of $\mathbb{Q}$ (and k is any subfield of K).

It is possible to provide examples of explicit application of Corollary 5.2.8 that show that the Rubin-Stark elements of relevant subfields of K/k have stronger integrality properties than that predicted by the relevant case of the Rubin-Stark element. One such ex-

ample, which extends the numerical investigation of the extension $\mathbb{Q}(\sqrt{5}, \sqrt{-7}, \sqrt{-11})/\mathbb{Q}$ by Erickson in [15, 16], is discussed in greater generality in Proposition 6.2.1.

Remark 5.2.9. In [21] McGown, Sands and Vallières have developed a systematic method for providing numerical evidence for higher order abelian Stark conjectures. It seems likely that this method could be adapted to provide evidence for Conjecture 5.2.4.

5.3 The proof of Theorem 5.2.1

We assume throughout this section the notation and hypotheses of Theorem 5.2.1. For convenience we also set $\mathcal{F}_a := \mathrm{Fit}_G^a(\mathrm{Sel}_S^T(K))$ for each non-negative integer a.

5.3.1 Preliminary result

First we record a result that will allow us to treat a special case of Theorem 5.2.1 (and also justifies observations made in Remark 5.3.2).

Lemma 5.3.1. The following claims are valid for each non-negative integer a.

(i) For χ in $\widehat{G}_{(a),S} \setminus \widehat{G}_{a,S}$ the space $e_\chi(\mathbb{C} \otimes_{\mathbb{Z}} \mathcal{F}_a)$ vanishes.

(ii) If $|S| > a + 1$, then $e_{\mathbf{1}} \cdot \mathcal{F}_a$ vanishes.

(iii) If $v \in S \setminus S_{\min}^a$, then $(e_v - e_{\mathbf{1}})e_{(a),S} \cdot \mathcal{F}_a$ vanishes.

Proof. For χ in $\widehat{G}$, a finitely generated G-module Z and a non-negative integer a it is easy to see that the sublattice $e_\chi \cdot \mathrm{Fit}_G^a(Z)$ of $\mathbb{C}[G]$ will vanish whenever $\dim_{\mathbb{C}}(e_\chi(\mathbb{C} \otimes_{\mathbb{Z}} Z)) > a$.

Using this observation, claim (i) is true since Lemma 2.1.3 implies that if $a' > a$, then for each χ in $\widehat{G}_{a',S}$ one has $\dim_{\mathbb{C}}(e_\chi(\mathbb{C} \otimes_{\mathbb{Z}} \operatorname{Sel}_S^T(K))) = a' > a$. Claim (ii) is then an immediate consequence of claim (i) in the case $\chi = \mathbf{1}$ and $a' = |S| - 1$.

To derive claim (iii) from claim (i) it suffices to prove that if χ belongs to $\widehat{G}_{a,S} \setminus \{\mathbf{1}\}$, then G_v cannot be contained in $\ker(\chi)$ (and hence $e_\chi(e_v - e_\mathbf{1}) = 0$). This follows from the fact that, as v does not belong to $S_{\min}^a$, the inclusion $G_v \subseteq \ker(\chi)$ would imply that $|S_\chi| \geq 1 + a$. □

Remark 5.3.2. The above observations can be used to make the results of Theorem 5.2.1 and Corollary 5.2.2 more explicit.

(i) Lemma 2.1.3 implies both that $r \leq |S| - 1$ and that $\widehat{G}$ is equal to the union of $\widehat{G}_{a',S}$ for $a' \geq r$. The latter fact implies that $e_{(a),S} = 1$ for $a \leq r$ and therefore in such a case simplifies the computation of both $\mathfrak{n}_{S,T}^a(K/k)$ and $\mathfrak{m}_{S,T}^a(K/k)$.

(ii) By combining Lemma 2.1.3 with the result of Lemma 5.3.1(i) one can also check that both sides of (5.2) vanish if either $a < r$ or $a > |S|$. This shows that Theorem 5.2.1 is of interest only for integers a in the range $r \leq a \leq |S|$.

(iii) Assume $S \neq S_{\min}^a$. Then Lemma 5.3.1(iii) implies that $(e_v - e_\mathbf{1})\operatorname{Fit}_G^a(\operatorname{Sel}_S^T(K))$ vanishes for each v in $S \setminus S_{\min}^a$. In particular, for each such v the left hand side of (5.2) is equal to $e_{(a),S} \cdot \operatorname{Fit}_G^a(\operatorname{Sel}_S^T(K))$, and hence, by remark (i), to $\operatorname{Fit}_G^r(\operatorname{Sel}_S^T(K))$ if $a = r$. In particular, in this case one has $\mathfrak{n}_v^r(K/k) = \mathfrak{m}_v^r(K/k) = \mathbb{Z}[G]$ and hence also $\mathfrak{n}_{S,T}^r(K/k) = \mathfrak{m}_{S,T}^r(K/k) = \mathbb{Z}[G]$.

(iv) Assume $S = S_{\min}^a$ and $a = r$. This is the 'boundary case' that is identified by Emmons in [13, §5.4] and is excluded from the formulation of our conjectures in the previous chapter and from the formulation of other refined abelian Stark conjectures

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(see, for example, Conjecture 2.2.11, Conjecture 2.2.16, [15, 16]).

We note first that in this case $|S| > r + 1$. To see this we note that $|S| \geq r + 1$ (by remark (i)) and the discussion in Remark 5.1.2 shows that if $|S| = r+1$ then $S^r_{\min} \neq S$. Then, since $|S| > r + 1$, Lemma 5.3.1(ii) implies $e_1 \mathrm{Fit}^r_G(\mathrm{Sel}^T_S(K))$ vanishes. This in turn combines with the observation in Remark 5.2.3 to imply that for each I in $\wp^*_r(S)$ the ideals $\mathfrak{n}^r_I$ and $\mathfrak{m}^r_I$ each contain the greatest common divisor d_I of $|G_v|$ as v varies over $S \setminus I$ and can actually contain proper divisors of d_I depending on the structure of $\mathrm{Fit}^r_G(\mathrm{Sel}^T_S(K))$. In particular, in all cases both $\mathfrak{n}^r_{S,T}(K/k)$ and $\mathfrak{m}^r_{S,T}(K/k)$ contains the lowest common multiple of d_I for I in $\wp^*_r(S)$. For a concrete application of this observation see Example 5.2.6 and Proposition 6.2.1.

5.3.2 Special cases

By using Lemma 5.3.1 we can now quickly prove Theorem 5.2.1 in several special cases.

Proposition 5.3.3. Theorem 5.2.1 is valid if $K = k$.

Proof. In this case G is trivial so $\widehat{G} = \{\mathbf{1}\}$.

In particular, if $a < |S| - 1$, then Lemma 5.3.1(ii) implies $\mathcal{F}_a$ vanishes and so the equality (5.2) is valid trivially.

If $a \geq |S|$, then Lemma 2.1.3 implies $e_{(a),S} = 0$ and so the left hand side of (5.2) is zero. On the other hand, in this case $c^a_{S,v}$ is defined to be 0 and $S^a_{\min}$ to be empty and so the right hand side of (5.2) is also zero.

We assume finally that $a = |S|-1$ and recall $S^a_{\min}$ is then defined to be $S\setminus\{v^*\}$ for a fixed place v^* in S. We also note that in this case Lemma 2.1.3 implies $(1 - e_v + e_1)e_{(a),S} = 1$ (for each v in S) so that the left hand side of (5.2) is equal to $\mathcal{F}_a$.

If $a = 0$, then $S = \{v^*\}$ so $S^a_{\min}$ is empty. This in turn implies $c^a_{S,v^*} = 1 - 0 - 1 = 0$ and $\wp_a(S, v^*) = \{I\}$ with $I := \emptyset$ and, since $e_\emptyset = e_\mathbf{1}$, this shows that the right hand side of (5.2) is equal to $e_\emptyset \mathcal{F}_a = \mathcal{F}_a$, as required.

It therefore only remains to consider the case $a = |S| - 1 > 0$. In this case $S^a_{\min}$ is not empty. In particular, if $v \neq v^*$, then $v \in S^a_{\min}$ so $c^a_{S,v} = 0 - 1 - 0 = -1$ and $\wp_a(S, v) = \emptyset$, whilst if $v = v^*$, then $c^a_{S,v} = 1 - 1 - 0 = 0$ and $\wp_a(S, v) = \{I\}$ with $I := S \setminus \{v^*\}$ and $e'_I = 1$. Given these facts, it is easily checked that for each choice of v the two sides of (5.2) agree. $\qquad\square$

Lemma 5.3.4. Theorem 5.2.1 is valid if $a = 0$.

Proof. Following Proposition 5.3.3 we assume both $a = 0$ and G is not trivial.

In this case Lemma 5.3.1(i) and (iii) combine to imply that the left hand side of (5.2) is equal to $e_{0,S} \cdot \mathcal{F}_0$.

In addition, $S^a_{\min} = \emptyset$ and so for each v in S one has $c^0_{S,v} = 1 - 0 - 1 = 0$ and $\wp_0(S, v) = \{\emptyset\}$. If $|S| = 1$ then (5.1) implies $e_\emptyset = e_\mathbf{1} + \sum_{\chi \in \widehat{G}'_{0,S,\emptyset}} e_\chi = e_\mathbf{1} + \sum_{\chi \in \widehat{G}_{0,S} \setminus \mathbf{1}} e_\chi = \sum_{\chi \in \widehat{G}_{0,S}} e_\chi = e_{0,S}$ and if $|S| > 1$ then $e_\emptyset = e_\mathbf{1} + e_{0,S}$. Since $e_\mathbf{1} \cdot \mathcal{F}_0 = 0$ if $|S| > 1$ this shows that the right hand side of (5.2) is also equal to $e_{0,S} \cdot \mathcal{F}_0$, as required. $\qquad\square$

The next result deals with the case that a is 'large' (see Remark 5.3.2(ii)).

Proposition 5.3.5. Theorem 5.2.1 is valid if $a \geq |S| - 1$.

Proof. Following Remark 5.3.2(ii) it is enough to consider a equal to either $|S| - 1$ or $|S|$. Our argument then splits into several subcases, depending on the cardinality of $S^a_{\min}$. Following Lemma 5.3.4 we always assume $a > 0$.

We consider first the case that $S_{\min}^a = \emptyset$ and $a > 0$, and hence that $\widehat{G}_{a,S} \setminus \mathbf{1} = \emptyset$. In this case Lemma 5.3.1(i) and (ii) combine to imply $e_{(a),S} \cdot \mathcal{F}_a = e_{a,S} \cdot \mathcal{F}_a$ is equal to $e_{\mathbf{1}} \cdot \mathcal{F}_a$ if $a = |S| - 1$ and vanishes if $a = |S|$. One also has $\wp_a(S, v) = \emptyset$ (since $a > 0$) and so the right hand side of (5.2) is equal to $c_{S,v}^a e_{\mathbf{1}} \cdot \mathcal{F}_a$. The claimed equality is thus true in this case since $c_{S,v}^a$ is equal to $1 - 0 - 0 = 1$ if $a = |S| - 1$ and to 0 if $a = |S|$.

In the remainder of the argument we assume $S_{\min}^a \neq \emptyset$ and hence that $|S_{\min}^a|$ is equal to either $|S| - 1$ (so $a = |S| - 1$) or $|S|$ (so $S_{\min}^a = S$). There are then four separate cases to consider depending on whether $|S_{\min}^a| = |S| - 1$ and $v \in S_{\min}^a$, or $|S_{\min}^a| = |S| - 1$ and $v \notin S_{\min}^a$, or $S_{\min}^a = S$ and $a = |S| - 1$, or $a = |S|$ (and hence $S_{\min}^a = S$).

We consider first the case $|S_{\min}^a| = |S| - 1 = a$ and $v \in S_{\min}^a$. In this case $G_v \subseteq \ker(\chi)$ for all $\chi \in \widehat{G}_{a,S} \setminus \mathbf{1}$ so $(1 - e_v)e_{a,S} = 0$ and Lemma 5.3.1(i) and (ii) combine to imply the left hand side of (5.2) is equal to $e_{\mathbf{1}} \cdot \mathcal{F}_a$. Given this, the claimed equality follows from the fact that $\wp_a(S, v) = \emptyset$ and $c_{S,v}^a = 0 - 1 - 0 = -1$.

We next assume $|S_{\min}^a| = |S| - 1 = a$ and $v \notin S_{\min}^a$ so that $S_{\min}^a = S \setminus \{v\}$. In this case Lemma 5.3.1(i) and (iii) together imply that the left hand side of (5.2) is equal to $e_{a,S} \cdot \mathcal{F}_a$. In addition, one has $c_{S,v}^a = 1 - 1 - 0 = 0$ and the unique element of $\wp_a(S, v)$ is equal to $I := S_{\min}^a$ so $e_I' = e_{a,S}$ and the right hand side of (5.2) is also equal to $e_{a,S} \cdot \mathcal{F}_a$.

We now consider the case $S_{\min}^a = S$ and $a = |S| - 1$. For each v in S we write $G(v)$ for the subgroup generated by $G_{v'}$ as v' varies over $S \setminus \{v\}$. Then, in this case, for each χ in $\widehat{G}_{a,S} \setminus \mathbf{1}$ one has $e_\chi e_v = 0$ if and only if $G(v) \subseteq \ker(\chi)$. This implies $(1 - e_v)e_{a,S} = (e_{G(v)} - e_{\mathbf{1}})e_{a,S}$ and hence that the left hand side of (5.2) is equal to $e_{G(v)}e_{a,S} \cdot \mathcal{F}_a$. The claimed equality thus follows from the fact that, in this case, $c_{S,v}^a = 0$, $\wp_a(S, v)$ has a single element $I_v := S \setminus \{v\}$ and it is straightforward to check that e_{I_v} is

equal to $e_{G(v)}e_{a,S}$.

Finally, we assume $a = |S|$ and $S^a_{\min} \neq \emptyset$ (and hence that both $S^a_{\min} = S$ and G is not trivial). In this case one verifies that $e_{(a),S} = e_{a,S} = e_{G_S} - e_{\mathbf{1}}$ with G_S the subgroup of G generated by G_v as v varies over S. For each v in S one therefore has $(1 - e_v + e_{\mathbf{1}})e_{(a),S} = (1 - e_v)e_{G_S} = 0$, where the last equality is valid since $G_v \subseteq G_S$, and so the left hand side of (5.2) vanishes. On the other hand, in this case the right hand side of (5.2) vanishes since for any v in S both $c^a_{S,v} = 0$ and $\wp_a(S, v)$ is empty. This completes the proof of Theorem 5.2.1 in the case that $a \geq |S| - 1$. $\qquad\square$

5.3.3 Main case

Following Propositions 5.3.3 and 5.3.5 and Lemma 5.3.4 we assume in the remainder of the argument that G is not trivial and that $0 < a < |S| - 1$.

We note that in this case Lemma 5.3.1(ii) implies that the term $1 - e_v + e_{\mathbf{1}}$ on the left hand side of (5.2) can be replaced by $1 - e_v$ and that the first summand on the right hand side of (5.2) can be omitted.

Since $\mathcal{O}^\times_{K,S,T}$ is assumed to be torsion-free, [6, Lem. 2.8] implies that for each non-negative integer a one has

$$\mathrm{Fit}^a_G(\mathrm{Sel}^T_S(K)) = \mathrm{Fit}^a_G(\mathrm{Sel}^T_S(K)^{\mathrm{tr}})^\#. \tag{5.5}$$

As a final preparatory remark, we note that it suffices to prove the equality of Theorem 5.2.1 after localising at p for every prime p.

We therefore fix a prime p. We then fix a place v in S and construct a convenient

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resolution of $\mathrm{Sel}_S^T(K)_{(p)}^{\mathrm{tr}}$. We construct a free $\mathbb{Z}[G]$ module of rank d_v using the method described in §3.3.1, but with v playing the role of v_0. After localising at p and applying Lemma 3.3.1 to the exact sequence (3.3) we obtain the presentation

$$\mathbb{Z}_{(p)}[G]^{d_v} \xrightarrow{\psi_v} \mathbb{Z}_{(p)}[G]^{d_v} \xrightarrow{\pi_v} \mathrm{Sel}_S^T(K)_{(p)}^{\mathrm{tr}} \longrightarrow 0 \tag{5.6}$$

where we write $\mathbb{Z}_{(p)}[G]^{d_v}$ for the direct sum of d_v copies of $\mathbb{Z}_{(p)}[G]$.

If we write M_v for the matrix of ψ_v with respect the standard basis of $\mathbb{Z}_{(p)}[G]^{d_v}$, then the equality (5.5) combines with the very definition of a-th Fitting ideals to imply that

$$\mathrm{Fit}_G^a(\mathrm{Sel}_S^T(K))_{(p)} = \sum_{J \in \wp_a(d_v)} m_v(J) \tag{5.7}$$

with $\wp_a(d_v) := \wp_a(\{i \in \mathbb{Z} : 1 \leq i \leq d_v\})$ and $m_v(J)$ denoting the ideal of $\mathbb{Z}_{(p)}[G]$ that is generated by the determinants of all $(d_v - a) \times (d_v - a)$ minors of the $d_v \times (d_v - a)$ matrix $M_v(J)^{\#}$ obtained by deleting all columns of M_v corresponding to integers in J. Next we note that if $\chi \in \widehat{G} \setminus \{\mathbf{1}\}$, then one has

$$v \notin S_\chi \iff G_v \not\subseteq \ker(\chi) \iff (1 - e_v)e_\chi \neq 0$$

and hence that for each non-negative integer a one has

$$(1 - e_v)e_{a,S} = \sum_{I \in \wp_a(S,v)} (e_I' - e_{\mathbf{1}}).$$

This equality combines with (5.7) to imply that

$$(1-e_v)e_{(a),S} \cdot \text{Fit}^a_G(\text{Sel}^T_S(K))_{(p)} = (1-e_v)e_{a,S} \cdot \text{Fit}^a_G(\text{Sel}^T_S(K))_{(p)}$$

$$= \left(\sum_{I \in \wp_a(S,v)} (e'_I - e_{\mathbf{1}}) \right) \cdot \left(\sum_{J \in \wp_a(d_v)} m_v(J) \right)$$

$$= \sum_{I \in \wp_a(S,v)} (e'_I - e_{\mathbf{1}}) \cdot m_v(I)$$

$$= \sum_{I \in \wp_a(S,v)} (e'_I - e_{\mathbf{1}}) \cdot \text{Fit}^a_G(\text{Sel}^T_S(K))_{(p)}.$$

The first equality here is true as Lemma 5.3.1(i) implies that $e_{(a),S} \cdot \text{Fit}^a_G(\text{Sel}^T_S(K))$ is equal to $e_{a,S} \cdot \text{Fit}^a_G(\text{Sel}^T_S(K))$ and the third and fourth equalities follow directly from the result of Lemma 5.3.6 below.

To deduce the equality (5.2) in the case that $a < |S| - 1$ it thus suffices to observe firstly that the last sum in the above formula is direct since for distinct elements I_1 and I_2 of $\wp^*_a(S)$ the idempotents $e_{I_1} - e_{\mathbf{1}}$ and $e_{I_2} - e_{\mathbf{1}}$ are orthogonal, and then that for each such I Lemma 5.3.1(ii) implies $(e'_I - e_{\mathbf{1}}) \cdot \text{Fit}^a_G(\text{Sel}^T_S(K)) = e'_I \cdot \text{Fit}^a_G(\text{Sel}^T_S(K))$.

In the following result we again use the correspondence $v_i \leftrightarrow i$ to identify $\wp_a(S \setminus \{v\})$ with the subset $\wp_a(d_1)$ of $\wp_a(d_v)$.

Lemma 5.3.6. Take $\chi \in \widehat{G}_{a,S} \setminus \{\mathbf{1}\}$ with $v \notin S_\chi$. Then S_χ belongs to $\wp_a(d_v)$ and for each J in $\wp_a(d_v) \setminus \{S_\chi\}$ one has $e_\chi \cdot m_v(J) = 0$.

Proof. Set $G_\chi := G/\ker(\chi)$ and $F^\chi := F^{\ker(\chi)}$.

Then to compute $e_\chi \cdot m_v(J)$ one can replace $m_v(J)$ by its image in $\mathbb{Z}[G_\chi]$. Note also that the exact sequence (5.6) implies that each column of the image $M_v(J)^\#_\chi$ in $\mathbb{Z}[G_\chi]$

of $M_v(J)^{\#}$ that corresponds to an integer in $S_\chi \setminus J$ must vanish since the corresponding place in $S \setminus \{v\}$ splits completely in F^χ/k.

In particular, if $J \neq S_\chi$, then $M_v(J)_\chi^{\#}$ has at least one column of zeroes and so the determinant of any of its $(d_v - a) \times (d_v - a)$ minors must vanish. It follows that $e_\chi \cdot m_v(J) = 0$, as claimed. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

5.4 Proof of Theorem 5.2.7

At the outset we fix an integer a with $0 \leq a < |S|$ and assume $\mathrm{LTC}(K/k)$. Then by Lemma 3.2.7 the leading term conjecture is also true for K^H/k for every subgroup H of G. We again abbreviate $\mathrm{Fit}_G^a(\mathrm{Sel}_S^T(K))$ to $\mathcal{F}_a$ and in addition now set $\mathcal{F}_a^{\mathrm{tr}} := \mathrm{Fit}_G^a(\mathrm{Sel}_S^T(K)^{\mathrm{tr}})$.

By (5.5) one has $\mathcal{F}_a^{\mathrm{tr}} = \mathcal{F}_a^{\#}$. Thus, to derive the claimed decomposition (5.3) from that in Theorem 5.2.1 it is enough to show that

$$
e_{\mathbf{1}} \cdot \mathcal{F}_a^{\mathrm{tr}} = e_{\mathbf{1}} \cdot \left\{ \Phi_G(\eta_k^a) : \Phi \in \bigwedge_G^a \mathrm{Hom}_G(\mathcal{O}_{K,S,T}^\times, \mathbb{Z}[G]) \right\} \tag{5.8}
$$

and

$$
e_I' \cdot \mathcal{F}_a^{\mathrm{tr}} = e_I \cdot \left\{ \Phi(\eta^I) : \Phi \in \bigwedge_G^a \mathrm{Hom}_G(\mathcal{O}_{K,S,T}^\times, \mathbb{Z}[G]) \right\} \quad \text{for all } I \in \wp_a(S, v). \tag{5.9}
$$

To prove (5.8) we first recall that $\mathrm{Sel}_S^T(K)^{\mathrm{tr}}$ is defined in Definition 3.2.3 to be the cohomology in degree -1 of a complex $C_{K,S}$ that is acyclic in degrees greater than -1 and, since S contains all places that ramify in K/k, is also such that $\mathbb{Z} \otimes_{\mathbb{Z}[G]} C_{K,S}$

identifies with $C_{k,S}$. This fact induces an identification of $\mathbb{Z} \otimes_{\mathbb{Z}[G]} \mathrm{Sel}_S^T(K)^{\mathrm{tr}}$ with $\mathrm{Sel}_S^T(k)^{\mathrm{tr}}$ and hence, by standard functorial properties of higher Fitting ideals, implies that

$$e_{\mathbf{1}} \cdot \mathcal{F}_a^{\mathrm{tr}} = e_{\mathbf{1}} \cdot \mathrm{Fit}^a(\mathrm{Sel}_S^T(k)^{\mathrm{tr}}).$$

It is thus enough to prove

$$\mathrm{Fit}^a(\mathrm{Sel}_S^T(k)^{\mathrm{tr}}) = \left\{ \Phi_G(\eta_k^a) : \Phi \in \bigwedge_G^a \mathrm{Hom}_G(\mathcal{O}_{K,S,T}^\times, \mathbb{Z}[G]) \right\}.$$

If $a < |S| - 1$, this equality is true since both sides vanish (the left hand side by Lemma 5.3.1(ii) and the right hand side since $\eta_k^a := 0$). If $a = |S| - 1$ the exact sequence (3.2) combines with the fact that $X_{k,S}$ is a free $\mathbb{Z}$-module of rank a to imply

$$\mathrm{Fit}^a(\mathrm{Sel}_S^T(k)^{\mathrm{tr}}) = \mathrm{Fit}^0(\mathrm{Cl}_S^T(k)) = |\mathrm{Cl}_S^T(k)| \cdot \mathbb{Z}$$

whilst, by unwinding the explicit definition of η_k^a, one finds that

$$\left\{ \Phi_G(\eta_k^a) : \Phi \in \bigwedge_G^a \mathrm{Hom}_G(\mathcal{O}_{K,S,T}^\times, \mathbb{Z}[G]) \right\} = \theta_{k/k,S,T}^a(0) \cdot R_{k,S,T}^{-1} \cdot \mathbb{Z}$$

with $R_{k,S,T}$ the determinant of the Dirichlet regulator isomorphism $\mathbb{R} \cdot \mathcal{O}_{k,S}^\times \cong \mathbb{R} \cdot X_{k,S}$ with respect to a choice of $\mathbb{Z}$-bases of $\mathcal{O}_{k,S,T}^\times$ and $X_{k,S}$ and so the required equality is equivalent, up to a sign, to the analytic class number formula of k.

It therefore suffices to prove (5.9). Suppose a is equal to $r = r_S(K/k)$. Then Theorem

4.4.6 implies that

$$e_{D_I} \cdot \mathcal{F}_r^{\mathrm{tr}} = \left\{ \varphi(\eta^I) : \varphi \in \bigwedge_G^r \mathrm{Hom}_G(\mathcal{O}_{K,S,T}^\times, \mathbb{Z}[G]) \right\} \qquad (5.10)$$

for all $I \in \wp_r(S, v)$. To see this we first note that if $I \in \wp_r(S, v)$ then $I \in \wp_r(S_{\min})$ since by Remark 5.1.2 we can always fix an ordering of S such that $S_{\min}^r \subseteq S_{\min}$. Secondly, the equality in Theorem 4.4.6 does not depend on the hypothesis $S \neq S_{\min}$ since, after fixing I, the derivation given in loc. cit. works with any place in $S \backslash I$ as a substitute for the specified place v_0. (Note, however, that the supplementary inclusion that is proved in Theorem 4.4.6, but which is not required here, does depend on the hypothesis $S \neq S_{\min}$). Thirdly, in all cases one has $e_I \cdot e_{r,S} = e_I' \cdot e_{r,S}$. If $a = r = |S| - 1$ and $I \in \wp_r(S, v)$ then $e_I = e_I'$ (since by Remark 5.1.2 $S_{\min}^r = S_{\min} = S_{\mathbf{1}}$ except in a single case where $r \neq 0$ and $S_{\min}^r = \emptyset$ and so we have no such I). If $a = r \neq |S| - 1$ then it is true since then one has both $e_I' = e_I + e_{\mathbf{1}}$ and $e_{\mathbf{1}} \cdot e_{r,S} = 0$.

It now only remains to discuss the equality (5.9) in the case $a \neq r$. Here one finds that

$$e_I' = \begin{cases} e_{a,S} \cdot e_{D_I} & \text{if } a = |S| - 1, \\[2mm] e_{\mathbf{1}} + e_{a,S} \cdot e_{D_I} & \text{if } a \neq |S| - 1 \end{cases}$$

and in both cases a simple computation shows that $e_I' \cdot e_{D_I} = e_I'$.

Given this, the required equality (5.9) follows directly from the analogue $e_{D_I} \cdot \mathcal{F}_a^{\mathrm{tr}} = \left\{ \varphi(\eta^I) : \varphi \in \bigwedge_G^a \mathrm{Hom}_G(\mathcal{O}_{K,S,T}^\times, \mathbb{Z}[G]) \right\}$ of (5.10). To verify the latter equality we first note that if we write L_I for the fixed field of K by D_I and set $\Gamma := G/D_I$ as in §4.4.1 then each place in I splits completely in L_I/k. We write $\epsilon_{L_I/k,S,T}^I$ for the Rubin-Stark

element in $\mathbb{C} \cdot \bigwedge_\Gamma^a \mathcal{O}_{L_I,S}^\times$ defined with respect to the places in I. Then we can prove in an analogous way to Lemma 4.4.2 that

$$\eta_{K/k,S,T}^I = |D_I|^{-a} \cdot \epsilon_{L_I/k,S,T}^I.$$

Taking this into account, we can mimic the justification of (5.10) given above (with r replaced by a).

Finally we note that (5.4) is derived from (5.3) by the same argument used to derive Corollary 5.2.2 from Theorem 5.2.1.

Chapter 6

Explicit examples

6.1 Non-minimal vanishing

For integers a with $a > r$ the predictions of (5.4) can still refine existing conjectures concerning Stark elements. As an example, the following proposition shows how we can obtain in this way predictions that are finer than the corresponding predictions of the Rubin-Stark conjecture for the relevant subextensions.

For a concrete example of an extension that satisfies all of the conditions that are described in (the proof of) this proposition see Example 6.1.3 below.

Proposition 6.1.1. There exists a family of abelian CM extensions K/k of totally real fields and suitable sets of places S of k (as in Conjecture 5.2.4) for which one has $r = 0$ and the conjectural containments of (5.4) for $a = 1$ refine the predictions of the Rubin-Stark conjecture for suitable subextensions of K/k.

Proof. Let k be totally real with $k \neq \mathbb{Q}$ and fix K to be a CM abelian extension of k

with Galois group $G := \mathrm{Gal}(K/k)$ of the form $J \times P$ where J has order 2 and P is non-cyclic of order p^2 for an odd prime p.

We assume that there exists a set of places S of k that satisfies all of the hypotheses of Conjecture 5.2.4 for K/k and also has the following three properties:

- Write $\mathcal{H}_1$ and $\mathcal{H}_0$ for the sets of subgroups of G of order p that are the decomposition subgroup of precisely one, respectively of no, place in S. We assume that the sets $\mathcal{H}_1$ and $\mathcal{H}_0$ are both non-empty.

- We assume that there exists a place v_0 in S for which G_{v_0} is a non-trivial subgroup of P that does not belong to $\mathcal{H}_1$ (so that either $G_{v_0} = P$ or G_{v_0} is the decomposition subgroup of at least two places in S).

- We assume that no place in $S \setminus S_\infty$ splits completely in K^J/k.

We set

$$e_- := (1 - \tau)/2$$

with τ the non-trivial element of J and note that a straightforward calculation using Lemma 2.1.3 (and the above assumptions) shows that

$$e_{0,S} = e_- \left(\sum_{H \in \mathcal{H}_0} (e_H - e_P) \right) \neq 0.$$

In particular, we note here that the idempotent e_- occurs in this formula since both $|S| > 1$ and each archimedean place of k (of which there are at least two since $k \neq \mathbb{Q}$) splits in K^J/k and so the formula of Lemma 2.1.3 implies that $r_S(\chi) > 0$ for any χ that is trivial on J.

It is then clear that

$$e_{(1),S} = 1 - e_{0,S}$$

and we can check directly that

$$S^1_{\min} = S_1 := \{v \in S : G_v \in \mathcal{H}_1\}$$

and

$$\wp_1^*(S) = \{I_v\}_{v \in S_1}$$

with $I_v := \{v\}$.

We next set $\mathcal{F}_1 := \mathrm{Fit}^1_G(\mathrm{Sel}^T_S(K))$ and write $I(P)$ for the augmentation ideal of $\mathbb{Z}[P]$ and claim that

$$e_- \cdot \mathcal{F}_1 \subseteq e_- \cdot I(P). \tag{6.1}$$

To show this it is enough to prove $e_P \mathcal{F}_1$ vanishes. To check this we note that $e_P \mathcal{F}_1$ identifies with $\mathrm{Fit}^1_{G/P}(\mathrm{Sel}^T_S(K^P))$ and hence that it suffices to show that for each character χ of G/P the rank of the χ-component of $\mathrm{Sel}^T_S(K^P)$ is at least two.

Now the group $G/P = J$ has only two characters χ and the rank of the corresponding component of $\mathrm{Sel}^T_S(K^P)$ is equal to $|S| - 1$ if χ is trivial, and to $\#\{v \in S : G_v \subseteq P\}$ if χ is not trivial. The required inclusion (6.1) is thus true because both

$$|S| \geq |S_\infty(k) \cup \{v_0\}| > 2$$

and any place v in $S_1 \cup \{v_0\}$ is such that $G_v \subseteq P$.

We now fix a non-trivial element h of a subgroup in $\mathcal{H}_0$ and claim that the element of

$\mathbb{Z}[G]$ obtained by setting

$$x := (1 - \tau)(h - 1)^{p-2} \qquad (6.2)$$

belongs to the ideal $\mathfrak{n}^1_{S,T}(K/k)$.

To prove this we need to show that $x \cdot (1 - e_{v_0} + e_{\mathbf{1}})e_{(1),S} \cdot \mathcal{F}_1$ is contained in $\mathbb{Z}[G]$. In

particular, since $(\tau - 1) \times e_{\mathbf{1}} = 0$ it is enough to check that elements in each of the sets

(a) $x \cdot e_{v_0}\mathcal{F}_1$,

(b) $x \cdot e_{v_0}(e_H - e_P)\mathcal{F}_1$,

(c) $x \cdot (e_H - e_P)\mathcal{F}_1$

all belong to $\mathbb{Z}[G]$ for each subgroup H in $\mathcal{H}_0$.

Since $h \in P$ we have $x \cdot e_P = 0$. Therefore it is enough show that elements in the sets

(a') $x \cdot e_{v_0}\mathcal{F}_1$,

(b') $x \cdot e_{v_0}e_H\mathcal{F}_1$,

(c') $x \cdot e_H\mathcal{F}_1$

all belong to $\mathbb{Z}[G]$ for $H \in \mathcal{H}_0$.

In case (b') since $H \in \mathcal{H}_0$ we have $G_{v_0} \neq H$. Therefore $e_{v_0}e_H$ is a multiple of e_P and

therefore the element in (b') is zero.

We now prove case (a'). If h is in G_{v_0}, then all elements in (a') are zero so we assume

$h \notin G_{v_0}$. This means G_{v_0} has order p and so we set $\Gamma := G/G_{v_0}$ and work in this group

(so $e_{v_0} = 1$ and we need to prove that the elements in (a') are divisible by p).

But the inclusion (6.1) combines with the definition (6.2) of x to imply that every element of $x \cdot \mathcal{F}_1$ lies in $(\tau - 1)I(P)^{p-1}$ and hence has image in $e_- \mathbb{Z}[\Gamma]$ of the form $e_-(p \cdot y' + d \cdot T)$ with $y' \in \mathbb{Z}[\Gamma]$, $d \in \mathbb{Z}$ and $T := \sum_\gamma \gamma$. It is thus enough to note that $Te_{v_0}e_-\mathcal{F}_1 = 0$ since $e_-\mathcal{F}_1 \subseteq e_-I(P)$ and Te_{v_0} is a multiple of e_P.

To prove (c') we again argue as above, but note that we can assume $h \notin H$ since $h \in H$ implies $x \cdot e_H$ is obviously zero and take $\Gamma = P/H$. This then completes the proof that x belongs to $\mathfrak{n}_{S,T}^1(K/k)$.

For each v in S_1 this fact combines with the analogue of Lemma 4.4.2 for the element $\eta_{K/k,S,T}^{I_v}$ to show that the conjectural prediction (5.4) implies the Rubin-Stark element ϵ_v for the data $(K^{G_v}/k, S, T, I_v)$ is such that

$$x(\epsilon_v) \in (K^\times)^p,$$

and hence, if K^P contains no primitive p-th root of unity, that

$$x(\epsilon_v) \in (K^{G_v,\times})^p.$$

To show that this prediction is finer than the Rubin-Stark conjecture for the data $(K^{G_v}/k, S, T, I_v)$ we need to show that the image of x under the natural projection

$$\mathbb{Z}[G] \to \mathbb{Z}[G]/\mathrm{Ann}_{\mathbb{Z}[G]}(\epsilon_v)$$

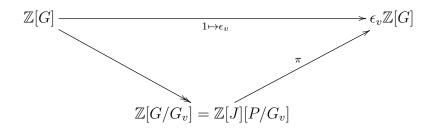is not divisible by p and this follows easily from the lemma below. $\qquad\square$

Lemma 6.1.2. For any place v in S_1 the G-module spanned by the Rubin-Stark element

ϵ_v for the data $(K^{G_v}/k, S, T, I_v)$ is isomorphic to the quotient

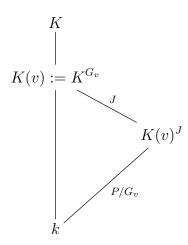$$e_-(\mathbb{Z}[P/G_v]/\left(\sum_{\gamma \in P/G_v} \gamma\right))$$

of $\mathbb{Z}[G]$.

Proof. Let $v \in S_1$ as above. Since ϵ_v belongs to K^{G_v} there is a commutative diagram of G-modules of the form

and hence the G-module $\epsilon_v\mathbb{Z}[G]$ is isomorphic to $\mathbb{Z}[J][P/G_v]/\ker(\pi)$.

To make this description explicit we consider the field diagram

If χ is a character of $J \times P/G_v$ then $\chi = \chi_1 \times \chi_2$ for some $\chi_1 \in \widehat{J}$ and $\chi_2 \in \widehat{P/G_v}$.

In addition, for any such χ the definition of ϵ_v implies that

$$\chi(\epsilon_v) \neq 0 \iff L^1_{K(v)/k,S,T}(\chi, 0) \neq 0 \iff r_S(\chi) = 1. \tag{6.3}$$

Now, as P/G_v has cardinality p and p is odd, both v and all archimedean places of k (of which there are at least two, as $k \neq \mathbb{Q}$) split completely in $K(v)^J/k$. Since also $|S| > 2$, the result of Lemma 2.1.3 implies that $r_S(\chi) > 1$, and hence that $\chi(\epsilon_v) = 0$, for any character χ that is trivial on J. This implies that $1 + \tau \in \ker(\pi)$ and hence that the natural map

$$\mathbb{Z}[J][P/G_v]/\ker(\pi) \to e_-(\mathbb{Z}[J][P/G_v]/\ker(\pi)) = e_-(\mathbb{Z}[P/G_v])/e_-(\ker(\pi))$$

is bijective.

It remains to show that $e_-(\ker(\pi))$ is generated by the element $e_-(\sum_{\gamma \in P/G_v} \gamma)$, or equivalently that if χ_{sgn} is the non-trivial character of J and χ_2 any character of P/G_v, then one has $r_S(\chi_{\text{sgn}} \times \chi_2) = 1$ if and only if χ_2 is non-trivial.

This follows from the formula of Lemma 2.1.3 and the fact that, as G_v is assumed to belong to $\mathcal{H}_1$, each non-archimedean place v' in $S \setminus \{v\}$ is such that $G_{v'} \neq G_v$ and hence has full decomposition subgroup in P/G_v. $\qquad \square$

Example 6.1.3. For primes $p > 3$ concrete extensions that satisfy all of the hypotheses that occur in the proof of Proposition 6.1.1 can be constructed as follows. Fix primes ℓ_1 and ℓ_2 with $\ell_1 \equiv \ell_2 \equiv 1 \pmod{4p}$, $\ell_1 \equiv 2 \pmod 3$, ℓ_1 is a p-th power residue modulo ℓ_2 and ℓ_2 is not a p-th power residue modulo ℓ_1. Then the extension of $k = \mathbb{Q}(\sqrt{3})$

generated by a root of unity of order $4\ell_1\ell_2$ contains subextensions K/k that satisfy all of the hypotheses described above with S taken to be the set of places of k dividing either ∞, ℓ_1 or ℓ_2, S_1 the unique place of k above ℓ_1 and v_0 any place of k above ℓ_2. The primes $p = 5$, $l_1 = 41$ and $l_2 = 101$ are one example that satisfy these congruences.

6.2   Stark elements at the boundary

The following proposition shows a family of a abelian extensions of number fields K/k and set S of places of k where $S = S_{\min}$. Thus it does not meet the conditions of Conjecture 4.2.5. However it does meet the conditions of Conjecture 5.2.4 for $a = r = 1$. Crucially, we show that it allows us to make integrality predictions beyond that can be made by looking at Rubin-Stark elements of subfields of K/k and applying the Rubin-Stark conjecture or Theorem 2.2.19.

The family of extensions considered here extends the numerical investigation of the extension $\mathbb{Q}(\sqrt{5}, \sqrt{-7}, \sqrt{-11})/\mathbb{Q}$ that is discussed in detail by Erickson in [15, 16].

Proposition 6.2.1. Let p_1, p_2 and p_3 be distinct prime numbers satisfying

$$p_1 \equiv -p_2 \equiv -p_3 \equiv 1 \pmod 4$$

and such that the respective Legendre symbols satisfy

$$\left(\frac{p_1}{p_2}\right) = \left(\frac{p_2}{p_3}\right) = -\left(\frac{p_3}{p_1}\right) = -1.$$

Let $K := \mathbb{Q}(\sqrt{p_1}, \sqrt{-p_2}, \sqrt{-p_3})$ so that $K/\mathbb{Q}$ is a Galois extension with Galois group

isomorphic to $(\mathbb{Z}/2\mathbb{Z})^3$ and

$$S = \{\infty, p_1, p_2, p_3\}.$$

Then the first, respectively second, integrality prediction in (5.4) implies that Rubin-Stark elements of subfields of K/k have stronger integrality properties than those predicted by the relevant case of the Rubin-Stark Conjecture (Conjecture 2.2.4), respectively of the refined Rubin-Stark Conjecture given in Theorem 2.2.19.

Proof. The extension $K/\mathbb{Q}$ has exactly seven quadratic subfields shown in the table below. These quadratic subfields correspond to the kernels of the seven non-trivial characters of $\widehat{G}$

We recall that a prime p of $\mathbb{Q}$ ramifies if and only if p divides the discriminant Δ. For each quadratic extension $\mathbb{Q}(\sqrt{D})$ of $\mathbb{Q}$ shown above we have $D \equiv 1 \mod p$ and thus $\Delta = D$. This, combined with the fact that K is complex implies that S is equal to all the places of k that ramify in K

If p is odd and does not divide the discriminant, then it splits if and only if the discriminant is a square modulo p. We can therefore use the Legendre symbol to work out the splitting of each of the primes in S for each of the quadratic subfields of K. These are given in the table below.

Field	Split	Ramified	Inert	Δ
$\mathbb{Q}(\sqrt{p_1})$	p_3, ∞	p_1	p_2	p_1
$\mathbb{Q}(\sqrt{-p_1 p_2})$	p_3	p_1, p_2, ∞		$-p_1 p_2$
$\mathbb{Q}(\sqrt{-p_2})$	p_3	p_2, ∞	p_1	$-p_2$
$\mathbb{Q}(\sqrt{p_2 p_3})$	∞	p_2, p_3	p_1	$p_2 p_3$
$\mathbb{Q}(\sqrt{-p_3})$	p_1	p_3, ∞	p_2	$-p_3$
$\mathbb{Q}(\sqrt{-p_1 p_3})$	p_2	p_1, p_3, ∞		$-p_1 p_3$
$\mathbb{Q}(\sqrt{p_1 p_2 p_3})$	∞	p_1, p_2, p_3		$p_1 p_2 p_3$

There is at least one prime of S that splits in each of the quadratic subfields above. Hence all the corresponding L-functions vanish to order at one at $z = 0$ except the one corresponding to $\mathbb{Q}(\sqrt{p_1})$, which vanishes to order two. Let the latter correspond to the character χ of $\widehat{G}$. Furthermore since $|S| = 4$ the L-function of the trivial character vanishes to order three. Thus the table shows that $r_S(K/\mathbb{Q}) = 1$ and $e_{1,S} = 1 - e_{\mathbf{1}} - e_{\chi}, e_{2,S} = e_{\chi}$ and $e_{3,S} = e_{\mathbf{1}}$. As each of the primes in S contribute to the order of vanishing of at least one L-function it is clear that $S^1_{\min} = S$.

We let T be any finite set of primes that is disjoint from S such that $\mathcal{O}^{\times}_{K,S,T}$ is $\mathbb{Z}$-torsion free. Then the abelian extension $K/\mathbb{Q}$ satisfies the hypotheses of Conjecture 5.2.4 (but does not satisfy the conditions of Conjecture 4.2.5 since $S = S_{\min}$). Corollary 5.2.8 therefore implies that Conjecture 5.2.4 is valid in this case.

Then the table above shows that two of the places in S have decomposition subgroups of order 4 and two have decomposition subgroups of order 2. Therefore the observations in Remark 5.3.2(iv) show that the ideals $\mathfrak{n}^1_{S,T}(K/\mathbb{Q})$ and $\mathfrak{m}^1_{S,T}(K/\mathbb{Q})$ both contain 2.

This combines with Lemma 4.4.2 to give the required result. □

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